

Artificial boundary conditions to compute correctors in linear elasticity

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1 Introduction

This work is motivated by the asymptotic analysis for a boundary singular perturbation in an elliptic boundary value problem. Let Ω_0 be a bounded domain of $\mathbb{R} \times \mathbb{R}^+$, with the origin $\mathbf{0}$ being a regular point on its boundary. In the following, the first coordinate axis coincides with the tangent direction of $\partial\Omega_0$ at point $\mathbf{0}$. We denote by ω another bounded domain containing $\mathbf{0}$. The perturbed domain Ω_ε is obtained from Ω_0 by removing a rescaled version of ω at size ε : $\Omega_\varepsilon = \Omega_0 \setminus (\varepsilon\bar{\omega})$. The problem we focus on is the following

$$\begin{cases} -\mu\Delta\mathbf{u}_\varepsilon - (\lambda + \mu)\mathbf{grad}\operatorname{div}\mathbf{u}_\varepsilon = \mathbf{f} & \text{in } \Omega_\varepsilon, \\ \mathbf{u}_\varepsilon = \mathbf{u}^d & \text{on } \Gamma_d, \\ \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{g} & \text{on } \Gamma_t, \end{cases} \quad (1)$$

where Γ_d and Γ_t are the Dirichlet and Neumann boundaries, respectively. It concerns Navier equations of *linear elasticity*. The perturbation point $\mathbf{0}$ lies on the Neumann boundary. Besides, the volumic load \mathbf{f} and the traction \mathbf{g} are assumed to vanish near the perturbation. Problem (1) naturally involves two scales: the size of the structure (*scale 1*), and the characteristic length ε of the perturbation (*scale ε*). At scale 1, the domain Ω_ε tends to Ω_0 as $\varepsilon \rightarrow 0$, while the limit domain after rescaling (i.e. limit of $\Omega_\varepsilon/\varepsilon$) is the semi-infinite domain \mathbf{H}_∞ defined as $\mathbf{H}_\infty = (\mathbb{R} \times \mathbb{R}^+) \setminus \bar{\omega}$. We briefly present the leading terms in the asymptotic expansion of the solution \mathbf{u}_ε to Problem (1) obtained by following the methods presented in [5]. This description requests two variables: x (*slow variable*) and x/ε (*fast variable*), corresponding to scale 1 and scale ε , respectively. The behavior of \mathbf{u}_ε in the fast variable relies on *profiles*, which are normalized functions defined in \mathbf{H}_∞ and contributing to the expansion in variable x/ε . Let us introduce the profile basis $(\mathbf{V}_\ell)_{\ell=1,2}$ as the solutions of

$$\begin{cases} -\mu\Delta\mathbf{V}_\ell - (\lambda + \mu)\mathbf{grad}\operatorname{div}\mathbf{V}_\ell = \mathbf{0} & \text{in } \mathbf{H}_\infty, \\ \boldsymbol{\sigma}(\mathbf{V}_\ell) \cdot \mathbf{n} = \mathbf{G}_\ell & \text{on } \partial\mathbf{H}_\infty, \end{cases} \quad (2)$$

with $\mathbf{G}_1 = (n_1, 0)$, $\mathbf{G}_2 = (0, n_1)$ and $\mathbf{n} = (n_1, n_2)$ the outer normal on $\partial\mathbf{H}_\infty$. Regularity of the solution of Navier equation has been studied, for example, in [4]. We emphasize that Problem (2) depends exclusively on the geometry. In particular, \mathbf{V}_ℓ is independent of the loading of Problem (1).

The expansion of \mathbf{u}_ε takes the form

$$\mathbf{u}_\varepsilon(\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) - \varepsilon \left[\alpha_1 \mathbf{V}_1 \left(\frac{\mathbf{x}}{\varepsilon} \right) + \alpha_2 \mathbf{V}_2 \left(\frac{\mathbf{x}}{\varepsilon} \right) \right] + \mathbf{r}_\varepsilon(\mathbf{x}), \quad (3)$$

with

- \mathbf{u}_0 solves Problem (1) for $\varepsilon = 0$ (i.e. is solution in the unperturbed domain Ω_0),
- the coefficients α_1 and α_2 are the stress values associated with \mathbf{u}_0 at point $\mathbf{0}$, namely

$$\alpha_1 = \sigma_{11}(\mathbf{u}_0)(\mathbf{0}) \quad \text{and} \quad \alpha_2 = \sigma_{12}(\mathbf{u}_0)(\mathbf{0}),$$

