A SPECTRAL LOWER BOUND FOR THE DIVISORIAL GONALITY OF METRIC GRAPHS

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Abstract. Let Γ be a compact metric graph, and denote by Δ the Laplace operator on Γ with the first non-trivial eigenvalue λ_1. We prove the following Yang-Li-Yau type inequality on divisorial gonality γ_{div} of Γ. There is a universal (explicit) constant C such that

$$\gamma_{\mathrm{div}}(\Gamma) \geq C\frac{\mu(\Gamma).\ell^{\mathrm{geo}}_{\min}(\Gamma).\lambda_1(\Gamma)}{d_{\max}},$$

where the volume $\mu(\Gamma)$ is the total length of the edges in Γ, $\ell^{\mathrm{geo}}_{\min}$ is the minimum length of edges in the minimal model of Γ, and $d_{\max}$ is the largest valency of points of Γ. Along the way, we also establish discrete versions of the above inequality concerning finite simple graph models of Γ and their spectral gaps.

1. Introduction

Let M be a compact Riemann surface, equipped with a metric of constant curvature in its conformal class, and denote by $\lambda_1(M)$ and $\mu(M)$ the first non-trivial eigenvalue of the Laplacian and the volume of M, respectively. Denote by $\gamma(M)$ the gonality of M, which is by definition, the minimum degree of a branched covering $M \to \mathbb{P}^1(\mathbb{C})$. It follows from the work of Yang-Yau [43] (see Li-Yau [34] for a refinement concerning the conformal invariant of Riemannian manifolds) that for any Riemann surface M, the following inequality holds

$$\lambda_1\, \mu(M) \leq 8\pi\gamma(M).$$

This result has been quite useful for applications in arithmetic geometry, for instance in the study of rational points of bounded degree on smooth proper curves over a number field K, see for example [1, 26]. Indeed a theorem of Faltings-Frey [28] implies that curves of large gonality have only finite number of points defined over finite extensions of K of bounded degree, and the Yang-Li-Yau inequality above provides a practical lower bound on the gonality in terms of geometric invariants of a complexification of the curve.

It is quite desirable to have analogous type of estimates for smooth proper curves defined over other base fields, e.g., over global function fields. A geometric object manageable to work with is the analytification of the curve over any place of the global field. For any non-Archimedean place ν of a global field K, the Berkovich analytification of the curve over the completion of the algebraic closure of K with respect to ν is a separated compact path-wise connected topological space, which deformation retracts to a finite metric graph, which is called a skeleton of the Berkovich analytification [14]. This finite metric graph captures important arithmetic and geometric aspects of the original curve, see e.g. [44, 17, 15, 42, 9, 5] for background on arithmetical and algebraic geometric properties of the skeleton of Berkovich analytic curves, and applications.

For metric graphs there are two different notions of gonality; geometric gonality, which is formulated in terms of finite harmonic maps from Γ to a metric tree T, and divisorial gonality, which is defined in terms of the divisor theory. In this paper we prove a Yang-Li-Yau type inequality for the divisorial gonality in general metric graphs. Since the divisorial gonality is a lower bound for geometric gonality of metric graphs, our results improve the previous Yang-Li-Yau type inequality of [18], and provides a generalization to arbitrary metric graphs. The result in [18] was already used there to obtain a linear lower bound in the genus for the gonality of Drinfeld modular curves, which allowed to lower bound the modularity of elliptic curves over function fields, to obtain finiteness results of rational points of bounded degree on Drinfeld modular curves, and to get uniform bounds on isogenies and torsion points of Drinfeld modules.

The proof of our result is built on the fundamental notion of tree-decomposition in graph minor theory. The link between gonality of graphs and their tree-decompositions was conjectured in [25]. This could be an indication that graph decompositions methods and minors could be useful for further understanding of algebraic geometry of metric graphs, and for potential applications in arithmetic geometry.

Our theorem can be stated as follows. Let Γ be a metric graph, and denote by γ_{div} the divisorial gonality of Γ, which is by definition, the smallest integer d such that there exists a divisor of degree d and rank one on Γ (we review all the basic definitions later in this introduction).

Let Δ be the (continuous) Laplacian of Γ, and λ_1 the first non-trivial eigenvalue of Δ. Denote by $\mu(\Gamma)$ the total length of Γ, and by $d_{\max}$ the maximum valency of points of Γ (which is the maximum degree of any simple graph model of Γ). For a simple graph model G of Γ, let $\ell_{\min}(G)$ be the minimum length of edges in G, and define $\ell_{\min}(\Gamma)$ as the maximum of $\ell_{\min}(G)$ over all simple graph models G of Γ.

Theorem 1.1. *There exists a constant C such that for any compact metric graph Γ of total length $\mu(\Gamma)$ with first non-trivial eigenvalue $\lambda_1(\Gamma)$ of the Laplacian Δ, the following holds*

$$\gamma_{\text{div}}(\Gamma) \geq C\frac{\lambda_1(\Gamma)\ell_{\min}(\Gamma)\mu(\Gamma)}{d_{\max}}.$$

Our method gives a constant C which is equal to $\frac{1}{1024}$, however, in order to simplify the presentation, we do not try to optimize the constant.
Note also that we have $\frac{1}{2}\ell_{\min}^{\text{geo}} \leq \ell_{\min} \leq \ell_{\min}^{\text{geo}}$, for $\ell_{\min}^{\text{geo}}$ defined as the minimum length of edges in the minimal model of Γ, which gives the statement in the abstract up to a change in the value of the constant.

The proof of our theorem goes as follows. Generalizing a result of J. van Dobben de Bruyn [25], we first prove a theorem which relates the divisorial gonality of a metric graph to the existence of a particular structure in the metric graph that we call a topological bramble. We then relate topological brambles to structures called strong brambles in simple graph models of Γ. We provide a dual notion for strong brambles, which is a relaxed version of tree-decomposition of graphs, that we call weak tree-decompositions. An inequality between the weak tree-width and the tree-width of finite graphs allows to use a spectral lower bound on tree-width to get a spectral lower bound for the divisorial gonality in terms of a finite simple graph model of Γ. A particular choice of a finite simple graph model G of Γ allows us to finish the proof of our theorem.

As a consequence of our methods, we get the following theorem, c.f. Section 5.2.

Theorem 1.2.
- *The divisorial gonality of a random Erdös-Rényi graph $G(n,p)$ is $\Theta(n)$ asymptotically almost surely in the range $p >> \frac{1}{n}$. More generally, the divisorial gonality of any metric graph whose model is a random $G(n,p)$ is $\Theta(n)$.*
- *The divisorial gonality of a random d-regular graph is $\Theta(n)$ asymptotically almost surely, for $d \geq 3$.*

(Recall that the notation $f = \Theta(g)$ for two functions f and g means the existence of constants $c_1, c_2 > 0$ such that $c_1 g \leq f \leq c_2 g$.)

Finally, we would like to mention two independent simultaneous works [19] and [20]. In [19] the authors prove the above mentioned conjecture of [25], and they extend it to metric graphs. Namely they show that the divisorial gonality of a metric graph Γ is lower bounded by the treewidth of any simple graph model of Γ. Combining their theorems and our methods, it is possible to improve the current constant in Theorem 1.1 by a factor of two. Moreover, in [20], the authors obtain a sharper estimate on the divisorial gonality of random Erdös-Rény graphs. More precisely, they show that the divisorial gonality of a random $G(n,p)$ is asymptotically almost surely $n - o(n)$, assuming $pn \to \infty$, with an estimate they provide on the error term.

In the rest of this introduction, we provide necessary definitions and background on algebraic geometry and harmonic analysis on metric graphs, and recall the concept of tree-decompositions. We also discuss some direct consequences of our main theorem.

1.1. Algebraic geometry of metric graphs.

In this section, we provide some background on algebraic geometry of metric graphs. More details can be found in the survey papers [2, 10], or [11, 37, 9, 5, 6].

1.1.1. *Metric graphs.* A metric graph (Γ, ℓ) is a compact connected metric space, such that for every $p \in \Gamma$ there is a non-negative integer n and a radius $r_p \in \mathbb{R}_{>0}$ and a neighborhood U_p around p which is isometric to the star shaped domain $S(n, r_p) := \{re^{2\pi i m/n} : 0 \leq r < r_p, 1 \leq m \leq n\} \subset \mathbb{C}$ equipped with the path-metric. We set $S(0, r_p) := \{0\}$. The integer n is called the valency of p and is denoted by $\mathrm{val}(p)$. A point of valency different from 2 is called an essential vertex of Γ: they are of two types, v with $\mathrm{val}(v) \geq 3$ which are called branching points, and v for which $\mathrm{val}(v) = 1$ which are called ends of Γ. We will usually drop the metric ℓ from the notation and simply refer to Γ as the metric graph. We use the notation $T_p(\Gamma)$ to denote the set of all unit tangent vectors emanating from p in Γ (which gets identified with the unit vectors $e^{2\pi i m/\mathrm{val}(p)}$ in $\mathbb{C}$ under the isometry of U_p with $S(\mathrm{val}(p), r_p)$).

For a function $f : \Gamma \to \mathbb{C}$, a point $p \in \Gamma$ and a unit tangent vector $w \in T_p(\Gamma)$, the directional derivative $d_w f(x)$ of f at p in the direction of w, which we simply call the (outgoing) slope of f at p along w, is defined by:

$$d_w f(x) = \lim_{t \downarrow 0} \frac{f(x + tw) - f(x)}{t},$$

if the limit exists. Note that the above expression makes sense by (isometrically) identifying a small enough neighborhood U_p of p with a star shaped domain $S(\mathrm{val}(p), r_p)$ in $\mathbb{C}$, and by restricting f to $U_p = S(\mathrm{val}(p), r_p)$.

