ÉTALE COHOMOLOGY OF ALGEBRAIC VARIETIES OVER STEIN COMPACTA

OLIVIER BENOIST

Abstract. We prove a comparison theorem between the étale cohomology of algebraic varieties over Stein compacta and the singular cohomology of their analytifications. We deduce that the field of meromorphic functions in a neighborhood of a connected Stein compact subset of a normal complex space of dimension n has cohomological dimension n. As an application of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant variants of these results, we obtain a quantitative version of Hilbert's 17th problem on compact subsets of real-analytic spaces.

Introduction

0.1. **Hilbert's 17th problem.** Hilbert discovered in [Hil88] that real polynomials $f \in \mathbb{R}[x_1, \dots, x_n]$ that are positive semidefinite (i.e., that only take nonnegative values on $\mathbb{R}^n$) may not always be written as sums of squares of polynomials. He however conjectured, in his celebrated 17th problem, that this issue may be resolved by allowing denominators: the polynomial f should always be a sum of squares of rational functions. This problem was solved positively by E. Artin [Art27, Satz 4] in 1927. Forty years later, Pfister [Pfi67, Theorem 1] obtained a quantitative improvement of Artin's theorem bounding the number of squares required. It states that a positive semidefinite $f \in \mathbb{R}[x_1, \dots, x_n]$ is a sum of 2^n squares in $\mathbb{R}(x_1, \dots, x_n)$.

0.2. **The real-analytic variant.** As an application of the main results of this article, we prove a quantitative theorem à la Pfister in real-analytic geometry.

Theorem 0.1 (Theorem 6.5)**.** *Let M be a normal real-analytic variety of pure dimension n. Let $K \subset M$ be compact. Any nonnegative real-analytic function on M is a sum of 2^n squares of real-analytic meromorphic functions in a neighborhood of K.*

Much less is known on Hilbert's 17th problem in the real-analytic setting, first considered in [BR75] (see [ABF22] for an up-to-date survey), than in the algebraic case. A nonnegative real-analytic function on a normal and pure-dimensional real-analytic variety M is a sum of squares of real-analytic meromorphic functions in dimension ≤ 2 (see [Jaw82, Corollary 2] and [ADCR03, Theorem 1]), under a compactness hypothesis (see [Rui85, Theorem 1] and [Jaw86, Theorem 1]) or, when M is a manifold, under a discreteness hypothesis on the zeros of f [BKS81, Theorem 1]. However, whether a nonnegative real-analytic function on $\mathbb{R}^n$ is a sum of squares of real-analytic meromorphic functions is still an open problem for $n \geq 3$.

In addition, quantitative results are known in dimension ≤ 2. A nonnegative real-analytic function on a normal real-analytic surface M is a sum of 3 squares of real-analytic functions if M is a manifold [Jaw82, Corollary 2], and a sum of 5 squares of real-analytic meromorphic functions in general (see [ABFR05, Theorem 1.2] or the more general [Fer21, Theorem 1.3]). No bounds on the required number of squares were known if $n \geq 3$ (except in the local case, for which see [Ben20, Theorem 0.2]). Theorem 0.1 rectifies this situation, under a compactness hypothesis.

Although the qualitative content of Theorem 0.1 is not new (see [Jaw86, Theorem 1]), the quantitative bounds it provides (more precisely, that these bounds only depend on the dimension of M) do imply new cases of (the qualitative version of) the real-analytic Hilbert's 17th problem. Indeed, work of Acquistapace, Broglia, Fernando and Ruiz [ABFR10, Proposition 1.8] readily implies the following.

Corollary 0.2. *Let f be a nonnegative real-analytic function on a real-analytic manifold. If the zero set of f is a disjoint union of compact sets, then f is a finite sum of squares of real-analytic meromorphic functions.*

The best results to date in this vein handled the case where the zero locus of f is the union of a compact set and of a discrete set (see [Jaw92, Theorem 2] or [ABFR10, Corollary 1.10]), or concerned infinite sums of squares [ABFR10, Corollary 1.9]. We also refer to [ABF14, Theorem 1.1] for quantitative results in this direction.

0.3. **From real-analytic to complex-analytic geometry.** In real algebraic geometry, it is very important to consider not only the sets of real points of real algebraic varieties, but also their sets of complex points, endowed with the action of $G := \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ by complex conjugation. For the same reason, it is crucial for our proof of Theorem 0.1 to work in the setting of G-equivariant complex-analytic geometry (see §5.3 for our conventions) instead of real-analytic geometry.

This point of view also leads to a more general theorem on sums of squares, from which Theorem 0.1 is easily derived using the successive works of Cartan [Car57b], Grauert [Gra58] and Tognoli [Tog67] on complexifications of real-analytic spaces. Recall that a compact subset of a complex space is said to be Stein if it admits a basis of Stein open neighborhoods.

Theorem 0.3 (Theorem 6.4)**.** *Let S be a reduced G-equivariant Stein space of dimension n. Let $K \subset S$ be a G-invariant Stein compact subset. Any G-invariant holomorphic function on S which is nonnegative on S^G is a sum of 2^n squares of G-invariant meromorphic functions in a neighborhood of K.*

We believe that this G-equivariant variation on the analytic Hilbert's 17th problem is novel, and that the new statements it comprises (for instance when $S = \mathbb{C}^n$ and $K \subset S$ is the closed unit ball) are of interest. We insist that Theorem 0.3 is already interesting and nontrivial when $S^G = \varnothing$, in which case the nonnegativity hypothesis is automatically satisfied.

0.4. **Cohomological dimension of fields of meromorphic functions.** It has been known since Voevodsky's proof of the Milnor conjectures [Voe03] that quadratic forms over a field are largely governed by the Galois cohomology of the field. As a consequence of these results, the validity of Theorem 0.3 is controlled by the vanishing of a single Galois cohomology class of degree $n+1$ (see [Ben20, Proposition 2.1]). Using this point of view, Theorem 0.3 is a consequence of an upper bound for the cohomological dimension of fields of G-invariant meromorphic functions on G-equivariant Stein compacta (Corollary 5.8). In this introduction, let us only state a non-G-equivariant version of this theorem.

Theorem 0.4 (Corollary 5.11 and Remark 5.12 (ii))**.** *Let K be a connected Stein compact subset of a normal Stein space S of dimension n. The field $\mathcal{M}(K)$ of germs of meromorphic functions in a neighborhood of K has cohomological dimension n.*

In dimension 1, Theorem 0.4 is attributed to M. Artin by Guralnick ([Gur88, Proposition 3.7], see also [BW21, Proposition A.6]), with no compactness hypothesis. It is new in dimension ≥ 2. Theorem 0.4 is an algebraic statement concerning a field of analytic origin, and its proof uses a mixture of analytic and algebraic tools. As such, it is deeper than its algebraic counterpart [Ser94, II, Proposition 11] bounding the cohomological dimension of function fields of algebraic varieties.

We also obtain bounds on the étale cohomological dimension of affine varieties over (possibly G-equivariant) Stein compacta (see Theorems 5.6 and 5.9). We refer to the work of Bhatt and Mathew [BM21, Theorem 7.3 and Remark 7.4] for related bounds in non-archimedean analytic geometry.

0.5. **Étale and singular cohomology.** By a complex-analytic incarnation of the weak Lefschetz theorem due to Andreotti and Frankel [AF59, §2] in the nonsingular case and to Hamm [Ham83, Satz 1] in general, a Stein space of dimension n has the homotopy type of a CW complex of dimension $\leq n$. Its singular cohomology therefore vanishes in degree $> n$. Our strategy of proof of Theorem 0.4 is to transfer this topological information on the singular cohomology of a Stein compactum to algebraic information on the étale cohomology of its ring of holomorphic functions, and eventually on the Galois cohomology of its field of meromorphic functions.

In algebraic geometry, such a transfer tool exists in the shape of M. Artin's comparison theorem [SGA43, XVI, Théorème 4.1] between the étale cohomology of a complex algebraic variety X and the singular cohomology of its analytification X^{an}. In order to implement the strategy described above, we prove the following analogue of Artin's comparison theorem in analytic geometry.

Theorem 0.5 (Theorem 5.1)**.** *Let S be a Stein space. Let X be an $\mathcal{O}(S)$-scheme of finite type and let $\mathbb{L}$ be a torsion étale abelian sheaf on X. Assume that X is proper over $\mathcal{O}(S)$ or that $\mathbb{L}$ is constructible. If one lets U run over all Stein open neighborhoods of a Stein compact subset K of S, the change of topology morphisms*

$$\operatorname*{colim}_{K\subset U} H^k_{\text{ét}}(X_{\mathcal{O}(U)}, \mathbb{L}_{\mathcal{O}(U)}) \to \operatorname*{colim}_{K\subset U} H^k((X_{\mathcal{O}(U)})^{\mathrm{an}}, \mathbb{L}^{\mathrm{an}})$$

are isomorphisms for $k \geq 0$.

Theorem 0.5 is our main result. The bulk of this article is dedicated to its proof. For applications to sums of squares, we really need a G-equivariant extension of Theorem 0.5 (see Theorem 5.5), which we deduce from Theorem 0.5 by means of the Hochschild–Serre spectral sequence.

0.6. **Proof of the comparison theorem.** Two proofs of Artin's comparison theorem appear in [SGA43]. The first [SGA43, XI, §4] (which only works for smooth varieties and locally constant coefficients) compares the étale and the classical topology by means of a Leray spectral sequence. It exploits the fact that points in smooth complex varieties admit good neighborhoods, which are iterated fibrations in smooth affine curves. The second [SGA43, XVI, §4] uses extensive dévissage and fibration arguments to reduce to the case of smooth projective curves.

None of these proofs adapt to the setting of Theorem 0.5 as such fibration arguments cannot be successfully implemented in Stein geometry. However, the dévissage argument of the second proof can indeed be used in the complex-analytic setting to yield the following easier *relative* comparison theorem over Stein spaces.

Theorem 0.6 (Theorem 3.7)**.** *Let S be a Stein space. Let $f : X \to Y$ be a morphism of $\mathcal{O}(S)$-schemes of finite type and let $\mathbb{L}$ be a torsion étale abelian sheaf on X. Assume that f is proper or that $\mathbb{L}$ is constructible. Then the base change morphisms $\mathrm{R}^k f_*(\mathbb{L})^{\mathrm{an}} \to \mathrm{R}^k f_*^{\mathrm{an}}(\mathbb{L}^{\mathrm{an}})$ are isomorphisms for $k \geq 0$.*

In contrast, our proof of Theorem 0.5 initially takes the point of view of the first proof of Artin's theorem. Reductions based on Theorem 0.6 allow us to assume that $X = \mathrm{Spec}(\mathcal{O}(S))$ and $\mathbb{L} = \mathbb{Z}/m$. We then consider the Leray spectral sequence comparing the étale and the classical topology. In order to show that this spectral sequence degenerates, one has to prove that singular cohomology classes with $\mathbb{Z}/m$ coefficients on analytifications of étale $\mathcal{O}(S)$-schemes are étale-locally trivial.

We proceed in two steps. First, we show that such cohomology classes become unramified after pull-back to a finite ramified covering (Proposition 4.3). This uses Theorem 0.6 in an essential way, as well as a vanishing theorem of Bhatt [Bha12, Theorem 1.1]. Second, we show that such unramified cohomology classes may be killed by a further finite ramified covering (Proposition 2.4). This second step is analytic in nature. It relies on the Oka–Weil approximation theorem and on Grauert's bump method as developed by Henkin and Leiterer [HL98] and Forstnerič [For17].

This procedure unfortunately only shows that the relevant singular cohomology classes are killed on possibly ramified coverings, not on étale coverings. To overcome this difficulty, our proof of Theorem 0.5 makes use of Voevodsky's qfh topology [Voe96, §3.1], which is finer than the étale topology and allows for such coverings.

Our relative comparison theorem (Theorem 0.6) is an analogue for étale sheaves of the relative GAGA theorem of Hakim [Hak72, VIII, Théorème 3.2] (see also [Bin76b, Theorem 4.2] or [AT19, Appendix C]) for coherent sheaves. In the coherent context, there is no need for an *absolute* comparison theorem such as Theorem 0.5, as higher cohomology groups of algebraic coherent sheaves on affine schemes, or of analytic coherent sheaves on Stein spaces, vanish. This is a marked difference with the étale setting, in which the absolute comparison theorem lies deeper.

In this article, we make extensive use of relative algebraic geometry over complex spaces as initiated by Hakim [Hak72], and developed by Bingener [Bin76b] when the base is Stein. We refer to §3.1 for our conventions which follow [Bin76b]. In particular, we use the above-mentioned relative GAGA theorem, through an application of [Bin76b, (7.2)], in Step 4 of the proof of Theorem 0.5.

0.7. **Structure of the article.** Section 1 gathers general results concerning Stein spaces. Most of this material is included to fix our conventions, for the convenience of the reader, or for lack of appropriate references. Specialists of Stein geometry might want to skip it. Section 2 contains the main analytic input of our work: a procedure to kill a singular cohomology class with torsion coefficients on a Stein compactum, after pull-back to a finite ramified covering. We deduce Theorem 0.6 from Artin's comparison theorem and its proof in Section 3, and we use it to kill the ramification of singular cohomology classes with torsion coefficients on finite ramified coverings in Section 4. The above results are combined in Section 5 to prove Theorem 0.5, and to derive cohomological dimension bounds including Theorem 0.4. Finally, applications to sums of squares problems are given in Section 6.

0.8. **Acknowledgements.** I thank Jean-Benoît Bost for a useful conversation, and for having made [Bin76b] available to me, and José F. Fernando and Marco Maculan for useful comments.

1. Geometry of Stein spaces

Among the many general facts on Stein spaces and their compact subsets that are collected in this section, let us put forward a descent result for $\mathcal{O}(S)$-convexity along finite surjective holomorphic maps (Proposition 1.6), and the correspondence between finite ramified coverings of connected normal Stein spaces and finite extensions of their meromorphic function fields (Proposition 1.14).

1.1. **Complex spaces and coherent sheaves.** A *complex space* is a $\mathbb{C}$-ringed space that is locally isomorphic to a model complex space defined by a finitely generated ideal sheaf in a domain of $\mathbb{C}^N$ (see [GR84, 1, §1.5]). We assume them to be second countable, but not necessarily Hausdorff, finite-dimensional or reduced.

As in [GR79], a complex space S is said to be *Stein* if $H^k(S,\mathcal{F})=0$ for all $k\geq 1$ and all coherent sheaves $\mathcal{F}$ on S. We refer to this property as Cartan's Theorem B and to its consequence that coherent sheaves on Stein spaces are globally generated as Cartan's Theorem A. Stein spaces are Hausdorff (holomorphic functions separate distinct $s,s'\in S$ as $H^1(S,\mathcal{I}_{\{s,s'\}})=0$). A complex space S is Stein if and only if its reduction S^{red} is [GR79, V, §4.3, Theorem 5]. The following lemma sometimes allows us to reduce problem about Stein spaces to the case of Stein manifolds.

Lemma 1.1. *Let S be a Stein space of dimension n. Then there exist a Stein manifold S' of dimension $2n+1$ and a proper injective holomorphic map $i:S\to S'$ such that $i(S)$ is a strong deformation retract of S'.*

Proof. Let $i:S^{\mathrm{red}}\to\mathbb{C}^{2n+1}$ be a proper injective holomorphic map (see [Nar60, Theorem 5]). As $\mathcal{O}(S)\to\mathcal{O}(S^{\mathrm{red}})$ is onto (by Cartan's Theorem B), one may extend it to a proper injective holomorphic map $i:S\to\mathbb{C}^{2n+1}$. Its image $i(S)\subset\mathbb{C}^{2n+1}$ is a Stein analytic subset (by [GR84, 3, §1.3, Proposition] and [GR79, V, §1, Theorem 1 b)]). The existence of S' now follows from [Mih96, Theorem 3.1]. $\square$

If S is a complex space and $\mathcal{F}$ is a coherent sheaf on S, we endow $H^0(S,\mathcal{F})$ with the canonical Fréchet topology defined in [GR79, V, §6] and use freely its properties listed in [GR79, V, §6.4]. When S is reduced and $\mathcal{F}=\mathcal{O}_S$, it coincides with the topology of uniform convergence on compact subsets [GR79, V, §6.6, Theorem 8].

1.2. **Runge domains.** If $K\subset S$ is a compact subset of a complex space, its $\mathcal{O}(S)$-*convex hull* is $\widehat{K}_{\mathcal{O}(S)}:=\{s\in S\mid |f(s)|\leq\sup_{t\in K}|f(t)| \text{ for all } f\in\mathcal{O}(S)\}$. By [GR79, V, §4.2, Theorem 3], a complex space S is Stein if and only if

(i) the global holomorphic functions $\mathcal{O}(S)$ separates the points of S, and
(ii) the $\mathcal{O}(S)$-convex hull $\widehat{K}_{\mathcal{O}(S)}\subset S$ of any compact subset $K\subset S$ is compact.

An open subset Ω in a Stein space S is said to be *Runge* if it satisfies any of the equivalent properties of the following proposition. As we could not find a precise discussion in the literature when S is possibly nonreduced, we include a brief proof. Such a generality will be useful in the proof of Proposition 1.6.

Proposition 1.2. *Let $\Omega\subset S$ be an open subset in a Stein space. The following assertions are equivalent, and hold for (S,Ω) if and only if they hold for $(S^{\mathrm{red}},\Omega^{\mathrm{red}})$.*

(i) For all compact subsets $K\subset\Omega$, one has $\widehat{K}_{\mathcal{O}(\Omega)}=\widehat{K}_{\mathcal{O}(S)}$.
(ii) The image of the restriction map $\mathcal{O}(S)\to\mathcal{O}(\Omega)$ is dense, and Ω is Stein.
(iii) For any coherent sheaf $\mathcal{F}$ on S, the restriction map $H^0(S,\mathcal{F})\to H^0(\Omega,\mathcal{F})$ has dense image, and Ω is Stein.

Proof. Write $(*)^{\mathrm{red}}$ for assertion $(*)$ for the pair $(S^{\mathrm{red}}, \Omega^{\mathrm{red}})$. Then (i)$^{\mathrm{red}}$, (ii)$^{\mathrm{red}}$ and (iii)$^{\mathrm{red}}$ are equivalent by [GR65, VII, A, Corollary 9 and VIII, A, Theorem 11]. The implication (iii)$\Rightarrow$(ii) is obvious. If (ii) holds, the continuity and surjectivity of the restriction map $\mathcal{O}(\Omega) \to \mathcal{O}(\Omega^{\mathrm{red}})$ show that (ii)$^{\mathrm{red}}$ also holds. That (i)$^{\mathrm{red}}$ implies (i) follows from the surjectivity of $\mathcal{O}(S) \to \mathcal{O}(S^{\mathrm{red}})$. Conversely, if (i) holds, then Ω is Stein ($\mathcal{O}(\Omega)$ separates the points of Ω because $\mathcal{O}(S)$ separates the points of S) and (i)$^{\mathrm{red}}$ follows from the surjectivity of $\mathcal{O}(\Omega) \to \mathcal{O}(\Omega^{\mathrm{red}})$.

To prove (iii)$^{\mathrm{red}}$ $\Rightarrow$(iii), we argue as in [NS77, Lemma 1.10]. The characterization [GR79, V, §6.2, Theorem 4] of the topology on $H^0(\Omega, \mathcal{F})$ shows that it is the initial topology with respect to the restriction maps $H^0(\Omega, \mathcal{F}) \to H^0(\Omega', \mathcal{F})$ with $\Omega' \subset \Omega$ relatively compact. By [Nar62, (1.1)], we may restrict to those Ω' such that $(\Omega')^{\mathrm{red}}$ is Runge in Ω^{red}. We may thus replace Ω with such an Ω' and assume that Ω is relatively compact in S. Let $\mathcal{N} \subset \mathcal{O}_S$ be the nilradical of $\mathcal{O}_S$. As Ω is relatively compact in S, there exists $k \geq 0$ such that $\mathcal{N}^k = 0$ on Ω. Then $\mathcal{F}$ is a successive extension of $\mathcal{F}/\mathcal{N}\mathcal{F}, \ldots, \mathcal{N}^{k-1}\mathcal{F}/\mathcal{N}^k\mathcal{F}$ and $\mathcal{N}^k\mathcal{F}$. As the set of coherent sheaves satisfying (iii) is stable by extensions (use [NS77, Lemma 1.9]), we may suppose that $\mathcal{N}\mathcal{F} = 0$ on Ω. As $H^0(S, \mathcal{F}) \to H^0(S, \mathcal{F}/\mathcal{N}\mathcal{F})$ is onto by Cartan's Theorem B, we may replace $\mathcal{F}$ with $\mathcal{F}/\mathcal{N}\mathcal{F}$. Now $\mathcal{F}$ may be identified with a coherent sheaf on S^{red} and the required density statement follows from (iii)$^{\mathrm{red}}$. □

1.3. **$\mathcal{O}(S)$-convex compact subsets.** A compact subset K of a complex space S is said to be $\mathcal{O}(S)$-*convex* if $\widehat{K}_{\mathcal{O}(S)} = K$. In the following lemma, $\mathcal{C}^\infty$ strongly plurisubharmonic (psh) functions are meant in the sense of [Nar61, §2].

