On generalized Ventcel's type boundary conditions for Laplace operator in a bounded domain.

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Abstract

Ventcel boundary conditions are second order differential conditions that appear in asymptotic models. Like Robin boundary conditions, they lead to wellposed variational problems under a sign condition of a coefficient. Nevertheless situations where this condition is violated appeared in several works. The wellposedness of such problems was still open. This manuscript establishes, in the generic case, the existence and uniqueness of the solution for the Ventcel boundary value problem without the sign condition. Then, we consider perforated geometries and give conditions to remove the genericity restriction.

Keywords: Ventcel boundary condition, pseudodifferential operators, asymptotic analysis, shape dependency.

1 Introduction

In various situations, an artificial boundary condition is introduced to replace the effect of a more complex geometry. We can mention the approximate boundary conditions in the framework of thin layers [4, 10, 14] or rough boundaries [1, 16, 2]. For exterior problems, absorbing (or transparent) conditions are another example, see [9, 13, 15].

These boundary conditions are generally simple differential conditions obtained from an asymptotic analysis with respect to a characteristic length: the thickness of the layer, the scale of the roughness, the diameter of the artificial boundary, for the previous three examples respectively. A hierarchy of models is obtained, depending on the target order of accuracy. Basically, we find Dirichlet or Neumann type condition at order 0, Robin/Fourier at order 1 (interpreted as a flow across the boundary), and Ventcel at order 2 (understood as a surface diffusion).

Precisely, let Ω be a smooth bounded domain of $\mathbb{R}^d$ with $d \geq 2$, and α, β denote two real numbers. The Ventcel boundary value problem for the Laplace operator reads

$$\left\{ \begin{array}{rcll} -\Delta u & = & f & \text{in } \Omega, \\ \partial_n u + \alpha u + \beta \Delta_\tau u & = & 0 & \text{on } \partial\Omega. \end{array} \right. \tag{1.1}$$

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This boundary value problem has been intensively studied ever since the pioneering works of Feller and Ventcel, [11, 12, 23, 22]. Under a natural assumption on the sign of β, a variational approach is available. Define the bilinear form A and the linear form B by

$$A(u,v) = \int_{\Omega} \nabla u \nabla v + \int_{\partial\Omega} \alpha\, uv - \beta\, \nabla_{\tau} u . \nabla_{\tau} v \quad \text{and} \quad B(v) = \int_{\Omega} fv,$$

on the variational space

$$\mathcal{H}(\Omega) = \left\{ u \in \mathrm{H}^1(\Omega),\ u_{|\partial\Omega} \in \mathrm{H}^1(\partial\Omega) \right\}.$$

Endowed with the norm

$$\|u\|^2_{\mathcal{H}(\Omega)} = \|u\|^2_{\mathrm{H}^1(\Omega)} + \|u\|^2_{\mathrm{H}^1(\partial\Omega)},$$

the space $\mathcal{H}(\Omega)$ is Hilbertian. The weak formulation of problem (1.1) then reads

$$\text{Find } u \in \mathcal{H}(\Omega) \text{ such that for all } v \in \mathcal{H}(\Omega),\ A(u,v) = B(v).$$

When $\alpha > 0$ and $\beta < 0$, the bilinear form A is coercive and existence and uniqueness of a solution to (1.1) follow from Lax-Milgram lemma. A large literature has been devoted to that case of great importance: the condition $\beta < 0$ is generally satisfied in the applications. For the specific case of the Laplace operator, we refer to [3] and [8].

In the case $\beta > 0$, the quadratic form $u \mapsto A(u,u)$ is neither positive, nor negative. Then, existence and uniqueness of the solution are not ensured by Lax-Milgram lemma. We precisely address this question in the present work. To the best of our knowledge, this condition $\beta > 0$ appears for the first time in a recent work of D. Bresch and V. Milisic [6].

Such a boundary condition also appears when looking for a transparent boundary condition for an exterior boundary value problem in planar linear elasticity. The goal is to bound the infinite domain by a large "box" to make numerical approximations possible (typically a large ball of radius R). The solution of the problem set in this bounded domain has to be close to the original solution ; the convergence is expected as R goes to infinity. Precisely, in [5], one considers the case of a linear elastic material in the exterior of a bounded domain on the boundary of which the displacement is prescribed. In that case, cancelling the leading singular parts at infinity of the solution leads to the approximate boundary condition

$$\sigma(\vec{u})\vec{n} + \frac{1}{R}\,\frac{E}{1+\nu} \begin{bmatrix} \dfrac{1}{1-\nu} & 0 \\ 0 & 1 \end{bmatrix} \vec{u} + \frac{1}{R}\,\frac{E(1-\nu)}{2(1+\nu)(1-2\nu)} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Delta_{\tau}\vec{u} = 0, \qquad (1.2)$$

set on the circle of radius R. In the former equation, $\vec{u}$ denotes the displacement decomposed into its radial and tangential components and σ stands for the stress tensor. The physical parameters E and ν are such that the quantity in front of the Laplace-Beltrami operator is nonnegative: Young's modulus E is nonnegative and Poisson's coefficient ν takes values in the interval $(-1, 0.5)$.