Let Γ be a metric graph. A vertex set $V(\Gamma)$ is a finite subset of the points of Γ which contains all the essential points of Γ. An element of a fixed vertex set $V(\Gamma)$ is called a vertex of Γ, and the closure of a connected component of $\Gamma \setminus V(\Gamma)$ is called an edge of Γ. We denote by $E(\Gamma)$ the set of all edges of Γ with respect to the vertex set $V(\Gamma)$. The (combinatorial) graph $G = (V(\Gamma), E(\Gamma))$ is called a model of Γ. A model G of Γ is simple if there is no loop edge in E. Since Γ is a metric graph, we can associate to each edge e of a model $G = (V, E)$ its length $\ell(e) \in \mathbb{R}_{>0}$.

The model $G = (V, E)$ of a metric graph Γ with V the set of all essential points of Γ is called the minimal model of Γ. In the case that Γ is a circle, and so there are no essential vertices, we define the minimal model to be a vertex (any point of Γ) with a loop edge. We denote by $\ell^{\mathrm{geo}}_{\min}$ the minimum length of the edges in the minimal model of Γ. The volume $\mu(\Gamma)$ of Γ is the sum of the edge lengths in any model G of Γ. We denote by $d_{\max}$ the maximum valency of points of Γ.

1.1.2. *Divisor theory on metric graphs and divisorial gonality.* We recall some basic definitions concerning the divisor theory of metric graphs and the notion of divisorial gonality. See [11, 37] for more details.

For a metric graph Γ, let $\mathrm{Div}(\Gamma)$ be the free abelian group on points of Γ. An element D of $\mathrm{Div}(\Gamma)$ is called a divisor on Γ and can uniquely be written as

$$D = \sum_{v \in \Gamma} a_v(v), \text{ with } a_v \in \mathbb{Z},$$

where all but finitely many a_v are zero. The degree of D is $\deg(D) = \sum_{v\in\Gamma} a_v$. A divisor D is effective if $D(v) \geq 0$ for all $v \in \Gamma$.

The set of points v for which a_v is nonzero is called the support of D and is denoted by $\mathrm{supp}(D)$.

A rational function on Γ is a continuous piecewise linear function on Γ whose slopes are all integers. The set of all rational functions on Γ is denoted by $R(\Gamma)$. The order of a rational function f at a point p of Γ, denoted by $\mathrm{ord}_p(f)$, is the sum of the slopes of f along the tangent directions in $T_p(\Gamma)$. As f is piecewise linear, and Γ is compact, the order of f is zero on all but finitely many points of Γ, and one gets a map

$$\mathrm{div} : R(\Gamma) \to \mathrm{Div}(\Gamma), f \mapsto \sum_p \mathrm{ord}_p(f)(p).$$

A divisor in the image of div is called a principal divisor. Two divisors, D and D' are called linearly equivalent, written $D \sim D'$, if they differ by a principal divisor, i.e., there is a rational function such that $D = \mathrm{div}(f) + D'$. The (complete) linear system $|D|$ of a divisor D is defined to be the set of all effective divisors which are linearly equivalent to D:

$$|D| := \{E \in \mathrm{Div}(\Gamma) : E \geq 0, E \sim D\}.$$

We denote by $R(D) := \{f \in \mathrm{R}(\Gamma) : D + \mathrm{div}(f) \geq 0\}$ the "set of all global sections of D". Note that $R(D)$ is closed under addition by constants and under taking maximum, i.e., for

$f, g \in R(D)$ and $c \in \mathbb{R}$, one has $c + f \in R(D)$ and $\max(f, g) \in R(D)$, in other words, $R(D)$ is a so called tropical semi-module.

The rank of a divisor D, denoted by $r(D)$ is defined by

$$r(D) := \min_{\{E: E \geq 0, |D-E| = \emptyset\}} \deg(E) - 1.$$

The divisorial gonality $\gamma_{div}(\Gamma)$ of a metric graph Γ is defined by

$$\gamma_{div}(\Gamma) := \min\{d : \text{ there exists a } D \in \mathrm{Div}(\Gamma), \text{ with } \deg(D) = d \text{ and } r(D) = 1\}.$$

1.1.3. *Reduced divisors.* Basic technical tool in the study of divisors on metric graphs are reduced divisors that we recall now.

A closed and connected subset X of Γ is called a cut in Γ. We denote by ∂X, the boundary of X: the finite set of points of X which are in the closure of the complement of X in Γ. For a point $p \in \partial X$, we denote by $\deg_X^{out}(p)$ the number of tangent directions in $T_p(\Gamma)$ leaving X at p; in other words, this is the maximum number of disjoint segments in $U_p \setminus X$ whose closures have p as an endpoint, where U_p is a neighborhood of p in Γ.

A boundary point p of a cut X is called saturated with respect to a divisor $D \in \mathrm{Div}(\Gamma)$ if $\deg_X^{out}(p) \leq D(p)$.

A divisor D is called reduced with respect to a fixed point $p_0 \in \Gamma$ if it satisfies the following properties:

(1) for all $p \neq p_0, D(p) \geq 0$,
(2) for every cut $X \subset \Gamma$ such that $p_0 \notin X$, there exist a $p \in \partial X$ which is not saturated.

Every divisor on a metric graph is equivalent to a unique p_0-reduced divisor, see e.g., [3, Theorem 2].

Note that if the rank of a divisor D is non-negative, then for any $p \in \Gamma$ the reduced divisor D_p is effective.

1.1.4. *Geometric gonality of metric graphs.* We refer to [5] for standard definitions regarding the morphisms between metric graphs and the corresponding tropical curves.

Recall that a tropical curve C is called d-gonal if there exists a tropical morphism $C \to \mathbb{TP}^1$ of degree d. A metric graph Γ has geometric gonality d, if the tropical curve associated to Γ is d-gonal, and d is the smallest integer satisfying this condition. The geometric gonality of a metric graph is denoted by $\gamma_{gm}(\Gamma)$.

It is easy to see that the fibers of any finite harmonic morphism from a metric graph Γ to a finite tree are linearly equivalent, and define a linear equivalence class of divisors on Γ of rank at least one. It thus follows that

$$\gamma_{gm}(\Gamma) \geq \gamma_{\mathrm{div}}(\Gamma)$$

for any metric graph Γ. Our Theorem 1.1 thus provides a spectral lower bound for the geometric gonality of a metric graph.

1.1.5. *Specialization of divisors from curves to metric graphs.* Let X be a smooth proper curve over an algebraically closed complete non-Archimedean field K with a non-trivial valuation. Recall (c.f. [13], see also [14, 23, 24, 41]) that a semistable vertex set of the Berkovich analytic curve X^{an} is a finite subset V of type-2 points of X^{an} such that $X^{\mathrm{an}} \setminus V$ is a disjoint union of open balls and (a finite number of) open annuli. (Semistable vertex sets are in bijection with semistable models of X over the valuation ring of K.) To each semistable vertex set, a skeleton $\Sigma(X,V)$ of the Berkovich curve X^{an} is associated, which is a finite metric graph. These metric graphs are tropically equivalent, and thus varying the semistable vertex set defines a tropical curve C associated to X [6].

Fixing a semistable vertex V for X^{an}, one gets a deformation retraction $\tau : X^{\mathrm{an}} \to \Sigma(X,V)$. Identifying $X(K)$ with points of type 1 on X^{an}, this induces a morphism $\tau_* : \mathrm{Div}(X) \to \mathrm{Div}(\Sigma(X,V))$ which is called the specialization map, and which coincides with the definition of the specialization map without reference to the analytification in [17, 44, 9].

Let X be a smooth proper curve over K and let Γ be a metric graph associated to X. Baker's specialization lemma [9] states that for any divisor D on X one has $r(D) \leq r(\tau_*(D))$. (Formulated in terms of the analytification of the curve, the statement is a consequence of the Poincaré-Lelong formula [13, 42, 44], see [4].)

In particular, it follows that the gonality of a smooth proper curve X over a non-Archimedean field K is bounded below by the divisorial gonality of the corresponding metric graph. Applying our main theorem, we get

Theorem 1.3. *Let X be a smooth proper curve over a non-Archimedean field K, and let Γ be a metric graph associated to X. We have*

$$\gamma(X) \geq C\frac{\mu(\Gamma)\ell_{\min}(\Gamma)\lambda_1(\Gamma)}{d_{\max}}.$$

Here C is the constant provided by Theorem 1.1.

1.2. **Harmonic analysis on metric graphs.** We recall the definitions of the Laplacian on a metric graph, and refer to [44, 12, 27] for more details on harmonic analysis on metric graphs.

For a metric graph Γ with a model $G = (V,E)$, one has a Lebesgue measure on each edge which gives rise to a well-defined Lebesgue measure on Γ denoted by dx. The Lebesgue measure does not depend on the choice of the model.

The space $\mathrm{Zh}(\Gamma)$ is the space of all continuous functions $f : \Gamma \to \mathbb{R}$ where f is piecewise C^2 and $f''(x) \in L^1(\Gamma)$. The subspace $\mathrm{Zh}_0(\Gamma) \subset \mathrm{Zh}(\Gamma)$ consists of all functions f which satisfy $\int_\Gamma f dx = 0$. The Laplacian Δ is the measure valued operator on $\mathrm{Zh}(\Gamma)$ whose value on a function $f \in \mathrm{Zh}(\Gamma)$ is the measure

$$\Delta(f) := -f''(x)dx - \sum_{p\in\Gamma}\left(\sum_{w\in T_p(\Gamma)} d_w f(p)\right)\delta_p(x),$$

where δ_p the Dirac measure at the point p.

The eigenvalues of Δ form a discrete subset $\lambda_0 = 0 < \lambda_1 < \lambda_2 < \dots$ of $\mathbb{R}_{\geq 0}$. The behavior of $\lambda_i(\Gamma)$ under the scaling of the edge lengths of a model G is easily seen to be as follows: if the metric graph Γ' with the same model G is obtained from Γ by scaling the length in Γ of each edge $e \in E(G)$ with a factor $\beta \in \mathbb{R}_{>0}$, then $\lambda_i(\Gamma') = \frac{1}{\beta^2}\lambda_i(\Gamma)$ [12].