Lemma 1.3. *Let $\rho : S \to \mathbb{R}$ be a $\mathcal{C}^\infty$ strongly psh function on a reduced Stein space.*

(i) For $c \in \mathbb{R}$, the open subset $\{s \in S \mid \rho(s) < c\}$ of S is Runge.

(ii) For $c \in \mathbb{R}$, if $S_{\leq c} := \{s \in S \mid \rho(s) \leq c\}$ is compact, then it is $\mathcal{O}(S)$-convex.

Proof. Assertion (i) is [Nar61, Theorem 3]. The compactness of $S_{\leq c}$ and (i) together imply that $S_{\leq c}$ has a basis of Runge neighborhoods in S, hence is $\mathcal{O}(S)$-convex. □

We also include the next lemma, proven in [Nar62, (1.1)] for later reference.

Lemma 1.4. *Let S be a Stein space and let $K \subset S$ be an $\mathcal{O}(S)$-convex compact subset. Then K admits a basis of Runge open neighborhoods in S.*

We now analyze how $\mathcal{O}(S)$-convexity behaves with respect to finite morphisms. Our goal is Proposition 1.6 (ii) which will be used in the proof of Proposition 2.4.

Lemma 1.5. *Let $p : T \to S$ be a finite surjective holomorphic map between connected normal Stein spaces. Let $K \subset S$ be a compact subset. Then*

$$\widehat{p^{-1}(K)}_{\mathcal{O}(T)} = p^{-1}(\widehat{K}_{\mathcal{O}(S)}).$$

Proof. The inclusion $\widehat{p^{-1}(K)}_{\mathcal{O}(T)} \subset p^{-1}(\widehat{K}_{\mathcal{O}(S)})$ is obvious. To prove the converse inclusion, we use the *norm morphism* $\mathrm{N}_p : p_*\mathcal{O}_T \to \mathcal{O}_S$ of p (it is defined on the locus over which p is an unramified finite covering of manifolds by summing over the fibers and it extends to all of S by Riemann's extension theorem [GR84, 7, § 4.2]).

Choose $t \in T \setminus \widehat{p^{-1}(K)}_{\mathcal{O}(T)}$ and set $s := p(t)$. Let δ be the maximal cardinality of the fibers of p over some compact neighborhood of K. By the version [For17, Theorem 2.8.4] of the Oka–Weil approximation theorem, there exists $g \in \mathcal{O}(T)$ with $|g| \leq 1$ on $\widehat{p^{-1}(K)}_{\mathcal{O}(T)}$, with $g = 0$ on $p^{-1}(\{s\}) \setminus \{t\}$ and with $g(t) = \delta + 1$.

Then $f := \mathrm{N}_p(g) \in \mathcal{O}(S)$ satisfies $|f| \leq \delta$ on K, and $|f(s)| > \delta$ (to see it, use that p is open by [GR84, 3, §3.2, Criterion of Openness]). Hence $s \notin \widehat{K}_{\mathcal{O}(S)}$. $\square$

Proposition 1.6. *Let $p : T \to S$ be a finite holomorphic map between Stein spaces, and let $K \subset S$ be a compact subset.*

(i) If K is $\mathcal{O}(S)$-convex, then $p^{-1}(K)$ is $\mathcal{O}(T)$-convex.
(ii) If $p^{-1}(K)$ is $\mathcal{O}(T)$-convex and p is surjective, then K is $\mathcal{O}(S)$-convex.

Proof. Assertion (i) is immediate from the definitions and we now prove (ii). Using (i), one may replace T with its normalization and hence assume that T is normal. We may also replace S with its reduction and assume that it is reduced.

Choose $s \in S \setminus K$. To construct $f \in \mathcal{O}(S)$ such that $|f| < 1$ on K and $|f(s)| = 1$, we may first construct it in restriction to the (finite) union of irreducible components of S intersecting $K \cup \{s\}$ and then extend it to S using Cartan's Theorem B. We may thus assume that S has finite dimension n and argue by induction on n.

Let $\widetilde{S}$ be the normalization of S. Using the natural factorization $T \to \widetilde{S} \to S$ of p, we may either assume that S is normal or that p is a normalization morphism. In the first case, we may suppose that S is connected, and replace T by any of its connected components dominating S. The result then follows from Lemma 1.5.

We now deal with the second case where p is a normalization morphism. Let $\mathcal{I} \subset \mathcal{O}_S$ be the annihilitor of the cokernel of $\mathcal{O}_S \hookrightarrow p_*\mathcal{O}_T$ (the *conductor* of p). It is a coherent sheaf of $\mathcal{O}_S$-ideals by [GR84, Annex, §4.4] which is also, in view of its definition, a sheaf of $p_*\mathcal{O}_T$-ideals. We let $S' \subset S$ and $T' \subset T$ be the (possibly nonreduced) complex subspaces defined by $\mathcal{I}$. We get a commutative exact diagram of coherent sheaves on S:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I} \arrow[r] & p_*\mathcal{O}_T \arrow[r] & p_*\mathcal{O}_{T'} \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{I} \arrow[r] \arrow[u, equal] & \mathcal{O}_S \arrow[r] \arrow[u] & \mathcal{O}_{S'} \arrow[r] \arrow[u] & 0. \end{tikzcd} \tag{1.1}$$

By (i) applied to the inclusion $T' \hookrightarrow T$, the subset $p^{-1}(S' \cap K) = T' \cap p^{-1}(K)$ of T' is $\mathcal{O}(T')$-convex. The induction hypothesis applied to $p|_{T'} : T' \to S'$ then shows that $S' \cap K$ is $\mathcal{O}(S')$-convex. Use Lemma 1.3 to choose a Runge neighborhood Ω of $S' \cap K$ in S' such that $s \notin \Omega$. Apply Lemma 1.3 again to construct a relatively compact Runge neighborhood Θ of $p^{-1}(K)$ in T that is disjoint from $p^{-1}(\{s\})$, and such that $T' \cap \Theta \subset p^{-1}(\Omega)$.

The restriction map $\mathcal{O}(\Theta) \to \mathcal{O}(T' \cap \Theta)$ is continuous, and surjective by Cartan's Theorem B. It thus follows from the open mapping theorem for Fréchet spaces that there exists a neighborhood $U \subset \mathcal{O}(T' \cap \Theta)$ of the origin such that for all $a \in U$, there exists $b \in \mathcal{O}(\Theta)$ such that $b|_{T' \cap \Theta} = a$ and $|b| < 1$ on $p^{-1}(K)$. Let $V \subset \mathcal{O}(\Omega)$ be the inverse image of U by the (continuous) pull-back map $\mathcal{O}(\Omega) \to \mathcal{O}(T' \cap \Theta)$.

Set $\mathcal{F} := \mathcal{O}_{S'}$ if $s \notin S'$ and $\mathcal{F} := \mathcal{I}_{\{s\}} \subset \mathcal{O}_{S'}$ if $s \in S'$. By Proposition 1.2 (iii), there exists $c \in H^0(S', \mathcal{F}) \subset \mathcal{O}(S')$ with $(1 + c)|_\Omega \in V$. Set $d := 1 + p^*c \in \mathcal{O}(T')$ so that $a := d|_{T' \cap \Theta} \in U$. By our choice of U, there exists $b \in \mathcal{O}(\Theta)$ with $b|_{T' \cap \Theta} = d|_{T' \cap \Theta}$ and $|b| < 1$ on $p^{-1}(K)$. The Oka–Weil approximation theorem [For17, Theorem 2.8.4] now shows the existence of $e \in \mathcal{O}(T)$ with $e|_{T'} = d$, and such that $|e| < 1$ on $p^{-1}(K)$ and $e = 1$ on $p^{-1}(\{s\})$.

Diagram (1.1) remains exact after taking global sections by Cartan's Theorem B. A diagram chase in the resulting diagram shows the existence of $f \in \mathcal{O}(S)$ with $p^*f = e \in \mathcal{O}(T)$. One has $f(s) = 1$ and $|f| < 1$ on K, so K is $\mathcal{O}(S)$-convex. $\square$

Remarks 1.7. (i) In Proposition 1.6 (ii), one cannot remove the assumption that S and T are Stein, even if f is a reduction morphism (see [Sch70, (8.5)]) or a normalization morphism (see [Mar75, Theorem 3]).

(ii) In the setting of Proposition 1.6, the equality $\widehat{p^{-1}(K)}_{\mathcal{O}(S)} = p^{-1}(\widehat{K}_{\mathcal{O}(T)})$ does not hold in general, for instance if $T = \{(x, y) \in \mathbb{C}^2 \mid xy = 0\}$, if $p : S \to T$ if the normalization morphism, and if $K = \{(x, 0) \in \mathbb{C}^2 \mid |x| = 1\} \subset T$.

1.4. **Stein compact subsets.** If $\mathcal{F}$ is a sheaf on a complex space S and $K \subset S$ is closed, we let $\mathcal{F}(K)$ denote the set of germs of sections of $\mathcal{F}$ in a neighborhood of K.

A compact subset of a complex space S is said to be *Stein* if it admits a basis of Stein open neighborhoods. A *Stein compactum* is the germ of a complex space along a Stein compact subset. As the intersection of two Stein open subsets is Stein (see [GR79, p. 127]), an intersection of Stein compact subsets is again Stein. By Lemma 1.3, an $\mathcal{O}(S)$-convex compact subset of a Stein space is Stein.

Lemma 1.8. *Let S be a Stein space and let $K \subset L \subset S$ be Stein compact subsets. Then the ring morphisms $\mathcal{O}(S) \to \mathcal{O}(K)$ and $\mathcal{O}(L) \to \mathcal{O}(K)$ are flat.*

Proof. That $\mathcal{O}(S) \to \mathcal{O}(K)$ is flat is proven in [Kuc05, Proof of Lemma 2.2]. There, the compact subset K is assumed to be $\mathcal{O}(S)$-convex, but only the fact that it is Stein is used. In addition, the Stein space S is assumed to be a connected manifold. The proof extends to our more general setting, replacing references to Hörmander's book with applications of Cartan's Theorems A and B (in the generality of [GR79]).

Let $(U_i)_{i\in\mathbb{N}}$ be a decreasing basis of Stein open neighborhoods of L in S. By the above, the ring $\mathcal{O}(K)$ is flat over $\mathcal{O}(U_i)$ for all $i \in \mathbb{N}$. By the flatness criterion [Mat80, Theorem 1 (6)], the ring $\mathcal{O}(K)$ is flat over $\mathrm{colim}_{i\in\mathbb{N}}(\mathcal{O}(U_i)) = \mathcal{O}(L)$. □

We will say that a Stein compact subset K of a Stein space S is *excellent* if the ring $\mathcal{O}(K)$ is noetherian. By a theorem of Siu [Siu69, Theorem 1], this is the case if and only if for every germ Z of closed analytic subset along K, the subset $Z \cap K$ of K has finitely many connected components. This holds in particular if K is semianalytic (see [Fri67, Théorème I.9]). Excellent Stein compact subsets are plentiful as the next lemma shows.

Lemma 1.9. *Let S be a Stein space and let $K \subset S$ be a compact subset.*

(i) The subset K has a semianalytic $\mathcal{O}(S)$-convex compact neighborhood in S.
(ii) If K is Stein, it has a basis of semianalytic Stein compact neighborhoods in S.

Proof. After replacing it with S^{red}, we may assume that S is reduced. Let $\rho : S \to \mathbb{R}$ be a real-analytic strongly psh exhaustion function (see [Nar61, Lemma p. 358]). For $c \gg 0$, the semianalytic compact subset $\{s \in S \mid \rho(s) \leq c\}$ is a neighborhood of K which is $\mathcal{O}(S)$-convex by Lemma 1.3 (ii), proving (i).

If K is Stein, it admits a basis of Stein open neighborhoods each of which is exhausted by semianalytic Stein compact subsets by (i). This proves (ii). □

Lemma 1.10. *Let K be a Stein compact subset of a Stein space S. There exists a basis of Stein compact neighborhoods $(L_j)_{j\in J}$ of K in S such that L_j admits a basis $(U_i)_{i\in I_j}$ of Stein open neighborhoods and L_j is $\mathcal{O}(U_i)$-convex for $j \in J$ and $i \in I_j$.*

Proof. Let U be a Stein open neighborhood of K in S. By Lemma 1.9 (i), there exists a compact neighborhood L of K in U that is $\mathcal{O}(U)$-convex. By Lemma 1.3, the subset L admits a basis of Runge open neighborhoods $(U_i)_{i\in I}$ in U. It follows that L is $\mathcal{O}(U_i)$-convex for all $i \in I$. This concludes. □

If K is a compact subset of Stein space S, we define $\mathcal{S}_K \subset \mathcal{O}(S)$ to be the subset of holomorphic functions that do not vanish on K, and $\mathcal{O}(S)_K := \mathcal{O}(S)[\mathcal{S}_K^{-1}]$. Such rings are studied in [Kuc05] and [ABF19], at least if S is reduced.

Lemma 1.11. *Let S be a Stein space and let $K \subset S$ be a compact subset.*

(i) If K is $\mathcal{O}(S)$-convex, the ring morphism $\mathcal{O}(S)_K \to \mathcal{O}(K)$ is faithfully flat.
(ii) The ring $\mathcal{O}(S)_K$ is excellent.
(iii) If K is Stein and excellent, the ring $\mathcal{O}(K)$ is excellent.

Proof. By Lemma 1.8, the ring $\mathcal{O}(K)$ is $\mathcal{O}(S)$-flat, hence $\mathcal{O}(S)_K$-flat. If K is $\mathcal{O}(S)$-convex, the description given in [ABF19, Corollary 3.3 (v)] of the maximal ideals of $\mathcal{O}(S)_K$ now implies assertion (i) (use [Mat80, Theorem 2]).

Assertion (ii) is proven in [ABF19, Theorem 3.5]. As the noetherianity of $\mathcal{O}(S)_K$ is not explicitly proven there, we give an argument. If K' is an excellent $\mathcal{O}(S)$-convex compact neighborhood of K in S (which exists by Lemma 1.9 (i)), then $\mathcal{O}(S)_K$ is a localization of $\mathcal{O}(S)_{K'}$. We may thus assume that K is excellent and $\mathcal{O}(S)$-convex. As $\mathcal{O}(K)$ is noetherian, so is $\mathcal{O}(S)_K$ by (i) and [Bou06, I, §3.5, Corollaire]. To dispel any doubt as to whether the excellence of $\mathcal{O}(S)_K$ holds if S is possibly nonreduced, one can reduce to the reduced case using [KS21, Main Theorem 1].

Assertion (iii) is proven in [Bin76a, Bemerkung pp.152-153] (there, the Stein compact subset K is assumed to be semianalytic but only its excellence is used). □

Let K be a Stein compact subset in a Stein space S. By [Zam76, Theorem 4.7], for every prime ideal $\mathfrak{p} \subset \mathcal{O}(K)$, there exist $s \in K$ and a germ Z of closed analytic subset along K which is *essentially irreducible at s* in the sense of [Zam76, p. 118], and such that $\mathfrak{p} = \{f \in \mathcal{O}(K) \mid f|_Z \text{ vanishes identically near } s\}$.

Lemma 1.12. *With the above notation, the following are equivalent.*

(i) The ring $\mathcal{O}(K)_{\mathfrak{p}}$ is regular.
(ii) The germ of Z at s is not included in the singular locus of S.

Proof. Assume first that $\mathcal{O}(K)_{\mathfrak{p}}$ is regular. Use Lemma 1.9 (ii) to find an excellent Stein compact neighborhood L of K such that Z extends to a germ of closed analytic subset along L. We still denote by $\mathfrak{p}$ the corresponding ideal of $\mathcal{O}(L)$. As $\mathcal{O}(L)_{\mathfrak{p}} \to \mathcal{O}(K)_{\mathfrak{p}}$ is flat by Lemma 1.8, the ring $\mathcal{O}(L)_{\mathfrak{p}}$ is regular (see [Mat80, Theorem 51 (i)]). As $\mathcal{O}(L)$ is excellent by Lemma 1.11 (iii), the singular locus of $\mathrm{Spec}(\mathcal{O}(L))$ is closed, defined by an ideal $I \subset \mathcal{O}(L)$. One can thus choose $f \in I \setminus \mathfrak{p}$. As $f \notin \mathfrak{p}$ and $s \in \mathring{L}$, any neighborhood of s in S contains a point $t \in Z \cap L$ with $f(t) \neq 0$. If $\mathfrak{m} \subset \mathcal{O}(L)$ is the ideal of functions vanishing at t, the ring $\mathcal{O}(L)_{\mathfrak{m}}$ is regular because $f \in I$. It is moreover of dimension $d := \dim_t(Z)$ (see [Zam76, Proof of Corollary 4.9]). As $\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\sim} \mathfrak{m}_{S,t}/\mathfrak{m}_{S,t}^2$ (see [Kie67, below Lemma 2]), it follows that $\mathcal{O}_{S,t}$ is regular hence that S is nonsingular at t (see [GR84, 6, §2.1]).

Assume now that (ii) holds. Let Z' be an irreducible component of the germ of Z at s which is not included in S^{sing}. If $\mathfrak{p}' \subset \mathcal{O}_{S,s}$ is the ideal of functions vanishing on Z', then $(\mathcal{O}_{S,s})_{\mathfrak{p}'}$ is regular by [Hou61, Théorème 3]. Since $\mathcal{O}(K)_{\mathfrak{p}} \to (\mathcal{O}_{S,s})_{\mathfrak{p}'}$ is a flat local morphism of local noetherian rings (by Lemma 1.8 and [Zam76, Proposition 4.11]), the ring $\mathcal{O}(K)_{\mathfrak{p}}$ is regular by [Mat80, Theorem 51 (i)]. □

Recall that a ring morphism $A \to B$ is *regular* if it is flat and the induced morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ has locally noetherian and geometrically regular fibers (see [EGA42, Définition 6.8.1]). Lemma 1.13 is used in the proof of Lemma 3.5.

Lemma 1.13. *Let S be a Stein space and let $K \subset L \subset S$ be excellent Stein compact subsets. Then the ring morphisms $\mathcal{O}(L) \to \mathcal{O}(K)$ and $\mathcal{O}(S)_K \to \mathcal{O}(K)$ are regular.*

Proof. The first assertion is proven in [Bin76b, (2.2)] (there, Lemmas 1.8 and 1.12 are used as well-known facts, and K and L are assumed to be semianalytic but only their excellence is used). We prove the second assertion following the arguments of *loc. cit.* As $\mathcal{O}(K)$ is $\mathcal{O}(S)_K$-flat by Lemma 1.8, and in view of [EGA42, Proposition 6.7.4 d)], it suffices to show that the fibers of $\mathrm{Spec}(\mathcal{O}(K)) \to \mathrm{Spec}(\mathcal{O}(S)_K)$ are regular. Fix $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}(K))$ and let $\mathfrak{q}$ be its image in $\mathrm{Spec}(\mathcal{O}(S))$. We will show the regularity of $\mathcal{O}(K)/\mathfrak{q}\,\mathcal{O}(K)$ at $\mathfrak{p}$. Set $T := \{s \in S \mid f(s) = 0 \text{ for all } f \in \mathfrak{q}\}$.

The restriction morphism $\mathcal{O}(K)/\mathfrak{q}\,\mathcal{O}(K) \to \mathcal{O}(T \cap K)$ is surjective by Cartan's Theorem B. It is also injective, as for any relatively compact Stein open neighborhood U of K in S, one can find $(f_i)_{1 \leq i \leq k}$ in $\mathfrak{q}$ such that $\mathcal{O}_U^{\oplus k} \xrightarrow{(f_i)} \mathcal{I}_T|_U$ is onto, and apply Cartan's Theorem B to show that $\mathcal{I}_T(K) = \mathfrak{q}\,\mathcal{O}(K)$. After replacing S with T and K with $T \cap K$, we may thus assume that $\mathfrak{q} = 0$ (and S is reduced).

Let $s \in K$ and Z be as in Lemma 1.12. As $\mathfrak{q} = 0$, no nonzero element of $\mathcal{O}(S)$ vanishes on the germ of Z at s. As S is reduced, S^{sing} has a nonzero equation in $\mathcal{O}(S)$, and hence the germ of Z at s is not included in S^{sing}. By Lemma 1.12, the ring $\mathcal{O}(K)_{\mathfrak{p}}$ is regular, as wanted. □

1.5. **Meromorphic functions and ramified coverings.** We let $\mathcal{M}$ denote the sheaf of germs of meromorphic functions on a complex space (see [GR84, 6, §3.1]). If S is a reduced and irreducible Stein space, then $\mathcal{M}(S)$ is the fraction field of the domain $\mathcal{O}(S)$ (because the sheaf of denominators of $h \in \mathcal{M}(S)$ defined in [GR84, 6, §3.2] is coherent and nonzero, hence admits a nonzero global section). The following proposition is classical in dimension 1, see e.g. [Sho94, 1, §4.14, Corollaries 4 and 5].

