SUBMODULAR PARTITION FUNCTIONS

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Abstract. — Adapting the method introduced in Graph Minors X, we propose a new proof of the duality between the bramble-number of a graph and its tree-width. Our approach is based on a new definition of submodularity on partition functions which naturally extends the usual one on set functions. The proof does not rely on Menger's theorem, and thus generalises the original one. It thus provides a dual for matroid tree-width. One can also derive all known dual notions of other classical width-parameters from it.

1. Introduction

In their seminal paper Graph Minors X [**10**], Robertson and Seymour introduced the notion of branch-width of a graph and its dual notion of tangle. Their method is based on bias and tree-labellings. Later, Seymour and Thomas [**11**] found a dual notion to tree-width, the bramble number (named by Reed [**8**]). The proof of the bramble-number/tree-width duality makes use of Menger's theorem to reconnect partial tree-decompositions, see for instance the textbook of Diestel [**2**]. Our aim in this paper is to show how the classical dual notions of width-parameters can be deduced from the original method of Graph Minors X.

In this paper, E will always denote a finite set with at least two elements. A *partitioning tree* on E is a tree T with at least three nodes in which the leaves are identified with the elements of E in a one-to-one way. Therefore, every internal node v of T, if any, corresponds to the partition T_v of E whose parts are the set of leaves of the subtrees obtained by deleting v.

An obvious way of defining a partitioning tree is simply to add a node adjacent to every element of E — a partitioning star. But what if we are not permitted to do so? Precisely, assume that a restricted set of partitions of E called *admissible partitions* is given. Is it possible to form an admissible partitioning tree? (i.e., such that every partition T_v for each internal node v is admissible.) An obstruction to the existence of such a tree is the dual notion of bramble.

An *admissible bramble* is a nonempty set of pairwise intersecting subsets of E which contains a part of every admissible partition of E. It is routine to define an admissible bramble: just

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pick an element e of E, and collect, for every admissible partition, the part that contains e. Such a bramble is called *principal.* The crucial fact is that if there is a non-principal admissible bramble $\mathcal{B}$, then there is no admissible partitioning tree. To see this, assume for contradiction that T is an admissible partitioning tree. For every internal node u of T, there is an element X of T_u that belongs to $\mathcal{B}$. Let v be the neighbour of u that belongs to the component of $T \setminus u$ having set of labels X. Orient the edge uv of T from u to v. Note that every internal node becomes the origin of an oriented edge. Observe also that an edge of T incident to a leaf never gets an orientation since $\mathcal{B}$ is non-principal. The contradiction follows from the fact that some edge of T receives two orientations, which is impossible since the elements of $\mathcal{B}$ are pairwise intersecting.

Unfortunately, there can be no admissible partitioning tree and no non-principal admissible bramble. Indeed, this is the case for $E = \{a, b, c, d, e\}$ and with $\big\{\{a, b\}, \{c\}, \{d\}, \{e\}\big\}$ and $\big\{\{a\}, \{b\}, \{c\}, \{d, e\}\big\}$ as admissible partitions.

In the first part of this paper, we prove that for some particular families of admissible partitions (e.g. generated by a submodular partition function) there exists an admissible partitioning tree if and only if no non-principal admissible bramble does. The second part of the paper is devoted to the translation of this result into the different notions of width-parameters.

2. Submodular Partition Functions

The *complement* of a subset X of E is the set $X^c := E \setminus X$. A *partition* of E is a set $\mathcal{X} = \{X_1, \dots, X_n\}$ ($n \geq 1$) of subsets of E satisfying $X_1 \cup \cdots \cup X_n = E$ and $X_i \cap X_j = \emptyset$ for all $i \neq j$. The order in which the parts appear is irrelevant. We allow degenerate partitions (i.e. the sets X_i can be empty).

The partition obtained from $\mathcal{X}$ by *pushing X_i to a subset F of E* is

$$\mathcal{X}_{X_i \to F} := \{X_1 \cap F^c, \dots, X_{i-1} \cap F^c, X_i \cup F, X_{i+1} \cap F^c, \dots, X_n \cap F^c\}.$$

A *partition function* is a function Φ defined from the set of partitions of E into $\mathbb{R} \cup \{\infty\}$ with $\mathbb{R}$ the set of reals. Let $\mathcal{X}$ be a partition of E. We call $\Phi(\mathcal{X})$ the Φ-*width*, or simply the *width*, of $\mathcal{X}$. Let $k \in \mathbb{R} \cup \{\infty\}$. A *$k$-partition* is a partition of width at most k. A partition function Φ is *submodular* if for every pair of partitions $\mathcal{X} = \{X_1, \dots, X_n\}$ and $\mathcal{Y} = \{Y_1, \dots, Y_l\}$ and for every $1 \leq i \leq n$ and $1 \leq j \leq l$, we have:

$$\Phi(\mathcal{X}) + \Phi(\mathcal{Y}) \geq \Phi(\mathcal{X}_{X_i \to Y_j^c}) + \Phi(\mathcal{Y}_{Y_j \to X_i^c}).$$

To justify *a posteriori* our terminology, observe that for bipartitions, partition submodularity gives

$$\begin{aligned}
\Phi(A, A^c) + \Phi(B, B^c) &= \Phi(A, A^c) + \Phi(B^c, B) \\
&\geq \Phi(A \cup (B^c)^c, A^c \cap B^c) + \Phi(B^c \cup A^c, B \cap A) \\
&\geq \Phi(A \cup B, A^c \cap B^c) + \Phi(A \cap B, A^c \cup B^c).
\end{aligned}$$

This corresponds to the usual notion of submodularity when setting $\Phi(F) := \Phi(F, F^c)$ for every subset F of E.

Unfortunately, some natural partition functions lack submodularity, and so we have to define a relaxed version of it. If Φ is partition submodular, then for any pair of partitions $\mathcal{X} = \{X_1, \dots, X_n\}$ and $\mathcal{Y} = \{Y_1, \dots, Y_l\}$, $\Phi(\mathcal{X}_{X_i \to Y_j^c}) \leq \Phi(\mathcal{X})$ or $\Phi(\mathcal{Y}_{Y_j \to X_i^c}) \leq \Phi(\mathcal{Y})$ ($1 \leq$

$i \leq n$ and $1 \leq j \leq l$). To define weakly submodular partition function, we strengthen this condition slightly. More precisely, a partition function Φ is *weakly submodular* if for every pair of partitions $\mathcal{X} = \{X_1, \ldots, X_n\}$ and $\mathcal{Y} = \{Y_1, \ldots, Y_l\}$ and every $1 \leq i \leq n$, $1 \leq j \leq l$, at least one of the following holds:

1. There exists F such that $X_i \subseteq F \subseteq (Y_j \setminus X_i)^c$ and $\Phi(\mathcal{X}) > \Phi(\mathcal{X}_{X_i \to F})$;
2. $\Phi(\mathcal{Y}) \geq \Phi(\mathcal{Y}_{Y_j \to X_i^c})$.