The smallest non-zero eigenvalue of Δ, $\lambda_1(\Gamma)$, has the following variational characterization

$$\lambda_1(\Gamma) = \inf_{f\in\mathrm{Zh}_0(\Gamma)} \frac{\int_\Gamma |f'|^2 dx}{\int_\Gamma f^2 dx}.$$

1.3. **Tree-decompositions.** Let $G = (V, E)$ be a connected graph. A tree-decomposition of G is a pair $(T, \mathcal{X})$ where T is a finite tree on a set of nodes I, and $\mathcal{X} = \{X_i : i \in I\}$ is a collection of subsets of V, subject to the following three conditions:

(1) $V = \cup_{i\in I} X_i$,
(2) for any edge e in G, there is a set $X_i \in \mathcal{X}$ which contains both end-points of e,
(3) for any triple i_1, i_2, i_3 of nodes of T, if i_2 is on the path from i_1 to i_3 in T, then $X_{i_1} \cap X_{i_3} \subseteq X_{i_2}$, or equivalently, for any vertex v in G, the set of nodes i of T with $v \in X_i$ form a connected subtree of T.

Note that the point (3) in the above definition simply means that the subgraph of T induced by all the nodes i which contain a given vertex v of the graph G is connected. (The vertices of T are called nodes in order to distinguish them from the vertices of G.)

The width of a tree-decomposition $(T, \mathcal{X})$ is defined as $w(T, \mathcal{X}) = \max_{i\in I} |X_i| - 1$. The tree-width of G, denoted by $tw(G)$, is the minimum width of any tree-decomposition of G.

There is a useful duality theorem concerning the tree-width which allows in practice to bound the tree-width of graphs. The dual notion for tree-width is called bramble (as named by Reed [38]): a bramble in a finite graph $G = (V, E)$ is a collection of connected subsets of V (i.e., those inducing a connected subgraph) such that the union of any two of these subsets form again a connected subset of G. The order of a bramble $\mathcal{F}$ in G is the minimum size of a hitting set for $\mathcal{F}$, i.e., the minimum size of a subset of vertices which has non-empty intersection with any element of $\mathcal{F}$. The bramble number of G denoted by $bn(G)$ is the maximum order of any bramble in G.

Theorem 1.4 (Seymour-Thomas [40])**.** *For any graph G, $tw(G) = bn(G)$.*

To give an example of the applications of the duality theorem, let H be an $n \times n$ grid. It is easy to see that $bn(H) = n$ by taking the bramble consisting of all crosses in the grid. This shows that grid graphs can have large tree-width, and so the tree-width can take arbitrary large values on planar graphs.

Duality theorems are part of Robertson-Seymour graph minor theory [39]. For a discussion of the different duality theorems and diverse generalizations see [8, 22].

2. Topological brambles and divisorial gonality

In this section we provide a lower bound on the divisorial gonality in terms of a topological variant of the notion of bramble in finite graphs.

Definition 2.1 (Topological bramble)**.** Let Γ be a metric graph. A topological bramble (or simply top-bramble) in Γ is a finite family $\mathcal{F}$ of non-empty closed connected metric subgraphs of Γ such that any two elements X and Y in $\mathcal{F}$ have a non-empty intersection. The order of a top-bramble $\mathcal{F}$ is the minimum size of a hitting set for $\mathcal{F}$, i.e., the minimum size of a set $S \subset \Gamma$ such that $S \cap X \neq \emptyset$ for any $X \in \mathcal{F}$. The topological bramble number of Γ denoted by $tbn(\Gamma)$ is the maximum order of any topological bramble on Γ.

Theorem 2.2. *Let Γ be a metric graph. The divisorial gonality of Γ is lower bounded by its top-bramble number $tbn(\Gamma)$.*

Remark 2.3. In the next section, we will provide a link between topological brambles and a special kind of brambles in finite simple graph models of Γ, called strong brmables. In view of this link, this theorem can be seen as a generalization of a theorem of J. van Dobben de Bruyn [25] to metric graphs.

Proof. In order to prove the theorem, we need to show that there cannot exist any divisor of degree k and rank at least one in Γ provided that there exists a top-bramble $\mathcal{F}$ in Γ of order $k+1$. For the sake of a contradiction, let $\mathcal{F}$ be a top-bramble of order $k+1$ for Γ and assume there exists a divisor D with $\deg(D) = k$ and $r(D) \geq 1$. In particular, for any point x of Γ, the reduced divisor D_x has x in its support.

Consider the linear equivalence class $|D|$ of D. By replacing D by another divisor $E \in |D|$ if necessary, we can assume that D is effective, and, in addition, that D is a divisor in the linear equivalence class $|D|$ whose support $\mathrm{supp}(D)$ has the maximum number of non-empty intersections $\mathrm{supp}(D) \cap X$ with elements X in $\mathcal{F}$.

Since $\mathrm{ord}(\mathcal{F}) = k+1 > |\mathrm{supp}(D)|$, there exists an element X in $\mathcal{F}$ such that $X \cap \mathrm{supp}(D) = \emptyset$. Let v be an arbitrary point of X, and consider the unique v-reduced divisor $D_v \sim D$. Let f be a rational function on Γ which gives $\mathrm{div}(f) + D = D_v$. Since $r(D) \geq 1$, we have $v \in \mathrm{supp}(D_v)$, while by the choice of X, we have $v \notin \mathrm{supp}(D)$.

The structure of the proof is as follows: we consider a specific path D_t in $|D|$ (parameterized by t) from D_v to D. Since $\mathrm{supp}(D_v) \cap X \neq \emptyset$ while $\mathrm{supp}(D) \cap X = \emptyset$, by compactness of X, there exists a maximum value h of t such that $\mathrm{supp}(D_h) \cap X \neq \emptyset$. Denote by $X_1, \dots, X_n$ all the different elements of $\mathcal{F}$ which have non-empty intersections with $\mathrm{supp}(D)$. Note that the support of D_h intersects X, and X is not among the X_i's, so by the choice of D, as the one maximizing the number of non-empty intersections with elements in $\mathcal{F}$, there should exist an element $Y = X_i$ such that $Y \cap \mathrm{supp}(D_h) = \emptyset$. We will show that $Y \cap X = \emptyset$, which will be in contradiction with the definition of a topological bramble.

For any real number t, define a function f_t on Γ by $f_t(x) := \max(f(x), t)$ for any $x \in \Gamma$. Since both f, and the constant function t lie in $R(D)$ and $R(D)$ is a tropical semi-module it follows that $f_t \in R(D)$ for all t. In other words, the divisor $D_t := D + \mathrm{div}(f_t)$ is in $|D|$. Denote now by $\min f$ and $\max f$ the minimum and maximum value of f on Γ respectively. Define the map $[\min f, \max f] \to |D|$ which assigns to any point t in the interval $[\min f, \max f]$, the divisor $D + \mathrm{div}(f_t)$. Since $f_{\min f} = f$ and $f_{\max f}$ is the constant function $\max f$, this defines a path in $|D|$ from D_v to D.
For any real number t denote by $f^{-1}(t) := \{x \in \Gamma \mid f(x) = t\}$ the level set at t, and define the upper level set $\Gamma_t := \{x \in \Gamma \mid f(x) \geq t\}$.

Next, we prove the following claim.

Claim (1)**.** *For any real number t, we have $\partial f^{-1}(t) \subseteq \mathrm{supp}(D_t)$.*

Proof of Claim (1): First note that writing $f = (f - f_t) + f_t$, we have $D_v = D + \mathrm{div}(f) = \mathrm{div}(f - f_t) + \mathrm{div}(f_t) + D = \mathrm{div}(f - f_t) + D_t$, which shows that $f - f_t \in R(D_t)$. Note also that $f - f_t$ is constant on Γ_t and coincides with $f - t$ outside Γ_t. Consider now a point $x \in \partial f^{-1}(t)$, in other words, f does not restrict to a constant function on any neighborhood of x. If $x \in \partial \Gamma_t$ since f (and so $f - f_t$) is strictly decreasing along any out-going branch e from Γ_t at x, the slope of $f - f_t$ at x along e is strictly negative. Since $f - f_t \in R(D_t)$, this shows that $x \in \mathrm{supp}(D_t)$, and the claim follows. If $x \notin \partial\Gamma_t$, since f is not a constant function locally at x, then f is strictly increasing along one of the branches at x, and so again, since f_t takes its minimum value at x, D is effective, and $D_t = D + \mathrm{div}(f_t)$, we conclude that $x \in \mathrm{supp}(D_t)$, and the claim follows.

Consider now the path in $|D|$ defined by all the divisors $D_t = D + \mathrm{div}(f_t)$, for the values of $t \in [\min f, \max f]$. Let h be the maximum value in $[\min f, \max f]$ such that $\mathrm{supp}(D_h) \cap X \neq \emptyset$.

Claim (2). *We have $X \cap \Gamma_h \subset f^{-1}(h)$.*

Proof of Claim (2): The claim trivially holds if f is constant on X. So suppose that $f_{|X}$ is not constant. It will be enough to show that $\max f_{|X} = h$. First, note that for any $t > \max f_{|X}$, since $\mathrm{supp}(D_t)\backslash\mathrm{supp}(D) \subseteq \Gamma_t$, and $\Gamma_t \cap X = \mathrm{supp}(D) \cap X = \emptyset$, the intersection $\mathrm{supp}(D_t) \cap X$ is empty. This shows that $h \leq \max f_{|X}$. Now, since $-f \in R(D_v)$ it follows from [3, Lemma 7] that the minimum of f is taken at v. Hence, any path P in X from v to a point in $f^{-1}(\max f_{|X}) \cap X$ intersects $\partial f^{-1}(\max f_{|X})$, which combined with Claim (1), implies that $\mathrm{supp}(D_{\max f_{|X}})$ intersects X. This shows that $\max f_{|X} = h$, and the claim follows.