Proposition 1.14. *Let S be a connected normal Stein space. Then the functor*

$$\left\{ \begin{array}{c} \textit{finite surjective holomorphic maps} \\ p : T \to S \textit{ of connected normal Stein spaces} \end{array} \right\} \to \left\{ \begin{array}{c} \textit{finite field extensions} \\ \mathcal{M}(S) \subset F \end{array} \right\}$$

which associates with $p : T \to S$ the extension $\mathcal{M}(S) \subset \mathcal{M}(T)$ of meromorphic function fields is an equivalence of categories.

Proof. We first show that the functor is well-defined. Let S^{sing} and T^{sing} be the singular loci of S and T. Then $Z := S^{\mathrm{sing}} \cup p(T^{\mathrm{sing}})$ is a closed analytic subset of S which has codimension ≥ 2 by [GR84, 6, §5.3]. Set $U := S \setminus Z$. The locus $W \subset T \setminus p^{-1}(Z)$ where p is not a local biholomorphism is an analytic subset of pure codimension 1 because it is the zero locus of the Jabobian of p. Over $U \setminus p(W)$, the map p is a finite local biholomorphism. As $U \setminus p(W)$ is connected by [GR84, 9, §1.2], the map $p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is a d-sheeted analytic covering of complex manifolds in the sense of [GR84, 7, §2.1], for some $d \geq 1$. It then follows from [GR84, 7, §3.1, Corollary 2] that the field extension $\mathcal{M}(U) \subset \mathcal{M}(p^{-1}(U))$ is finite (of degree $\leq d$). As $\mathcal{M}(S) = \mathcal{M}(U)$ and $\mathcal{M}(T) = \mathcal{M}(p^{-1}(U))$ by [GR84, 9, §5.2], the field extension $\mathcal{M}(S) \subset \mathcal{M}(T)$ is finite (of degree $\leq d$).

Let $p_1 : T_1 \to S$ and $p_2 : T_2 \to S$ be two finite surjective holomorphic maps of connected normal Stein spaces. It follows from [Iss66, Theorem II] that any morphism $\mathcal{M}(T_1) \subset \mathcal{M}(T_2)$ of $\mathbb{C}$-algebras is induced by a unique holomorphic map $f : T_2 \to T_1$, and that the former is a morphism of $\mathcal{M}(S)$-algebras if and only if $p_2 = p_1 \circ f$. This shows that the functor is full and faithful.

We finally prove that the functor is essentially surjective. Let F be a finite extension of degree d of $\mathcal{M}(S)$. Let $P(x) = x^d + \sum_{i=0}^{d-1} a_i x^i \in \mathcal{M}(S)[x]$ be an irreducible polynomial with splitting field F. Let $b \in \mathcal{O}(S)$ be a nonzero element with $ba_i \in \mathcal{O}(S)$ for $i \in \{0, \ldots, d-1\}$, and let $R \subset \mathbb{P}^1(\mathbb{C}) \times S$ be the zero locus of $bX^d + \sum_{i=0}^{d-1} ba_i X^i Y^{d-i}$, where $[X : Y]$ are homogeneous coordinates of $\mathbb{P}^1(\mathbb{C})$. Let R' be an irreducible component of R dominating S, let $R' \to R'' \to S$ be the Stein factorization of the projection $R' \to S$ (see [GR84, 10, §6.1]), and let T be the normalization of R'', with projection $p : T \to S$. As the inverse image by $R \to S$ of a general point of S has cardinality $\leq d$, the same holds for the inverse image by π of a general point of S. We deduce from the first paragraph of this proof that the field extension $\mathcal{M}(S) \subset \mathcal{M}(T)$ has degree $\leq d$. As the element $X/Y \in \mathcal{M}(T)$ is annihilated by P, we get inclusions $\mathcal{M}(S) \subset F \subset \mathcal{M}(T)$. A degree argument now shows that F and $\mathcal{M}(T)$ are isomorphic extensions (of degree d) of $\mathcal{M}(S)$. $\square$

Remark 1.15. It follows from the proof of Proposition 1.14 that a degree d finite extension $\mathcal{M}(S) \subset F$ corresponds to a d-sheeted analytic covering $p : T \to S$.

We deduce at once from Proposition 1.14 the following corollary.

Corollary 1.16. *Let S be a normal Stein space of pure dimension n with finitely many connected components. Associating with $p : T \to S$ the $\mathcal{M}(S)$-algebra $\mathcal{M}(T)$ induces an equivalence of categories*

$$\left\{ \begin{array}{c} \textit{finite surjective holomorphic maps } p : T \to S \\ \textit{of normal Stein spaces of pure dimension } n \end{array} \right\} \to \left\{ \begin{array}{c} \textit{finite étale} \\ \mathcal{M}(S)\textit{-algebras} \end{array} \right\}.$$

The next lemma will be used in the proofs of Propositions 2.1 (ii) and 4.2.

Lemma 1.17. *Let $q : R \to S$ be a finite surjective holomorphic map of connected normal Stein spaces. There exists a finite surjective holomorphic map $p' : T \to R$ of connected normal Stein spaces and a finite group Γ acting on T and acting trivially on S, such that $p := q \circ p'$ is Γ-equivariant and Γ acts transitively on the fibers of p.*

Proof. The field extension $\mathcal{M}(S) \subset \mathcal{M}(R)$ is finite by Proposition 1.14. Let $\mathcal{M}(S) \subset F$ be its Galois closure, with Galois group Γ. Let $p' : T \to R$ be the holomorphic map associated with $\mathcal{M}(R) \subset F$ through the equivalence of categories of Proposition 1.14. By functoriality, the finite group Γ acts on T and $p := q \circ p'$ is Γ-equivariant. The quotient complex space T/Γ (see [Car57a, Theorem 4]) is connected, normal and admits a natural finite holomorphic map $T/\Gamma \to S$. One deduces from the field inclusions $\mathcal{M}(S) \subset \mathcal{M}(T/\Gamma) \subset \mathcal{M}(T)^\Gamma = \mathcal{M}(S)$ that $\mathcal{M}(S) = \mathcal{M}(T/\Gamma)$. In view of Proposition 1.14, the projection $T/\Gamma \to S$ is a biholomorphism. Consequently, the group Γ acts transitively on the fibers of p. $\square$

2. Killing cohomology classes on finite coverings

The goal of this section is Proposition 2.4, proven in §2.3 by induction on the degree of a cohomology class. We cover the base case of the induction in §2.1 and the induction procedure relies on Grauert's bump method described in §2.2.

2.1. **Finite coverings of $\mathcal{O}(S)$-convex compact subsets.** The following proposition will be key in dealing with degree 1 cohomology classes.

Proposition 2.1. *Let S be a Stein manifold of dimension n, let $K \subset S$ be an $\mathcal{O}(S)$-convex compact subset, let $U \subset S$ be an open neighborhood of K and let $f : \widetilde{U} \to U$ be a surjective finite local biholomorphism.*

(i) After maybe shrinking U, there exist a finite surjective holomorphic map $q : R \to S$ and an open embedding $h : \widetilde{U} \hookrightarrow R$ such that $q \circ h = f$.
(ii) After maybe shrinking U, there exist a finite surjective holomorphic map $p : T \to S$ and a holomorphic map $g : p^{-1}(U) \to \widetilde{U}$ with $f \circ g = p|_{p^{-1}(U)}$.

Proof of (i). By Lemma 1.3, we may assume that U is Stein after shrinking it, hence that so is $\widetilde{U}$ by [GR79, V, §1, Theorem 1 d)]. By [Nar60, Theorem 3], one can then find an embedding $i : \widetilde{U} \hookrightarrow \mathbb{C}^N$ (with $N = 2n + 1$). Consider the embedding $j : \widetilde{U} \hookrightarrow \mathbb{C}^N \times U$ defined by $j(z) = (i(z), f(z))$. Write $\mathrm{pr}_1 : \mathbb{C}^N \times U \to \mathbb{C}^N$ and $\mathrm{pr}_2 : \mathbb{C}^N \times U \to U$ for the projections. For any continuous $\varepsilon : U \to]0, +\infty[$, define

$$\Theta_\varepsilon := \{(a, b) \in \mathbb{C}^N \times U \mid \exists\, c \in \widetilde{U} \text{ with } f(c) = b \text{ and } |a - i(c)| < \varepsilon(b)\}. \tag{2.1}$$

Choose ε sufficiently small so that the element c in (2.1) is always unique. The subset $\Theta_\varepsilon \subset \mathbb{C}^N \times U$ is then a tubular neighborhood of $j(\widetilde{U})$. Define $\pi : \Theta_\varepsilon \to \widetilde{U}$ such that $\pi(a, b) = c$. By [CM86, Proposition 2.1], there exists a neighborhood Ω of $j(\widetilde{U})$ in Θ_ε that is Runge in $\mathbb{C}^N \times U$. Finally, choose $\delta : U \to]0, +\infty[$ continuous sufficiently small so that $\overline{\Theta_\delta} \subset \Omega$.

Define a holomorphic map $\phi : \Omega \to \mathbb{C}^N$ by setting $\phi(z) = \mathrm{pr}_1(z) - i \circ \pi(z)$. The map ϕ is submersive along $\overline{\Theta_\delta}$, and its zero locus in $\overline{\Theta_\delta}$ is included in Θ_δ and projects biholomorphically to $\widetilde{U}$ by π. All these properties persist (after maybe shrinking U) if one replaces ϕ with a holomorphic map $\phi' : \Omega \to \mathbb{C}^N$ close enough to ϕ on $\overline{\Theta_\delta} \cap \mathrm{pr}_2^{-1}(K)$ in the C^0-topology (hence in the C^1-topology by the Cauchy estimates). We construct such a ϕ' as follows. First approximate ϕ by the restriction of a holomorphic map $\mathbb{C}^N \times U \to \mathbb{C}^N$ using that Ω is Runge in $\mathbb{C}^N \times U$. Expand this map as N power series in N variables with coefficients in $\mathcal{O}(U)$, truncate these power series to turn them into N polynomials in N variables with coefficients in $\mathcal{O}(U)$, and then use the Oka–Weil approximation theorem [GR65, VII, A, Theorem 6] to approximate these coefficients on K by restrictions of elements of $\mathcal{O}(S)$. Let $\psi : \mathbb{C}^N \times S \to \mathbb{C}^N$ be the holomorphic map (polynomial in the first variable) obtained in this way and set $\phi' := \psi|_\Omega$.

Let $\Psi_1, \ldots, \Psi_N \in \mathcal{O}(S)[X_0, \ldots, X_N]$ be the homogenizations of the components of ψ, and let $\widehat{R} \subset \mathbb{P}^N(\mathbb{C}) \times S$ be the locus where they all vanish, with projection $\hat{q} : \widehat{R} \to S$. The map $\hat{q}$ is surjective by [Har77, I, Theorem 7.2]. Hence so is the finite holomorphic map $q : R \to S$ appearing in the Stein factorization [GR84, 10, §6.1] of $\hat{q}$. By our choice of ϕ', the intersection of $\widehat{R}$ with Θ_δ is biholomorphic (via the map π) to $\widetilde{U}$. The resulting open embedding $\hat{h} : \widetilde{U} \to \widehat{R}$ satisfies $\hat{q} \circ \hat{h} = f$.

The uniqueness of the Stein factorization shows that it is compatible with base change by open embeddings. As $\hat{h}(\widetilde{U})$ is a connected component of $\hat{q}^{-1}(U)$ such that $\hat{q}|_{\hat{h}(\widetilde{U})} : \hat{h}(\widetilde{U}) \to U$ is finite (as f is), we deduce that the composition of $\hat{h}$ with the natural map $\widehat{R} \to R$ is an open embedding $h : \widetilde{U} \to R$ such that $q \circ h = f$. □

Proof of (ii). Let $q : R \to S$, and $h : \widetilde{U} \hookrightarrow Y$ be as in (i). We may assume that R is normal after normalizing it, and that S is connected after replacing it with one of its connected components. After shrinking U, we may suppose that it has finitely many connected components $U_1, \dots, U_r$. If $p_i : T_i \to S$ solves our problem for $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ instead of f, then we may choose T to be the fiber product $T_1 \times_S \cdots \times_S T_r$. We may thus assume that U is connected. Replacing $\widetilde{U}$ with one of its connected components, we may also suppose that $\widetilde{U}$ is connected. Finally replacing R with its connected component containing $h(\widetilde{U})$, we reduce to the case where R is connected.

Let $p' : T \to R$ and Γ be as in Lemma 1.17, and set $p := q \circ p'$. Let V be a connected component of $p^{-1}(U)$. By the transitivity of the action of Γ on the fibers of p, one may choose $\gamma_V \in \Gamma$ such that $\gamma_V(V) \subset p'^{-1}(\widetilde{U})$. One then defines $g(x) := p' \circ \gamma_V(x)$ for $x \in V$. $\square$

2.2. **Grauert's bump method.** To prove Proposition 2.4, we will use Grauert's bump method as developed by Henkin and Leiterer [HL98]. In Proposition 2.3 below, we sum up what we exactly need, relying on the exposition by Forstnerič [For17].

Lemma 2.2. *Let S be a Stein manifold of dimension n. Any point $s \in S$ admits an $\mathcal{O}(S)$-convex compact neighborhood $B \subset S$ such that B admits a basis of contractible open neighborhoods in S.*

Proof. By [Nar60, Theorem 3], one may view S as a submanifold of $\mathbb{C}^N$ for $N \gg 0$. Then $\rho : S \to \mathbb{R}$ defined by $\rho(x) := |x - s|^2$ is a $\mathcal{C}^\infty$ strongly psh exhaustion function, and $B := \{\rho \leq \varepsilon\}$ works for $0 < \varepsilon \ll 1$ (see Lemma 1.3 (ii)). $\square$

Proposition 2.3. *Let S be a Stein manifold of dimension n. Then there exist sequences $(A_0, A_1, \dots)$ and $(B_0, B_1, \dots)$ of compact subsets of S such that:*

(i) the subsets A_i and B_i are $\mathcal{O}(S)$-convex;
(ii) the subset B_i has a basis of contractible open neighborhoods;
(iii) one has $A_0 = \varnothing$ and $A_{i+1} \subset A_i \cup B_i$;
(iv) for any compact subset $K \subset S$, one has $K \subset A_i$ for $i \gg 0$.

Proof. Let $\rho : S \to \mathbb{R}$ be a $\mathcal{C}^\infty$ strongly psh exhaustion function (see [Nar61, Lemma p. 358]). By [GG73, Proposition 6.13] and [For17, Lemma 3.10.3], we may assume that its critical points are nondegenerate, have distinct images by ρ, and are nice in the sense of [For17, Definition 3.10.2].

We adopt the following nonstandard terminology. We say that a pair (A, A') of $\mathcal{O}(S)$-convex compact subsets of S is a *bump* if there exists an $\mathcal{O}(S)$-convex compact subset $B \subset S$ with a basis of contractible neighborhoods with $A' \subset A \cup B$. The proof has two parts, which, combined, prove the proposition.

In the first part (the *noncritical extension case*), we show that if $c < c'$ are such that ρ has no critical values in $[c, c']$, then one can go from $A := \{\rho \leq c\}$ to $A' := \{\rho \leq c'\}$ by a sequence of bumps containing A. This follows at once from [For17, Proof of Lemma 5.10.3]. Indeed, it is explained there how to go from A to A' by an increasing sequence of sublevel sets of (varying) $\mathcal{C}^\infty$ psh exhaustion functions (which are $\mathcal{O}(S)$-convex by Lemma 1.3 (ii)), in a way that the difference between two successive ones is included in an open subset of any fixed open covering of A'. In view of Lemma 2.2, one can ensure that this difference is included in an $\mathcal{O}(S)$-convex compact subset with a basis of contractible neighborhoods.

In the second part (the *critical extension case*), we fix a critical value of ρ (assumed without loss of generality to be equal to 0), with associated critical point $p_0 \in S$. We show that there exist $c < 0 < c'$ such that one can go from $A := \{\rho \leq c\}$

to $A' := \{\rho \leq c'\}$ by a sequence of bumps containing A. To do so, use Lemma 2.2 to choose a compact $\mathcal{O}(S)$-convex neighborhood of p_0 in S with a basis of contractible open neighborhoods. Apply the constructions of [For17, p. 102] with $q = 1$ and $U \subset B$ an open neighborhood of p_0. In particular, choose $c_0 > 0$ small enough so that [For17, Lemma 3.11.4] applies, and let $0 < t_0 < t_1$ and $\tau : \{\rho < 3c_0\} \to \mathbb{R}$ be as in *loc. cit.* Set $c := -t_0$ and $c' := c_0$. One can go from A to $\{\tau \leq t_1 - t_0\}$ by a bump by [For17, Lemma 3.11.4 (c)]. Arguing as in the noncritical extension case (using the $\mathcal{C}^\infty$ strongly psh function τ on $\Omega := \{\rho < 3c_0\}$, which is legitimate by [For17, Proposition 3.11.4 (d)]), one sees that one can go from $\{\tau \leq t_1 - t_0\}$ to $\{\tau \leq 2c_0\}$ by a sequence of bumps (the involved compact $\mathcal{O}(\Omega)$-convex sets are also $\mathcal{O}(S)$-convex because Ω is Runge in S by Lemma 1.3 (i)). Finally, one can go from $\{\tau \leq 2c_0\}$ to A' by a bump by [For17, Lemma 3.11.4 (b)]. ☐

2.3. **Induction on the cohomological degree.** We finally reach our goal.

Proposition 2.4. *Let S be a Stein space, let $K \subset S$ be an $\mathcal{O}(S)$-convex compact subset, and let $U \subset S$ be an open neighborhood of K. Fix integers $k, m \geq 1$. For all $\alpha \in H^k(U, \mathbb{Z}/m)$, there exists a finite surjective holomorphic map $p : T \to S$ and an open neighborhood V of K in U such that $\alpha|_{p^{-1}(V)} = 0$ in $H^k(p^{-1}(V), \mathbb{Z}/m)$.*

Proof. We argue by induction on $k \geq 1$, which we fix. Using Proposition 1.6 (i), we may replace S with an irreducible component of its reduction, and assume that S is irreducible, hence of finite dimension n. We further reduce to the case where S is a manifold as follows. Let $i : S \to S'$ be as in Lemma 1.1, fix a continuous retraction $r : S' \to i(S)$ and replace S, K, U and α by S', $i(K)$, $r^{-1}(U)$ and $r^*\alpha$, noting that $i(K)$ is $\mathcal{O}(i(S))$-convex by Proposition 1.6 (ii), hence $\mathcal{O}(S')$-convex.

Assume first that $k = 1$. Let $f : \widetilde{U} \to U$ be the degree m unramified cyclic covering associated with α. After maybe shrinking U, choose $p : T \to S$ and $g : p^{-1}(U) \to \widetilde{U}$ with $f \circ g = p|_{p^{-1}(U)}$ as in Proposition 2.1 (ii). As $f^*\alpha = 0$ by choice of f, one has $\alpha|_{p^{-1}(U)} = g^*f^*\alpha = 0$, as wanted.

Assume now that $k \geq 2$. After shrinking U, we may assume that it is a Runge domain in S (see Lemma 1.3). Let $(A_0, A_1, \dots)$ and $(B_0, B_1, \dots)$ be as in Proposition 2.3 applied to the Stein manifold U. As U is Runge in X, it follows from Proposition 2.3 (i) that the A_i and the B_i are $\mathcal{O}(S)$-convex compact subsets of S.

We will now show by induction on $i \geq 0$ that there exists an open neighborhood V_i of A_i in U and a finite surjective holomorphic map $p_i : T_i \to S$ such that $\alpha|_{p_i^{-1}(V_i)} = 0$ in $H^k(p_i^{-1}(V_i), \mathbb{Z}/m)$. When $i = 0$, one can take $V_0 = \varnothing$.