It is straightforward to check that submodular partition functions are weakly submodular. (To see this, it suffices to consider $F = (Y_j \setminus X_i)^c$.) Let us illustrate these notions by some examples. In what follows, $\mathcal{X} = \{X_1, \ldots, X_n\}$ is a partition of E.

- The key example of a submodular partition function is the function *border size* defined on the set of partitions of the edge set E of a graph $G = (V, E)$. The *border* of a partition $\mathcal{X}$ of edges is the set $\Delta(\mathcal{X})$ of vertices that are incident with edges in at least two parts of $\mathcal{X}$. The *border size* of $\mathcal{X}$ is then $\delta(\mathcal{X}) = |\Delta(\mathcal{X})|$. For a subset F of E we will often write $\Delta(F)$ and $\delta(F)$ instead of $\Delta(F, F^c)$ and $\delta(F, F^c)$. The proof of the submodularity of border functions is postponed to Section 5.1. As we will see, the function δ leads to the tree-width of G.
- Let f be a submodular function on 2^E (the set of subsets of E). We define a submodular partition function Σ_f by letting $\Sigma_f(\mathcal{X}) = \sum_{i \in I} f(X_i)$. The submodularity of this function is proved in Section 5.2. This corresponds to the tree-width of matroids.
- Let f be a symmetric submodular function on 2^E, that is a submodular function satisfying $f(A) = f(A^c)$ for all $A \subseteq E$. The function $\max_{i \in \{1, \ldots, n\}} f(X_i)$, which is a limit of weakly submodular functions, gives the notion of branch-width and its relatives like rank-width. It is treated in Section 5.3.
- Let Φ be a weakly submodular partition function and $p \geq 2$ be an integer. We define a weakly submodular partition function by letting $\Phi_p(\mathcal{X}) = \Phi(\mathcal{X})$ when the number of parts of $\mathcal{X}$ is at most p, and $+\infty$ otherwise (or any large integer constant). This kind of functions allows us to describe the branch-width.
- Let Φ be a weakly submodular partition function and $p \geq 2$ be an integer. By letting $\Phi'_p(\mathcal{X}) = \Phi(\mathcal{X})$ when the number of X_i with at least two elements is at most p, and $+\infty$ otherwise (or any large constant integer), we obtain a partition function which gives, in particular, the notion of path-width. This is a weakly submodular partition function if we only push subsets that are non-singletons.

3. Search-Trees

A *bidirected tree* is a directed graph obtained from an undirected tree by replacing every edge with an oriented circuit of length two.

A *search-tree* T on E is an arc-labelled bidirected tree on at least three nodes such that:

- The arcs of T are labelled by subsets of E, and we denote by l the labelling function;
- If u is an internal node of T, the sets $l(uv)$, for all outneighbours v of u, define a partition of E that we denote by T_u;
- The labels of a 2-circuit are disjoint, i.e., $l(uv) \cap l(vu)\emptyset$.

Let Φ be a partition function on E. The Φ-*width* of a search-tree T is the maximum of $\Phi(T_u)$, taken over the internal nodes u. If there is no risk of confusion, we just speak of the *width* of T. A k-*search-tree* is a search-tree of width at most k. A 2-circuit uv is *exact* if

$l(uv) \cup l(vu) = E$. A search-tree T is *exact* if all its 2-circuits are exact. The label of an arc leaving a leaf of T is called a *leaf-label*.

Proposition 1. — *In an exact search-tree T, the set of labels of the arcs entering the leaves of T is a partition of E.*

Proof. — Let T be an exact search-tree. We prove the proposition by induction on the number of internal nodes of T. If T has one internal node, it satisfies the proposition. Otherwise, let u and v be two adjacent internal nodes of T, and uv be the corresponding 2-circuit. Let T^u (resp. T^v) be the exact search-tree obtained by removing from T all the components of $T \setminus \{v\}$ (resp. of $T \setminus \{u\}$) not containing u (resp. v). By induction, the set $\mu_{uv} = \{l(uv), A_1, \ldots, A_p\}$ of labels of the arcs entering the leaves of T^u is a partition of E. Similarly, the set $\mu_{vu} = \{l(vu), B_1, \ldots, B_q\}$ of labels of the arcs entering the leaves of T^v is a partition of E. Since T is exact, $l(uv) = l(vu)^c$, hence the set of labels of the arcs entering the leaves of T is the partition $\{A_1, \ldots, A_p, B_1, \ldots, B_q\}$. □

Since the labels of the arcs entering the leaves are exactly the complements of the leaf-labels, we obtain the following corollary.

Corollary 2. — *No two leaves of an exact search-tree can have the identical leaf-labels other than E.*

When this partition consists of singletons and empty sets, T is a *partitioning* k-search-tree. (In which case T provides naturally a partitioning tree on E.)

A search-tree T is *compatible* with a set $\mathcal{F}$ of subsets of E if every leaf-label of T contains an element of $\mathcal{F}$ as a subset (recall that a leaf-label is the label of an arc leaving a leaf). Let uv be a 2-circuit of T with u an internal node and let F be such that $l(uv) \subseteq F \subseteq l(vu)^c$. A key fact is that replacing the partition T_u in T by $(T_u)_{l(uv)\to F}$ (in the obvious one-to-one way) gives a new search-tree that is still compatible with $\mathcal{F}$ since its leaf-labels are unchanged.