Let now Y be an element of $\mathcal{F}$ with the property that $Y \cap \mathrm{supp}(D_h) = \emptyset$ while $Y \cap \mathrm{supp}(D) \neq \emptyset$. The following claim implies that $Y \cap X = \emptyset$, which contradicts the definition of a topological bramble, and our theorem follows.

Claim (3). *We have $Y \subseteq \Gamma_h \setminus f^{-1}(h)$, in other words, $f_{|Y} > h$.*

We make the following observation which will be used in the proof of the above claim.

Observation. Let y be a point in the intersection $Y \cap \mathrm{supp}(D)$ (which is by assumption non-empty). Since $y \notin \mathrm{supp}(D_h)$, the point y is not a local minimum of f_h, i.e., not all the slopes of f_h along the adjacent branches at y can be non-negative. In particular, Y cannot be entirely contained in $f^{-1}(h)$.

Proof of Claim (3): First, note that $Y \cap \Gamma_h \neq \emptyset$: indeed, otherwise, this would mean that the restriction $f|_Y < h$, and so any point of Y would be a local minimum for f_h, contradicting the above observation. Second, we show that $Y \subset \Gamma_h$, i.e., $\min f_{|Y} \geq h$. Otherwise, there would exist a point $z \in Y$ such that $f(z) < h$. By connectivity of Y, and by taking a path $P \subset Y$ from z to a point in $\Gamma_h \cap Y$, we would obtain that $\partial f^{-1}(h) \cap Y \neq \emptyset$, which by Claim (2) would lead to $\mathrm{supp}(D_h) \cap Y \neq \emptyset$, contradicting the choice of Y. Combining this with the above observation, since Y is not entirely contained in $f^{-1}(h)$, we infer that $Y \cap (\Gamma_h \setminus f^{-1}(h)) \neq \emptyset$. We finish the proof of Claim (3) by showing that $Y \cap f^{-1}(h) = \emptyset$: suppose there is an element in $Y \cap f^{-1}(h)$, then, by connectivity of Y, there would exist a path P in Y from that element to a point in $Y \cap (\Gamma_h \setminus f^{-1}(h))$. This path would contain a point in $\partial f^{-1}(h)$, which by Claim (2) would imply that $\mathrm{supp}(D_h) \cap Y \neq \emptyset$, contradicting again the choice of Y. This finishes the proof of the claim, and our theorem follows. □

3. Topological brambles in metric graphs vs strong brambles in graphs

Let $G = (V, E)$ be a finite simple graph on vertex set V and with edge set E. We gave the definition of a bramble in Section 1.3, and mentioned that it provides a dual notion for tree-width. A strong bramble is a specific kind of bramble in a finite graph defined as follows:

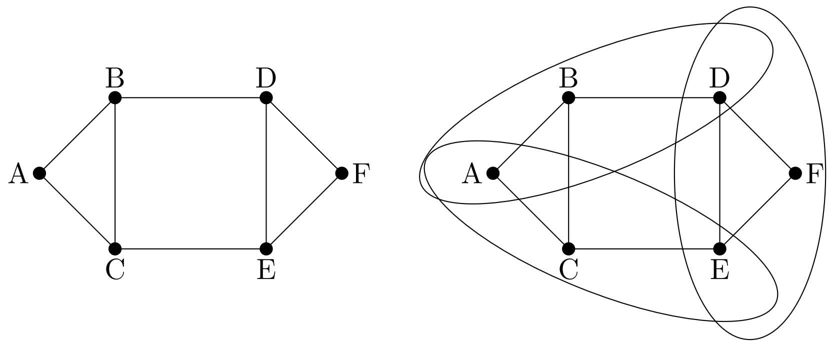

FIGURE 1. An example of a graph together with a strong bramble $\mathcal{F} = \{\{A, B, D\}, \{A, C, E\}, \{D, E, F\}\}$. The minimum hitting set is $\{D, E\}$, and the strong bramble number is 2.

Definition 3.1 (Strong bramble)**.** A strong bramble in G is a finite collection $\mathcal{F}$ of connected subsets of G such that for any two elements B and C, one has $B \cap C \neq \emptyset$. Figure 1 illustrates an example of a graph with a strong bramble.

Note in particular that for any two elements B and C in $\mathcal{F}$, the union $B \cup C$ is obviously connected. In other words, a strong bramble is a bramble.

The order of a strong bramble is its order as a bramble, i.e., the minimum size of a hitting set for $\mathcal{F}$ in V. The strong bramble number of a finite graph G, denoted by $sbn(G)$, is the maximum order of any strong bramble in G.

Let Γ be a metric graph with a simple graph model $G = (V, E)$. For any subset X of V, we denote by $G[X]$, resp. $\Gamma[X]$, the subgraph of G, resp. the metric subgraph of Γ, defined by X: it contains all the edges of G, resp. all the metric edges of Γ, which connect a vertex in X to another vertex of X. (Note that in general neither $G[X]$ nor $\Gamma[X]$ are connected.)

The link between strong and topological brambles is provided in the following proposition.

Proposition 3.2. (1) *Let Γ be a metric graph with a simple graph model $G = (V, E)$. For any strong bramble $\mathcal{F}$ in G, the collection $\Gamma[\mathcal{F}] = \{\Gamma[X] \mid X \in \mathcal{F}\}$ is a topological bramble for Γ of the same order.*

(2) *For any topological bramble $\mathcal{F}_\Gamma$ for Γ, there exists a simple graph model G of Γ and a strong bramble $\mathcal{F}$ for G such that $\mathcal{F}_\Gamma = \Gamma[\mathcal{F}]$.*

(3) *The topological bramble number of Γ is given by $\sup_G sbn(G)$ where the supremum is over all simple graph models G of Γ and $sbn(G)$ is the strong bramble number of G.*

In addition, the topological bramble number of any metric graph is finite, in other words, the supremum is a maximum.

Proof. (1) Let $\mathcal{F}$ be a strong bramble for a simple graph model G of Γ. Obviously, for any $X \in \mathcal{F}$, the subset $\Gamma[X]$ is a connected metric subgraph of Γ. In addition, for any two elements $X, Y \in \mathcal{F}$, we have $\Gamma[X] \cap \Gamma[Y] \supset X \cap Y \neq \emptyset$ which shows that $\Gamma[\mathcal{F}]$ is a topological bramble for Γ. To see that $\mathrm{ord}(\Gamma[\mathcal{F}]) = \mathrm{ord}(\mathcal{F})$, note first that since any hitting set for $\mathcal{F}$ is also a hitting set for $\Gamma[\mathcal{F}]$, we obviously have $\mathrm{ord}(\Gamma[\mathcal{F}]) \leq \mathrm{ord}(\mathcal{F})$. To prove the equality, it is enough to show there exists a hitting set S for $\Gamma[\mathcal{F}]$ of minimum size such that $S \subset V(G)$ (S

is then a hitting set for $\mathcal{F}$). Let S be a hitting set for $\Gamma[F]$ of minimum size. For any point x in $S \setminus V$ which lies in the interior of an edge e_x of G, choose one of the two extremities v_x of e_x and replace x with v_x to obtain a set $\tilde{S}$ of the same size $|S|$. Note that a metric subgraph of the form $\Gamma[X]$ in $\Gamma[\mathcal{F}]$ which contains the point $x \in S \setminus V$ contains both the end-points of the edge e_x, and has non-empty intersection with $\tilde{S}$. An element of $\Gamma[\mathcal{F}]$ which intersects $S \cap V$ has also non-empty intersection with $\tilde{S}$. It follows that $\tilde{S}$ is a hitting set for $\Gamma[\mathcal{F}]$, and the claim follows.

(2) Let $\mathcal{F}_\Gamma$ be a topological bramble for Γ, and let $G_0 = (V_0, E_0)$ be a simple graph model for Γ. Consider the set $\cup_{X \in \mathcal{F}_\Gamma} \partial X$ of all points of Γ which lie on the boundary of a set $X \in \mathcal{F}_\Gamma$ and define $V_1 = V_0 \cup \cup_{X \in \mathcal{F}_\Gamma} \partial X$. Consider the model $G_1 = (V_1, E_1)$ of Γ defined by V_1. Finally, subdivide each edge e of G_1 by adding a new vertex in the middle of e to obtain a model $G = (V, E)$ of Γ. Let $\mathcal{F} = \{V \cap X \mid X \in \mathcal{F}_\Gamma\}$. It is not hard to see that $\mathcal{F}$ is a strong bramble and $\Gamma[\mathcal{F}] = \mathcal{F}_\Gamma$.

(3) The equality of the topological bramble number and the supremum $\sup_G sbn(G)$, for G a model of Γ, formally follows from the two assertions (1) and (2). Finiteness of the topological bramble number of a metric graph is a consequence of Theorem 2.2, since by the Brill-Noether bound (or simply Riemann-Roch) for metric graphs, the divisorial gonality of any metric graph is finite. □

4. Strong brambles and weak tree-decompositions

In this section, we provide the dual notion to strong brambles: we introduce a new class of graph decompositions that we call weak tree-decompositions. We will then show that strong brambles of given order are the dual obstructions for the existence of weak tree-decompositions of given order, see Theorem 4.3 below for a precise formulation.

Let $G = (V, E)$ be a connected graph. A weak tree-decomposition of G, illustrated in Figure 2, is a pair $(T, \mathcal{S})$ where T is a finite tree on a set of nodes I, and $\mathcal{S} = \{S_i : i \in I\}$ is a collection of subsets of V, subject to the following three conditions:

(1) $\cup_{i \in I} S_i = V$,
(2) for any edge e in G with extremities v and w, there is an edge $\{i, j\}$ in T such that $\{v, w\} \subset S_i \cup S_j$.
(3) for any vertex v in G, the set of nodes i of T with $v \in S_i$ form a connected subtree of T.