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- the remainder \mathbf{r}_ε satisfies the following estimate in the energy norm

$$\|\mathbf{r}_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C\varepsilon^2. \quad (4)$$

As regards the computation of \mathbf{u}_ε for small values of ε , asymptotics expansion (3) leads to an alternative numerical strategy to adaptive mesh refinement. Indeed the knowledge of the unperturbed solution \mathbf{u}_0 and of the profiles \mathbf{V}_ℓ gives a reasonable approximation of \mathbf{u}_ε , see [1]. To approximate the profile defined by, we compute a solution of a problem posed in a bounded domain. To that end, we introduce an artificial boundary at $\partial B_R = \{|x| = R, R > 0\}$ and we need to impose a boundary condition on this boundary in the spirit of [3]. Since, the corrector decreases at infinity, the most natural artificial condition is the Dirichlet one. However, this condition is not precise.

In this note, we present the derivation of a higher order artificial boundary condition, we explain that this condition sparks theoretical questions and finally present some numerical simulations.

2 Seeking an artificial boundary condition

To derive artificial boundary conditions for the linear elasticity, the domain under consideration here is the upper half plane $\mathbb{R} \times \mathbb{R}^+$. Since we aim at writing artificial conditions on a circle of radius R , it is more convenient to work with the polar coordinates and we define $\mathbf{u} = u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta$. To approximate more accurately the profile, we cancel the leading singular parts at infinity of the solution in polar coordinates.

2.1 Computing the leading terms at infinity

Let $L = \mu\Delta + (\lambda + \mu)\mathbf{grad}\mathbf{div}$, then

$$\begin{aligned} L\mathbf{u} &= \left((\lambda + 2\mu) \left[\partial_r^2 u_r + \frac{1}{r} \partial_r u_r - \frac{1}{r^2} u_r \right] + \frac{\mu}{r^2} \partial_\theta^2 u_r - \frac{\lambda + 3\mu}{r^2} \partial_\theta u_\theta + \frac{\lambda + \mu}{r} \partial_{r\theta}^2 u_\theta \right) \mathbf{e}_r \\ &+ \left(\mu \left[\partial_r^2 u_\theta + \frac{1}{r} \partial_r u_\theta - \frac{1}{r^2} u_\theta \right] + \frac{\lambda + 2\mu}{r^2} \partial_\theta^2 u_\theta + \frac{\lambda + \mu}{r} \partial_{r\theta}^2 u_r + \frac{\lambda + 3\mu}{r^2} \partial_\theta u_r \right) \mathbf{e}_\theta. \end{aligned}$$

In polar coordinates, the stress tensor takes the form:

$$\sigma(\mathbf{u}) = \begin{bmatrix} (\lambda + 2\mu) \partial_r u_r + \frac{\lambda}{r} (u_r + \partial_\theta u_\theta) & \mu \left(\frac{1}{r} (\partial_\theta u_r - u_\theta) + \partial_r u_\theta \right) \\ \mu \left(\frac{1}{r} (\partial_\theta u_r - u_\theta) + \partial_r u_\theta \right) & (\lambda + 2\mu) \frac{1}{r} (\partial_\theta u_\theta + u_r) + \lambda \partial_r u_r \end{bmatrix}. \quad (5)$$

We seek the displacement under the form $\mathbf{u} = r^s [\varphi_r(\theta), \varphi_\theta(\theta)]^T$. Consequently, we obtain

$$L\mathbf{u} = r^{s-2} \begin{bmatrix} \mu \varphi_r'' + (\lambda + 2\mu)(s^2 - 1) \varphi_r + [(\lambda + \mu)s - (\lambda + 3\mu)] \varphi_\theta' \\ (\lambda + 2\mu) \varphi_\theta'' + \mu(s^2 - 1) \varphi_\theta + [(\lambda + \mu)s + (\lambda + 3\mu)] \varphi_r' \end{bmatrix}, \quad (6)$$

$$\sigma(\mathbf{u}) = r^{s-1} \begin{bmatrix} \lambda \varphi_\theta' + ((\lambda + 2\mu)s + \lambda) \varphi_r & \mu(\varphi_r' + (s - 1) \varphi_\theta) \\ \mu(\varphi_r' + (s - 1) \varphi_\theta) & (\lambda + 2\mu) \varphi_\theta' + (\lambda s + (\lambda + 2\mu)) \varphi_r \end{bmatrix}. \quad (7)$$