Theorem 3. — *Let $\mathcal{F}$ be a set of subsets of E. If Φ is a weakly submodular partition function on E and T is a k-search-tree compatible with $\mathcal{F}$, there is a relabelling of T that is an exact k-search-tree compatible with $\mathcal{F}$.*

Proof. — Choose any internal node r as a root of T. Among all relabellings of T that are k-search-trees compatible with $\mathcal{F}$, we minimise the sum of $\Phi(T_u)$, taken over all internal nodes u, and then we maximise the sum of the sizes of the labels of backward arcs of T. We claim that T is exact. If not, then let uv be a non-exact 2-circuit, with u closer to r than v. If v is an internal node, then the sum of $\Phi(T_u)$ being minimal, there is no F with $l(uv) \subseteq F \subseteq l(vu)^c$ for which $\Phi(T_u) > \Phi((T_u)_{l(uv)\to F})$. We can thus replace T_v by $(T_v)_{l(vu)\to l(uv)^c}$. If v is a leaf, then we replace $l(vu)$ by $l(uv)^c$. In any case, we get a new search-tree compatible with $\mathcal{F}$, and both replacements strictly increase the size of the label $l(vu)$, a contradiction. □

4. Tree-Bramble Duality

Let Φ be a weakly submodular partition function on E. Let $k \in \mathbb{R} \cup \{\infty\}$. Recall that a k-partition is a partition whose Φ-width is at most k. A *partitioning k-search-tree* is an exact search-tree of Φ-width at most k that is compatible with $\{E \setminus \{e\} \mid e \in E\}$. A *bias* is a nonempty family $\mathcal{B}$ of subsets of E such that $\bigcap_{X\in\mathcal{B}} X = \emptyset$. A *k-bramble* $\mathcal{B}$ is a nonempty family of subsets of E such that:

- For all $X, Y \in \mathcal{B}$, we have $X \cap Y \neq \emptyset$.
- For every k-partition $\mathcal{X} = \{X_1, \ldots, X_n\}$, there exists i such that $X_i \in \mathcal{B}$.

A k-bramble is *principal* if it contains a singleton. In particular, if a k-bramble is principal, then it is not a bias.

Theorem 4. — *Let Φ be a weakly submodular partition function on a set E and $k \in \mathbb{R}\cup\{\infty\}$. There exists a non-principal k-bramble if and only if there does not exist a partitioning k-search-tree.*

Proof. — If there is a partitioning k-search-tree, then every k-bramble is principal. The proof is given in the introduction in terms of admissible partitions.

Now let us show that if all k-brambles are principal, then there exists a partitioning k-search-tree. Let us therefore assume that all k-brambles are principal. First note that there exists at least one *non-trivial* k-partition (i.e. a partition that does not contain E) for otherwise $\{E\}$ is a non-principal k-bramble, a contradiction.

We claim that every bias has a compatible k-search-tree. If true, then the bias $\{E \setminus \{e\} \mid e \in E\}$ is compatible with a search tree T. By Theorem 3, we may assume that T is exact, and thus that T corresponds to a k-partitioning tree. Our claim thus implies the theorem. To prove it, assume for the sake of a contradiction that there exists a bias $\mathcal{B}$ which is compatible with no k-search-tree. Choose such a bias $\mathcal{B}$ maximal with respect to inclusion. That is, for any $X \notin \mathcal{B}$, if any, there exists a search-tree compatible with $\mathcal{B}\cup\{X\}$. Two cases can happen:

- The set $\mathcal{B}$ contains a part of every k-partition.

  We claim that $\mathcal{B}$ contains two disjoint sets B_1 and B_2, where B_1 is a part of a k-partition containing at least two parts. Indeed, remove from $\mathcal{B}$ all the elements that belong to no k-partition and call the resulting set $\mathcal{B}'$. Since there exists a non-trivial k-partition, $\mathcal{B}$ and thus $\mathcal{B}'$ are non-empty. If $\mathcal{B}'$ is a k-bramble, then it is principal. There thus exists $B_1 = \{e\} \in \mathcal{B}'$ that belongs to a k-partition with at least two parts; since $\mathcal{B}$ is a bias, it contains a set B_2 disjoint from $\{e\}$. If $\mathcal{B}'$ is not a k-bramble, then it contains two disjoint sets B_1 and B_2. Either $B_2 \neq \emptyset$ and $B_1 \in \mathcal{B}'$ or $B_2 = \emptyset$ and B_1 can be any part of a non-trivial k-partition. In both cases, we can suppose that B_1 belongs to a k-partition with at least two parts. This finishes the proof of the claim.

  Let $\{X_1, \ldots, X_n\}$ be a k-partition containing B_1 with $n \geq 2$ (say $B_1 = X_j$). Let T be the bidirected star T with n leaves $v_1, \ldots, v_n$ and one internal node x. Set $l(xv_i) := X_i$ and $l(v_ix) := X_i^c$ for all $i \leq n$. Since $B_1 \subseteq X_j^c$ and $B_2 \subseteq X_i^c$ for all $i \neq j$, T is a k-search-tree compatible with $\mathcal{B}$.
- The set $\mathcal{B}$ contains no part of a k-partition $\mathcal{X} = \{X_1, \ldots, X_n\}$.

  Suppose that $\mathcal{X}$ is trivial, say $X_i = E$. Any k-search-tree compatible with $\mathcal{B}\cup\{E\}$ is also compatible with $\mathcal{B}$, contradictory to our choice of $\mathcal{B}$. We can thus suppose that $\mathcal{X}$ is non-trivial and that $n \geq 2$.

  We claim that for each non-empty X_i, there exists a k-search-tree T_i having a unique leaf-label $l(v_ix_i)$ containing X_i.

  Before we prove the claim, suppose we have found the k-search-trees T_i for the non-empty X_i. If $X_i = \emptyset$, then let T_i be the two circuit x_iv_i with $l(x_iv_i) = \emptyset$ and $l(v_ix_i) = E$. We can "merge" the trees T_i to get a k-search-tree compatible with $\mathcal{B}$. Indeed, let T be the tree obtained from $\cup_{i=1}^n T_i$ by identifying each x_i in a new vertex z and setting

$l(zv_i) := X_i$ and $l(v_iz) := l(v_ix_i)$. Since each $l(v_iz) = l(v_ix_i)$ is disjoint from $l(x_iv_i)$ and $l(x_iv_i)$ contains X_i, $l(zv_i) \cap l(v_iz) = \emptyset$ and T is a k-search-tree which, by construction, is compatible with $\mathcal{B}$, a contradiction.

To obtain the tree T_i, it is tempting to consider the bias $\mathcal{B}_i = \mathcal{B} \cup \{X_i\}$, since the maximality of $\mathcal{B}$ implies that there is a k-search-tree T_i compatible with $\mathcal{B}_i$. The problem is that even if Corollary 2 ensures that they are different, T_i may have multiple leaf-labels containing X_i. To overcome this difficulty, for every X_i we choose an inclusion-wise maximal set X'_i that contains X_i and contains no element of $\mathcal{B}$. We then set $\mathcal{B}'_i := \mathcal{B} \cup \{X'_i\}$. Since $\mathcal{B}$ is maximal, Theorem 3 implies that there is an exact k-search-tree T_i compatible with $\mathcal{B}'_i$. Since $\mathcal{B}$ is compatible with no k-search-tree, T_i has a leaf-label containing X'_i and no element of $\mathcal{B}$. Since X'_i is maximal with this property, this leaf-label is exactly X'_i. Since $X'_i \neq E$, by Corollary 2, X'_i appears only once as a leaf-label of T_i, as required.