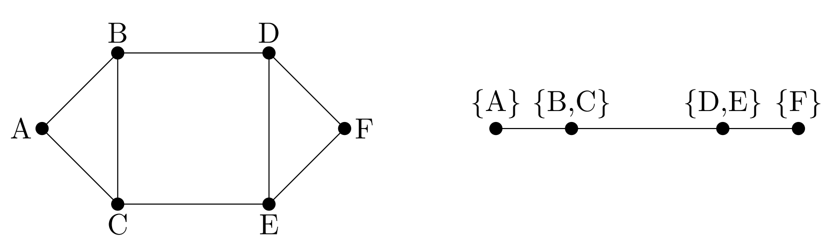

Figure 2. The same graph as above with a weak tree-decomposition of width 2.

For any vertex $v \in V$, we denote by T_v the (connected) subtree of T which is induced by all the nodes i of T with $v \in S_i$.

Note that the only difference with the usual definition of a tree-decomposition is in point (2) where we impose a weaker condition. In particular, it might happen that an edge e of G is not necessarily contained in any set $S_i \in \mathcal{S}$.

The width of a weak tree-decomposition $(T, \mathcal{S})$ is defined as $w(T, \mathcal{S}) = \max_{i \in I} |S_i|$. The weak tree-width of G, denoted by $wtw(G)$, is the minimum width of any weak tree-decomposition of G. Note that, similar as in the definition of tree-width, the weak tree-width is defined in such a way that trees themselves have weak tree-width equal to one.

Lemma 4.1. *Let $(T, \mathcal{S})$ be a weak tree-decomposition of a graph $G = (V, E)$. For any two adjacent vertices u and v, $T_u \cup T_v$ is connected. In particular, for any connected subset X of G, the union $\cup_{v \in X} T_v$ is a connected subtree of T.*

Proof. By property (2) of a weak tree-decomposition, the edge $\{u, v\}$ is contained in the union of two sets S_i and S_j in $\mathcal{S}$ for an edge $\{i, j\}$ of T. This means that either $T_u \cap T_v \neq \emptyset$ or i and j do not belong to the same tree among T_u and T_v. In any of the two cases, $T_u \cup T_v$ is connected. The second statement obviously follows by connectivity of X and the first assertion. $\square$

The following proposition is straightforward from the definition.

Proposition 4.2.

- *Let $(T, \mathcal{S})$ be a weak tree-decomposition of a graph $G = (V, E)$, and let U be a subset of V. The restriction of $(T, \mathcal{S})$ to U defined by replacing any S_i in the decomposition with $S_i \cap U$ is a weak tree-decomposition of $G[U]$.*
- *Let $(T, \mathcal{S})$ be a weak tree-decomposition of a graph $G = (V, E)$, and i and t two nodes of T. Let $v \in S_t$, and for any node j on the unique path between i and t, define $S'_j = S_j \cup \{v\}$. For all the other nodes j of T, define $S'_j = S_j$. Then $(T, \mathcal{S}' = \{S'_j\})$ is a weak tree-decomposition of G.*

Proof. The first assertion is obvious from the definition. For the second one, we only need to verify the property (3) for the vertex v. The tree T'_v associated to $(T, \mathcal{S}')$ is the union of T_v and the unique path in T between i and t. Since t also belongs to T_v, T'_v is connected. $\square$

The following theorem is a duality theorem, in the spirit of the duality theorems in graph minor theory, which relates strong brambles to weak tree-decompositions. It does not seem to follow from the generalized forms of duality established in [8, 22], so we provide a proof.

Theorem 4.3. *A finite graph G has a weak tree-decomposition of width k if and only if there is no strong bramble of order strictly larger than k in G. In other words, $wtw(G) = sbn(G)$.*

The proof mimics the well-known proof of the duality theorem between tree-width and bramble order [21, 40], and is based on the use of Menger's theorem in graph theory. We thus start by recalling the statement of Menger's theorem.

Let $G = (V, E)$ be a finite simple graph and consider two subsets $X, Y \subseteq V$. An (X, Y)-separator in G is a subset $S \subset V$ such that there is no path in $G \setminus S$ between any point in $X \setminus S$ and $Y \setminus S$.

The connectivity of the pair (X, Y) is the maximum number of vertex disjoint path between X and Y (a point $v \in X \cap Y$ is considered as a path between X and Y of length zero).

Consider a set of k vertex disjoint paths between X and Y. Obviously any (X,Y) separator in G should contain at least one point on each of the k paths. In other words, the size of any (X,Y) separator is at least the connectivity of the pair (X,Y) in G. Menger's theorem asserts the equality of these two quantities.

Theorem 4.4 (Menger [36])**.** *The connectivity of a pair (X,Y), $X,Y \subseteq V$, is equal to the minimum size of an (X,Y)-separator in G.*

We are now ready to prove Theorem 4.3.

Proof of Theorem 4.3. We first show that $sbn(G) \leq wtw(G)$. Let $(T,\mathcal{S})$ be a weak tree-decomposition of G. Consider a strong bramble $\mathcal{F}$ for G. We show that there exists a node i such that S_i intersects any element $X \in \mathcal{F}$, i.e., S_i is a hitting set for $\mathcal{F}$. This proves the claimed inequality. For the sake of a contradiction, suppose this is not the case. This means that for any node i in T, there exists $X_i \in \mathcal{F}$ such that $X_i \cap S_i = \emptyset$, in other words, T_v does not contain node i for any $v \in X_i$. Thus, the union $T_i = \cup_{v\in X_i} T_v$ does not contain i. In addition, by Lemma 4.1, T_i is a connected subtree of T. This implies that T_i is entirely included in one of the connected components of $T \setminus i$. Let j be the unique node of this connected component which is adjacent to i, and give the orientation ij to the edge $\{i,j\}$ of T. Doing this for any node i of T, we give the size of $V(T)$ orientations to the edges of T. Since T contains $|V(T)|-1$ edges, there exists an edge $\{i,j\}$ which gets both the orientations ij and ji. This precisely means that the two trees T_i and T_j are disjoint, which implies that $X_i \cap X_j = \emptyset$, contradicting the defining property of a strong bramble.

To prove $wtw(G) \leq sbn(G)$, we show the existence of a weak tree-decomposition of G of order at most $k := sbn(G)$. The proof goes by an induction procedure as follows. We claim that for any graph G and for any strong bramble $\mathcal{F}$ in G, there exists a weak tree-decomposition $(T,\mathcal{S})$ such that for any node i of T, either $|S_i| \leq k$ or, otherwise if $|S_i| > k$, then S_i is not a hitting set for $\mathcal{F}$. The proof of this latter statement is by a reverse induction on $|\mathcal{F}|$ for any strong bramble $\mathcal{F}$ in G. Once this has been proved, for the empty strong bramble $\mathcal{F} = \emptyset$, since every set is a hitting set for $\mathcal{F}$, we get a weak tree-decomposition of G of width at most k, and the theorem follows.

Since there is no strong bramble of size larger than $2^{|G|}$ in G, the base of the induction holds trivially for strong brambles of size $2^{|G|}+1$ (which do not exist). Suppose that the statement is true for an integer N and any strong bramble $\mathcal{F}$ of size $|\mathcal{F}| = N$ in G, we show that it also holds for all strong brambles of size $N-1$.

For the sake of a contradiction, suppose that the statement does not hold for $N-1$. Let $\mathcal{F}$ be a strong bramble with $|\mathcal{F}| = N-1$ for which the statement does not hold. Consider a hitting set S of $\mathcal{F}$ in G of size equal to the order of $\mathcal{F}$. Note that $|S| \leq k$.

Let $C_1,\dots,C_l$ be all the connected components of $G \setminus S$, and for any integer $1 \leq a \leq l$, consider the induced subgraph G^a of G with vertex set $S \cup V(C_a)$. We will show that there exists a weak tree-decomposition $(T^a, \mathcal{S}^a = \{S^a_j\}_{j\in V(T^a)})$ of G^a such that

- (i) there is a node i_a in T^a such that $S^a_{i_a} = S$;
- (ii) for any node j of T^a with $|S^a_j| > k$, S^a_j is not a hitting set for $\mathcal{F}$.

Once this statement is proved, we obtain a weak tree-decomposition of the whole graph by gluing the trees T^a on the node i_a to obtain a tree T, and by defining for any node j of the tree T, which is thus a node of one of the trees T^a for some a, $S_j := S^a_j$. Since $S^a_{i_a} = S$, these sets are well-defined, and it is easy to verify that they form a weak tree-decomposition of the whole graph G. In addition, by Property (ii) above, any S_j of size strictly larger than k belongs is not a hitting set for $\mathcal{F}$. Thus, $(T, \mathcal{S})$ is a weak tree-decomposition for G which satisfies the required property with respect to $\mathcal{F}$, and this leads to a contradiction with our choice of $\mathcal{F}$.

We are thus left to prove the above claim. Consider one of the graphs G_a. There are two cases to consider:

(1) Either C_a is not a hitting set for $\mathcal{F}$.

In this case, we get a weak tree-decomposition $(T^a, \mathcal{S}^a)$ of G^a by taking a path of length two T^a on two vertices i_a and j, and by defining $S^a_{i_a} = S$ and $S^a_j = C_a$. Obviously (i) and (ii) are verified.

(2) Or C_a is a hitting set for $\mathcal{F}$.

Since C_a is connected, this precisely means that $\mathcal{F}_a = \mathcal{F} \cup \{C_a\}$ is a strong bramble in G. Note that S is a hitting set for $\mathcal{F}$ and $S \cap C_a = \emptyset$, so $\mathcal{F}_a \neq \mathcal{F}$, and thus $|\mathcal{F}_a| = |\mathcal{F}| + 1 = N$. By the hypothesis of our induction, there exists a weak tree-decomposition $(T, \mathcal{S})$ for G such that any S_i, for a node i in T, with $|S_i| > k$ is not a hitting set for $\mathcal{F}_a$. By the choice of $\mathcal{F}$, the weak tree-decomposition $(T, \mathcal{S})$ has a node i_a in T with $|S_{i_a}| > k$ such that S_{i_a} is a hitting set for $\mathcal{F}$. We must have $S_{i_a} \cap C_a = \emptyset$ (otherwise, S_{i_a} would be a hitting set for $\mathcal{F}_a$).