Assume that we have proven the statement for i and let us prove it for $i+1$. Let W_i be a contractible open neighborhood of B_i in U (see Proposition 2.3 (ii)). Consider the boundary map

$$\partial : H^{k-1}(p_i^{-1}(V_i \cap W_i), \mathbb{Z}/m) \to H^k(p_i^{-1}(V_i \cup W_i), \mathbb{Z}/m)$$

in the Mayer–Vietoris exact sequence. As $\alpha|_{p_i^{-1}(V_i)} = 0$ by the induction hypothesis on i and $\alpha|_{p_i^{-1}(W_i)} = (p_i|_{p_i^{-1}(W_i)})^*(\alpha|_{W_i}) = 0$ because W_i is contractible, there exists $\beta \in H^{k-1}(p_i^{-1}(V_i \cap W_i), \mathbb{Z}/m)$ such that $\partial(\beta) = \alpha|_{p_i^{-1}(V_i \cup W_i)}$. By the induction hypothesis on k (applied to the $\mathcal{O}(T_i)$-convex compact subset $p_i^{-1}(A_i \cap B_i)$ of T_i, see Proposition 1.6 (i)), there exists a finite surjective holomorphic map $q_i : T_{i+1} \to T_i$ and an open neighborhood Ω_i of $p_i^{-1}(A_i \cap B_i)$ in $p_i^{-1}(V_i \cap W_i)$ such that $\beta|_{q_i^{-1}(\Omega_i)} = 0$. Let V_i' and W_i' be open neighborhoods of A_i and B_i in V_i and W_i respectively, such that $p_i^{-1}(V_i' \cap W_i') \subset \Omega_i$. Set $p_{i+1} := p_i \circ q_i$ and $V_{i+1} := V_i' \cup W_i'$. Then $\alpha|_{p_{i+1}^{-1}(V_{i+1})}$ is the image of $\beta|_{p_{i+1}^{-1}(V_i' \cap W_i')}$ by the boundary map

$$\partial' : H^{k-1}(p_{i+1}^{-1}(V_i' \cap W_i'), \mathbb{Z}/m) \to H^k(p_{i+1}^{-1}(V_i' \cup W_i'), \mathbb{Z}/m)$$

of the Mayer–Vietoris exact sequence, by compatibility of ∂ and ∂'. As $\beta|_{p_{i+1}^{-1}(V_i' \cap W_i')}$ vanishes (because $\beta|_{q_i^{-1}(\Omega_i)} = 0$), it follows that $\alpha|_{p_{i+1}^{-1}(V_{i+1})} = 0$, as required.

In view of Proposition 2.3 (iv), one can take $V = V_i$ and $p = p_i$ for $i \gg 0$. □

3. A relative comparison theorem

After studying analytifications of algebraic varieties over Stein algebras in §3.1, and étale sheaves on them in §3.2, we prove the relative Artin comparison theorem in Stein geometry in §3.3 (Theorem 3.7).

3.1. **Analytification of algebraic varieties over Stein algebras.** Fix a Stein space S. Beware that the ring $\mathcal{O}(S)$ is in general not noetherian. Let (P) be a property of morphisms of schemes. A morphism $f : X \to Y$ of $\mathcal{O}(S)$-schemes is said to *interiorly* satisfy (P) if there exists an open covering $(U_i)_{i \in I}$ of S such that the morphisms $f_{\mathcal{O}(U_i)} : X_{\mathcal{O}(U_i)} \to Y_{\mathcal{O}(U_i)}$ satisfy (P). We say that an $\mathcal{O}(S)$-scheme X interiorly satisfies (P) if so does the structural morphism $X \to \mathrm{Spec}(\mathcal{O}(S))$.

Lemma 3.1. *Let (P) be a property of morphisms of schemes that is stable by base change and fpqc local on the base. Let S be a Stein space and let $f : X \to Y$ be a morphism of $\mathcal{O}(S)$-schemes that interiorly satisfies (P).*

(i) For $K \subset S$ compact, $f_{\mathcal{O}(K)} : X_{\mathcal{O}(K)} \to Y_{\mathcal{O}(K)}$ satisfies (P).
(ii) For $K \subset S$ compact, $f_{\mathcal{O}(S)_K} : X_{\mathcal{O}(S)_K} \to Y_{\mathcal{O}(S)_K}$ satisfies (P).
(iii) For $U \subset S$ open and relatively compact, $f_{\mathcal{O}(U)} : X_{\mathcal{O}(U)} \to Y_{\mathcal{O}(U)}$ satisfies (P).

Proof. To prove (i) and (ii), we may replace K with $\widehat{K}_{\mathcal{O}(S)}$ (by base change), and consequently assume that K is $\mathcal{O}(S)$-convex, hence Stein. Any $s \in S$ has a basis of Stein compact neighborhoods (see Lemma 1.9 (ii)). By hypothesis on f and the base change property, each $s \in K$ has a Stein compact neighborhood K_s such that $f_{\mathcal{O}(K_s)}$ satisfies (P). By compactness of K, we may extract a finite family $(K_i)_{1 \leq i \leq k}$ covering K. Replacing K_i with $K_i \cap K$ (and using the base change property again), we may assume that $K_i \subset K$ for $1 \leq i \leq k$. The morphism $\mathcal{O}(K) \to \prod_{i=1}^k \mathcal{O}(K_i)$ is flat by Lemma 1.8, hence faithfully flat by [Mat80, Theorem 2 (i)⇔(iii)] and the description of the maximal ideals of $\mathcal{O}(K)$ given in [Zam76, Corollary 3.3]. Assertion (i) follows since (P) is fpqc local on the base. So does (ii) by Lemma 1.11 (i). Assertion (iii) follows from (i) applied with $K = \overline{U}$ (by base change). □

An $\mathcal{O}(S)$-scheme interiorly locally of finite type is interiorly locally of finite presentation (use Lemmas 1.9 (ii) and 3.1 (i)), i.e., belongs to the category $\mathbb{K}$ in the terminology of Bingener [Bin76b, p. 23]. With such an $\mathcal{O}(S)$-scheme X, one can associate its analytification X^{an} (see [Bin76b, Satz 1.1] when X is of finite presentation over $\mathcal{O}(S)$; the construction and its properties extend under our more general hypotheses as indicated in [Bin76b, p. 23]). It is a (possibly nonseparated) complex space over S endowed with a map $i_X : X^{\mathrm{an}} \to X$ of locally ringed spaces such that the map

$$\begin{array}{ccc}\mathrm{Hom}_S(T,X^{\mathrm{an}}) & \to & \mathrm{Hom}_{\mathcal{O}(S)}(T,X)\\ h & \mapsto & i_X\circ h.\end{array}$$

is bijective for all complex spaces T over S. In particular, with each point $x\in X^{\mathrm{an}}$ corresponds a closed point of X with complex residue field, also denoted by x.

This construction is functorial [Bin76b, p. 2] (and we let $f^{\mathrm{an}}:X^{\mathrm{an}}\to Y^{\mathrm{an}}$ denote the holomorphic map induced by a morphism $f:X\to Y$ of $\mathcal{O}(S)$-schemes interiorly locally of finite type), commutes with the formation of fiber products [Bin76b, p. 3], and is compatible with change of the base Stein space [Bin76b, (1.2)].

Lemmas 3.2 and 3.3 give examples of $\mathcal{O}(S)$-schemes interiorly of finite type.

Lemma 3.2. *Let Z be a closed analytic subspace of a Stein space S. Then there exists an open immersion $j:V\hookrightarrow\mathrm{Spec}(\mathcal{O}(S))$ interiorly of finite type such that j^{an} identifies with $S\setminus Z\hookrightarrow S$. If Z is set-theoretically defined by the vanishing of finitely many elements of $\mathcal{O}(S)$, one may moreover choose j to be quasi-compact.*

Proof. Set $V:=\mathrm{Spec}(\mathcal{O}(S))\setminus\mathrm{Spec}(\mathcal{O}(S)/\mathcal{I}_Z(S))$, where $\mathcal{I}_Z$ be the ideal sheaf of Z in S. Let $U\subset S$ be a relatively compact Stein open subset. By Lemma 3.4 below,

$$V_U:=V\times_{\mathrm{Spec}(\mathcal{O}(S))}\mathrm{Spec}(\mathcal{O}(U))=\mathrm{Spec}(\mathcal{O}(U))\setminus\mathrm{Spec}(\mathcal{O}(U)/\mathcal{I}_Z(U))$$

and $\mathcal{I}_Z(U)$ is finitely generated over $\mathcal{O}(U)$. We deduce that V_U is an $\mathcal{O}(U)$-scheme of finite type. The description of j^{an} follows from [Bin76b, Proof of Satz 1.1].

If Z is set-theoretically defined by the vanishing of finitely many elements of $\mathcal{O}(S)$, we apply the above construction after replacing Z with the complex subspace defined by the vanishing of these equations. This ensures that $\mathcal{I}_Z(S)$ is a finitely generated ideal of $\mathcal{O}(S)$ and hence that j is quasi-compact. □

Lemma 3.3. *If $p:T\to S$ is a finite holomorphic map of Stein spaces, then the induced morphism $f:\mathrm{Spec}(\mathcal{O}(T))\to\mathrm{Spec}(\mathcal{O}(S))$ is interiorly finite and $f^{\mathrm{an}}=p$.*

Proof. Let $U\subset S$ be a relatively compact Stein open subset. By Lemma 3.4 below applied to $\mathcal{F}:=p_*\mathcal{O}_T$, the $\mathcal{O}(U)$-module $\mathcal{O}(T)\otimes_{\mathcal{O}(S)}\mathcal{O}(U)$ is finitely generated. It follows that f is interiorly finite. As $\mathcal{O}(T)\otimes_{\mathcal{O}(S)}\mathcal{O}(U)\xrightarrow{\sim}\mathcal{O}(p^{-1}(U))$ by Lemma 3.4, that $f^{\mathrm{an}}=p$ may be verified locally on S. We may thus assume that S is finite-dimensional, in which case it follows from [Bin76b, (1.3)]. □

Lemma 3.4. *Let U be a relatively compact Stein open subset of a Stein space S. Let $\mathcal{F}$ be a coherent sheaf on S. The $\mathcal{O}(U)$-module $\mathcal{F}(U)$ is generated by finitely many elements of $\mathcal{F}(S)$, and $\mathcal{F}(S)\otimes_{\mathcal{O}(S)}\mathcal{O}(U)\to\mathcal{F}(U)$ is an isomorphism.*

Proof. As U is relatively compact, Cartan's Theorem A implies that there exist $a_1,\dots,a_r\in\mathcal{F}(S)$ generating $\mathcal{F}$ on U. The first assertion and the surjectivity of $\mathcal{F}(S)\otimes_{\mathcal{O}(S)}\mathcal{O}(U)\to\mathcal{F}(U)$ follow from the vanishing of $H^1(U,\mathrm{Ker}(\mathcal{O}_U^{\oplus r}\xrightarrow{a_i}\mathcal{F}|_U))$.

Now, let $\sum_{j=1}^s b_j\otimes c_j\in\mathcal{F}(S)\otimes_{\mathcal{O}(S)}\mathcal{O}(U)$ be such that $\sum_{j=1}^s b_jc_j=0$ in $\mathcal{F}(U)$. Consider the exact sequence $0\to\mathcal{N}\to\mathcal{O}_S^{\oplus s}\xrightarrow{b_j}\mathcal{F}$, so $(c_j)\in\mathcal{N}(U)$. By the surjectivity result applied to $\mathcal{N}$, there exist $(d_{j,1}),\dots,(d_{j,t})\in\mathcal{N}(S)\subset\mathcal{O}(S)^{\oplus s}$ and $e_1,\dots,e_t\in\mathcal{O}(U)$ with $\sum_{k=1}^t e_kd_{j,k}=c_j$ for $1\leq j\leq s$. Then

$$\sum_{j=1}^s b_j\otimes c_j=\sum_{j=1}^s\sum_{k=1}^t b_j\otimes e_kd_{j,k}=\sum_{k=1}^t\Big(\sum_{j=1}^s b_jd_{j,k}\Big)\otimes e_k=0.$$ □

3.2. **Étale sheaves on algebraic varieties over Stein algebras.** Let S be a Stein space, and let X be an $\mathcal{O}(S)$-scheme interiorly locally of finite type. There is a commutative diagram of morphisms of sites

$$\begin{tikzcd} (X^{\mathrm{an}})_{\mathrm{cl}} \arrow[r, "\delta"] \arrow[d, "\varepsilon"] & X^{\mathrm{an}} \arrow[d, "i_X"] \\ X_{\text{ét}} \arrow[r] & X, \end{tikzcd} \tag{3.1}$$

where $X_{\text{ét}}$ is the small étale site of the scheme X, and $X^{\mathrm{an}}_{\mathrm{cl}}$ is the site of local isomorphisms of the topological space X^{an} (see [SGA43, XI, §4.0]). Indeed, if $f : Y \to X$ is étale, then $f^{\mathrm{an}} : Y^{\mathrm{an}} \to X^{\mathrm{an}}$ is a local homeomorphism by [Bin76b, Satz 3.1 (1)]. It follows from [SGA41, III, Théorème 4.1] that δ_* induces an equivalence of topoi.

If $\mathbb{L}$ is an abelian sheaf on $X_{\text{ét}}$, we define $\mathbb{L}^{\mathrm{an}} := \delta_*\varepsilon^*\mathbb{L}$. For $x \in X^{\mathrm{an}}$, one has $\mathbb{L}^{\mathrm{an}}_x = \mathbb{L}_x$. Finally, we say that $\mathbb{L}$ is *interiorly constructible* if there exists an open covering $(U_i)_{i\in I}$ of S such that the étale sheaves $\mathbb{L}|_{X_{\mathcal{O}(U_i)}}$ are constructible.

Lemma 3.5. *Let K be an excellent Stein compact subset of a Stein space S. Let $f : X \to Y$ be a morphism interiorly of finite type of $\mathcal{O}(S)$-schemes interiorly locally of finite type. Fix a torsion abelian sheaf $\mathbb{L}$ on $X_{\text{ét}}$. Then the base change morphisms $(\mathrm{R}^k f_*\mathbb{L})|_{Y_{\mathcal{O}(K)}} \to \mathrm{R}^k f_{\mathcal{O}(K),*}(\mathbb{L}|_{X_{\mathcal{O}(K)}})$ are isomorphisms for $k \geq 0$.*

Proof. Fix $k \geq 0$. As $\mathcal{O}(S) \to \mathcal{O}(S)_K$ is a localization, the description [SGA42, VIII, Théorème 5.2] of the fibers of higher push-forwards in étale cohomology implies that the base change morphism

$$\mathrm{R}^k f_*(\mathbb{L})|_{Y_{\mathcal{O}(S)_K}} \to \mathrm{R}^k f_{\mathcal{O}(S)_K,*}(\mathbb{L}|_{X_{\mathcal{O}(S)_K}}) \tag{3.2}$$

is an isomorphism. By Néron–Popescu desingularization (see [Swa98, Theorem 1.1]) and Lemma 1.13, the ring $\mathcal{O}(K)$ is a filtered colimit of smooth $\mathcal{O}(S)_K$-algebras. By the smooth base change theorem [SGA43, Exp. XVI, Corollaire 1.2] and the commutation of étale cohomology with filtered limits of schemes with affine transition maps [SGA42, Exp. VII, Corollaire 5.8], the base change morphism

$$(\mathrm{R}^k f_{\mathcal{O}(S)_K,*}(\mathbb{L}|_{X_{\mathcal{O}(S)_K}}))|_{Y_{\mathcal{O}(K)}} \to \mathrm{R}^k f_{\mathcal{O}(K),*}(\mathbb{L}|_{X_{\mathcal{O}(K)}}) \tag{3.3}$$

is also an isomorphism. The lemma follows from (3.2) and (3.3). □

Lemma 3.6. *Let S be a Stein space. Let $f : X \to Y$ be a morphism interiorly of finite type of $\mathcal{O}(S)$-schemes interiorly locally of finite type. Let $\mathbb{L}$ be an interiorly constructible sheaf on $X_{\text{ét}}$. Then $\mathrm{R}^k f_*\mathbb{L}$ is interiorly constructible for $k \geq 0$.*

Proof. Working locally on Y, we may suppose that X and Y are interiorly of finite type over $\mathcal{O}(S)$. Fix $s \in S$. Let K be an excellent Stein compact neighborhood of s in S chosen small enough so that $\mathbb{L}|_{X_{\mathcal{O}(K)}}$ is constructible (see Lemma 1.9 (ii)). As $f_{\mathcal{O}(K)} : X_{\mathcal{O}(K)} \to Y_{\mathcal{O}(K)}$ is a morphism of $\mathcal{O}(K)$-schemes of finite type (see Lemma 3.1 (i)), which are excellent noetherian schemes by Lemma 1.11 (iii), the sheaf $\mathrm{R}^k f_{\mathcal{O}(K),*}(\mathbb{L}|_{X_{\mathcal{O}(K)}})$ is constructible by [SGA43, XIX, Théorème 5.1] and [Tem08, Theorem 1.1]. It follows from Lemma 3.5 that $(\mathrm{R}^k f_*\mathbb{L})|_{X_{\mathcal{O}(U)}}$ is constructible for any open neighborhood U of s in K. □

3.3. **A relative Artin comparison theorem over Stein spaces.** Here is the statement of our relative comparison theorem. In the constructible case, its proof is entirely parallel to that of [EGA43, XVI, Théorème 4.1].

Theorem 3.7. *Let S be a Stein space. Let $f : X \to Y$ be a morphism interiorly of finite type of $\mathcal{O}(S)$-schemes interiorly locally of finite type. Let $\mathbb{L}$ be a torsion abelian sheaf on $X_{\text{ét}}$. If either f is interiorly proper or $\mathbb{L}$ is interiorly constructible, the base change morphisms $\mathrm{R}^k f_*(\mathbb{L})^{\mathrm{an}} \to \mathrm{R}^k f^{\mathrm{an}}_*(\mathbb{L}^{\mathrm{an}})$ are isomorphisms for $k \geq 0$.*

Proof. Fix $y \in Y^{\mathrm{an}}$ and let s be the image of y in S. We proceed in several steps.

Step 1. We first deal with the case where f is interiorly proper.

By Lemma 3.1 (i), the base change $f_{\mathcal{O}_{S,s}} : X_{\mathcal{O}_{S,s}} \to Y_{\mathcal{O}_{S,s}}$ of f is proper. By Lemma 3.5, proper base change [SGA43, XII, Théorème 5.1], and Artin's comparison theorem [SGA43, XVI, Théorème 4.1], the base change morphisms

$$(\mathrm{R}^k f_*\mathbb{L})_y \to \mathrm{R}^k f_{\mathcal{O}_{S,s},*}(\mathbb{L}|_{X_{\mathcal{O}_{S,s}}})_y \to H^k_{\text{ét}}(X_y, \mathbb{L}|_{X_y}) \to H^k_{\text{ét}}(X^{\mathrm{an}}_y, \mathbb{L}^{\mathrm{an}}|_{X^{\mathrm{an}}_y}) \tag{3.4}$$

are isomorphisms. As f^{an} is proper by [Bin76b, Satz 3.1 (5)], proper base change in topology (see [God58, II, Théorème 4.11.1]) implies that the base change morphism

$$\mathrm{R}^k f^{\mathrm{an}}_*(\mathbb{L}^{\mathrm{an}})_y \to H^k_{\text{ét}}(X^{\mathrm{an}}_y, \mathbb{L}^{\mathrm{an}}|_{X^{\mathrm{an}}_y}) \tag{3.5}$$

is also an isomorphism. Combining (3.4) and (3.5) shows that the base change morphism $\mathrm{R}^k f_*(\mathbb{L})^{\mathrm{an}}_y \to \mathrm{R}^k f^{\mathrm{an}}_*(\mathbb{L}^{\mathrm{an}})_y$ is an isomorphism, as wanted.

Step 2. Setup of the proof when $\mathbb{L}$ is interiorly constructible.

By Lemma 3.5, if $U \subset S$ is a Stein open neighborhood of s, the base change morphism $(\mathrm{R}^k f_*\mathbb{L})_y \to (\mathrm{R}^k f_{\mathcal{O}(U),*}(\mathbb{L}|_{X_{\mathcal{O}(U)}}))_y$ is an isomorphism. Therefore, to check that $\mathrm{R}^k f_*(\mathbb{L})^{\mathrm{an}}_y \to \mathrm{R}^k f^{\mathrm{an}}_*(\mathbb{L}^{\mathrm{an}})_y$ is an isomorphism, we may at any time replace S with a Stein open neighborhood of s in S. As one can moreover work locally on Y, one may assume that f is a morphism of finite type between $\mathcal{O}(S)$-schemes of finite presentation, that $\mathbb{L}$ is constructible, that Y is separated, and that X^{an} is finite-dimensional. We argue by induction on the dimension of X^{an}.

Step 3. We reduce to the case where f is an open immersion of reduced separated $\mathcal{O}(S)$-schemes of finite presentation such that $f_{\mathcal{O}_{S,s}}$ has dense image, and $\mathbb{L} = \mathbb{Z}/m$.