□

5. Examples of Submodular Partition Functions

In this section we prove that the partition functions given in Section 2 to illustrate the notions of (weak) partition submodularity are indeed (weakly) partition submodular.

5.1. The Submodular Border Partition Function δ.

— Recall that for a partition $\mathcal{X}$ of the edge set of a graph, $\Delta(\mathcal{X})$ is the set of vertices incident with edges in at least two distinct parts of $\mathcal{X}$, and $\delta(\mathcal{X}) = |\Delta(\mathcal{X})|$.

Proposition 5. — *The border function δ is submodular.*

Proof. — Let $G = (V, E)$ be a graph. Let $\mathcal{X} = \{X_1, \dots, X_n\}$ and $\mathcal{Y} = \{Y_1, \dots, Y_l\}$ be some partitions of E. We want to prove that:

$$\begin{aligned}\delta(\mathcal{X}) + \delta(\mathcal{Y}) &\geq \delta(\mathcal{X}_{X_1 \to Y_1^c}) + \delta(\mathcal{Y}_{Y_1 \to X_1^c}) \\ &\geq \delta(X_1 \cup Y_1^c, X_2 \cap Y_1, \dots, X_n \cap Y_1) + \\ &\qquad \delta(Y_1 \cup X_1^c, Y_2 \cap X_1, \dots, Y_l \cap X_1)\end{aligned}$$

Let x be a vertex of G. Two cases can happen:

- The contribution of x in the right-hand term of the previous inequality is 1. The vertex x belongs to, say, $\Delta(\mathcal{X}_{X_1 \to Y_1^c})$. There are edges e_x and f_x containing x with e_x in some $Y_1 \cap X_i$ and f_x not in $Y_1 \cap X_i$ ($i \geq 2$). If $f_x \notin Y_1$, then x belongs to $\Delta(Y_1)$. Otherwise, $f_x \notin X_i$ and $x \in \Delta(X_i)$. In both cases, the contribution of x to the left-hand term is at least 1.
- Assume now that x belongs both $\Delta(\mathcal{X}_{X_1 \to Y_1^c})$ and $\Delta(\mathcal{Y}_{Y_1 \to X_1^c})$. Since x belongs to $\Delta(\mathcal{X}_{X_1 \to Y_1^c})$, there is an edge e_x containing x in some $X_i \cap Y_1$ ($i \geq 2$). Similarly there is an edge f_x containing x in some $Y_j \cap X_1$ ($j \geq 2$). Since $e_x \in X_i$ and $f_x \in X_1$, x belongs to $\Delta(\mathcal{X})$. Similarly x belongs to $\Delta(\mathcal{Y})$, and thus contributes also twice to the left-hand term.

□

5.2. The Submodular Partition Function Σ_f. — Let f be a submodular function on 2^E. Recall that $\Sigma_f(\mathcal{X}) = \sum_{X\in\mathcal{X}} f(X)$.

Lemma 6. — 1. *Let X and Y be two disjoint subsets of E. If $X_1 \subseteq X$ and $Y_1 \subseteq Y$, we have:*

$$f(X) + f(Y) - f(X_1) - f(Y_1) \geq f(X \cup Y) - f(X_1 \cup Y_1)$$

2. *More generally, if $X_1, \ldots, X_r$ are pairwise disjoint subsets of E, and $X'_i \subseteq X_i$ for all $i = 1, \ldots, r$, then we have:*

$$\sum_{i=1}^{r} \big(f(X_i) - f(X'_i)\big) \geq f\Big(\bigcup_{i=1}^{r} X_i\Big) - f\Big(\bigcup_{i=1}^{r} X'_i\Big)$$

Proof. — 1. Apply first the submodularity of f to the subsets $A = X \cup Y_1$ and $B = Y$. Since $A \cap B = Y_1$ and $A \cup B = X \cup Y$, we obtain:

$$f(X \cup Y_1) + f(Y) \geq f(X \cup Y) + f(Y_1) \tag{1}$$

Apply then the submodularity of f to the subsets $A = X_1 \cup Y_1$ and $B = X$. Since $A \cap B = X_1$ and $A \cup B = X \cup Y_1$, we obtain:

$$f(X_1 \cup Y_1) + f(X) \geq f(X \cup Y_1) + f(X_1) \tag{2}$$

The conclusion follows from (1)+(2).

2. Follows by induction on r.

□

Proposition 7. — *The function Σ_f is a submodular partition function.*

Proof. — Let $\mathcal{X} = \{X_1, \ldots, X_n\}$ and $\mathcal{Y} = \{Y_1, \ldots, Y_l\}$ be two partitions of E. We want to prove that $\Sigma_f(\mathcal{X}) + \Sigma_f(\mathcal{Y}) \geq \Sigma_f(\mathcal{X}_{X_1 \to Y_1^c}) + \Sigma_f(\mathcal{Y}_{Y_1 \to X_1^c})$. We must then prove:

$$\begin{aligned}\sum_{i=1}^{n} f(X_i) + \sum_{j=1}^{l} f(Y_j) \geq f(X_1 \cup Y_1^c) &+ \sum_{i=2}^{n} f(Y_1 \cap X_i) \\ &+ f(Y_1 \cup X_1^c) + \sum_{j=2}^{l} f(X_1 \cap Y_j).\end{aligned} \tag{3}$$

Since $X_2 \cup \cdots \cup X_n = X_1^c$, by applying Lemma 6 to X_i's and X'_i's with $X'_i = Y_1 \cap X_i$ for $i = 2, \ldots, n$, we have:

$$\sum_{i=2}^{n} f(X_i) - \sum_{i=2}^{n} f(Y_1 \cap X_i) \geq f(X_1^c) - f(Y_1 \cap X_1^c) \tag{4}$$

Similarly we obtain:

$$\sum_{j=2}^{l} f(Y_j) - \sum_{j=2}^{l} f(X_1 \cap Y_j) \geq f(Y_1^c) - f(X_1 \cap Y_1^c) \tag{5}$$