In this manuscript, we study the case of the Laplace operator as a model problem. In Section 2, we present a study of existence and uniqueness of solutions to Ventcel boundary value problem without sign condition. We start by two simple examples where explicit computations can be done. Then, the general case is treated thanks to reformulation of the boundary value problem (1.1) into a nonlocal equation on the boundary $\partial\Omega$. This idea is by now standard in the study of boundary problems (see e.g. [7]). In our case, pseudodifferential and spectral techniques lead to existence and uniqueness results apart from exceptional cases. This study is valid in any dimension of space.

Section 3 refers to the framework of transparent boundary conditions for an exterior problem in the plane. As mentioned above, in the Laplace case, the coefficient β generally has the right sign $\beta < 0$, but we consider here $\beta > 0$. The geometric setting is the following: let ω be a smooth bounded domain in $\mathbb{R}^2$ and we introduce a larger domain Ω such that $\omega \subset \Omega$. We study the model Ventcel boundary value problem

$$\left\{ \begin{array}{rcll} -\Delta u & = & 0 & \text{in } \Omega \setminus \overline{\omega}, \\ \partial_n u + \alpha u + \beta \Delta_\tau u & = & 0 & \text{on } \partial\Omega, \\ u & = & g & \text{on } \partial\omega, \end{array} \right. \tag{1.3}$$

where g is a fixed right hand side in $\mathrm{H}^{1/2}(\partial\omega)$ and $\beta > 0$. In a first approach, one can choose the bounding domain Ω as a ball centered in ω. The general theory presented in Section 2 applies but cannot answer the question: for a given Ω, is boundary value problem (1.3) wellposed ? Therefore, we address and give positive answers to the two following questions:

Q1: Is there some continuity with respect to the shape ? Precisely, if (1.3) has a unique solution, does it remain the case when ω is slightly modified ?

Q2: Does (1.3) has a unique solution if Ω is a ball of radius R chosen large enough ?

2 About existence and uniqueness of solution

We focus in this section on the Ventcel boundary value problem for Laplace operator such as

$$\left\{ \begin{array}{rcll} -\Delta u & = & f & \text{in } \Omega, \\ \partial_n u + \alpha u + \beta \Delta_\tau u & = & 0 & \text{on } \partial\Omega \end{array} \right.$$

under the unusual condition $\beta > 0$. We address the question of existence of a solution and, if this problem admits solutions, of the uniqueness.

2.1 Analytic examples

2.1.1 Laplace Ventcel problem in a disk

Before studying the general case, we first consider what happens in the unit disk in $\mathbb{R}^2$:

$$\left\{ \begin{array}{rcll} -\Delta u & = & 0 & \text{in } B(0,1), \\ \partial_n u \;+\; \alpha\, u \;+\; \beta\, \partial^2_{\theta\theta} u & = & f & \text{on } \partial B(0,1). \end{array} \right. \tag{2.1}$$

We seek the solution under the form

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)\, r^n,$$

thanks to Poisson kernel. We obtain the totally decoupled equations

$$a_n(-\beta n^2 + n + \alpha) = a_n(f) \quad \text{and} \quad b_n(-\beta n^2 + n + \alpha) = b_n(f),$$

where $a_n(f)$ and $b_n(f)$ denote the Fourier coefficients of the boundary data f. The problem admits a unique solution as soon as

$$\alpha \notin \{\alpha_n = \beta n^2 - n,\ n \in \mathbb{N}\}. \tag{2.2}$$

2.1.2 Laplace Dirichlet-Ventcel problem in a ring

Now, we consider a situation with two boundaries (having in mind that the outer one may play the role of an artificial boundary). Let us consider a concentric annulus Ω of inner radius R_i and outer radius $R_e > R_i$ and a distribution $g \in \mathrm{H}^{-1/2}(\partial B_{R_i})$. We consider the model boundary value problem

$$\left\{ \begin{array}{rcll} -\Delta u &=& 0 & \text{in } \Omega, \\ \partial_n u &=& g & \text{on } \partial B_{R_i}, \\ \partial_n u + \dfrac{\alpha}{R_e} u + \dfrac{\beta}{R_e} \partial^2_{\theta\theta} u &=& 0 & \text{on } \partial B_{R_e}. \end{array} \right. \tag{2.3}$$