Let K be an excellent Stein compact neighborhood of s in S (see Lemma 1.9 (ii)). By [SGA43, IX, Proposition 2.14 (ii) and XIV, Théorème 1.1], the sheaf $\mathbb{L}|_{X_{\mathcal{O}(K)}}$ has a resolution by finite products of sheaves of the form $\pi_*\mathbb{M}$ with $\pi : Z \to X_{\mathcal{O}(K)}$ finite and $\mathbb{M}$ a constant constructible sheaf on $Z_{\text{ét}}$. Shrinking S, we may assume that $\mathbb{L}$ itself admits such a resolution (with finite morphisms $\pi : Z \to X$). By the spectral sequence of a resolution, we may suppose that $\mathbb{L}$ is of the form $\pi_*\mathbb{M}$. As the higher direct images of morphisms induced by π and π^{an} vanish by [SGA42, VIII, Proposition 5.5] and [God58, II, Théorème 4.11.1], and as $\pi_*(\mathbb{M})^{\mathrm{an}} \xrightarrow{\sim} \pi^{\mathrm{an}}_*(\mathbb{M}^{\mathrm{an}})$ by Step 1, we may further assume that $\mathbb{L}$ is constant (after replacing X with Z and $\mathbb{L}$ with $\mathbb{M}$), and then that $\mathbb{L} \simeq \mathbb{Z}/m$.

Let $\mathcal{U} = (U_i)_{i \in I}$ be a finite affine covering of X. Computing cohomology using Čech spectral sequences associated with $\mathcal{U}$, we may replace f with $f|_{U_J}$, where $U_J := \cap_{i \in J} U_i$ for $J \subset I$. Since X was quasi-separated (as is any $\mathcal{O}(S)$-scheme of finite presentation), this trick reduces us to the case where X is separated.

As the étale and classical topologies are insensitive to nilpotents (see [SGA42, VIII, Théorème 1.1]), we may assume that X and Y are reduced.

Write $f_{\mathcal{O}_{S,s}}$ as the composition $X_{\mathcal{O}_{S,s}} \xrightarrow{g_s} \overline{X}_s \xrightarrow{h_s} Y_{\mathcal{O}_{S,s}}$ of an open immersion g_s with dense image and of a proper morphism h_s using Nagata's compactification theorem [SP, Theorem 0F41]. Use a limit argument based on [EGA43, Théorèmes 8.8.2

and 8.10.5] to extend, after shrinking S, these morphisms of schemes to a diagram $X \xrightarrow{g} \overline{X} \xrightarrow{h} Y$ of reduced separated $\mathcal{O}(S)$-schemes of finite presentation, where g is an open immersion and h is proper. As Theorem 3.7 holds for h (and any torsion étale sheaf) by Step 1, the composite functors spectral sequence computing $\mathrm{R}^k(h \circ g)_*\mathbb{Z}/m$ reduces us to proving it for g.

Step 4. We reduce to the case where moreover $Y_{\mathcal{O}_{S,s}}$ is regular.

Let $\mu_s : \widetilde{Y}_s \to Y_{\mathcal{O}_{S,s}}$ be a resolution of singularities (use [Tem08, Theorem 1.1] and Lemma 1.11 (iii)). Set $\widetilde{X}_s := X \times_{Y_{\mathcal{O}_{S,s}}} \widetilde{Y}_s$, let $j_s : U_s \hookrightarrow X_s$ be a dense open subset over which μ_s is an isomorphism, and let $\tilde{j}_s : U_s \to \widetilde{X}_s$ be the lift of j_s. By a limit argument based on [EGA43, Théorèmes 8.8.2 and 8.10.5], the above schemes and morphisms arise by base change, after shrinking S, from a cartesian diagram

$$\begin{tikzcd} U \arrow[r, "\tilde{j}"] \arrow[d, equal] & \widetilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "\mu_X"] & \widetilde{Y} \arrow[d, "\mu"] \\ U \arrow[r, "j"] & X \arrow[r, "f"] & Y, \end{tikzcd}$$

whose horizontal arrows are quasi-compact open immersions and whose vertical arrows are proper of finite presentation. Let $i : Z \to X$ and $\tilde{i} : \widetilde{Z} \to \widetilde{X}$ be the inclusions of the complements of U in X and $\widetilde{X}$ respectively. As j_s and $\tilde{j}_s$ have dense image, both $\dim(Z^{\mathrm{an}})$ and $\dim((\widetilde{Z})^{\mathrm{an}})$ are $< \dim(X^{\mathrm{an}})$ after possibly shrinking S by [Bin76b, Aussage 2.8]. The induction hypothesis thus shows that Theorem 3.7 holds for $(f \circ i, \mathbb{Z}/m)$ and $(\tilde{f} \circ \tilde{i}, \mathbb{Z}/m)$, hence for $(f, i_*\mathbb{Z}/m)$ and $(\tilde{f}, \tilde{i}_*\mathbb{Z}/m)$.

The five lemma applied to long exact sequences induced by the short exact sequence $0 \to j_!\,\mathbb{Z}/m \to \mathbb{Z}/m \to i_*\mathbb{Z}/m \to 0$ now shows that, to prove Theorem 3.7 for $(f, \mathbb{Z}/m)$, it suffices to prove it for $(f, j_!\,\mathbb{Z}/m)$.

One computes at once using proper base change [SGA43, XII, Théorème 5.1] that $\mu_{X,*}(\tilde{j}_!\,\mathbb{Z}/m) = j_!\,\mathbb{Z}/m$ and that $\mathrm{R}^q\mu_{X,*}(\tilde{j}_!\,\mathbb{Z}/m) = 0$ if $q > 0$. Using the composite functors spectral sequence computing $\mathrm{R}^k(f \circ \mu_X)_*$, we deduce that it suffices to prove Theorem 3.7 for $(\tilde{f}, \tilde{j}_!\,\mathbb{Z}/m)$. By another application of the five lemma based on the short exact sequence $0 \to \tilde{j}_!\,\mathbb{Z}/m \to \mathbb{Z}/m \to \tilde{i}_*\mathbb{Z}/m \to 0$, it remains to prove Theorem 3.7 for $(\tilde{f}, \mathbb{Z}/m)$.

Step 5. We reduce to the case where $Y_{\mathcal{O}_{S,s}} \setminus f_{\mathcal{O}_{S,s}}(X_{\mathcal{O}_{S,s}})$ is moreover regular.

The noetherian scheme $Y_{\mathcal{O}_{S,s}} \setminus f_{\mathcal{O}_{S,s}}(X_{\mathcal{O}_{S,s}})$ is excellent by Lemma 1.11 (iii). Stratify it by its regular locus, the regular locus of its singular locus, etc. Using [EGA43, Théorèmes 8.8.2 and 8.10.5], lift this stratification to a stratification of $Y \setminus f(X)$, after maybe shrinking S. One can thus write $f = f_r \circ \cdots \circ f_1$ where $f_i : X_{i-1} \to X_i$ is a quasi-compact open immersion (with $X_0 := X$ and $X_r := Y$), so that $f_{i,\mathcal{O}_{S,s}}$ has dense image and $(X_i \setminus f_i(X_{i-1}))_{\mathcal{O}_{S,s}}$ is regular. Use [Bin76b, Aussage 2.8] to ensure after shrinking S that $\dim((X_i \setminus X_{i-1})^{\mathrm{an}}) < \dim(X^{\mathrm{an}})$.

By [SGA43, XIX, Théorème 2.1], one has $\mathbb{Z}/m \xrightarrow{\sim} f_{i,\mathcal{O}_{S,s},*}\mathbb{Z}/m$. It thus follows from Lemma 3.5 that $\mathbb{Z}/m \to f_{i,*}\mathbb{Z}/m$ becomes an isomorphism after restriction to $(X_i)_{\mathcal{O}_{S,s}}$. Using [SGA43, Corollaire 2.7.4], we may suppose after shrinking S that $\mathbb{Z}/m \to f_{i,*}\mathbb{Z}/m$ is an isomorphism.

Assume that Theorem 3.7 holds for all the $(f_i, \mathbb{Z}/m)$. We prove by induction on i that it holds for $(f_i \circ \cdots \circ f_1, \mathbb{Z}/m)$. To do so, it suffices to make use of the composite functors spectral sequence computing $\mathrm{R}^k(f_i \circ (f_{i-1} \circ \cdots \circ f_1))_*\mathbb{Z}/m$, since

$\mathbb{Z}/m \xrightarrow{\sim} f_{i,*}\mathbb{Z}/m$, and noting that Theorem 3.7 holds for $(f_{i-1}\circ\cdots\circ f_1, \mathrm{R}^q f_{i,*}\mathbb{Z}/m)$ when $q > 0$ by the induction hypothesis, as $\mathrm{R}^q f_{i,*}\mathbb{Z}/m$ is interiorly constructible by Lemma 3.6, and supported on $X_i \setminus X_{i-1}$.

Step 6. We apply purity to conclude.

As the theorem is trivial if $y \in f^{\mathrm{an}}(X^{\mathrm{an}})$, we assume that $y \in Y^{\mathrm{an}} \setminus f^{\mathrm{an}}(X^{\mathrm{an}})$. After shrinking Y, we may assume that $Y_{\mathcal{O}_{S,s}} \setminus f_{\mathcal{O}_{S,s}}(X_{\mathcal{O}_{S,s}}) \to Y_{\mathcal{O}_{S,s}}$ is a regular immersion of pure codimension $c \geq 1$. Using [Bin76b, (2.7) and (2.8)], one can ensure after shrinking S that $Y^{\mathrm{an}} \setminus f^{\mathrm{an}}(X^{\mathrm{an}}) \to Y^{\mathrm{an}}$ is an embedding of pure codimension c of complex manifolds. It follows from Lemma 3.5 and [SGA43, XIX, Théorèmes 3.2 and 3.4] that $(\mathrm{R}^k f_*\mathbb{Z}/m)_y = \mathbb{Z}/m$ if $k \in \{0, 2c-1\}$ and vanishes otherwise. A direct topological computation shows that $(\mathrm{R}^k f^{\mathrm{an}}_*\mathbb{Z}/m)_y = \mathbb{Z}/m$ if $k \in \{0, 2c-1\}$ and vanishes otherwise. Inspecting the construction of these isomorphisms shows that they are compatible, and concludes the proof. □

Remark 3.8. Theorem 3.7 fails if one drops both the properness and the constructibility hypotheses, already if S is a point, if $f : \mathbb{A}^1_{\mathbb{C}} \to \mathrm{Spec}(\mathbb{C})$ is the structural morphism, if $k = 0$, and if $\mathbb{L}$ is a direct sum of skyscraper sheaves at the integers.

4. Killing ramification on finite coverings

The goal of this section is result is Proposition 4.3. It will be used in conjunction with Proposition 2.4 in the proof of our Theorem 5.1.

4.1. **Killing ramification locally.** We first apply Theorem 3.7 to kill the ramification of cohomology classes in the local analytic setting.

Proposition 4.1. *Let U be an open subset of a connected normal Stein space S. Choose $s \in U$. Let $Z \subset S$ be a closed analytic subset. Fix $\alpha \in H^k(U \setminus (Z \cap U), \mathbb{Z}/m)$ with $k, m \geq 1$. There exist a finite surjective holomorphic map $q : R \to S$ of connected normal Stein spaces, a point $r \in R$ with $q(r) = s$, and an open neighborhood Θ of r in $q^{-1}(U)$ with $\alpha|_{\Theta\setminus(q^{-1}(Z)\cap\Theta)} = 0$ in $H^k(\Theta \setminus (q^{-1}(Z) \cap \Theta), \mathbb{Z}/m)$.*

Proof. We may assume that Z is nowhere dense in S (otherwise $Z = S$ and the lemma is trivial). Let $\mathcal{I}_Z \subset \mathcal{O}_S$ be the ideal sheaf of Z. Use Cartan's Theorem A to choose finitely many elements $(f_i)_{i\in I}$ of $\mathcal{I}_Z(S)$ that generate $\mathcal{I}_{Z,s}$. After replacing Z with the vanishing locus of the f_i, we may apply Lemma 3.2 to find a quasi-compact open immersion $j : V \hookrightarrow \mathrm{Spec}(\mathcal{O}(S))$ such that j^{an} identifies with $S \setminus Z \hookrightarrow S$. Theorem 3.7 then shows that the base change morphism

$$(\mathrm{R}^k j_*\mathbb{Z}/m)^{\mathrm{an}} \to \mathrm{R}^k j^{\mathrm{an}}_*\mathbb{Z}/m \tag{4.1}$$

is an isomorphism. Let α_s be the image of α in $(\mathrm{R}^k j^{\mathrm{an}}_*\mathbb{Z}/m)_s$ and consider its inverse image $\beta_s \in (\mathrm{R}^k j_*\mathbb{Z}/m)_s$ by the isomorphism (4.1).

Let $\beta \in H^k_{\text{ét}}(X_V, \mathbb{Z}/m)$ be a representative of β_s, where $f : X \to \mathrm{Spec}(\mathcal{O}(S))$ is an integral affine étale neighborhood of $s \in \mathrm{Spec}(\mathcal{O}(S))$ and $X_V := X \times_{\mathrm{Spec}(\mathcal{O}(S))} V$. The scheme $(X_V)_s := X_V \times_{\mathrm{Spec}(\mathcal{O}(S))} \mathrm{Spec}(\mathcal{O}(S)_s)$ is étale of finite type over $\mathcal{O}(S)_s$ (use Lemma 3.1 (ii)), hence an excellent noetherian scheme by Lemma 1.11 (ii). By [Bha12, Theorem 1.1], there exists a finite surjective morphism $\pi_s : Y_s \to (X_V)_s$ such that $\beta|_{Y_s} = 0$ in $H^k_{\text{ét}}(Y_s, \mathbb{Z}/m)$ (note that Bhatt's proof simplifies in our situation, as the group scheme $\mathbb{Z}/m$ is étale, see [Bha12, Remark 3.3]).

By a limit argument based on [EGA43, Théorèmes 8.8.2 and 8.10.5], there exist an affine open neighborhood W of s in $\mathrm{Spec}(\mathcal{O}(S))$ and a finite surjective morphism of finite presentation $\pi : Y \to (X_V) \times_{\mathrm{Spec}(\mathcal{O}(S))} W$ such that the base change of π by $\mathrm{Spec}(\mathcal{O}(S)_s) \to W$ identifies with π_s. By [SGA42, Exp. VII, Corollaire 5.8], one may assume after shrinking W that $\beta|_Y = 0$ in $H^k_{\text{ét}}(Y, \mathbb{Z}/m)$. After replacing X with $X \times_{\mathrm{Spec}(\mathcal{O}(S))} W$, we may finally view π as a finite surjective morphism of finite presentation $\pi : Y \to X_V$ such that $\pi^*\beta = 0$. As X_V is integral, we may assume that so is Y (after replacing it with an irreducible component that dominates X_V).

One may factor $f : X \to \mathrm{Spec}(\mathcal{O}(S))$ as the composition $X \xrightarrow{a} \overline{X} \xrightarrow{b} \mathrm{Spec}(\mathcal{O}(S))$ of a quasi-compact open immersion a and of a finite morphism b, by Zariski's Main Theorem [EGA43, Théorème 8.12.6]. A second application of Zariski's Main Theorem allows us to factor the natural morphism $Y \to \overline{X}$ as the composition $Y \xrightarrow{c} \overline{Y} \xrightarrow{d} \overline{X}$ of a quasi-compact open immersion c and a finite morphism d. After replacing $\overline{X}$ with the closure of X in $\overline{X}$ and $\overline{Y}$ with the closure of Y in $\overline{Y}$, one may assume that $\overline{X}$ and $\overline{Y}$ are integral. Here is a diagram summarizing the situation:

$$\begin{tikzcd}
Y \arrow[rr, "c"] \arrow[d, "\pi"'] & & \overline{Y} \arrow[d, "d"] \\
X_V \arrow[r] \arrow[d] & X \arrow[r, "a"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "b"] \\
V \arrow[rr, "j"'] & & \mathrm{Spec}(\mathcal{O}(S)).
\end{tikzcd}$$

As the natural morphism $Y \to \overline{Y} \times_{\overline{X}} X_V$ is a proper open immersion with nonempty source and integral target, it is an isomorphism. Choose $x \in X^{\mathrm{an}}$ with $f^{\mathrm{an}}(x) = s$. As f is étale, the map f^{an} is a local biholomorphism at x (see [Bin76b, Satz 3.1 (1)]) so that X^{an} is normal at x, hence locally irreducible at x. As $(X_V)^{\mathrm{an}}$ is dense in X^{an} (because Z is nowhere dense in S), and as π^{an} is finite surjective and d^{an} is finite by [Bin76b, Satz 3.1 (2) (5)], the restriction $(\overline{Y} \times_{\overline{X}} X)^{\mathrm{an}} \to X^{\mathrm{an}}$ of d^{an} is finite and surjective. One may thus choose $y \in (\overline{Y} \times_{\overline{X}} X)^{\mathrm{an}} \subset (\overline{Y})^{\mathrm{an}}$ such that $d^{\mathrm{an}}(y) = x$ and moreover, by local irreducibility of X^{an} at x, such that the image in $(\overline{X})^{\mathrm{an}}$ of some irreducible component C of $(\overline{Y})^{\mathrm{an}}$ through y contains a neighborhood of x.

Consider the natural commutative diagram

$$\begin{tikzcd}
(\mathrm{R}^k j_*\mathbb{Z}/m)_s \arrow[r] \arrow[d] & (\mathrm{R}^k j^{\mathrm{an}}_*\mathbb{Z}/m)_s \arrow[d] \\
(\mathrm{R}^k c_*\mathbb{Z}/m)_y \arrow[r] & (\mathrm{R}^k c^{\mathrm{an}}_*\mathbb{Z}/m)_y.
\end{tikzcd} \tag{4.2}$$

As $\pi^*\beta = 0$, the image of β_s in $(\mathrm{R}^k c_*\mathbb{Z}/m)_y$ vanishes. By commutativity of (4.2), the image of α_s in $(\mathrm{R}^k c^{\mathrm{an}}_*\mathbb{Z}/m)_y$ also vanishes. This exactly means that there exists an open neighborhood Θ' of y in $(b^{\mathrm{an}} \circ d^{\mathrm{an}})^{-1}(U) \subset (\overline{Y})^{\mathrm{an}}$ such that $\alpha|_{Y^{\mathrm{an}} \cap \Theta'} = 0$. If Θ' is small enough to be included in $(d^{\mathrm{an}})^{-1}(X^{\mathrm{an}})$, one has the equality

$$Y^{\mathrm{an}} \cap \Theta' = \Theta' \setminus ((b^{\mathrm{an}} \circ d^{\mathrm{an}})^{-1}(Z) \cap \Theta').$$

To conclude, it suffices to take R to be the normalization of $C \subset \overline{Y}^{\mathrm{an}}$, to let r be any preimage of y in R, and to choose Θ to be the inverse image of Θ' in R. □

Proposition 4.2. *Let S be a Stein space, let $U \subset S$ be open, and choose $s \in U$. Let $Z \subset S$ be a closed analytic subset. Fix $\alpha \in H^k(U \setminus (Z \cap U), \mathbb{Z}/m)$ with $k, m \geq 1$. Then there exist a finite surjective holomorphic map $p : T \to S$ and an open neighborhood Ω of s in U such that $\alpha|_{p^{-1}(\Omega \setminus (Z \cap \Omega))} = 0$ in $H^k(p^{-1}(\Omega \setminus (Z \cap \Omega)), \mathbb{Z}/m)$.*

Proof. Let $\nu : \widetilde{S} \to S$ be the normalization map and let $(\tilde{s}_i)_{i\in I}$ be the preimages of s by ν. Assume that the maps $p_i : T_i \to \widetilde{S}$ prove the proposition for $(\widetilde{S}, \nu^{-1}(U), \tilde{s}_i, \nu^{-1}(Z), \nu^*\alpha)$ using disjoint open neighborhoods Ω_i of $\tilde{s}_i$ in $\widetilde{S}$. Then the fiber product $p : T \to S$ of the maps $\nu \circ p_i$ solves our original problem for any neighborhood Ω of s in S such that $\nu^{-1}(\Omega) \subset \cup_{i\in I}\Omega_i$. We may thus assume that S is normal. After replacing S with its connected component containing s, we may also suppose that it is connected.

Choose $q : R \to S$, a point $r \in R$ and $\Theta \subset q^{-1}(U)$ as in Proposition 4.1. Let $p' : T \to R$ and Γ be given by Lemma 1.17, and set $p := q \circ p'$. Let $(t_j)_{j\in J}$ be the preimages of s by p. As p' is surjective and Γ acts transitively on the fibers of p, one can find, for each $j \in J$, an element $\gamma_j \in \Gamma$ such that $p' \circ \gamma_j(t_j) = r$. Let Ω_j be a neighborhood of t_j in T such that $p' \circ \gamma_j(\Omega_j) \subset \Theta$. After shrinking the Ω_j, we may assume that they are disjoint. Choosing Ω to be a neighborhood of s in U such that $p^{-1}(\Omega) \subset \cup_{j\in J}\Omega_j$ concludes the proof. $\square$

4.2. **Killing ramification globally.** We finally globalize Proposition 4.2 thanks to the Čech-to-derived spectral sequence.