We would like to restrict this weak tree-decomposition to G^a, which by Lemma 4.2, is a weak tree-decomposition of G^a. However, the restriction does not necessarily verify properties (i) and (ii), in particular, $S_{i_a} \neq S$, so we use Menger's theorem in order to slightly modify the restriction of $(T, \mathcal{S})$ to G^a, making it satisfy (i) and (ii).

By applying Menger's theorem, we first show that there are $|S|$ vertex disjoint paths from S to S_{i_a} in G. Consider thus an (S, S_{i_a})-separator A in G. We are reduced to proving that $|A| \geq |S|$. Suppose this is not the case and so $|A| < |S|$. Since the order of $\mathcal{F}$ is equal to $|S|$, A is not a hitting set for $\mathcal{F}$, and there exists an element $X \in \mathcal{F}$ such that $A \cap X = \emptyset$. To obtain a contradiction, note that X is connected and thus there exists a path in X from a vertex in $X \cap S \neq \emptyset$ to a vertex in $X \cap S_{i_a} \neq \emptyset$. This path does not contain any point of A contradicting the choice of A as an (S, S_{i_a})-separator.

We thus have a collection of $|S|$ vertex disjoint paths between S and S_{i_a} in G. Denote the unique path with endpoint $v \in S$ with P_v. Note that since the number of paths is $|S|$ and they are vertex disjoint, we have $S \cap P_v = \{v\}$.

Since the other end-point of P_v is in S_{i_a} and $S_{i_a} \cap C_a = \emptyset$, this in particular shows that the path P_v intersects G_a only at v.

We now define the weak tree-decomposition $(T^a, \mathcal{S}^a)$ as follows. Let $T^a = T$, and for any $v \in S$, pick a node t_v of T with $t_v \in T_v$ (i.e., $v \in S_{t_v}$), and for any node j of T on the unique path from i_a to t_v, add v to S_j. By Proposition 4.2, this leads to a weak tree-decomposition $(T, \mathcal{S}')$ of G. Define $(T^a, \mathcal{S}^a)$ as the restriction of $(T, \mathcal{S}')$ to G^a, which, again by Proposition 4.2, is a weak tree-decomposition of G^a.

Note that since $S'_{i_a} = S_{i_a} \cup S$ and $S_{i_a} \cap C_a = \emptyset$, we have $S^a_{i_a} = S'_{i_a} \cap V(G_a) = S$, and so Property (i) holds. We now show that Property (ii) above holds too, which finishes the proof of our theorem.

We first show that for any node j, $|S_j| \geq |S_j^a|$, or equivalently, $|S_j \cap S| \geq |S_j^a \cap S|$. Since the paths P_v for $v \in S \cap S_j^a$ are vertex disjoint and intersect $V(G^a)$ only at $S \cap S_j^a$, in order to prove $|S_j \cap S| \geq |S_j^a \cap S|$, it will be enough to show that for any $v \in S \cap S_j^a$, S_j contains at least one vertex in P_v. Consider a vertex $v \in S_j^a \cap S$. By the definition of S_j^a, this means that either $v \in S_j$, or j lies on the unique path between i_a and t_v in T. In the former case, we obviously have $\{v\} \subset S_j \cap P_v$. In the latter case, both the sets S_{i_a} and S_{t_v} have non-empty intersections with P_v (which we recall is a path from $v \in S_{t_v}$ to a vertex in S_{i_a}). This means that i_a and t_v both belong to $\bigcup_{u \in P_v} T_u$, which is a connected subtree of T by Lemma 4.1. This in particular means that the unique path between i_a and t_v is contained in $\bigcup_{u \in P_v} T_u$, in other words, there is a vertex u on P_v such that $j \in T_u$. Reformulating this, we get $u \in S_j \cap P_v$, which is what we wanted to prove.

To show (ii), let now j be a node of T with S_j^a of size strictly larger than k. Since $|S| \leq k$, this means $S_j^a \setminus S \neq \emptyset$. Since $S_j^a \subset C_a \cup S$, this shows that $S_j^a \cap C_a \neq \emptyset$. By what we just proved, $|S_j| \geq |S_j^a|$, so S_j is not a hitting set for $\mathcal{F}_a = \mathcal{F} \cup \{C_a\}$. On the other hand, $S_j^a \subseteq S_j \cup S$, and S_j^a has non-empty intersection with C_a, which shows that $S_j \cap C_a \neq \emptyset$. This means there exists $X \in \mathcal{F}$ such that $S_j \cap X = \emptyset$. We show that $S_j^a \cap X = \emptyset$, which implies that S_j^a is not a hitting set for $\mathcal{F}$.

Suppose this is not true, and so $S_j^a \cap X \neq \emptyset$. Since $S_j \cap X = \emptyset$, and $S_j^a \subset S_j \cup S$, a point v in $S_j^a \cap X$ should belong to S. In other words, j is on the unique path between i_a and t_v in T. To get a contradiction, note that X intersects S_{i_a} (S_{i_a} is a hitting set for $\mathcal{F}$), it intersects S_{t_v} ($v \in X \cap S_{t_v}$), but it does not intersect S_j. This is in contradiction with Lemma 4.1.

The proof of Theorem 4.3 is now complete. □

5. Proof of the spectral lower bound on divisorial gonality

Using the results of the previous section, we are now ready to give the proof of Theorems 1.1 and 1.2.

The following is a direct corollary of Theorem 4.3, Proposition 3.2, and Theorem 2.2.

Corollary 5.1. *Let Γ be a metric graph. The divisorial gonality of Γ is lower bounded by the weak tree-width of any simple graph model $G = (V, E)$ of Γ.*

Proposition 5.2. *Let G be a simple finite graph. We have $2wtw(G) \geq tw(G) + 1$.*

Proof. Let $(T, \mathcal{S})$ be a weak tree-decomposition of G of order $wtw(G)$, i.e., $|S_i| \leq wtw(G)$ for any node i of T. We build a tree-decomposition (T, Y) for G of width at most $2wtw(G) - 1$ out of $(T, \mathcal{S})$.

Fix a node $\mathfrak{r}$ of T, and consider T as being rooted at $\mathfrak{r}$. Any node $i \neq \mathfrak{r}$ has a unique parent p_i in the $\mathfrak{r}$-rooted tree T, which, we recall, is the unique neighbor of i in the unique path from i to $\mathfrak{r}$ in T. For any node i of T different from $\mathfrak{r}$, define $Y_i := X_i \cup X_{p_i}$. Furthermore, define $Y_{\mathfrak{r}} = X_{\mathfrak{r}}$. Let $\mathcal{Y} = \{Y_i\}_{i \in V(T)}$. It is easy to check that $(T, \mathcal{Y})$ is a tree-decomposition of G. In addition, we have $|Y_i| \leq 2wtw(G)$, from which the proposition follows. □

5.1. Spectral lower bound for tree-width.

We will need the following slight simplification of a spectral lower bound for tree-width proved by Chandran-Subramanian [16].

Recall the definition of the discrete Laplacian L_G on a connected graph $G = (V, E)$. For a function $f : V \to R$, $L_G(f)$ is the real-valued function on V whose value at a given vertex v

is given by

$$L_G(f)(v) = \sum_{u \sim v} f(v) - f(u),$$

where the sum is taken over all vertices u adjacent to v. Denote by $\lambda_1(G)$ the smallest non-trivial eigenvalue of L_G.

Theorem 5.3 (Chandran-Subramanian [16])**.** *For any connected graph $G = (V, E)$, the following holds*

$$tw(G) + 1 \geq \frac{|V|\lambda_1(G)}{12\, d_{\max}},$$

where $d_{\max}$ is the maximum valency of vertices of G.

For the sake of completeness, we include the short proof of the above theorem. First recall the following variational characterization of λ_1:

$$\lambda_1 = \inf_{f:V\to\mathbb{R} \text{ with } \sum_v f(v)=0} \frac{\sum_{uv\in E}\big(f(u)-f(v)\big)^2}{\sum_{v\in V} f(v)^2}.$$

Let Y and Z be two disjoint non-empty subsets of V. Applying this to the (test) function f defined by $f(z) = \frac{1}{|Z|}$ for $z \in Z$, $f(y) = -\frac{1}{|Y|}$ for $y \in Y$ and $f(w) = 0$ for any $w \in V \setminus (Y \cup Z)$, which satisfies $\sum_{v\in V} f(v) = 0$, we get

$$\lambda_1 \leq \Big(|E| - |E(Y)| - |E(Z)|\Big)\Big(\frac{1}{|Y|} + \frac{1}{|Z|}\Big), \tag{1}$$

where $E(A)$ denotes the set of all edges with both endpoints in A. This is used repeatedly in the proof.

Proof of Theorem 5.3. Denote by n the number of vertices of G. For the sake of a contradiction, assume the inequality does not hold and let $(T, \mathcal{S} = \{S_i\})$ be a tree decomposition of G such that

$$|S_i| \leq \frac{n\lambda_1}{12\, d_{\max}}.$$

for any vertex i of T. Denote by ρ the quantity in the right hand side of the above equation. The following argument, used also in the proof of Theorem 4.3, shows the existence of a subset $S_i \in \mathcal{S}$ (thus, of size at most ρ) such that each component of $G \setminus S_i$ has size at most $\frac{2(n-|S_i|)}{3}$. Suppose such a set S_i does not exist. Consider a node i of T. For each neighbor j of i in the tree, let $A_i(j)$ be the union of all the S_k with k being a node in the subtree of $T \setminus \{i\}$ which contains j. Our assumption implies one of the sets $A_i(j) \setminus S_i$, for j adjacent to i in T, has size strictly larger than $\frac{2(n-|S_i|)}{3}$. Give the orientation $i \to j$ to the edge $e = \{i, j\}$ of T. Doing this for any node i, we give orientations to the edges of the tree exactly $|V(T)|$ times. Since T has $|V(T)| - 1$ edges, at least one edge $\{i, j\}$ gets orientated twice, which means $|A_i(j) \setminus S_i| > \frac{2(n-|S_i|)}{3}$ and $|A_j(i) \setminus S_j| > \frac{2(n-|S_j|)}{3}$. Since $\rho \leq \frac{n}{12}$, this leads to a contradiction: indeed, the union of the two disjoint sets $A_i(j) \setminus S_i$ and $A_j(i) \setminus S_j$ would have more than $\frac{4n}{3} - \frac{4\rho}{3} > n$ vertices.