Using (6), we reduce the second order system $L\mathbf{u} = 0$ to a bigger system of first order: Define $\psi_r = \varphi_r'$ and $\psi_\theta = \varphi_\theta'$, and $\mathbf{U} = (\varphi_r, \varphi_\theta, \psi_r, \psi_\theta)^T$, then

$$\mathbf{U}' = A\mathbf{U},$$

with

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{(\lambda + 2\mu)(1 - s^2)}{\mu} & 0 & 0 & \frac{(\lambda + 3\mu) - (\lambda + \mu)s}{\mu} \\ 0 & \frac{\mu(1 - s^2)}{\lambda + 2\mu} & -\frac{(\lambda + 3\mu) + (\lambda + \mu)s}{\lambda + 2\mu} & 0 \end{bmatrix}.$$

The eigenvalues of A are $\pm i(s \pm 1)$. Hence, the polar functions $\varphi_r, \varphi_\theta$ belong to the space generated by $\cos((s \pm 1)\theta), \sin((s \pm 1)\theta)$. We look for coefficients $A_r, B_r, A_\theta, B_\theta$ so that

$$\begin{aligned}\varphi_r &= A_r \cos((s-1)\theta) + B_r \sin((s-1)\theta) + C_r \cos((s+1)\theta) + D_r \sin((s+1)\theta), \\ \varphi_\theta &= A_\theta \cos((s-1)\theta) + B_\theta \sin((s-1)\theta) + C_\theta \cos((s+1)\theta) + D_\theta \sin((s+1)\theta).\end{aligned}$$

Writing $L\mathbf{u} = 0$ gives:

$$\begin{cases} \alpha A_r + B_\theta = 0 \\ \alpha B_r - A_\theta = 0 \\ C_r + D_\theta = 0 \\ D_r - C_\theta = 0 \end{cases} \quad \text{with} \quad \alpha = \frac{(\lambda + \mu)s + (\lambda + 3\mu)}{(\lambda + \mu)s - (\lambda + 3\mu)}. \quad (8)$$

Consequently, the functions φ_r and φ_θ satisfy:

$$\varphi_r = A_r \cos((s-1)\theta) + B_r \sin((s-1)\theta) + C_r \cos((s+1)\theta) + D_r \sin((s+1)\theta), \quad (9)$$

$$\varphi_\theta = \alpha B_r \cos((s-1)\theta) - \alpha A_r \sin((s-1)\theta) + D_r \cos((s+1)\theta) - C_r \sin((s+1)\theta). \quad (10)$$

The boundary conditions read: $\sigma(\mathbf{u}) \cdot \mathbf{e}_\theta = \mathbf{0}$ for $\Theta = 0, \pi$, that is to say

$$\begin{cases} \varphi_r'(\Theta) + (s-1)\varphi_\theta(\Theta) = 0, \\ (\lambda + 2\mu)\varphi_\theta'(\Theta) + ((\lambda + 2\mu)s + \lambda)\varphi_r(\Theta) = 0. \end{cases}$$

Boundary conditions at $\theta = 0$ reads

$$\begin{cases} (s-1)B_r + (s+1)D_r + (s-1)(A_\theta + C_\theta) = 0, \\ (\lambda + 2\mu)((s-1)B_\theta + (s+1)D_\theta) + (\lambda s + \lambda + 2\mu)(A_r + C_r) = 0. \end{cases}$$

Boundary conditions at $\theta = \pi$ reads

$$\begin{cases} \sin s\pi[(s-1)(A_r - B_\theta - D_\theta) + (s+1)C_r] - \cos s\pi[(s-1)(A_\theta + B_r + C_\theta) + (s+1)D_r] = 0 \\ \sin s\pi((\lambda + 2\mu)[(s-1)(A_\theta + (s+1)C_\theta) - (\lambda + 2\mu + \lambda s)(B_r + D_r)] \\ - \cos s\pi((\lambda + 2\mu)[(s-1)B_\theta + (s+1)D_\theta] + (\lambda s + \lambda + 2\mu)(A_r + C_r)) = 0 \end{cases}$$