An harmonic function u in Ω can be written as a Laurent's series:

$$u(r,\theta) = d + c \ln r + \sum_{n=1}^{\infty} \left(a_n r^n + a_{-n} r^{-n}\right) \cos n\theta + \left(b_n r^n + b_{-n} r^{-n}\right) \sin n\theta.$$

Inserting the previous expression into problem (2.3) leads to the following linear systems for the coefficients a_n, a_{-n}, and b_n, b_{-n}:

$$\begin{bmatrix} R_i^{n-1} n & -R_i^{-n-1} n \\ R_e^{n-1}\left(\alpha + n - \beta n^2\right) & R_e^{-n-1}\left(\alpha - n - \beta n^2\right) \end{bmatrix} \begin{bmatrix} a_n \\ a_{-n} \end{bmatrix} = \begin{bmatrix} -a_n(g) \\ 0 \end{bmatrix}, \tag{2.4}$$

$$\begin{bmatrix} R_i^{n-1} n & -R_i^{-n-1} n \\ R_e^{n-1}\left(\alpha + n - \beta n^2\right) & R_e^{-n-1}\left(\alpha - n - \beta n^2\right) \end{bmatrix} \begin{bmatrix} b_n \\ b_{-n} \end{bmatrix} = \begin{bmatrix} -b_n(g) \\ 0 \end{bmatrix}. \tag{2.5}$$

Here, $a_n(g)$ and $b_n(g)$ denote the Fourier coefficients of the boundary data g. These linear systems admit a unique solution if

$$n R_e^{-n-1} R_i^{n-1}\left(\alpha - n - \beta n^2\right) + n R_e^{n-1} R_i^{-n-1}\left(\alpha + n - \beta n^2\right) \neq 0,$$

i.e. if

$$\alpha \notin \left\{ \alpha_n = \beta n^2 - n + \frac{2n}{1 + (R_e/R_i)^{2n}},\ n \in \mathbb{N} \right\}. \tag{2.6}$$

An other point of view is the following : for fixed α and β, the forbidden values of the radii R_i and R_e are characterized by the existence of n such that

$$\left(\frac{R_e}{R_i}\right)^{2n} = -\frac{\alpha - n - \beta n^2}{\alpha + n - \beta n^2} = -1 + \frac{2n}{\alpha + n - \beta n^2}.$$

Since $R_e/R_i > 1$ by definition, a necessary condition on n is

$$-1 + \frac{2n}{\alpha + n - \beta n^2} > 1 \quad \Leftrightarrow \quad \frac{n}{\alpha + n - \beta n^2} > 1 \quad \Leftrightarrow \quad \left\{ \begin{array}{rcl} \alpha + n &>& \beta n^2, \\ \alpha &<& \beta n^2. \end{array} \right.$$

Thus, the forbidden values γ_n for the ratio R_e/R_i are in finite number. If

$$\frac{R_e}{R_i} \notin \left\{ \gamma_n = \left(-\frac{\alpha - n - \beta n^2}{\alpha + n - \beta n^2} \right)^{1/2n} \quad \text{for } \sqrt{\frac{\alpha}{\beta}} < n < \frac{1 + \sqrt{1 + 4\alpha\beta}}{2\beta} \right\}, \tag{2.7}$$

then the boundary value problem (2.3) is wellposed with a unique solution corresponding to the coefficients a_n, a_{-n} and b_n, b_{-n} satisfying (2.4)-(2.5).

Remark 2.1 *Having in mind either the application to transparent boundary conditions ($R_e \to \infty$), or the case of perforated domains ($R_i \to 0$, see Section 3.2 here), we check that condition* (2.7) *is satisfied if R_e/R_i is large enough. Note also that in the limiting case $R_e/R_i = +\infty$ we recover in* (2.6) *the forbidden values of α_n corresponding to the case of a disk appearing in* (2.2).