Proposition 4.3. *Let S be a Stein space, let $K \subset S$ be a compact subset and let U be an open neighborhood of K in S. Let $Z \subset S$ be a closed analytic subset, set $V := S \setminus Z$, and fix $\alpha \in H^k(V \cap U, \mathbb{Z}/m)$ for some $k, m \geq 1$. Then, after maybe shrinking U, there exist a finite surjective holomorphic map $p : T \to S$ and a class $\beta \in H^k(p^{-1}(U), \mathbb{Z}/m)$ with $\alpha|_{p^{-1}(V\cap U)} = \beta|_{p^{-1}(V\cap U)}$ in $H^k(p^{-1}(V \cap U), \mathbb{Z}/m)$.*

Proof. Let $\mathcal{H}^s$ be the presheaf on S defined by $\mathcal{H}^s(\Omega) := H^s(\Omega, \mathbb{Z}/m)$ for $\Omega \subset S$ open. Note that the presheaf $\mathcal{H}^0$ is the constant sheaf $\mathbb{Z}/m$. Let $\mathcal{U} = (U_i)_{i\in I}$ be a finite open covering of U and consider the Čech-to-derived spectral sequence

$$E_2^{r,s} = \check{H}^r(V \cap \mathcal{U}, \mathcal{H}^s) \implies H^{r+s}(V \cap U, \mathbb{Z}/m)$$

associated with the open covering $V\cap\mathcal{U} = (V\cap U_i)_{i\in I}$ of $V\cap U$ ([SP, Lemma 03AZ], see [SP, Definition 03AM] for the definition of Čech cohomology). Let $F^\bullet$ be the filtration on its abutment. Let $l \geq 0$ be maximal with $\alpha \in F^l H^k(V \cap U, \mathbb{Z}/m)$.

Assume first that $l < k$. If $J = (i_0, \ldots, i_l) \in I^{l+1}$, write $U_J := \cap_{t=0}^l U_{i_t}$. Then α is induced by a class in $\check{H}^l(V\cap\mathcal{U}, \mathcal{H}^{k-l})$, represented by a cocycle $(\gamma_J)_{J\in I^{l+1}}$, where

$$\gamma_J \in \mathcal{H}^{k-l}(V \cap U_J) = H^{k-l}(V \cap U_J, \mathbb{Z}/m).$$

For each $s \in K$ and $J \in I^{l+1}$, one can apply Proposition 4.2 to find an open neighborhood $\Omega_{s,J}$ of s in U_J and a finite surjective holomorphic map $p_{s,J} : T_{s,J} \to S$ with $\gamma_J|_{p_{s,J}^{-1}(V\cap\Omega_{s,J})} = 0$. Intersecting the $\Omega_{s,J}$ and taking the fiber product of the $p_{s,J} : T_{s,J} \to S$ over S for varying J yields an open neighborhood Ω_s of s in U and a finite surjective holomorphic map $p_s : T_s \to S$. Extract from $(\Omega_s)_{s\in K}$ a finite covering of K (which we view, after shrinking U, as a finite covering $\mathcal{U}'$ of U which refines $\mathcal{U}$) and let $p : T \to S$ be the fiber product over S of the corresponding maps $p_s : T_s \to S$. Our choices ensure that the image in $\check{H}^l(p^{-1}(V)\cap p^{-1}(\mathcal{U}'), \mathcal{H}^{k-l})$ of $[\gamma_J] \in \check{H}^l(V\cap\mathcal{U}, \mathcal{H}^{k-l})$ is represented by the zero cocycle and hence vanishes. As a consequence, after replacing S, K, V and $\mathcal{U}$ with T, $p^{-1}(K)$, $p^{-1}(V)$ and $p^{-1}(\mathcal{U}')$, we have managed to increase the value of l.

Repeating this procedure finitely many times, we may assume that $l = k$, hence that α is induced by a class $\tilde{\alpha} \in \check{H}^k(V\cap\mathcal{U}, \mathcal{H}^0) = \check{H}^k(V\cap\mathcal{U}, \mathbb{Z}/m)$. We may suppose that S is normal and connected (replace it with a connected component of its normalization) and that Z is nowhere dense in S (otherwise $Z = S$ and $\alpha = 0$). Then, if $j : V \hookrightarrow S$ is the inclusion morphism, the adjunction morphism $\mathbb{Z}/m \to j_*j^*\mathbb{Z}/m$ is an isomorphism (use [GR84, 9, §1.2, Theorem vii) $\Leftrightarrow$ viii)]). As a consequence, the restriction map $\check{H}^k(\mathcal{U}, \mathbb{Z}/m) \to \check{H}^k(V \cap \mathcal{U}, \mathbb{Z}/m)$ is an isomorphism, and we let $\tilde{\beta} \in \check{H}^k(\mathcal{U}, \mathbb{Z}/m)$ be the inverse image of $\tilde{\alpha}$. The image $\beta \in H^k(U, \mathbb{Z}/m)$ of $\tilde{\beta}$ by the natural morphism $\check{H}^k(\mathcal{U}, \mathbb{Z}/m) \to H^k(U, \mathbb{Z}/m)$ then has the required property that $\beta|_{V \cap U} = \alpha$ in $H^k(V \cap U, \mathbb{Z}/m)$. □

5. An absolute comparison theorem

After fixing in §5.1 our conventions concerning the Grothendieck topologies that we will use, we prove our main comparison theorem (Theorem 5.1) in §5.2, we extend it to the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant setting in §5.3 and §5.4, and we give applications to cohomological dimension bounds in §5.5.

5.1. **The qfh topology.** Our proof of Theorem 5.1 requires the use of a Grothendieck topology for schemes that is finer than the étale topology: Voevodsky's qfh topology whose coverings are given by surjective quasi-finite morphisms. The precise definition that we use is the one given in [Cho22, §6.6]. As explained in *loc. cit.*, it coincides with Voevodsky's [Voe96, Definition 3.1.2] in the noetherian case.

Let S be a Stein space, and let X be a separated $\mathcal{O}(S)$-scheme of finite type. We will consider the following commutative diagram of sites:

$$\begin{tikzcd} (X^{\mathrm{an}})_{\mathrm{qc}} \arrow[r] \arrow[d, "\zeta"] & (X^{\mathrm{an}})'_{\mathrm{cl}} \arrow[d, "\varepsilon"] \\ X_{\mathrm{qfh}} \arrow[r] & (X_{\text{ét}})'. \end{tikzcd} \tag{5.1}$$

The site $(X_{\text{ét}})'$ is the variant of the site $X_{\text{ét}}$ of (3.1) where one restricts the objects to those étale X-schemes that are separated and of finite type over $\mathcal{O}(S)$, and where one only considers coverings involving finitely many arrows. As objects of $X_{\text{ét}}$ admit finite Zariski coverings by affine hence separated schemes, the topoi associated with $(X_{\text{ét}})'$ and $X_{\text{ét}}$ are equivalent by [SP, Lemma 03A0]. We may thus use them interchangeably for cohomological computations.

The site $(X^{\mathrm{an}})'_{\mathrm{cl}}$ is the variant of the site $(X^{\mathrm{an}})_{\mathrm{cl}}$ of (3.1) where one restricts the objects to those topological spaces endowed with a local homeomorphism to X^{an} that are Hausdorff (but we impose no further constraints on coverings). As objects of $(X^{\mathrm{an}})_{\mathrm{cl}}$ admit open coverings by Hausdorff spaces, the topoi associated with $(X^{\mathrm{an}})'_{\mathrm{cl}}$ and $(X^{\mathrm{an}})_{\mathrm{cl}}$ are equivalent, again by [SP, Lemma 03A0].

The site X_{qfh} has as objects those X-schemes that are separated and of finite type over $\mathcal{O}(S)$, and is endowed with the qfh topology [Cho22, §6.6], restricting to coverings involving only finitely many arrows.

The site $(X^{\mathrm{an}})_{\mathrm{qc}}$ has as objects the Hausdorff and locally compact topological spaces over X^{an} and is endowed with the qc topology [SP, Definition 09X0].

The morphisms of sites in (5.1) are the obvious ones (étale coverings are qfh coverings by [SP, Lemma 0ETV] and analytifications of qfh coverings are qc coverings as a consequence of [SP, Lemma 09X5]).

5.2. **Artin's comparison theorem over Stein compact sets.** We may now state and prove the main theorem of this article.

Theorem 5.1. *Let S be a Stein space. Let $f : X \to \mathrm{Spec}(\mathcal{O}(S))$ be an $\mathcal{O}(S)$-scheme interiorly of finite type. Fix a torsion abelian sheaf $\mathbb{L}$ on $X_{\text{ét}}$. Assume that f is interiorly proper or that $\mathbb{L}$ is interiorly constructible. If one lets U run over all Stein open neighborhoods of a Stein compact subset K of S, the base change morphisms*

$$\operatorname*{colim}_{K\subset U} H^k_{\text{ét}}(X_{\mathcal{O}(U)}, \mathbb{L}_{\mathcal{O}(U)}) \to \operatorname*{colim}_{K\subset U} H^k((X_{\mathcal{O}(U)})^{\mathrm{an}}, \mathbb{L}^{\mathrm{an}}) \tag{5.2}$$

are isomorphisms for $k \geq 0$.

Proof. We split the proof in several steps.

Step 1. We first reduce to the case where $X = \mathrm{Spec}(\mathcal{O}(S))$.

By Lemma 1.9 (ii), the morphism (5.2) identifies with the natural morphism

$$\operatorname*{colim}_{K\subset L} H^k_{\text{ét}}(X_{\mathcal{O}(L)}, \mathbb{L}_{\mathcal{O}(L)}) \to \operatorname*{colim}_{K\subset L} H^k((X_{\mathcal{O}(L)})^{\mathrm{an}}, \mathbb{L}^{\mathrm{an}}), \tag{5.3}$$

where L runs over all excellent Stein compact neighborhoods of K in S, and $(X_{\mathcal{O}(L)})^{\mathrm{an}} := (f^{\mathrm{an}})^{-1}(L)$. Both sides of (5.2) or equivalently (5.3) are computed by (colimits of) Leray spectral sequences, whose $E_2^{p,q}$ terms read respectively

$$\operatorname*{colim}_{K\subset L} H^p_{\text{ét}}(\mathrm{Spec}(\mathcal{O}(L)), (\mathrm{R}^q f_*\mathbb{L})_{\mathcal{O}(L)}) \text{ and } \operatorname*{colim}_{K\subset U} H^p(U, (\mathrm{R}^q f_*\mathbb{L})^{\mathrm{an}}),$$

where we used Lemma 3.5 and Theorem 3.7 applied with $Y = \mathrm{Spec}(\mathcal{O}(S))$. These terms are isomorphic by the $X = \mathrm{Spec}(\mathcal{O}(S))$ case of Theorem 5.1 applied to the sheaves $\mathrm{R}^q f_*\mathbb{L}$.

Step 2. We further reduce to the case where $\mathbb{L} = \mathbb{Z}/m$ for some $m \geq 1$.

If K' is a fixed excellent Stein compact neighborhood of K in S (which exists by Lemma 1.9 (ii)), the sheaf $\mathbb{L}|_{\mathrm{Spec}(\mathcal{O}(K'))}$ is the filtered colimit of its constructible subsheaves by [SGA43, IX, Proposition 2.9 (iii)]. Consequently, after shrinking S, we may assume that $\mathbb{L}$ is the filtered colimit of its constructible subsheaves.

As both the étale cohomology of $\mathrm{Spec}(\mathcal{O}(L))$ and the sheaf cohomology of L commute with filtered colimits (see [SGA42, VII, Proposition 3.3] and [God58, II, Théorème 4.12.1]), we may assume that $\mathbb{L}$ is constructible.

Fix again an excellent Stein compact neighborhood K' of K in S. By [SGA43, IX, Proposition 2.14 (ii) and XIV, Théorème 1.1], the sheaf $\mathbb{L}|_{\mathrm{Spec}(\mathcal{O}(K'))}$ admits a resolution by finite products of sheaves of the form $\pi_*\mathbb{M}$, where $\pi : Z \to \mathrm{Spec}(\mathcal{O}(K'))$ is finite and $\mathbb{M}$ is a constant constructible sheaf on $Z_{\text{ét}}$. After shrinking S, we may assume that $\mathbb{L}$ itself admits such a resolution (with morphisms $\pi : Z \to \mathrm{Spec}(\mathcal{O}(S))$). Using the spectral sequence of a resolution, we may assume that $\mathbb{L}$ is of the form $\pi_*\mathbb{M}$. As the higher direct images of morphisms induced by π and π^{an} vanish by [SGA42, VIII, Proposition 5.5] and [God58, II, Théorème 4.11.1], and as $\pi_*(\mathbb{M})^{\mathrm{an}} \xrightarrow{\sim} \pi^{\mathrm{an}}_*(\mathbb{M}^{\mathrm{an}})$ by Theorem 3.7, we may assume that $\mathbb{L}$ is constant (after replacing S with Z^{an} and $\mathbb{L}$ with $\mathbb{M}$), and then that $\mathbb{L} \simeq \mathbb{Z}/m$, as wanted.

Step 3. We finally assume that $X = \mathrm{Spec}(\mathcal{O}(S))$ and $\mathbb{L} = \mathbb{Z}/m$ for some $m \geq 1$.

For any Stein open neighborhood U of K in S (resp. any excellent Stein compact neighborhood L of K in S), write $X_U := \mathrm{Spec}(\mathcal{O}(U))$ (resp. $X_L := \mathrm{Spec}(\mathcal{O}(L))$). Our goal is to show that the morphisms

$$\operatorname*{colim}_{K\subset U} H^k_{\text{ét}}(X_U, \mathbb{Z}/m) \xrightarrow{\varepsilon^*} \operatorname*{colim}_{K\subset U} H^k(X_U^{\mathrm{an}}, \mathbb{Z}/m) \tag{5.4}$$

are isomorphisms. For any excellent Stein compact neighborhood L of K in S, the change of topology morphisms $H^k_{\text{ét}}(X_L, \mathbb{Z}/m) \to H^k((X_L)_{\text{qfh}}, \mathbb{Z}/m)$ are isomorphisms by [Voe96, Theorem 3.4.4]. Taking the colimit over all such L and using Lemma 1.9 (ii) shows that $\operatorname*{colim}_{K\subset U} H^k_{\text{ét}}(X_U, \mathbb{Z}/m) \to \operatorname*{colim}_{K\subset U} H^k((X_U)_{\text{qfh}}, \mathbb{Z}/m)$ is an isomorphism. As the pull-back morphisms $H^k(X_U^{\text{an}}, \mathbb{Z}/m) \to H^k((X_U^{\text{an}})_{\text{qc}}, \mathbb{Z}/m)$ are also isomorphisms by [SP, Lemma 09X4], it follows from diagram (5.1) that (5.4) may be identified with the morphism

$$\operatorname*{colim}_{K\subset U} H^k((X_U)_{\text{qfh}}, \mathbb{Z}/m) \xrightarrow{\zeta^*} \operatorname*{colim}_{K\subset U} H^k((X_U^{\text{an}})_{\text{qc}}, \mathbb{Z}/m). \tag{5.5}$$

The colimit over U of the Leray spectral sequences of ζ reads

$$E_2^{p,q} = \operatorname*{colim}_{K\subset U} H^p((X_U)_{\text{qfh}}, \mathrm{R}^q\zeta_*\mathbb{Z}/m) \implies \operatorname*{colim}_{K\subset U} H^{p+q}(U_{\text{qc}}, \mathbb{Z}/m). \tag{5.6}$$

We will prove that the adjunction morphism $\operatorname*{colim}_{K\subset U} H^p((X_U)_{\text{qfh}}, \mathbb{Z}/m) \to E_2^{p,0}$ is an isomorphism in Step 4 and that $E_2^{p,q} = 0$ for $q > 0$ in Step 5. It follows that the spectral sequence (5.6) degenerates and that (5.5) hence (5.4) are isomorphisms.

Step 4. One has $\operatorname*{colim}_{K\subset U} H^p((X_U)_{\text{qfh}}, \mathbb{Z}/m) \xrightarrow{\sim} \operatorname*{colim}_{K\subset U} H^p((X_U)_{\text{qfh}}, \zeta_*\mathbb{Z}/m)$.

Let U_0 be a fixed Stein open neighborhood of K in S and let Y be a separated $\mathcal{O}(U_0)$-scheme of finite type. For any Stein open neighborhood U of K in U_0, write $Y_U := Y \times_{X_{U_0}} X_U$. Consider the adjunction morphism

$$\operatorname*{colim}_{K\subset U\subset U_0} H^0((Y_U)_{\text{qfh}}, \mathbb{Z}/m) \to \operatorname*{colim}_{K\subset U\subset U_0} H^0((Y_U)_{\text{qfh}}, \zeta_*\mathbb{Z}/m). \tag{5.7}$$

As the étale sheaf $\mathbb{Z}/m$ is already a qfh sheaf (see [SP, Lemma 0EW8]), the left-hand side of (5.7) is isomorphic to $\operatorname*{colim}_{K\subset U\subset U_0} H^0_{\text{ét}}(Y_U, \mathbb{Z}/m)$. In addition, as the constant sheaf $\mathbb{Z}/m$ for the usual topology is already a qc sheaf (see [SP, Lemma 09X3]), the right-hand side of (5.7) is isomorphic to $\operatorname*{colim}_{K\subset U\subset U_0} H^0((Y_U)^{\text{an}}, \mathbb{Z}/m)$. All in all, the morphism (5.7) may be identified with

$$\operatorname*{colim}_{K\subset U\subset U_0} H^0_{\text{ét}}(Y_U, \mathbb{Z}/m) \to \operatorname*{colim}_{K\subset U\subset U_0} H^0((Y_U)^{\text{an}}, \mathbb{Z}/m),$$

and hence is an isomorphism by [Bin76b, (7.2)].

The computation of the cohomology of the site $(X_U)_{\text{qfh}}$ using hypercoverings [SP, Proposition 09VZ], the fact that any given qfh hypercovering of X_U involves only finitely many separated $\mathcal{O}(U)$-schemes of finite type at each simplicial level (as coverings in $(X_U)_{\text{qfh}}$ involve only finitely many arrows) and the fact that (5.7) is an isomorphism for such schemes, together imply that the natural morphism

$$\operatorname*{colim}_{K\subset U} H^p((X_U)_{\text{qfh}}, \mathbb{Z}/m) \to \operatorname*{colim}_{K\subset U} H^p((X_U)_{\text{qfh}}, \zeta_*\mathbb{Z}/m)$$

is an isomorphism, which is what we wanted to prove.

Step 5. One has $\operatorname*{colim}_{K\subset U} H^p((X_U)_{\text{qfh}}, \mathrm{R}^q\zeta_*\mathbb{Z}/m) = 0$ for $q > 0$.

Fix $q > 0$. We claim that for any Stein open neighborhood U_0 of K in S, and any separated quasi-finite $\mathcal{O}(U_0)$-scheme of finite presentation Y, the group

$$\operatorname*{colim}_{K\subset U\subset U_0} H^0((Y_U)_{\text{qfh}}, \mathrm{R}^q\zeta_*\mathbb{Z}/m) \tag{5.8}$$

vanishes. Taking this claim for granted, the computation of the cohomology of the site $(X_U)_{\mathrm{qfh}}$ using hypercoverings [SP, Proposition 09VZ], the fact that any given qfh hypercovering of X_U involves only finitely many separated quasi-finite $\mathcal{O}(U)$-schemes of finite presentation at each simplicial level, and the fact that (5.8) vanishes for such schemes, together imply that $\operatorname*{colim}_{K\subset U} H^p((X_U)_{\mathrm{qfh}}, \mathrm{R}^q\zeta_*\mathbb{Z}/m) = 0$.

We now prove the claim. Using Lemma 1.10, we may assume that K admits a basis $(U_i)_{i\in I}$ of Stein open neighborhoods such that K is $\mathcal{O}(U_i)$-convex for all $i \in I$. We may thus restrict the colimit (5.8) to such neighborhoods. Fix a Stein open neighborhood U of K in U_0 such that K is $\mathcal{O}(U)$-convex and choose a class α in $H^0((Y_U)_{\mathrm{qfh}}, \mathrm{R}^q\zeta_*\mathbb{Z}/m)$. There exists a qfh covering of Y_U, which we may choose to be given by a single morphism $W \to Y_U$, such that α is induced by a class $\beta \in H^q(W^{\mathrm{an}}, \mathbb{Z}/m)$. By Zariski's Main Theorem (see [EGA43, Théorème 8.12.6]), there exists a factorization $W \xrightarrow{j} \overline{W} \xrightarrow{\pi} X_U$ of $W \to X_U$ where j is an open immersion and π is finite.