Using (8), we deduce that the coefficients A_r, B_r, C_r, D_r have to satisfied

$$M(A_r, B_r, C_r, D_r)^T = 0,$$

with

$$M = \begin{bmatrix} 0 & \gamma & 0 & 2s \\ \beta & 0 & -2\mu s & 0 \\ \gamma \sin s\pi & -\gamma \cos s\pi & 2s \sin s\pi & -2s \cos s\pi \\ -\beta \cos s\pi & -\beta \sin s\pi & 2\mu s \cos s\pi & 2\mu s \sin s\pi \end{bmatrix},$$

and

$$\beta = (1 + \alpha)(\lambda + 2\mu) + 2(\lambda - \alpha(\lambda + 2\mu)), \quad \gamma = (s-1)(1 + \alpha).$$

We compute

$$\det M = -4(\sin s\pi)^2(\mu\gamma + \beta)^2.$$

Therefore, $s \mapsto \det M$ cancels only for $s \in \mathbb{Z}$ and for s such that $\mu\gamma + \beta = 0$ that is to say

$$s = \frac{\mu - \lambda + \sqrt{\lambda^2 + 6\lambda\mu + 25\mu^2}}{2\mu} > 2 \quad \text{and} \quad s = \frac{\mu - \lambda - \sqrt{\lambda^2 + 6\lambda\mu + 25\mu^2}}{2\mu} < -2.$$

Then the leading term is obtained for $s = -1$. For $s = -1$, we have

$$\alpha = -\frac{\mu}{\lambda + 2\mu}, \quad \beta = 0, \quad \gamma = -\frac{2(\lambda + \mu)}{\lambda + 2\mu}.$$

In this case, system $M(A_r, B_r, C_r, D_r)^T = 0$ and (8) read

$$\gamma B_r - 2D_r = 0, \quad 2\mu C_r = 0, \quad C_\theta = D_r.$$

Consequently, setting $A = A_r$ and $B = -B_r$ in (9), the functions φ_r and φ_θ take the form

$$\begin{cases} \varphi_r = A \cos 2\theta + B \sin 2\theta, \\ \varphi_\theta = \frac{\mu}{\lambda + 2\mu}(B \cos 2\theta - A \sin 2\theta) + B \frac{\lambda + \mu}{\lambda + 2\mu}. \end{cases}$$

2.2 Using the singular profiles to obtain a transparent boundary condition for the half plane

We go back to the expression for $\sigma(\mathbf{u}) \cdot \mathbf{e}_r$ deduced from (7). When $s = -1$, we have

$$\sigma(\mathbf{u}) \cdot \mathbf{e}_r = r^{-2} \begin{bmatrix} \lambda\varphi'_\theta - 2\mu\varphi_r \\ \mu(\varphi'_r - 2\varphi_\theta) \end{bmatrix} = r^{-1} \begin{bmatrix} -\frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} & 0 \\ -\frac{2\mu(\lambda + \mu)\sin 2\theta}{\mu + (\lambda + \mu)\cos 2\theta} & \frac{2\mu(\lambda + \mu)(1 - \cos 2\theta)}{\mu + (\lambda + \mu)\cos 2\theta} \end{bmatrix} \mathbf{u}.$$

This means that the relation between \mathbf{u} and $\sigma(\mathbf{u}) \cdot \mathbf{n}$ on ∂B_R has variable (in the θ variable) coefficients, this fact makes the interpretation of this computation more difficult. Hence, we search a relation with the second order tangential derivative of the displacement: If such a relationship exists, the associated bilinear form still remains symmetric. Notice that

$$\varphi'_r = -\frac{\lambda + 2\mu}{2\mu} \varphi''_\theta,$$

we can also write

$$\sigma(\mathbf{u}) \cdot \mathbf{e}_r = r^{-2} \begin{bmatrix} -\frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} \varphi_r \\ \mu\varphi'_r - 2\mu\varphi_\theta \end{bmatrix} = \frac{1}{R} \begin{bmatrix} -\frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} u_r \\ -2\mu u_\theta - \frac{\lambda + 2\mu}{2} \partial_\theta^2 u_\theta \end{bmatrix}.$$