Remark 2.2 *In the case of Dirichlet boundary conditions on the inner boundary, we consider a distribution $f \in \mathrm{H}^{1/2}(\partial B_{R_i})$ and we have the boundary value problem*

$$\left\{\begin{array}{rcll} -\Delta u &=& 0 & \textit{in } \Omega, \\ u &=& f & \textit{on } \partial B_{R_i}, \\ \partial_n u \ + \ \dfrac{\alpha}{R_e}\, u \ + \ \dfrac{\beta}{R_e}\, \partial^2_{\theta\theta} u &=& 0 & \textit{on } \partial B_{R_e}. \end{array}\right. \tag{2.8}$$

Similar computations lead to the following conclusion: there exists a unique solution in the case when

$$\alpha \notin \left\{ \alpha_n = \beta n^2 - n + \frac{2n}{1 - (R_e/R_i)^{2n}} \ n \in \mathbb{N} \right\}, \tag{2.9}$$

and an infinite number of values γ_n are forbidden for the ratio R_e/R_i. These values equal

$$\gamma_n = \left(\frac{\beta n^2 + n - \alpha}{\beta n^2 - n - \alpha} \right)^{1/2n} \quad \textit{with } n \in \mathbb{N}. \tag{2.10}$$

Nevertheless, it is easy to check that $\gamma_n \to 1$ at infinity and that the forbidden ratios remain bounded. Hence the boundary value problem (2.8) *is wellposed when R_e/R_i is large enough.*

2.2 The general case

In this section, we consider Ω a bounded smooth domain of $\mathbb{R}^d$ (at least $\mathcal{C}^3$), with $d \geq 2$. For $f \in \mathrm{H}^1(\Omega)$ and $g \in \mathrm{L}^2(\partial\Omega)$, and for $\alpha, \beta > 0$, we consider:

$$\left\{\begin{array}{rcll} -\Delta u &=& f & \text{in } \Omega, \\ \partial_n u + \alpha u + \beta \Delta_\tau u &=& g & \text{on } \partial\Omega. \end{array}\right. \tag{2.11}$$

We shall see that the situation is similar to the particular example studied in Section 2.1, i.e. problem (2.11) is wellposed except for a countable set of values for the parameters.

2.2.1 An equivalent nonlocal equation

The leading idea is to look at this boundary value problem as a nonlocal equation set on the boundary. Let us obtain this equivalent problem. First, we remove the right hand side in the partial differential equation by a standard lift: let $F \in \mathrm{H}^1_0(\Omega)$ satisfying $-\Delta F = f$ in Ω. The function $W = u - F$ solves the boundary value problem

$$\left\{\begin{array}{rcll} -\Delta W &=& 0 & \text{in } \Omega, \\ \partial_n W \ + \ \alpha\, W \ + \ \beta\, \Delta_\tau W &=& \varphi & \text{on } \partial\Omega, \end{array}\right. \tag{2.12}$$

where $\varphi = g + \partial_n F + \alpha F + \beta \Delta_\tau F$ on $\partial\Omega$. Note that by elliptic regularity of F the trace φ is in fact in $H^{1/2}(\partial\Omega)$. Then, we introduce the Dirichlet-to-Neumann map Λ associated to the Laplace operator on Ω: this operator is defined from $\mathrm{H}^{1/2}(\partial\Omega)$ onto $\mathrm{H}^{-1/2}(\partial\Omega)$ by $\Lambda(\psi) = \partial_n U$ where U is the solution of the boundary value problem

$$\begin{cases} -\Delta U &= 0 \quad \text{in } \Omega, \\ U &= \psi \quad \text{on } \partial\Omega. \end{cases}$$

The introduction of the Dirichlet-to-Neumann map allows us to rewrite (2.12) as the surface equation

$$\beta\Delta_\tau w + \Lambda w + \alpha w = \varphi \text{ on } \partial\Omega. \tag{2.13}$$

The new unknown w in (2.13) is the trace of W on $\partial\Omega$. It is easy to check that (2.13) and (2.12) are equivalent in the sense that if w solves (2.13), then its harmonic extension W in Ω solves (2.12). Conversely, if W solves (2.12), then its trace on Ω solves (2.13). The computations made in the examples studied in Section 2.1 can be performed since the Laplace-Beltrami operator and the Dirichlet-to-Neumann are both diagonal on the Fourier basis when the domain is a disk or a centered annulus. For a general domain Ω, there is no particular reason for the Laplace-Beltrami operator and the Dirichlet-to-Neumann map to have a common diagonal basis.

2.2.2 On the resolubility of the nonlocal equation

In order to get results for the general case we will use some properties of the Dirichlet-to-Neumann map. To that end, we use the theory of pseudodifferential operators. We know that the domain $\partial\Omega$ is smooth and compact, so the general theory of pseudodifferential operators can be applied here.