By Proposition 4.3 applied to the Stein space $(\overline{W})^{\mathrm{an}}$, to the Stein compact subset $(\pi^{\mathrm{an}})^{-1}(K)$, and to the cohomology class β, there exist, after maybe shrinking U, a finite surjective holomorphic map $p : T \to (\overline{W})^{\mathrm{an}}$ and a class $\gamma \in H^q(T, \mathbb{Z}/m)$ with $\gamma|_{p^{-1}(W^{\mathrm{an}})} = \beta|_{p^{-1}(W^{\mathrm{an}})}$. As K is $\mathcal{O}(U)$-convex, the compact subset $(\pi^{\mathrm{an}} \circ p)^{-1}(K)$ is $\mathcal{O}(T)$-convex (see Proposition 1.6 (i)). One can thus apply Proposition 2.4 to ensure, after modifying $p : T \to (\overline{W})^{\mathrm{an}}$ and further shrinking U, that $\gamma = 0$ and hence that $\beta|_{p^{-1}(W^{\mathrm{an}})} = 0$.

By Lemma 3.3, the scheme $\mathrm{Spec}(\mathcal{O}(T))$ is interiorly finite with analytification isomorphic to T (both when viewed as a $\overline{W}$-scheme or as an $\mathcal{O}(U)$-scheme), and the structural morphism $g : \mathrm{Spec}(\mathcal{O}(T)) \to \overline{W}$ satisfies $g^{\mathrm{an}} = p$. Using Lemma 3.1 (iii), one may assume after shrinking U that g is finite of finite presentation. It follows that $g|_{g^{-1}(W)} : g^{-1}(W) \to W$ is a qfh covering. As $\beta|_{g^{-1}(W)^{\mathrm{an}}} = 0$, we conclude that $\alpha \in H^0((Y_U)_{\mathrm{qfh}}, \mathrm{R}^q\zeta_*\mathbb{Z}/m)$ is qfh-locally trivial and hence trivial. □

When f is interiorly proper, the statement of Theorem 5.1 takes a simpler form.

Corollary 5.2. *Let K be a Stein compact subset of a Stein space S and let X be an interiorly proper $\mathcal{O}(S)$-scheme. Fix a torsion abelian sheaf $\mathbb{L}$ on $X_{\text{ét}}$. For $k \geq 0$, there are canonical isomorphisms $H^k_{\text{ét}}(X_{\mathcal{O}(K)}, \mathbb{L}|_{X_{\mathcal{O}(K)}}) \to H^k((X_{\mathcal{O}(K)})^{\mathrm{an}}, \mathbb{L}^{\mathrm{an}})$.*

Proof. The isomorphism (5.2) has the required form by [SGA42, Exp. VII, Corollaire 5.8] and [God58, II, Théorème 4.11.1]. □

Remark 5.3. Bingener's [Bin76b, Theorem 7.4] implies the particular case of Theorem 5.1 where $k = 1$, the sheaf $\mathbb{L}$ is constant and $X = \mathrm{Spec}(\mathcal{O}(S))$.

5.3. **G-equivariant complex spaces.** Let $G := \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}/2$ be the Galois group of $\mathbb{R}$, generated by the complex conjugation $\sigma \in G$. We use the conventions of [BW21, Appendix A] concerning G-equivariant complex-analytic geometry. In particular, a G-equivariant complex space is a complex space endowed with an action of G (as a locally ringed space) such that $\sigma \in G$ acts $\mathbb{C}$-antilinearly.

A G-equivariant complex space is said to be Stein if so is its underlying complex space. A G-invariant Stein compact subset of a G-equivariant complex space S admits a basis of G-invariant Stein open neighborhoods in S (as the intersection of a Stein open neighborhood of K in S with its image by σ is Stein).

The *complex conjugate* S^σ of a complex space S with structural morphism $\mu : \mathbb{C} \to \mathcal{O}_S$ is the complex space which is equal to S as a locally ringed space, with structural morphism $\mu_{S^\sigma} := \mu \circ \sigma : \mathbb{C} \to \mathcal{O}_S$. A G-equivariant complex space can equivalently be described as a complex space S endowed with an isomorphism $\alpha : S^\sigma \xrightarrow{\sim} S$ such that $\alpha \circ \alpha^\sigma = \mathrm{Id}_S$ (see [BW21, §A.2]).

Note that $\mathcal{O}(S) = \mathcal{O}(S)^G \otimes_{\mathbb{R}} \mathbb{C}$. A morphism $f : X \to Y$ of $\mathcal{O}(S)^G$-schemes is said to interiorly have a property if such is the case for $f_{\mathcal{O}(S)} : X_{\mathcal{O}(S)} \to Y_{\mathcal{O}(S)}$. If X is an $\mathcal{O}(S)^G$-scheme interiorly locally of finite type, the descent datum on $X_{\mathcal{O}(S)} = X \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C})$ yields an isomorphism $\alpha : ((X_{\mathcal{O}(S)})^{\mathrm{an}})^\sigma \xrightarrow{\sim} (X_{\mathcal{O}(S)})^{\mathrm{an}}$ which endows $(X_{\mathcal{O}(S)})^{\mathrm{an}}$ with a structure of G-equivariant complex space: the analytification X^{an} of X. The site morphism $\varepsilon : (X^{\mathrm{an}})_{\mathrm{cl}} \to X_{\text{ét}}$ of (3.1) is G-equivariant for the actions of G by complex conjugation. In particular, the analytification $\mathbb{L}^{\mathrm{an}}$ of a sheaf $\mathbb{L}$ on $X_{\text{ét}}$ is naturally a G-equivariant sheaf on X^{an}. We will say that $\mathbb{L}$ is interiorly constructible if so is $\mathbb{L}|_{X_{\mathcal{O}(S)}}$.

We state for later use in the proof of Theorem 6.1 a G-equivariant analogue of Corollary 1.16, which appears in [BW21, Proposition A.5] when S has dimension 1.

Proposition 5.4. *Let S be a normal G-equivariant Stein space of pure dimension n with finitely many connected components. Associating with $p : T \to S$ the $\mathcal{M}(S)^G$-algebra $\mathcal{M}(T)^G$ induces an equivalence of categories*

$$\left\{ \begin{array}{c} \textit{finite surjective } G\textit{-equivariant holomorphic} \\ \textit{maps } p : T \to S \textit{ of normal } G\textit{-equivariant} \\ \textit{Stein spaces of pure dimension } n \end{array} \right\} \to \left\{ \begin{array}{c} \textit{finite étale} \\ \mathcal{M}(S)^G\textit{-algebras} \end{array} \right\}.$$

Proof. The proposition follows from Corollary 1.16 by using the above description of G-equivariant complex analytic spaces as complex analytic spaces S endowed with an isomorphism $\alpha : S^\sigma \xrightarrow{\sim} S$ such that $\alpha \circ \alpha^\sigma = \mathrm{Id}_S$. □

5.4. **Artin's comparison theorem over G-equivariant Stein compact sets.** Here is an extension of Theorem 5.1 to the G-equivariant setting, which goes back to Cox [Cox79, Theorem 1.1] when S is a point (see also [Sch94, (15.3.2)]).

Theorem 5.5. *Let S be a G-equivariant Stein space. Let $f : X \to \mathrm{Spec}(\mathcal{O}(S)^G)$ be an $\mathcal{O}(S)^G$-scheme interiorly of finite type. Fix a torsion abelian sheaf $\mathbb{L}$ on $X_{\text{ét}}$. Assume that f is interiorly proper or that $\mathbb{L}$ is interiorly constructible. If one lets U run over all G-invariant Stein open neighborhoods of a G-invariant Stein compact subset K of S, the base change morphisms*

$$\operatorname*{colim}_{K \subset U} H^k_{\text{ét}}(X_{\mathcal{O}(U)^G}, \mathbb{L}_{\mathcal{O}(U)^G}) \to \operatorname*{colim}_{K \subset U} H^k_G((X_{\mathcal{O}(U)^G})^{\mathrm{an}}, \mathbb{L}^{\mathrm{an}}) \tag{5.9}$$

are isomorphisms for $k \geq 0$.

Proof. The G-equivariant site morphism $\varepsilon : ((X_{\mathcal{O}(U)})^{\mathrm{an}})_{\mathrm{cl}} \to (X_{\mathcal{O}(U)})_{\text{ét}}$ induces a morphism between the Hochschild–Serre spectral sequences of [Sch94, Remark 10.9]

$$E_2^{p,q} = H^p(G, H^q_{\text{ét}}(X_{\mathcal{O}(U)}, \mathbb{L}_{\mathcal{O}(U)})) \implies H^{p+q}_{\text{ét}}(X_{\mathcal{O}(U)^G}, \mathbb{L}_{\mathcal{O}(U)^G}) \text{ and}$$
$$E_2^{p,q} = H^p(G, H^q((X_{\mathcal{O}(U)})^{\mathrm{an}}, \mathbb{L}^{\mathrm{an}})) \implies H^{p+q}_G((X_{\mathcal{O}(U)^G})^{\mathrm{an}}, \mathbb{L}^{\mathrm{an}}),$$

which is an isomorphism on page 2 by Theorem 5.1, and hence an isomorphism. □

5.5. **Cohomological dimension.** A scheme X is said to have *cohomological dimension* $\leq n$ if the étale cohomology of any torsion abelian sheaf on $X_{\text{ét}}$ vanishes in degree $> n$. Our next theorem controls the cohomological dimension of affine varieties over Stein compacta. It is a generalization in Stein geometry of the bounds on the cohomogical dimension of complex and real affine varieties obtained in [SGA43, XIV, Corollaire 3.2] and [Sch94, Corollary 7.21].

Theorem 5.6. *Let K be a G-invariant Stein compact subset in a G-equivariant Stein space S. Let X be an affine $\mathcal{O}(S)^G$-scheme interiorly of finite type with $(X^{\mathrm{an}})^G = \varnothing$. Then $X_{\mathcal{O}(K)^G}$ has étale cohomological dimension $\leq \dim(X^{\mathrm{an}})$.*

Proof. We may suppose that $\dim(X^{\mathrm{an}}) < \infty$. We wish to show the vanishing of the degree k cohomology of a torsion abelian sheaf on $(X_{\mathcal{O}(K)^G})_{\text{ét}}$ if $k > \dim(X^{\mathrm{an}})$. By [SP, Lemma 03SA (2)], any such sheaf is a filtered colimit of constructible sheaves. In view of [SGA42, VII, Proposition 3.3], we may thus only consider constructible sheaves. By [SGA43, IX, Corollaire 2.7.4], any constructible abelian sheaf on $(X_{\mathcal{O}(K)^G})_{\text{ét}}$ is of the form $\mathbb{L}|_{X_{\mathcal{O}(K)^G}}$ for some constructible abelian sheaf $\mathbb{L}$ on $X_{\text{ét}}$, after possibly shrinking S.

Let K' be an excellent Stein compact neighborhood of K in S (see Lemma 1.9 (i)). By [SGA42, IX, Proposition 2.14 (ii) and XIV, Théorème 1.1], the sheaf $\mathbb{L}|_{X_{\mathcal{O}(K')^G}}$ has a resolution by finite products of sheaves of the form π_*A, where A is a finite abelian group and $\pi : Y \to X_{\mathcal{O}(K')^G}$ is finite. After shrinking S, we may assume that $\mathbb{L}$ itself has such a resolution (with finite morphisms $\pi : Y \to X$). Using the spectral sequence of a resolution, we may assume that $\mathbb{L}$ is of the form π_*A. Letting U run over all G-invariant Stein open neighborhoods of K in S, one computes

$$\begin{aligned} H^k_{\text{ét}}(X_{\mathcal{O}(K)^G}, \mathbb{L}|_{X_{\mathcal{O}(K)^G}}) &= \operatorname*{colim}_{K\subset U} H^k_{\text{ét}}(X_{\mathcal{O}(U)^G}, \mathbb{L}|_{X_{\mathcal{O}(U)^G}}) \\ &= \operatorname*{colim}_{K\subset U} H^k_{\text{ét}}(Y_{\mathcal{O}(U)^G}, A) \\ &= \operatorname*{colim}_{K\subset U} H^k_G((Y_{\mathcal{O}(U)^G})^{\mathrm{an}}, A) \end{aligned} \tag{5.10}$$

where we used successively [SGA42, Exp. VII, Corollaire 5.8], the vanishing of the higher direct images of π [SGA42, VIII, Proposition 5.5], and Theorem 5.5. To conclude, we will show that $H^k_G((Y_{\mathcal{O}(U)^G})^{\mathrm{an}}, A) = 0$ for $k > \dim(X^{\mathrm{an}})$.

After shrinking S, we may assume that the $\mathcal{O}(S)^G$-scheme Y is affine of finite type, hence a closed subscheme of $\mathbb{A}^N_{\mathcal{O}(S)^G}$ for $N \gg 0$. It follows that $(Y_{\mathcal{O}(U)^G})^{\mathrm{an}}$ may be realized as a G-invariant closed complex subspace of $\mathbb{C}^N \times U$ (see [Bin76b, Proof of Satz 1.1]). It is thus a G-equivariant Stein space.

Let $\mathbb{Z}(1)$ be the G-module isomorphic to $\mathbb{Z}$ as an abelian group, on which $\sigma \in G$ acts by multiplication by -1. Set $A(1) := A \otimes_{\mathbb{Z}} \mathbb{Z}(1)$. The short exact sequences of G-modules $0 \to A(1) \to A[G] \to A \to 0$ and $0 \to A \to A[G] \to A(1) \to 0$ yield, for any $l \geq 0$, the exact sequences of G-equivariant cohomology

$$H^k((Y_{\mathcal{O}(U)})^{\mathrm{an}}, A) \to H^k_G((Y_{\mathcal{O}(U)^G})^{\mathrm{an}}, A) \to H^{k+1}_G((Y_{\mathcal{O}(U)^G})^{\mathrm{an}}, A(1)) \quad \text{and} \tag{5.11}$$

$$H^{k+1}((Y_{\mathcal{O}(U)})^{\mathrm{an}}, A) \to H^{k+1}_G((Y_{\mathcal{O}(U)^G})^{\mathrm{an}}, A(1)) \to H^{k+2}_G((Y_{\mathcal{O}(U)^G})^{\mathrm{an}}, A). \tag{5.12}$$

By a theorem of Hamm [Ham83, Satz 1], the Stein space $(Y_{\mathcal{O}(U)})^{\mathrm{an}}$ has the homotopy type of a CW complex of dimension $\leq \dim((Y_{\mathcal{O}(U)})^{\mathrm{an}}) \leq \dim(X^{\mathrm{an}})$. It follows that $H^k((Y_{\mathcal{O}(U)})^{\mathrm{an}}, A) = 0$ for $k > \dim(X^{\mathrm{an}})$. For all $k > \dim(X^{\mathrm{an}})$, this vanishing combined with (5.11) and (5.12) shows the existence of an injective map $H^k_G((Y_{\mathcal{O}(U)^G})^{\mathrm{an}}, A) \to H^{k+2}_G((Y_{\mathcal{O}(U)^G})^{\mathrm{an}}, A)$. We therefore only need to show that $H^k_G((Y_{\mathcal{O}(U)^G})^{\mathrm{an}}, A) = 0$ for all $k \gg 0$. To do so, note that

$$H^k_G((Y_{\mathcal{O}(U)^G})^{\mathrm{an}}, A) = H^k((Y_{\mathcal{O}(U)^G})^{\mathrm{an}}/G, A)$$

because $((Y_{\mathcal{O}(U)^G})^{\mathrm{an}})^G = \varnothing$. As the real-analytic space $(Y_{\mathcal{O}(U)^G})^{\mathrm{an}}/G$ has real dimension $\leq 2\dim(X^{\mathrm{an}})$, it has the homotopy type of a CW complex of finite dimension (combine [GMT86, VI, Theorem 2.7] and [Loj64, Theorem 2]), and the desired vanishing for $k \gg 0$ follows. $\square$

The following application of Theorem 5.6 extends an algebraic result [CTP90, Proposition 1.2.1] attributed to Ax by Colliot-Thélène and Parimala.

Theorem 5.7. *Let K be a G-invariant Stein compact subset of a G-equivariant Stein space S. Let X be an $\mathcal{O}(S)^G$-scheme interiorly of finite type with $(X^{\mathrm{an}})^G = \varnothing$. If $X_{\mathcal{O}(K)^G}$ is integral, its function field has cohomological dimension $\leq \dim(X^{\mathrm{an}})$.*

Proof. We may assume that X is affine (after replacing it with an affine open subset $X' \subset X$ such that $X'_{\mathcal{O}(K)^G}$ is nonempty). Let $(U_i)_{i\in\mathbb{N}}$ be a decreasing basis of G-invariant Stein open neighborhoods of K in S. If $i \in \mathbb{N}$ and $h \in \mathcal{O}(X_{\mathcal{O}(U_i)^G})$ is nonzero in restriction to $X_{\mathcal{O}(K)^G}$, we define $Y_{i,h}$ to be the distinguished affine open subset of $X_{\mathcal{O}(U_i)^G}$ where $h \neq 0$. Then the function field F of $X_{\mathcal{O}(K)^G}$ can be written as the filtered colimit of rings $F = \operatorname*{colim}_{i,h} \mathcal{O}((Y_{i,h})_{\mathcal{O}(K)^G})$. By Theorem 5.6 applied to the affine $\mathcal{O}(U_i)^G$-scheme $Y_{i,h}$, the scheme $(Y_{i,h})_{\mathcal{O}(K)^G}$ has cohomological dimension $\leq \dim(X^{\mathrm{an}})$. We deduce from [SP, Lemma 0F0R] that F has cohomological dimension $\leq \dim(X^{\mathrm{an}})$. $\square$

If $(X^{\mathrm{an}})^G \neq \varnothing$, one may sometimes apply Theorem 5.7 to a Zariski open subset X' of X such that $((X')^{\mathrm{an}})^G = \varnothing$. The next corollary is obtained in this way.

Corollary 5.8. *Let K be a G-invariant Stein compact subset of a G-equivariant Stein space S. If S^G is contained in a nowhere dense closed analytic subset $Z \subset S$ and $\mathcal{O}(K)^G$ is a domain, then* $\mathrm{Frac}(\mathcal{O}(K)^G)$ *has cohomological dimension* $\leq \dim(S)$.

Proof. Define $V := \mathrm{Spec}(\mathcal{O}(S)^G) \setminus \mathrm{Spec}(\mathcal{O}(S)^G/\mathcal{I}_Z(S)^G)$. By Lemma 3.2 and its proof, the $\mathcal{O}(S)^G$-scheme V is interiorly of finite type and the analytification of the open immersion $V \hookrightarrow \mathrm{Spec}(\mathcal{O}(S)^G)$ identifies with $S \setminus Z \hookrightarrow S$. It now suffices to apply Theorem 5.7 with $X = V$ to conclude. $\square$

If S is a complex space, then the disjoint union $T := S \sqcup S^\sigma$ may be endowed with a structure of G-equivariant complex space, by letting $\sigma \in G$ exchange the two factors (and act by the identity on the underlying locally ringed spaces). One then has $T^G = \varnothing$ and $\mathcal{O}(T)^G \simeq \mathcal{O}(S)$. Using this trick, one can formally deduce non-G-equivariant statements from G-equivariant ones. For instance, Theorems 5.6 and 5.7 and Corollary 5.8 immediately imply the following.

Theorem 5.9. *Let K be a Stein compact subset of a Stein space S. Let X be an affine $\mathcal{O}(S)$-scheme interiorly of finite type. Then $X_{\mathcal{O}(K)}$ has étale cohomological dimension $\leq \dim(X^{\mathrm{an}})$.*

Theorem 5.10. *Let K be a Stein compact subset of a Stein space S. Let X be an $\mathcal{O}(S)$-scheme interiorly of finite type. If $X_{\mathcal{O}(K)}$ is integral, then its function field has cohomological dimension $\leq \dim(X^{\mathrm{an}})$.*

Corollary 5.11. *Let K be a Stein compact subset of a Stein space S. If $\mathcal{O}(K)$ is a domain, then* $\mathrm{Frac}(\mathcal{O}(K))$ *has cohomological dimension* $\leq \dim(S)$.

Remarks 5.12. (i) One could have given direct proofs of Theorems 5.9 and 5.10 and Corollary 5.11, similar to the proofs of Theorems 5.6 and 5.7 and Corollary 5.8 (replacing Theorem 5.5 by Theorem 5.1).

(ii) In the setting of Corollary 5.11, if S has pure dimension n and K is nonempty, the cohomological dimension of $\mathrm{Frac}(\mathcal{O}(K))$ is equal to n. To see it, choose $s \in K$. The local ring morphism $\mathcal{O}(K)_s \to \mathcal{O}_{S,s}$ is faithfully flat by Lemma 1.8 and [Mat80, Theorem 2]. It thus follows from [Bou06, I, §3.5, Corollaire] and [Mat80, Theorem 19] that $\mathcal{O}(K)_s$ is noetherian of dimension n. By [SGA43, Corollaire 2.5], the field $\mathrm{Frac}(\mathcal{O}(K)) = \mathrm{Frac}(\mathcal{O}(K)_s)$ has cohomological dimension $\geq n$.