Let S be a set of size at most ρ such that all the connected components $Y_1, \ldots, Y_s$ of $G \setminus S$ has size at most $\frac{2(n-|X|)}{3}$. Applying Inequality (1) to the disjoint sets Y_i and $Z_i = V \setminus (X \cup Y_i)$, and noting that the quantity $|E|-|E(Y_i)|-|E(Z_i)|)$ is bounded by $|S|d_{\max}$, and $|Z_i| \geq \frac{n-|S|}{3} \geq \frac{|Y|}{2}$, we get $\lambda_1 \leq \rho d_{\max}\frac{3}{|Y|} = \frac{n\lambda_1}{4|Y|}$. This shows that each Y_i has size at most $\frac{n}{4}$.

Consider now the smallest j such that $\overline{Y}_j = Y_1 \cup \cdots \cup Y_j$ has size at least $\frac{n}{4}$. Since each Y_i has size at most $\frac{n}{4}$, $\overline{Y}_j$ has size at most $\frac{n}{2}$. Since $|S| \leq \rho \leq \frac{n}{12}$, we have $|\overline{Z}_j| \geq \frac{n}{2} - \frac{n}{12} = \frac{5n}{12}$. Applying now Inequality (1) to $\overline{Y}_j$ and $\overline{Z}_j = V \setminus (X \cup \overline{Y}_j)$, we get $\lambda_1 \leq \rho d_{\max}(\frac{4}{n} + \frac{12}{5n}) = \frac{n}{12}(\frac{4}{n} + \frac{12}{5n})\lambda_1$, which is a contradiction since λ_1 is strictly positive. □

The following result is a direct consequence of Corollary 5.1, Proposition 5.2 and Theorem 5.3.

Theorem 5.4. *Let Γ be a metric graph and $G = (V, E)$ a simple graph model of Γ. The divisorial gonality of Γ satisfies the inequality*

$$\gamma_{\text{div}}(\Gamma) \geq \frac{|V|\lambda_1(G)}{24d_{\max}}.$$

5.2. **Divisorial gonality of random graphs.** In this section we discuss some direct consequences of Theorem 5.4 above when the underlying graph of the metric graph is a random graph (according to some model), and the edge lengths are arbitrary.

Let $G = (V, E)$ be a simple graph, and Γ any metric graph with G as a model. The set of vertices of G form a rank-determining set for Γ [33, 35], it follows that the divisor $\sum_{v\in V}(v)$ has rank at least one. In other words, $\gamma_{\text{div}}(\Gamma) \leq n$. As we discuss now, in well-known classes of random graphs, we obtain that the divisorial gonality is $\Theta(n)$ with probability tending to one as n goes to infinity.

Let $G \in G(n,p)$ be a Erdös-Rényi random graph on n vertices where any pair of two vertices are joined with an edge independently with probability p. The threshold for the connectivity of G is $\frac{\log(n)}{n}$. For $p >> \frac{\log(n)}{n}$, a random graph in $G(n,p)$ is with high probability connected, has, by Chernoff bound, maximum degree $d_{\max} = O(np)$, and has $\lambda_1 \sim pn$ by [32]. Thus, it follows from our results that $\gamma_{\text{div}}(\Gamma) = \Theta(n)$

On the other hand, for $p < \frac{\log n}{n}$ the random graph $G \in G(n,p)$ is not necessarily connected. However, the threshold for the existence of a (unique) giant connected component in G (i.e., of size linear in n) is $\frac{1}{n}$. If in addition, we assume that $p >> \frac{1}{n}$, it follows from [31] that the tree-width of a random graph in $G(n,p)$ is greater than βn for some constant $\beta > 0$, which in particular implies that a random Erdös-Rényi random graph with $p >> \frac{1}{n}$ has divisorial gonality again $\Theta(n)$.

Corollary 5.5. *The divisorial gonality of an Erdös-Rényi random graph in $G(n,p)$ is $\Theta(n)$ with probability tending to one as n goes to infinity, provided that $pn >> 1$.*

It should be certainly possible to obtain sharper results. One might expect that when pn is above a certain threshold, the divisorial gonality of a random graph in $G(n,p)$ is $(1-o(1))n$ with high probability.

Let $d \geq 3$ be an integer, and let $\epsilon > 0$ be any small enough constant. A random d-regular graph on n vertices is asymptotically almost surely connected, and by Friedman's theorem [30], has λ_1 lower bounded by $d - 2\sqrt{d-1} - \epsilon$. It follows that

Corollary 5.6. *The divisorial gonality of a random d-regular graph is $\Theta(n)$ with probability tending to one as n goes to infinity.*

Again, it should be possible to obtain better bounds (and convergence theorems) for the divisorial gonality of a random d-regular graph as a function of the degree d.

5.3. **Proof of Theorem 1.1.** In this final section, we present the proof of our main theorem.

Let Γ be a metric graph and let G be a simple graph model of Γ with $\ell_{\min}(G) = \ell_{\min}(\Gamma)$.

Rescaling Γ by a factor of $\beta = \frac{1}{\ell_{\min}}$, we get a simple graph model G' of $\Gamma' = \beta\Gamma$ where each edge of G' has length at least one. Note that $\lambda_1(\beta\Gamma) = 1/\beta^2\lambda_1(\Gamma)$, while $\mu(\beta\Gamma) = \beta\mu(\Gamma)$ and $\ell_{\min}(\beta\Gamma) = \beta\ell_{\min}(\Gamma)$, so that the quantity $\lambda_1(.)\mu(.)\ell_{\min}(.)$ is scale free for a metric graph. The divisorial gonality of a metric graph is also easily seen to be scale free, which means in proving the inequality of Theorem 1.1, rescaling Γ with a factor of β if necessary, we can assume that $\ell_{\min}(\Gamma) = 1$, and the simple graph model G of Γ has minimum edge length equal to one.

We now subdivide the simple graph model $G = (V, E)$ of Γ in the following way. For any edge $e = \{u, v\}$ of G of length $\ell(e)$, let u_1 and u_2 be the two points of Γ on e at distance $\frac{1}{16\deg_G(u)}$ and $\frac{1}{16\deg_G(v)}$ from u and v, respectively. Consider a set of points A_e in the interval $[u_1, v_1]$ on the edge e, including u_1 and v_1, such that the distance between any two points of A_e in the interval is at least $\frac{1}{4}$. Taking A_e of maximum size, we see that $4\ell(e) - 1 \leq |A_e| \leq 4\ell(e) + 2 \leq 6\ell(e)$.

Let $\overline{G} = (\overline{V}, \overline{E})$ be the subdivision of G at all the points in the union of A_e, for e an edge of G. We see that

$$|\overline{V}| \geq \sum_{e\in E} |A_e| \geq \sum_{e\in E}(4\ell(e) - 1) \geq \sum_{e\in E} 3\ell(e) = 3\mu(\Gamma).$$

(Note that we also have

$$|\overline{V}| = |V| + \sum_{e\in E} |A_e| \leq 2|E| + \sum_{e\in E} |A_e| = \sum_{e\in E}(|A_e| + 2) \leq \sum_{e\in E}(4\ell(e) + 4) \leq \sum_{e\in E} 8\ell_e = 8\mu(\Gamma),$$

which together give $3\mu(\Gamma) \leq |\overline{V}| \leq 8\mu(\Gamma)$.)

We now claim that

Claim 5.7. *There is a constant c_1, independent of Γ, such that $\lambda_1(\Gamma) \leq c_1\lambda_1(\overline{G})$.*

Here, $\lambda_1(\overline{G})$ is the first non-trivial eigenvalue of the discrete Laplacian of $\overline{G}$. In the proof we will get $c_1 = 128$, however, we do not try to optimize the constant.

Once this has been proved, applying Theorem 5.4, we get

$$\gamma_{\mathrm{div}}(\Gamma) \geq \frac{|\overline{V}|\lambda_1(\overline{G})}{24d_{\max}} \geq \frac{3\mu(\Gamma)\lambda_1(\Gamma)}{24c_1d_{\max}}.$$

Since we assume $\ell_{\min}(\Gamma) = 1$, this leads to the proof of Theorem 1.1 for the constant $C = \frac{1}{1024}$.

We are thus left to prove the above claim.

Proof of Claim 5.7. Recall that

$$\lambda_1(\Gamma') = \inf_{f\in \mathrm{Zh}_0(\Gamma')} \frac{\int_{\Gamma'} |f'|^2 dx}{\int_{\Gamma'} f^2 dx}.$$

Recalling the variational characterization of $\lambda_1(\overline{G})$, let $g : \overline{V} \to \mathbb{R}$ with $\sum_{v\in\overline{V}} g(v) = 0$ and

$$\lambda_1(\overline{G}) = \frac{\sum_{e=\{u,v\}\in\overline{E}} \big(g(u)-g(v)\big)^2}{\sum_{v\in\overline{V}} g(v)^2}.$$

For each vertex v in $\overline{V}$ of degree $\deg(v)$, consider the disk $B(v)$ of radius $\frac{1}{16\deg(v)}$ around v in Γ. Note that $B(v)$ has volume $1/16$ for any vertex v in $\overline{G}$.

Note also that by the choice of A_e, $\Gamma \setminus \bigcup_{v\in\overline{V}} B(v)$ is a disjoint collection of segments of length at least $\frac{1}{8}$ (and at most $\frac{1}{2}$, by the maximality of each A_e).