By adding (4) and (5), we obtain

$$\sum_{j=2}^{l} f(Y_j) + \sum_{i=2}^{n} f(X_i) + f(X_1 \cap Y_1^c) + f(Y_1 \cap X_1^c) \geq \tag{6}$$
$$f(Y_1^c) + f(X_1^c) + \sum_{j=2}^{l} f(X_1 \cap Y_j) + \sum_{i=2}^{n} f(Y_1 \cap X_i)$$

By applying submodularity to X_1^c and Y_1 first and then to X_1 and Y_1^c, and adding the two inequalities, we obtain:

$$f(X_1) + f(Y_1) - f(X_1 \cap Y_1^c) - f(Y_1 \cap X_1^c) \geq f(X_1 \cup Y_1^c) + f(Y_1 \cup X_1^c) - f(Y_1^c) - f(X_1^c) \tag{7}$$

Adding (6) and (7), we obtain (3). Thus Σ_f is submodular. □

5.3. The Weakly Submodular Partition Function $\mathrm{Max}_f^\varepsilon$**.** — Let f be a symmetric submodular function on 2^E. The partition function $\max_f(\mathcal{X}) = \max_{X \in \mathcal{X}} f(X)$ may not be weakly submodular. Indeed, the partition function $\max_\delta$ is not weakly submodular. Let us consider the graph with vertex set $\{a, b, c, d, e, f\}$ and edge set $\{ab, bc, cd, de, ef, fa\}$. Set $X_1 := \{af, bc\}$, $X_2 := \{ab, de\}$, $X_3 := \{cd, ef\}$, $Y_2 := \{af, ab, bc\}$, $Y_1 := Y_2^c$ and consider the partitions $\mathcal{X} = \{X_1, X_2, X_3\}$ and $\mathcal{Y} = \{Y_1, Y_2\}$ (see Fig 1).

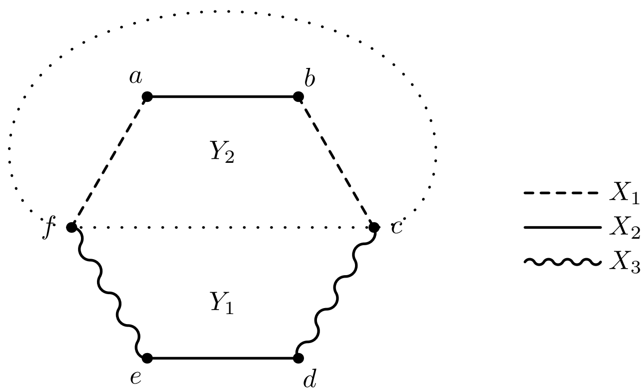

FIGURE 1. An example of two partitions for which $\max_\delta$ is not weakly submodular.

1. On one hand, there exists no F with $X_1 \subseteq F \subseteq Y_1^c$ and $\max_\delta(\mathcal{X}) > \max_\delta(\mathcal{X}_{X_1 \to F})$. Indeed, $F = X_1$ is clearly not good, but $F = X_1 \cup \{ab\}$ gives $\mathcal{X}_{X_1 \to F} = \big\{Y_2, \{ed\}, X_3\big\}$ and we still have $\max_\delta(\mathcal{X}_{X_1 \to F}) = \delta(X_3) = 4 = \max_\delta(\mathcal{X})$.
2. On the other hand, $\max_\delta(\mathcal{Y}) < \max_\delta(\mathcal{Y}_{Y_1 \to X_1^c})$. Indeed, since $\mathcal{Y}_{Y_1 \to X_1^c} = \{X_1, X_1^c\}$, we have $\max_\delta(\mathcal{Y}) = 2 < 4 = \mathcal{Y}_{Y_1 \to X_1^c}$ as claimed.

To overcome this subtlety when dealing with the function $\max_f$, we have to shift it a little to break ties. For any $\varepsilon > 0$ (which will be chosen arbitrarily small) , we consider instead the function:

$$\mathrm{Max}_f^\varepsilon(\mathcal{X}) \max_f(\mathcal{X}) + \varepsilon \Sigma_f(\mathcal{X})$$

Lemma 8. — *For every $\varepsilon > 0$, the function $\mathrm{Max}_f^\varepsilon$ is a weakly submodular partition function.*

Proof. — Let $\mathcal{X} = \{X_1, \ldots, X_n\}$ and $\mathcal{Y} = \{Y_1, \ldots, Y_l\}$ be two partitions of E and let $1 \leq i \leq n$ and $1 \leq j \leq l$. Let F be a set such that

$$X_i \setminus Y_j \subseteq F \subseteq (Y_j \setminus X_i)^c \tag{8}$$

so that $f(F)$ is minimum. We claim that $\max_f(\mathcal{X}) \geq \max_f(\mathcal{X}_{X_i \to F})$. To do so, we must prove that $f(X_i) \geq f(X_i \cup F)$ and that $f(X_k) \geq f(X_k \cap F^c)$ for every $k \neq i$.

– By submodularity, we have:

$$f(F) + f(X_i) \geq f(F \cap X_i) + f(F \cup X_i), \tag{9}$$

and since $X_i \cap F$ satisfies (8),

$$f(F \cap X_i) \geq f(F). \tag{10}$$

Adding (9) and (10), we get $f(X_i) \geq f(F \cup X_i)$.

– For every $k \neq i$, we have by submodularity of f:

$$f(X_k) + f(F^c) \geq f(X_k \cap F^c) + f(X_k \cup F^c) \tag{11}$$

Furthermore, $f(F)$ being minimum, $f(F) \leq f(F \setminus X_k)$, and since f is symmetric,

$$f(X_k \cup F^c) \geq f(F^c). \tag{12}$$

Adding (11) and (12), we obtain $f(X_k) \geq f(X_k \cap F^c)$.

This proves that $\max_f(\mathcal{X}) \geq \max_f(\mathcal{X}_{X_i \to F})$.