For the elasticity problem, we observe that the leading decreasing profile at infinity satisfies the following inequality on the circle of radius R :

$$\sigma(\mathbf{u}) \cdot \mathbf{n} + \frac{2\mu}{R} \begin{bmatrix} \frac{2(\lambda + \mu)}{\lambda + 2\mu} & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u} + \frac{\lambda + 2\mu}{2R} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \partial_\theta^2 \mathbf{u} = 0.$$

Lamé's coefficients are linked to Young's modulus and Poisson's coefficient through the relations:

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}, \quad \mu = \frac{E}{2(1 + \nu)} \quad \text{with} \quad E > 0, \quad -1 < \nu < 0.5.$$

So that, this boundary condition of Ventcel's type rewrites on $\partial B_R \cap \mathbb{R} \times \mathbb{R}^+$:

$$\sigma(\mathbf{u}) \cdot \mathbf{n} + \frac{1}{R} \frac{E}{1 + \nu} \begin{bmatrix} \frac{1}{1 - \nu} & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u} + \frac{1}{R} \frac{E(1 - \nu)}{2(1 + \nu)(1 - 2\nu)} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \partial_\theta^2 \mathbf{u} = 0. \quad (11)$$

The main difficulty here is that the coefficient $\frac{E(1 - \nu)}{2(1 + \nu)(1 - 2\nu)}$ is positive; meaning that the variational formulation of the boundary value problem for the profiles \mathbf{V}_ℓ

$$\begin{cases} -\mu \Delta \mathbf{V}_\ell - (\lambda + \mu) \mathbf{grad} \operatorname{div} \mathbf{V}_\ell = \mathbf{0} & \text{in } B_R \cap \mathbf{H}_\infty, \\ \sigma(\mathbf{V}_\ell) \cdot \mathbf{n} = \mathbf{G}_\ell & \text{on } \partial \mathbf{H}_\infty \cap B_R, \\ \sigma(\mathbf{V}_\ell) \cdot \mathbf{n} + \frac{1}{R} \frac{E}{1 + \nu} \begin{bmatrix} \frac{1}{1 - \nu} & 0 \\ 0 & 1 \end{bmatrix} \mathbf{V}_\ell + \frac{E(1 - \nu)}{2(1 + \nu)(1 - 2\nu)R} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \partial_\theta^2 \mathbf{V}_\ell = 0 & \text{on } \partial B_R \cap \mathbf{H}_\infty, \end{cases} \quad (12)$$

is not coercive. Therefore, the question of existence and uniqueness of the solution to (12) is open. For the model scalar case of

$$\begin{cases} -\Delta V = 0 & \text{in } \Omega, \\ \partial_n V + \alpha V + \beta \Delta_\Gamma V = G & \text{on } \partial\Omega, \end{cases}$$

with $\beta > 0$ and Δ_Γ denotes the Laplace-Beltrami operator on $\partial\Omega$, it has been shown in [2] that, for a fixed R , the boundary value problem has a unique solution if α/β avoid a countable set of values, and moreover, that it has a unique solution for R large enough. We expect similar results for the elasticity case and we work in that direction.

3 Numerical results

In order to investigate Problem (12) from a numerical point of view, we have considered the test situation where the domain ω is the top half of a ball: $\mathbf{H}_\infty = \mathbb{R} \times \mathbb{R}^+ \setminus B(0, 1)$. We have successively considered the following two questions:

- (i) resolvability of the Ventcel-type boundary condition (11),
- (ii) order of precision of this condition as an artificial boundary condition for Problem (2).

For the first question, we consider the norm of the inverse operator $(L_R)^{-1}$ corresponding to Problem (12):

$$L_R : \mathbf{V}_\ell \mapsto \mathbf{G}_\ell.$$

In Figure 1, we show the evolution of this norm with respect to the radius R . The computation have been performed with the finite element library MÉLINA (see [6]), the domain (a ring) has been meshed into 128 quadrangular elements of degree 8, and a \mathbb{Q}_{10} interpolation is used for the finite element method.