Define the function $f : \Gamma \to \mathbb{R}$ as follows: first for any vertex $v \in \overline{V}$, define f on the disk $B(v)$ to be the constant function taking value $g(v)$. Extend f to whole Γ by linear interpolation on any segment of $\Gamma \setminus \bigcup_v B(v)$. Let $m = \frac{1}{\mu(\Gamma)}\int_\Gamma f dx$ and consider the function $f-m$ which lies in $\mathrm{Zh}(\Gamma)$. We thus have

$$(2) \qquad \lambda_1(\Gamma) \le \frac{\int_\Gamma f'^2 dx}{\int_\Gamma (f-m)^2 dx}.$$

The function $f-m$ can be written as a sum f_1+f_2 where f_1 is the restriction of f to $\bigcup_v B(v)$ extended by zero to whole of Γ, and $f_2 = f - m - f_1$.

We have $\int_\Gamma f_1 dx = \sum_v \int_{B(v)} f_1 dx = \sum_v g(v)\mu(B(v)) = 1/16\sum_v g(v) = 0$, and so $\int_\Gamma f_2 = 0$, as well.

In addition, since f_2 restricts to the constant function $-m$ on $\bigcup_v B(v)$, we have $\int_\Gamma f_1.f_2 = 0$, which gives

$$(3) \qquad \int_\Gamma (f-m)^2 = \int_\Gamma f_1^2 + \int_\Gamma f_2^2 \ge \int_\Gamma f_1^2 = \sum_{v\in V} g(v)^2\mu(B(v)) = \frac{1}{16}\sum_v g(v)^2.$$

We now give an estimate of $\int_\Gamma f'^2$. Each connected component in $\Gamma \setminus \bigcup_v B(v)$ is a (unique) segment I_e lying in the interior of an edge $e = \{u,v\}$ $\overline{G}$, and is adjacent to the two disks $B(u)$ and $B(v)$.

The function f is affine linear with slope $\frac{g(u)-g(v)}{\ell(I)}$. Thus, we have

$$\int_\Gamma f'^2 = \sum_{e\in\overline{E}} \int_{I_e} f'^2 dx = \sum_{e=\{u,v\}\in\overline{E}} \frac{\big(g(u)-g(v)\big)^2}{\ell(I_e)}.$$

Given that the length of I_e is at least $\frac{1}{8}$, we get

$$(4) \qquad \int_{\Gamma'} f'^2 dx \le 8 \sum_{e=\{u,v\}\in\overline{E}} \big(g(u)-g(v)\big)^2.$$

Equations (2), (3) and (4) together give

$$\lambda_1(\Gamma) \le 128\lambda_1(\overline{G}),$$

which is what we wanted to prove. ☐

Remark 5.8. We refer to the paper of Cohen-Steiner and the first author [7] for a complement to Claim 5.7, and for an inequality in the other direction.

Acknowledgments: We thank the referees for very helpful comments which improved the presentation. O. A. likes to thank David Cohen-Steiner for his interest in the subject and for fruitful discussion and collaboration on related questions. Part of this research was conducted while the authors were visiting Max-Planck institute in mathematics in Bonn. They are grateful to the warm hospitality of the staff and excellent working conditions at MPI.

References

[1] D. Abramovich, *A linear lower bound on the gonality of modular curves*, Internat. Math. Res. Notices 20 (1996), 1005–1011.

[2] O. Amini, *Geometry of graphs and applications in arithmetic and algebraic geometry.* Preprint available at `www.math.ens.fr/~amini`.

[3] O. Amini, *Reduced divisors and embeddings of tropical curves*, Trans. Amer. Math. Soc. 365 (2013), 4851–4880.

[4] O. Amini and M. Baker, *Linear series on metrized complexes of algebraic curves*, Math. Annalen, to appear.

[5] O. Amini, M. Baker, E. Brugallé, and J. Rabinoff, *Lifting harmonic morphisms I: metrized complexes and Berkovich skeleta.* Research in the Mathematical Sciences, to appear.

[6] O. Amini, M. Baker, E. Brugallé, and J. Rabinoff, *Lifting harmonic morphisms II: tropical curves and metrized complexes and Berkovich skeleta.* Algebra & Number Theory 9 (2015), no.2, 267–315.

[7] O. Amini and D. Cohen-Steiner, *A transfer principle and applications to eigenvalue estimates for graphs.* Preprint available at `http://arxiv.org/abs/1409.4228`.

[8] O. Amini, F. Mazoit, N. Nisse, and S. Thomassé, *Submodular partition functions.* Disc. Math., 309 (2009), 6000-6008.

[9] M. Baker, *Specialization of linear systems from curves to graphs*, Algebra & Number Theory, 2 (2008), 613–653.

[10] M. Baker and D. Jensen, *Degeneration of linear series from the tropical point of view and applications.* Preprint available at `http://arxiv.org/abs/1504.05544`.

[11] M. Baker and S. Norine, *Riemann-Roch and Abel-Jacobi theory on a finite graph*, Advances in Mathematics 215 (2007), 766–788.

[12] M. Baker and R. Rumley, *Harmonic analysis on metrized graphs*, Canadian Journal of Mathematics 59 (2007), 225–275.

[13] M. Baker, S. Payne, J. Rabinoff, *Non-Archimedean geometry, tropicalization, and metrics on curves* (2011). Preprint available at http://arxiv.org/abs/1104.0320.

[14] G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs, vol. 33, American Mathematical Society, Providence, RI, 1990.

[15] Z. Cinkir, *Zhang's Conjecture and the Effective Bogomolov Conjecture over function fields*, Inventiones mathematicae, 183 (2011), 517–562.

[16] L.S. Chandran and C.R. Subramanian, *A spectral lower bound for the treewidth of a graph and its consequences.* Inf. Proc. Let. 87 (2003), 195–200.

[17] T. Chinburg and R. Rumely, *The capacity pairing*, J. reine angew. Math, 434 (1993), 1–44.

[18] G. Cornelissen, F. Kato, and J. Kool, *A combinatorial Li-Yau inequality and rational points on curves*, Math. Annalen 361 (2015) no.1, 211–258

[19] J. van Dobben de Bruyn and D. Gijswijt, *Treewidth is a lower bound on graph gonality.* Preprint available at http://arxiv.org/abs/1407.7055.

[20] A. Deveau, D. Jensen, J. Kainic, D. Mitropolsy, *Gonality of random graphs.* Preprint available at http://arxiv.org/abs/1409.5688.

[21] R. Diestel, *Graph theory*, Springer-Verlag, Graduate Texts in Mathematics 173, 4th edition 2010.

[22] R. Diestel and S-I. Oum, *Unifying duality theorems for width parameters in graphs and matroids. I. Weak and strong duality.* Preprint available at `http://arxiv.org/abs/1406.3797`.

[23] A. Ducros, *Triangulations et cohomologie étale sur une courbe analytique p-adique*, J. Algebraic Geom. 17 (2008), 503–575.

[24] A. Ducros, *La structure des courbes analytiques*, book available at http://www.math.jussieu.fr/ ducros/.

[25] J. van Dobben de Bruyn, *Reduced divisors and gonality in finite graphs*, manuscript.

[26] J. Ellenberg, C. Hall, and E. Kowalski, *Expander graphs, gonality and variation of Galois representations*, Duke Math. J. 161 (2012), 1233–1275.

[27] X. Faber, *Spectral convergence of the discrete Laplacian on models of a metrized graph*, New York J. Math.12 (2006), 97–121.

[28] G. Frey, *Curves with infinitely many points of fixed degree*, Israel J. Math. 85 (1994), 79–83.

[29] M. Fiedler *Algebraic connectivity of graphs*, Czechoslovak Math. J. 23(98) (1973), 298-305

[30] J. Friedman, *A proof of Alon's second eigenvalue conjecture and related problems*, Memoirs of the American Mathematical Society 195 (2008), 100pp.

[31] Y. Gao, *Treewidth of Erdös-Rényi random graphs, random intersection graphs, and scale-free random graphs*, Discrete Appl. Math. 160 (2012), 566–578.

[32] F. Juhász, *The asymptotic behaviour of Fiedler's algebraic connectivity for random graphs*, Discrete Mat. 96 (1991), 59–63

[33] J. Hladký, D. Kràl', and S. Norine, *Rank of divisors on tropical curves*, J. Comb. Theory. Series A. 120 (2013), 1521–1538.

[34] P. Li and S. T. Yau, *A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces*, Invent. math. 69 (1982), 269–291.

[35] Y. Luo, *Rank-determining sets of metric graphs*, J. Comb. Theory. Series A. 118 (2011), no. 6, 1775–1793.

[36] K. Menger, *Zur allgemeinen Kurventheorie*, Fund. Math. 10 (1927): 96–115.

[37] G. Mikhalkin and I. Zharkov, *Tropical curves, their Jacobians and theta functions*, Contemp. Math. 465 (2008), Amer. Math. Soc., Providence, RI, 203–230.

[38] B.A. Reed, *Tree width and tangles: A new connectivity measure and some applications*, Surveys in Combinatorics 241 (1997), 87–162.

[39] N. Robertson and P. Seymour, *Graph Minors. I - XXIII*, J. Comb. Theory Series B. 1983–2010.

[40] P.D. Seymour and R. Thomas, *Graph searching and a min-max theorem for tree-width.* J. Comb. Theory, Series B, 58 (1993), 22-33.

[41] M. Temkin, *Introduction to Berkovich analytic spaces.* Preprint available at http://arxiv.org/abs/1010.2235v1.

[42] A. Thuillier, *Théorie du potentiel sur les courbes en géométrie analytique non archimédienne. Applications l̀a théorie dArakelov.* Ph.D. thesis, University of Rennes, 2005. Preprint available at `http://tel.ccsd.cnrs.fr/documents/archives0/00/01/09/90/index.html`.

[43] P.C. Yang and S-T. Yau, *Eigenvalues of the Laplacian of compact Riemann surfaces and minimal submanifolds*, Annali della Scuola Normale Superiore di Pisa 7 (1980), 55–63.

[44] S. Zhang, *Admissible pairing on a curve.* Inventiones mathematicae, 112 (1993), 171–193.

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