By submodularity of $\sum_f$ applied to $\mathcal{X}$ and $\{F^c, F\}$, we obtain

$$\textstyle\sum_f(\mathcal{X}) + \sum_f(F^c, F) \geq \sum_f(\mathcal{X}_{X_i \to F}) + \sum_f(X_i^c, X_i). \tag{13}$$

Since X_i satisfies (8), $f(X_i) \geq f(F)$, and $\sum_f(F^c, F) \geq \sum_f(X_i^c, X_i)$, hence $\sum_f(\mathcal{X}) \geq \sum_f(\mathcal{X}_{X_i \to F})$ and thus, $\mathrm{Max}^\varepsilon_f(\mathcal{X}) \geq \mathrm{Max}^\varepsilon_f(\mathcal{X}_{X_i \to F})$. Now two cases can happen:

– If $f(X_i) > f(F)$, from (13), then we get $\sum_f(\mathcal{X}) > \sum_f(\mathcal{X}_{X_i \to F})$ and $\mathrm{Max}^\varepsilon_f(\mathcal{X}) > \mathrm{Max}^\varepsilon_f(\mathcal{X}_{X_i \to F})$. Now, since $\mathcal{X}_{X_i \to F} = \mathcal{X}_{X_i \to F \cup X_i}$, $F' := F \cup X_i$ is such that $X_i \subseteq F' \subseteq (Y_j \setminus X_i)^c$ and $\mathrm{Max}^\varepsilon_f(\mathcal{X}) > \mathrm{Max}^\varepsilon_f(\mathcal{X}_{X_i \to F'})$.
– If $f(X_i) = f(F)$, then we set $F := X_i$. By exchanging the roles of $\mathcal{X}$ and $\mathcal{Y}$, and since $f(F) = f(F^c)$, we obtain $\mathrm{Max}^\varepsilon_f(\mathcal{Y}) \geq \mathrm{Max}^\varepsilon_f(\mathcal{Y}_{Y_j \to X_i^c})$.

Thus $\mathrm{Max}^\varepsilon_f$ is a weakly submodular partition function. □

6. Width Parameters

We assume in this section that the reader is somehow familiar with the usual definitions of tree-decompositions (such as tree-width, branch-width, path-width, rank-width, ...). Our aim is just to associate a weakly submodular partition function to each of these parameters and show how to translate the exact partitioning k-search-tree into a decomposition, and the non-principal k-bramble into the known dual notion (if any). To avoid technicalities, we assume that k is at least two and that $G = (V, E)$ is a simple loopless graph with minimum degree at least two. In this section, if X is a set of vertices of G, then $E(X)$ denote the set of edges incident with at least one vertex in X.

6.1. Tree-Width of Graphs. — The duality between tree-decompositions and brambles was first proved in [**11**]. Brambles were renamed from their original name, screens, in [**8**]. The tree-width of G corresponds to the border function δ defined on partitions of $E(G)$. More precisely, the following property ([**10**], Theorem 5.1) links tree-decompositions and k-search-trees.

Proposition 9. — *Let G be a graph with minimum degree at least two. There exists a tree-decomposition of G of width at most $k-1$ if and only if there exists a partitioning k-search-tree for the partition submodular function δ.*

Theorem 4 gives a duality theorem between tree-decompositions and non-principal k-brambles. The next property links usual brambles with non-principal k-brambles. Recall that a bramble in a graph G is a set $\mathcal{B}$ of subsets of vertices such that:

- for every $X \in \mathcal{B}$, $G[X]$ is a connected subgraph of G;
- for any X, $Y \in \mathcal{B}$, X and Y *touch*, that is $G[X \cup Y]$ is a connected subgraph of G.

The *order* of a bramble $\mathcal{B}$ is the minimum size of one of its transversal.

Proposition 10. — *Let G be a graph with minimum degree at least two. There exists a bramble in G of order at least $k+1$ if and only if there exists a non-principal k-bramble for the partition submodular function δ.*

Proof. — The key idea behind this proof is that in a graph without isolated vertices, two sets of vertices X and Y touch if and only if $E(X)$ and $E(Y)$ intersect. Recall that $E(X)$ is the set of edges that are incident with at least one vertex in X.

Suppose that G has a bramble $\mathcal{B}$ of order $k+1$. Let $\mathcal{X} = \{X_1, \dots, X_p\}$ be a partition of E with border of size at most k. Since $\mathcal{B}$ has order $k+1$, there is an element B of $\mathcal{B}$ disjoint from $\Delta(\mathcal{X})$. Let X_i be the part of $\mathcal{X}$ containing $E(B)$. Note that X_i cannot be a singleton (since G has minimum degree at least two). Let $\mathcal{B}_k$ be the set of all these sets X_i (over all partitions $\mathcal{X}$ with $\delta(\mathcal{X}) \leq k$). We claim that $\mathcal{B}_k$ is a non-principal k-bramble. Indeed, let X and Y be some elements of $\mathcal{B}_k$. Assume that X and Y contain respectively $E(B_X)$ and $E(B_Y)$ with B_X and B_Y in $\mathcal{B}$. Since B_X and B_Y touch, $\emptyset \neq E(B_X) \cap E(B_Y) \subseteq X \cap Y$. This proves that $\mathcal{B}_k$ is a k-bramble. As already noted, no chosen X_i is a singleton; thus $\mathcal{B}_k$ is non-principal.

Assume now that E has a non-principal k-bramble $\mathcal{B}_k$. For any subset $S \subseteq V$ of size at most k, let $\{E_1, \dots, E_n\}$ be the partition of E in which the sets E_i are the (nonempty) sets of edges minimal with respect to inclusion for the property $\Delta(E_i) \subseteq S$. Since $\mathcal{B}_k$ is a non-principal k-bramble, one of the E_i, with at least two edges, is in $\mathcal{B}_k$. This means that $X_i = V(E_i) \setminus S$ is a nonempty connected set of vertices. Note that $E_i = E(X_i)$. Now let $\mathcal{B}$ be the set of these X_i (over all subsets $S \subseteq V$ of size at most k). We claim that $\mathcal{B}$ is a bramble of order at least $k+1$. Indeed let X_i, X_j be any two elements of $\mathcal{B}$. Since $E(X_i)(= E_i)$ and $E(X_j)(= E_j)$ both belong to the k-bramble $\mathcal{B}_k$, $E_i \cap E_j \neq \emptyset$ and thus X_i and X_j touch. Hence $\mathcal{B}$ is a bramble. Since any covering set of $\mathcal{B}$ has at least $k+1$ elements, the order of $\mathcal{B}$ is at least $k+1$. □

6.2. Branch-Width of Connectivity Functions. — Branch-decompositions and tangles were introduced in Graph Minors X [**10**] for hypergraphs. However, the general setting for these decompositions is in terms of *connectivity functions*, i.e., symmetric submodular functions. Indeed, Robertson and Seymour proved the duality between branch-width and tangle-number by explicitly using decompositions of connectivity functions [**10**]. Their theorem thus also applies to matroid branch-width (see for example [**4**]) or rank-width (see [**6**]).