6. Applications to Hilbert's 17th problem

We may now present our applications to sums of squares of analytic functions.

6.1. **Sums of squares.** The following theorem is an analogue of Artin's solution to Hilbert's 17th problem [Art27] on G-equivariant Stein compacta.

Theorem 6.1. *Let K be a G-invariant Stein compact subset of a reduced G-equivariant Stein space S. Let $f \in \mathcal{O}(K)^G$ be nonnegative on a neighborhood of K^G in S^G. Then f is a sum of squares in $\mathcal{M}(K)^G$.*

Proof. Using [GR84, 8, §1.3, Proposition], we may replace S by its normalization and hence assume that it is normal. Let K' be a Stein compact neighborhood of K in M such that $f \in \mathcal{O}(K')^G$ (see Lemma 1.9 (ii)). By compactness of K, we may assume that K' has finitely many connected components. Replacing K with a G-orbit of connected components of K', we may assume that K/G is connected.

We may suppose that $f \neq 0$. After shrinking S, we may assume that S/G is connected, that $f \in \mathcal{O}(S)^G$, and that f is nonnegative on S^G. Let $p : T \to S$ be the finite surjective G-equivariant holomorphic map of normal G-equivariant Stein spaces associated with the finite étale $\mathcal{M}(S)^G$-algebra $F := \mathcal{M}(S)^G[x]/\langle x^2 + f\rangle$ by Proposition 5.4. The nonnegativity hypothesis on f and the fact that $-f$ is a square in $\mathcal{M}(T)^G$, hence in $\mathcal{O}(T)^G$ by normality of T, imply that T^G is contained in the nowhere dense closed analytic subset $\{f = 0\}$ of T. Let $L \subset T$ be a G-orbit of connected components of $p^{-1}(K)$. It is a G-invariant Stein compact subset of T by [GR79, V, §1.1, Theorem 1 d)]. By Corollary 5.8, the field $\mathcal{M}(L)^G$ has finite cohomological dimension, and hence cannot be ordered.

As $\mathcal{M}(K)^G \subset \mathcal{M}(L)^G \subset \mathcal{M}(K)^G[x]/\langle x^2 + f\rangle$, one has $\mathcal{M}(L)^G = \mathcal{M}(K)^G$ or $\mathcal{M}(L)^G = \mathcal{M}(K)^G[x]/\langle x^2+f\rangle$. In the first case, f is a sum of squares in $\mathcal{M}(K)^G$ by [Lam05, VIII, Theorem 1.10 and Proposition 1.1 (2)]. In the second case, f is a sum of squares in $\mathcal{M}(K)^G$ by [Lam05, VIII, Theorem 1.10 and Basic Lemma 1.4]. □

Remark 6.2. Theorem 6.1 can be proven by the more elementary methods of [Jaw86], but one could not recover in this way the quantitative results of Theorem 6.4.

6.2. **Sums of few squares.** If A is a ring, we let $\mathrm{cd}_2(A)$ denote the cohomological 2-dimension of the étale site of $\mathrm{Spec}(A)$ (see [Sch94, Definition 7.1]).

Proposition 6.3. *Let F be a field such that $\mathrm{cd}_2(F[x]/\langle x^2 + 1\rangle) \leq n$. If $f \in F$ is a sum of squares in F, then it is a sum of 2^n squares in F.*

Proof. If $\mathrm{char}(F) = 2$, then sums of squares in F are squares in F, so we may assume that $\mathrm{char}(F) \neq 2$. We may also assume that $f \neq 0$. Denote by $\{a\} \in H^1(F, \mathbb{Z}/2)$ the class induced by $a \in F^*$ via the Kummer isomorphism $F^*/(F^*)^2 \xrightarrow{\sim} H^1(F, \mathbb{Z}/2)$.

As f is a sum of squares, it is positive with respect to all the field orderings of F. By a theorem of Arason [Ara75, Satz 3], the class $\{f\} \cdot \{-1\}^N \in H^{N+1}(F, \mathbb{Z}/2)$ vanishes for $N \gg 0$. Consider the étale F-algebra $F' := F[x]/\langle x^2+1\rangle$ and set $\Gamma := \mathrm{Gal}(F'/F)$. For all $N \geq 0$, the short exact sequence

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/2[\Gamma] \to \mathbb{Z}/2 \to 0$$

of Γ-modules induces an exact sequence

$$H^N(F', \mathbb{Z}/2) \to H^N(F, \mathbb{Z}/2) \xrightarrow{\cdot\{-1\}} H^{N+1}(F, \mathbb{Z}/2).$$

A decreasing induction on the degree shows that $\{f\} \cdot \{-1\}^N = 0$ for all $N \geq n$. The Milnor conjectures proven by Voevodsky [Voe03] now imply that f is a sum of 2^n squares in F (see [Ben20, Proposition 2.1]). $\square$

The next theorem is a quantitative improvement of Theorem 6.1 in the spirit of Pfister's theorem [Pfi67, Theorem 1].

Theorem 6.4. *Let K be a G-invariant Stein compact subset of a reduced G-equivariant Stein space S of dimension n. Let $f \in \mathcal{O}(K)^G$ be nonnegative on a neighborhood of K^G in S^G. Then f is a sum of 2^n squares in $\mathcal{M}(K)^G$.*

Proof. Arguing as in the proof of Theorem 6.1, we may assume that S is normal and that K/G is connected. Then $F := \mathcal{M}(K)^G$ is a field and $F[x]/\langle x^2+1\rangle \simeq \mathcal{M}(K)$ has cohomological dimension $\leq n$ by Corollary 5.11 applied to the connected components of K. Apply Theorem 6.1 and Proposition 6.3 to conclude. $\square$

6.3. **Real-analytic geometry.** Theorem 6.4 has applications to real-analytic variants of Hilbert's 17th problem. We follow the conventions of [GMT86] and refer to [GMT86, II, Definition 1.4] for the definitions of *real-analytic spaces* and *real-analytic varieties*. If K is a closed subset of a real-analytic space M, we denote by $\mathcal{O}(K)$ (resp. $\mathcal{M}(K)$) the ring of germs of real-analytic functions (resp. of real-analytic meromorphic functions) in a neighborhood of K.

Theorem 6.5. *Let M be a real-analytic space of dimension n with reduced local rings. Let $K \subset M$ be a compact subset. Let $f \in \mathcal{O}(K)$ be nonnegative on a neighborhood of K in M. Then f is a sum of 2^n squares in $\mathcal{M}(K)$.*

Proof. By [GMT86, III, Theorems 3.6 and 3.10], there exist a reduced G-equivariant Stein space S of dimension n, and an isomorphism $M \xrightarrow{\sim} S^G$ of real-analytic spaces. In addition, there exists a proper injective G-equivariant holomorphic map $i : S \to \mathbb{C}^N$ for $N \gg 0$ (see [GMT86, V, Theorem 3.7]).

The compact subset $i(K) \subset \mathbb{R}^N$ admits a basis of Stein neighborhoods in $\mathbb{C}^N$ (see [Siu69, Lemma 5]) which we may choose to be G-invariant as the intersection of two Stein open subsets is Stein (see [GR79, p. 127]). We deduce that K admits a basis of G-invariant Stein neighborhoods in S (use [GR79, V, §1.1, Theorem 1 d)]). By [GMT86, III, Proposition 1.8], one may choose one such neighborhood S with the property that f extends to a holomorphic map $g : S \to \mathbb{C}$. After replacing g with $(g + \overline{g \circ \sigma})/2$, we may assume that it is G-equivariant. One may now apply Theorem 6.4 to the function g to conclude. $\square$

Remark 6.6. Theorem 6.5 applies to normal real-analytic varieties of pure dimension by [GMT86, IV, Proposition 3.8], hence to real-analytic manifolds.

References

[ABF14] F. Acquistapace, F. Broglia and J. F. Fernando, *On Hilbert's 17th problem and Pfister's multiplicative formulae for the ring of real analytic functions*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **13** (2014), no. 2, 333–369.

[ABF19] ———, *Normalization of complex analytic spaces from a global viewpoint*, J. Geom. Anal. **29** (2019), no. 3, 2888–2930.

[ABF22] ———, *Topics in global real analytic geometry*, Springer Monographs in Math., Springer, Cham, 2022.

[ABFR05] F. Acquistapace, F. Broglia, J. F. Fernando and J. M. Ruiz, *On the Pythagoras numbers of real analytic surfaces*, Ann. Sci. ENS **38** (2005), no. 5, 751–772.

[ABFR10] ———, *On the finiteness of Pythagoras numbers of real meromorphic functions*, Bull. SMF **138** (2010), no. 2, 231–247.

[ADCR03] C. Andradas, A. Díaz-Cano and J. M. Ruiz, *The Artin–Lang property for normal real analytic surfaces*, J. Reine Angew. Math. **556** (2003), 99–111.

[AF59] A. Andreotti and T. Frankel, *The Lefschetz theorem on hyperplane sections*, Ann. of Math. (2) **69** (1959), 713–717.

[Ara75] J. K. Arason, *Primideale im graduierten Wittring und im* mod 2 *Cohomologiering*, Math. Z. **145** (1975), no. 2, 139–143.

[Art27] E. Artin, *Über die Zerlegung definiter Funktionen in Quadrate*, Abh. Math. Sem. Univ. Hamburg **5** (1927), no. 1, 100–115.

[AT19] D. Abramovich and M. Temkin, *Functorial factorization of birational maps for qe schemes in characteristic 0*, Algebra Number Theory **13** (2019), no. 2, 379–424.

[Ben20] O. Benoist, *Sums of squares in function fields over Henselian local fields*, Math. Ann. **376** (2020), no. 1-2, 683–692.

[Bha12] B. Bhatt, *Annihilating the cohomology of group schemes*, Algebra Number Theory **6** (2012), no. 7, 1561–1577.

[Bin76a] J. Bingener, *Holomorph-prävollständige Resträume zu analytischen Mengen in Steinschen Räumen*, J. Reine Angew. Math. **285** (1976), 149–171.

[Bin76b] ———, *Schemata über Steinschen Algebren*, Schr. Math. Inst. Univ. Münster (2), vol. 10, 1976.

[BKS81] J. Bochnak, W. Kucharz and M. Shiota, *On equivalence of ideals of real global analytic functions and the 17th Hilbert problem*, Invent. Math. **63** (1981), no. 3, 403–421.

[BM21] B. Bhatt and A. Mathew, *The arc-topology*, Duke Math. J. **170** (2021), no. 9, 1899–1988.

[Bou06] N. Bourbaki, *Éléments de mathématique. Algèbre commutative. Chapitres 1 à 4*, reprint of the 1985 original ed., Springer, Berlin, 2006.

[BR75] J. Bochnak and J.-J. Risler, *Le théorème des zéros pour les variétés analytiques réelles de dimension* 2, Ann. Sci. ENS **8** (1975), no. 3, 353–363.

[BW21] O. Benoist and O. Wittenberg, *The tight approximation property*, J. Reine Angew. Math. **776** (2021), 151–200.

[Car57a] H. Cartan, *Quotient d'un espace analytique par un groupe d'automorphismes*, Algebraic geometry and topology. A symposium in honor of S. Lefschetz, Princeton University Press, Princeton, N.J., 1957, pp. 90–102.

[Car57b] ———, *Variétés analytiques réelles et variétés analytiques complexes*, Bull. Soc. Math. France **85** (1957), 77–99.

[Cho22] C.-Y. Chough, *Proper base change for étale sheaves of spaces*, JEMS **24** (2022), no. 11, 3679–3702.

[CM86] M. Colţoiu and N. Mihalache, *On the homology groups of Stein spaces and Runge pairs*, J. Reine Angew. Math. **371** (1986), 216–220.

[Cox79] D. A. Cox, *The étale homotopy type of varieties over* $\mathbf{R}$, Proc. AMS **76** (1979), no. 1, 17–22.

[CTP90] J.-L. Colliot-Thélène and R. Parimala, *Real components of algebraic varieties and étale cohomology*, Invent. Math. **101** (1990), no. 1, 81–99.

[EGA42] A. Grothendieck, *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, II*, Publ. Math. IHES **24** (1965).

[EGA43] ———, *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, III*, Publ. Math. IHES **28** (1966).

[Fer21] J. F. Fernando, *Positive semidefinite analytic functions on real analytic surfaces*, J. Geom. Anal. **31** (2021), no. 12, 12375–12410.

[For17] F. Forstnerič, *Stein manifolds and holomorphic mappings*, second ed., Ergeb. Math. Grenzgeb. (3), vol. 56, Springer, Cham, 2017.

[Fri67] J. Frisch, *Points de platitude d'un morphisme d'espaces analytiques complexes*, Invent. Math. **4** (1967), 118–138.

[GG73] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Graduate Texts in Math., vol. 14, Springer-Verlag, New York-Heidelberg, 1973.

[GMT86] F. Guaraldo, P. Macrì and A. Tancredi, *Topics on real analytic spaces*, Advanced Lectures in Math., Friedr. Vieweg & Sohn, Braunschweig, 1986.

[God58] R. Godement, *Topologie algébrique et théorie des faisceaux*, Actualités Scientifiques et Industrielles, vol. 1252, Hermann, Paris, 1958, Publ. Math. Univ. Strasbourg. No. 13.

[GR65] R. C. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1965.

[GR79] H. Grauert and R. Remmert, *Theory of Stein spaces*, Grundlehren der Mathematischen Wissenschaften, vol. 236, Springer-Verlag, Berlin-New York, 1979.

[GR84] ———, *Coherent analytic sheaves*, Grundlehren der Mathematischen Wissenschaften, vol. 265, Springer-Verlag, Berlin, 1984.

[Gra58] H. Grauert, *On Levi's problem and the imbedding of real-analytic manifolds*, Ann. of Math. (2) **68** (1958), 460–472.

[Gro68] A. Grothendieck, *Le groupe de Brauer. III. Exemples et compléments*, Dix exposés sur la cohomologie des schémas, Adv. Stud. Pure Math., vol. 3, North-Holland, Amsterdam, 1968, pp. 88–188.

[Gur88] R. M. Guralnick, *Matrices and representations over rings of analytic functions and other one-dimensional rings*, Visiting scholars' lectures—1987 (Lubbock, TX), Texas Tech Univ. Math. Ser., vol. 15, Texas Tech Univ., Lubbock, TX, 1988, pp. 15–35.

[Hak72] M. Hakim, *Topos annelés et schémas relatifs*, Ergeb. Math. Grenzgeb., vol. 64, Springer-Verlag, Berlin-New York, 1972.

[Ham83] H. A. Hamm, *Zur Homotopietyp Steinscher Räume*, J. Reine Angew. Math. **338** (1983), 121–135.

[Har77] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Math., vol. 52, Springer-Verlag, New York-Heidelberg, 1977.

[Hil88] D. Hilbert, *Ueber die Darstellung definiter Formen als Summe von Formenquadraten*, Math. Ann. **32** (1888), no. 3, 342–350.

[HL98] G. Henkin and J. Leiterer, *The Oka–Grauert principle without induction over the base dimension*, Math. Ann. **311** (1998), no. 1, 71–93.

[Hou61] C. Houzel, *Géométrie analytique locale III*, Familles d'espaces complexes et fondements de la géométrie analytique, exposé 20, Séminaire H. Cartan E.N.S., vol. 13, 1960/61.

[Iss66] Hej Iss'sa, *On the meromorphic function field of a Stein variety*, Ann. of Math. (2) **83** (1966), 34–46.

[Jaw82] P. Jaworski, *Positive definite analytic functions and vector bundles*, Bull. Acad. Polon. Sci. Sér. Sci. Math. **30** (1982), 501–506.

[Jaw86] ———, *Extensions of orderings on fields of quotients of rings of real analytic functions*, Math. Nachr. **125** (1986), 329–339.

[Jaw92] ———, *The 17th Hilbert problem for noncompact real analytic manifolds*, Real algebraic geometry (Rennes, 1991), Lecture Notes in Math., vol. 1524, Springer, Berlin, 1992, pp. 289–295.

[Kie67] R. Kiehl, *Note zu der Arbeit von J. Frisch: "Points de platitude d'un morphisme d'espaces analytiques complexes"*, Invent. Math. **4** (1967), 139–141.

[KS21] K. Kurano and K. Shimomoto, *Ideal-adic completion of quasi-excellent rings (after Gabber)*, Kyoto J. Math. **61** (2021), no. 3, 707–722.

[Kuc05] W. Kucharz, *Meromorphic functions and factoriality*, Proc. AMS **133** (2005), no. 7, 2013–2021.

[Lam05] T. Y. Lam, *Introduction to quadratic forms over fields*, Graduate Studies in Math., vol. 67, Amer. Math. Soc., Providence, RI, 2005.

[Loj64] S. Lojasiewicz, *Triangulation of semi-analytic sets*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) **18** (1964), 449–474.

[Mar75] A. Markoe, *Invariance of holomorphic convexity under proper mappings*, Math. Ann. **217** (1975), no. 3, 267–270.

[Mat80] H. Matsumura, *Commutative algebra*, second ed., Math. Lecture Note Series, vol. 56, Benjamin/Cummings Publishing Co., Inc., Reading, Mass., 1980.

[Mih96] N. Mihalache, *Special neighbourhoods of subsets in complex spaces*, Math. Z. **221** (1996), no. 1, 49–58.

[Nar60] R. Narasimhan, *Imbedding of holomorphically complete complex spaces*, Amer. J. Math. **82** (1960), 917–934.

[Nar61] ——, *The Levi problem for complex spaces*, Math. Ann. **142** (1961), 355–365.

[Nar62] ——, *The Levi problem for complex spaces. II*, Math. Ann. **146** (1962), 195–216.

[NS77] F. Norguet and Y.-T. Siu, *Holomorphic convexity of spaces of analytic cycles*, Bull. Soc. Math. Fr. **105** (1977), no. 2, 191–223.

[Pfi67] A. Pfister, *Zur Darstellung definiter Funktionen als Summe von Quadraten*, Invent. Math. **4** (1967), 229–237.

[Rui85] J. M. Ruiz, *On Hilbert's 17th problem and real Nullstellensatz for global analytic functions*, Math. Z. **190** (1985), no. 3, 447–454.

[Sch70] H. W. Schuster, *Infinitesimale Erweiterungen komplexer Räume*, Comment. Math. Helv. **45** (1970), 265–286.

[Sch94] C. Scheiderer, *Real and étale cohomology*, Lecture Notes in Math., vol. 1588, Springer-Verlag, Berlin, 1994.

[Ser94] J.-P. Serre, *Cohomologie galoisienne*, fifth ed., Lecture Notes in Math., vol. 5, Springer-Verlag, Berlin, 1994.

[SGA41] A. Grothendieck, *Théorie des topos et cohomologie étale des schémas (SGA 4 I)*, Lecture Notes in Math., vol. 269, Springer-Verlag, Berlin-New York, 1972.

[SGA42] ——, *Théorie des topos et cohomologie étale des schémas (SGA 4 II)*, Lecture Notes in Math., vol. 270, Springer-Verlag, Berlin-New York, 1972.

[SGA43] ——, *Théorie des topos et cohomologie étale des schémas (SGA 4 III)*, Lecture Notes in Math., vol. 305, Springer-Verlag, Berlin-New York, 1973.

[Sho94] V. V. Shokurov, *Riemann surfaces and algebraic curves*, Algebraic geometry, I, Encyclopaedia Math. Sci., vol. 23, Springer, Berlin, 1994, pp. 1–166.

[Siu69] Y.-T. Siu, *Noetherianness of rings of holomorphic functions on Stein compact subsets*, Proc. AMS **21** (1969), 483–489.

[SP] A. J. de Jong et al., *The Stacks Project*, `https://stacks.math.columbia.edu`.

[Swa98] R. G. Swan, *Néron–Popescu desingularization*, Algebra and geometry (Taipei, 1995), Lect. Algebra Geom., vol. 2, Int. Press, Cambridge, MA, 1998, pp. 135–192.

[Tem08] M. Temkin, *Desingularization of quasi-excellent schemes in characteristic zero*, Adv. Math. **219** (2008), no. 2, 488–522.

[Tog67] A. Tognoli, *Proprietà globali degli spazi analitici reali*, Ann. Mat. Pura Appl. (4) **75** (1967), 143–218.

[Voe96] V. Voevodsky, *Homology of schemes*, Selecta Math. (N.S.) **2** (1996), no. 1, 111–153.

[Voe03] ——, *Motivic cohomology with* $\mathbf{Z}/2$*-coefficients*, Publ. Math. IHES (2003), no. 98, 59–104.

[Zam76] W. R. Zame, *Holomorphic convexity of compact sets in analytic spaces and the structure of algebras of holomorphic germs*, Trans. AMS **222** (1976), 107–127.

Département de mathématiques et applications, École normale supérieure, CNRS, 45 rue d'Ulm, 75230 Paris Cedex 05, France

Email address: `olivier.benoist@ens.fr`