Figure 1 shows some values of R for which the resolvent is unbounded, but these values remain close to 1. It seems that Problem (12) is uniquely solvable for R sufficiently large (which is the result shown for the scalar model problem).

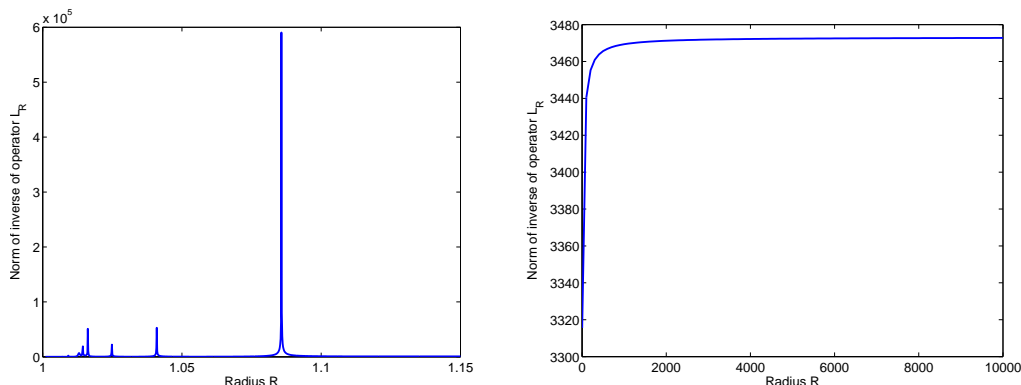


Figure 1: Norm of $(L_R)^{-1}$ with respect to R .

For question (ii), we consider Problem (12) where the data \mathbf{G}_ℓ is chosen so that the solution \mathbf{V}_ℓ is the sum of two dual singular solutions:

$$\mathbf{V}_\ell = \mathbf{s}^1 + \mathbf{s}^2,$$

where \mathbf{s}^1 is given by (2.1) and \mathbf{s}^2 has the form

$$\mathbf{s}^2(r, \theta) = r^{-2} [\varphi_r^2(\theta)\mathbf{e}_r + \varphi_\theta^2(\theta)\mathbf{e}_\theta],$$

with

$$\begin{aligned} \varphi_r^2(\theta) &= A' \cos(3\theta) - B' \sin(3\theta) + \frac{3\alpha\lambda - \lambda - 4\mu}{4\mu} A' \cos \theta + \frac{3}{4}(1 + \alpha)B' \sin \theta, \\ \varphi_\theta^2(\theta) &= \alpha B' \cos(3\theta) + \alpha A' \sin(3\theta) - \frac{3}{4}(1 + \alpha)B' \cos \theta + \frac{3\alpha\lambda - \lambda - 4\mu}{4\mu} A' \sin \theta, \end{aligned}$$

and

$$\alpha = \frac{-\lambda + \mu}{-3\lambda - 5\mu}.$$

With this choice, the function \mathbf{V}_ℓ solves Problem (2) in the infinite domain \mathbf{H}_∞ . In Figure 2, we show the H^1 -norm of the difference between the exact solution of (2) and its approximation with artificial boundary conditions (Dirichlet or Ventcel). It turns out that – as expected – the Dirichlet condition is of order 1 and the Ventcel condition of order 2.

Remark 1 *Writing the variational formulation of (12) makes appear punctual terms which need a specific treatment for the numerical approach with the finite element method.*

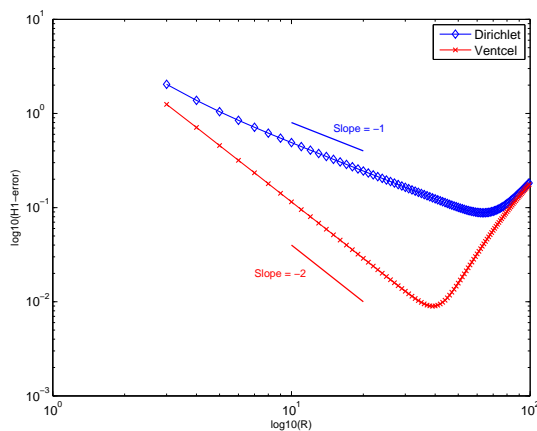


Figure 2: Precision of the artificial conditions with respect to R .

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