Let Ψ be a connectivity function on E. The branch-width of Ψ corresponds to the weakly submodular partition function $(\max_\Psi)_3$, where $(\max_\Psi)_3$ denotes the maximum Ψ-width of an element of a partition of E containing two or three parts. An exact partitioning k-search tree of E is precisely a branch-decomposition of E of width at most k. As noted in Section 5.3, the function $\max_\Psi$ is not weakly partition submodular. But since $\max_\Psi = \lim_{\varepsilon\to 0^+} \mathrm{Max}_\Psi^\varepsilon$, Theorem 4 also applies.

Let us now explain the correspondence between a non-principal k-bramble $\mathcal{B}$ and a tangle of E. Recall that a *tangle of order k* for a connectivity function Ψ (see [**7**]) is a set $\mathcal{T}$ of subsets of E such that:

- for every $A \subseteq E$ with $\Psi(A) \le k$, either $A \in \mathcal{T}$ or $A^c \in \mathcal{T}$;
- if A, B, $C \in \mathcal{T}$, then $A \cup B \cup C \ne E$;
- for every $e \in E$, $E \setminus \{e\} \notin \mathcal{T}$.

The difference between tangles and k-brambles is very simple. If $\Psi(A) \le k$, then the tangle contains the "small" part of $\{A, A^c\}$ while the bramble contains the "large" one. To link tangles and k-brambles, we need the following lemma.

Lemma 11. — *Let $\mathcal{B}$ be a k-bramble corresponding to the partition function* $(\max_\Psi)_3$. *For every A, B, C in $\mathcal{B}$, the intersection $A \cap B \cap C$ is non-empty.*

Proof. — Suppose for the sake of a contradiction that there exist A, B, $C \in \mathcal{B}$ with $A\cap B\cap C = \emptyset$. Choose A, B, C inclusion-wise maximal with this property. Since

$$\begin{aligned}\Psi(A \setminus B) + \Psi(B \setminus A) &= \Psi(A \cap B^c) + \Psi(B \cap A^c)\\ &= \Psi(A \cap B^c) + \Psi(B^c \cup A)\\ &\le \Psi(A) + \Psi(B^c) = \Psi(A) + \Psi(B)\end{aligned}$$

we can assume that $\Psi(A \setminus B) \le k$.

We now claim that $\Psi(A \cap C) \le k$. Indeed, let $C' = (A \setminus B) \cup C$.

- Suppose that $C = C'$, that is $A \setminus B \subseteq C$. Since $A \cap B \cap C = \emptyset$, $A \setminus B = A \cap C$, and the claim follows.
- Suppose that $C \subsetneq C'$. If $\Psi(C') \le k$, then $C' \in \mathcal{B}$. But this is impossible for $A\cap B\cap C' = \emptyset$ and A, B, C are maximal with this property. Thus $\Psi(C') > k$. By submodularity of Ψ, we have
$$2k \ge \Psi(A \setminus B) + \Psi(C) \ge \Psi(C') + \Psi((A \setminus B) \cap C).$$
Therefore $\Psi((A \setminus B) \cap C) \le k$. Finally, since $A \cap B \cap C = \emptyset$, $A \cap C = (A \setminus B) \cap C$ and the claim follows.

By the same calculation as above, we can suppose that $\Psi(A\setminus C) \le k$. The partition $\{A^c, A\cap C, A \setminus C\}$ is then a k-partition. This is impossible, since these three sets are respectively disjoint from A, B and C, which all belong to $\mathcal{B}$. □

We are now ready to prove the following:

Proposition 12. — *A tangle of order k exists if and only if a non-principal k-bramble for the partition function* $(\max_\Psi)_3$ *does.*

Proof. — Let $\mathcal{T}$ be a tangle of order k. We claim that $\mathcal{B} = \{A^c \mid A \in \mathcal{T}\}$ is a non-principal k-bramble. Observe first that if A, $B \in \mathcal{B}$, then A^c, $B^c \in \mathcal{T}$. Therefore, $A^c \cup B^c \neq E$ which proves that $A \cap B \neq \emptyset$. Remark now that, for every partition (A, B, C) with $(\max_\Psi)_3(A, B, C) \leq k$, exactly one set amongst A, B and C does not belong to $\mathcal{T}$ (since $A \cup B \cup C = E$). Therefore $\mathcal{B}$ contains an element of every k-partition. Finally, the third condition in the definition of tangle imposes that $\mathcal{B}$ is non-principal.

Now let $\mathcal{B}$ be a non-principal k-bramble. We claim that $\mathcal{T} = \{A^c \mid A \in \mathcal{B}\}$ is a tangle of order k. By construction, $\mathcal{T}$ satisfies the first tangle axiom. The second axiom follows directly from Lemma 11. Finally, $\mathcal{B}$ being non-principal imposes the third tangle condition. □

6.3. Path-Width of Graphs. — Path-decompositions were introduced in [**9**]. The duality theorem between path-decompositions and blockages appears in [**1**] (see also [**3**]). Recall that the partition function δ'_2 corresponds to the size of the border of partitions $\{X_1, \ldots, X_n\}$ of E with at most two parts with more than one element. We show in this section that the path-width of $G = (V, E)$ is the minimum k such that there exists a partitioning k-search-tree of δ'_2. The following analogue of Theorem 3 holds for partition functions Φ'_p, where Φ is a weakly submodular partition function, and $p \geq 2$ is some integer:

Theorem 13. — *If T is a k-search-tree (with respect to Φ'_p) compatible with $\mathcal{F}$, then there is a relabelling of a subtree of T which is an exact k-search-tree compatible with $\mathcal{F}$.*

Proof. — The proof is similar to the one of Theorem 3 except in one case: For u and v internal nodes of T, one cannot always push the part $l(uv)$ to $l(vu)$ in the partition T_u. Indeed, when $|l(uv)| \leq 1$, this could increase the number of parts of T_u with more than one element. In this case, we simply define a new tree T' by deleting the nodes of T which belong to the components of $T \setminus v$ not containing u. Now, v is a leaf of T', and we set $l(vu) = l(uv)^c$. Observe that T' is still compatible with $\mathcal{F}$. The reason for this is that $\bigcap_{X \in \mathcal{F}} X = \emptyset$, hence one of its element is included in $l(uv)^c$. □

It follows that Theorem 4 also holds for Φ'_p, and consequently for δ'_2. Using the same technique as in Proposition 9, one can prove:

Proposition 14. — *Let G be a graph with minimum degree at least two. There exists a path-decomposition of G of width at most k if and only if there exists a partitioning k-search-tree of E with respect to δ'_2.*

Let us now link blockages and non-principal k-brambles. A *k-cut* (V_1, V_2) is a pair of subsets of vertices with $|V_1 \cap V_2| \leq k$, $V_1 \cup V_2 = V$ and such that no edge of G joins $V_1 \setminus V_2$ to $V_2 \setminus V_1$. Let us recall that a *blockage* of order k in a graph $G = (V, E)$ is a set $\mathcal{B}$ of subsets of V such that:

i. for every $A \in \mathcal{B}$, $(A, A^c \cup N(A^c))$ is a k-cut;
ii. for every k-cut (A, B), $\mathcal{B}$ contains exactly one of A and B;
iii. if (A, B) is a k-cut and $C \in \mathcal{B}$ is such that $A \subseteq C$, then $A \in \mathcal{B}$.

We need the following lemma. Recall that for $X \subseteq V(G)$, $E(X)$ is the set of edges that are incident with at least one vertex in X:

Lemma 15. — *Let G be a graph with no isolated vertex and let $\mathcal{B}$ be a blockage of order k. For every U_1, $V_1 \in \mathcal{B}$, $E(U_1^c) \cap E(V_1^c) \neq \emptyset$.*

Proof. — Let $U_2 = U_1^c \cup N(U_1^c)$. The pair (U_1, U_2) is a k-cut. If $E(U_1^c) \cap E(V_1^c) = \emptyset$, then $U_2 \subseteq V_1$ which is impossible by the third blockage condition. The lemma follows. □

Proposition 16. — *Let $G = (V, E)$ be a graph with minimum degree at least two. There exists a blockage in G of order k if and only if there exists a non-principal k-bramble with respect to δ_2'.*

Proof. — We first establish a correspondence between k-cuts and partitions $\mathcal{X}$ such that $\delta_2'(\mathcal{X}) \leq k$. If $\mathcal{X} = \{X_1, \ldots, X_p\}$ is a partition of E with $\delta_2'(\mathcal{X}) \leq k$ (where $|X_1|$, $|X_2| \geq 2$), then $(V(X_2^c), V(X_1^c))$ is a k-cut. Conversely, if (V_1, V_2) is a k-cut, then the partition $\mathcal{X} = \{X_1, \ldots, X_p\}$ in which $X_1 = E(V_2^c)$, $X_2 = E(V_1^c)$, and $\{X_3, \ldots, X_p\} = E(V_1 \cap V_2)$ is such that $\delta_2'(\mathcal{X}) \leq k$.

Let $\mathcal{B}$ be a blockage of order k. Let $\mathcal{X} = \{X_1, \ldots X_p\}$ be such that $\delta_2'(\mathcal{X}) \leq k$ and let $(V(X_2^c), V(X_1^c))$ be the corresponding k-cut. Since $\mathcal{B}$ is a blockage, it contains a part $V(X_i^c)$ (for $i = 1$ or 2). We then select X_{3-i} to be in our bramble $\mathcal{B}_k$. We claim that $\mathcal{B}_k$ is a non-principal k-bramble. The fact that the elements of $\mathcal{B}_k$ are pairwise intersecting follows from Lemma 15. To prove that $\mathcal{B}_k$ is non-principal, note that $V_1 \neq V(G)$ for any $V_1 \in \mathcal{B}_k$. Otherwise, for any k-cut (U_1, U_2), both U_1 and U_2 would be subsets of V_1 and the second and third blockage conditions would be incompatible. Thus, by construction $\mathcal{B}_k$ is non-principal (since G has minimum degree at least two).

Let us assume that $\mathcal{B}_k$ is a non-principal k-bramble. Let (V_1, V_2) be a k-cut and let $\{X_1, \ldots, X_p\}$ be the corresponding partition. Since $\mathcal{B}_k$ is a non-principal k-bramble, $\mathcal{B}_k$ contains a part X_i (for $i = 1$ or 2). We then select V_{3-i} to be in $\mathcal{B}$. We claim that $\mathcal{B}$ is a blockage. It clearly satisfies the first and second conditions and the third one follows from the fact that the elements of $\mathcal{B}_k$ are pairwise intersecting. □

6.4. Tree-Width of Matroids. — Matroid tree-decompositions were introduced in [**5**] but no duality theorem was known for them. Let M be a matroid on ground set E with rank function r. We denote by r^c the submodular function such that $r^c(F) := r(F^c)$ for all subsets F of E. We also denote by Φ the partition function such that for any partition $\mathcal{X} = \{X_1, \ldots, X_l\}$,

$$\Phi(\mathcal{X}) = \Sigma_{r^c}(\mathcal{X}) - (l-1)r(E)$$

Since Σ_{r^c} is submodular by Proposition 7, and the number of parts in $\mathcal{X}$ and $\mathcal{X}_{X_1 \to F}$ are the same, Φ is also submodular. Note that since $\Phi(\mathcal{X} \cup \{\emptyset\})\Phi(\mathcal{X}) + r^c(\emptyset) - r(E) = \Phi(\mathcal{X})$, Σ_{r^c} remains submodular if we remove from $\mathcal{X}_{X_1 \to F}$ its empty sets.

A *tree-decomposition* of M (see Hliněný and Whittle [**5**]) is given by a tree T and a mapping $\tau : E \to V(T)$. Every node u of T corresponds to the partition $(F_0, \ldots, F_d)$ where $F_0 = \tau^{-1}(u)$ and $F_i = \tau^{-1}(T_i)$ where T_1, …, T_d are the components of $T \setminus u$. The *weight* of u is $\Sigma_{i=1}^d r^c(F_i) - (d-1)r(E)$. The *width* of T is the maximum weight of one of its nodes and the *tree-width* of M is the minimum width of one of its tree-decompositions.

Proposition 17. — *There exists a partitioning k-search-tree with respect to Φ if and only if there exists a tree-decomposition of width at most k.*

Proof. — Partitioning k-search-trees on E are indeed tree-decompositions of width at most k. This proves the forward implication.

For the backward implication, let T be any tree-decomposition of width k. We claim that T can be turned into a partitioning k-search-tree. To do so, first note that we can prune empty labelled leaves without changing the weight of the other nodes. Then, let u be either an internal node with a non-empty label or a leaf whose label is not a singleton. Let F_0 be its label. Attach $|F_0|$ new leaves to u and move each element of F_0 to one of these leaves. The contribution of a new leaf labelled by e to the weight of u is $r^c(e) - r(E) \leq 0$ so the weight of u does not increase. □

Non-principal brambles provide a dual notion to matroid tree-width.

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