First recall some standard definitions. For $N \in \mathbb{N}^*$, we introduce symbols which are functions defined on the cotangent bundle $T^*(\mathcal{O})$ where $\mathcal{O}$ is an open subset of $\mathbb{R}^N$: for $m \in \mathbb{R}$, this is the set denoted by $S^m(\mathcal{O})$ of functions a of $(x,\xi) \in T^*(\mathcal{O}) \longrightarrow \mathbb{C}$, which are $\mathcal{C}^\infty$ on $T^*(\mathcal{O})$ except perhaps on $\{\xi = 0\}$, and such that

$$\left|\partial_x^i \partial_\xi^j a(x,\xi)\right| \leq C_{i,j}(1+|\xi|^2)^{m/2-|j|/2}, \qquad \forall i,j \in \mathbb{N}^N, \quad (x,\xi) \in T^*(\mathcal{O})/\{\xi = 0\}.$$

In the definition m is called the order, and a symbol is called elliptic if there exist C' and $C'' > 0$ such that we have in addition

$$|a(x,\xi)| \geq C'(1+|\xi|^2)^{m/2} \qquad \text{for } |\xi| \geq C''.$$

To a symbol in $S^m(\mathcal{O})$ we associate a (pseudo-)differential operator a^w thanks to the following formula

$$a^w u(x) = (2\pi)^{-N} \iint e^{i(x-y).\xi} a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi \qquad \text{for } u \in \mathcal{C}_0^\infty(\mathcal{O}),$$

defined as an oscillatory integral for high orders.

By extension, we say that an operator is of order m and/or elliptic if its symbol has this property. We choose here the so-called Weyl quantization of symbols, for which real elliptic semi-bounded symbols give rise to selfadjoint extensions on L^2 (in fact formally selfadjoint on L^2 and closable when initially defined on $\mathcal{C}_0^\infty$, see e.g. [20]).

We won't here recall the definitions of symbols and pseudodifferential operators on a smooth manifold. The smooth manifold in question is $\partial\Omega$, which is locally diffeomorphic to $\mathbb{R}^{d-1}$. Let us just point out the fact that for $m \in \mathbb{R}$, the corresponding space of symbols $S^m(\partial\Omega)$ is invariantly defined modulo $S^{m-1}(\partial\Omega)$ thanks to local charts and according to the definitions in $\mathbb{R}^{d-1}$. The previous notions of order and ellipticity naturally extend to operators on a manifold.

One of the main property of (pseudo-)differential operators of order m is the following one: they map $\mathrm{H}^s(\partial\Omega)$ to $\mathrm{H}^{s-m}(\partial\Omega)$ for any $s \in \mathbb{R}$. Recall that they can *a priori* be defined without ambiguity on the space $\mathcal{S}'(\partial\Omega)$ of temperate distributions by duality, and that $\partial\Omega$ is compact. The following result is a summary of some basic spectral results we shall need later. We do not give the proof here and refer to standard books about the spectral theory of pseudodifferential operators (e.g. [21], [19] or [20]):

Proposition 2.3 *Let P be an elliptic semi-bounded and formally selfadjoint pseudodifferential operator of order $m > 0$ on $\partial\Omega$. Then*

1. *P is closable and has a compact resolvent as an unbounded operator on $\mathrm{H}^s(\partial\Omega)$;*
2. *P maps $\mathrm{H}^s(\partial\Omega)$ to $\mathrm{H}^{s-m}(\partial\Omega)$ for any $s \in \mathbb{R}$;*
3. *Its spectrum (as an unbounded operator on $\mathrm{H}^s(\partial\Omega)$) is independent of s, and made of a series of real eigenvalues growing to infinity;*
4. *The associated eigenspaces are finite dimensional and the eigenfunctions belong to $\mathcal{C}^\infty(\partial\Omega)$;*
5. *P is Fredholm of index 0.*

We just give some comments on the proof of the previous proposition, without giving details. Point *1.* and *2.* are related to the compact embedding $\mathrm{H}^s(\partial\Omega) \hookrightarrow \mathrm{H}^{s-m}(\partial\Omega)$ and to the existence of a pseudodifferential inverse (modulo regularizing operators) of P, called a parametrix, which symbol belongs to $S^{-m}(\partial\Omega)$. Point *3.* and *4.* are consequence of the $\mathrm{L}^2(\partial\Omega)$ case by elliptic regularity since any eigenfunction in $\mathrm{H}^s(\partial\Omega)$ is an eigenfunction in $\mathrm{L}^2(\partial\Omega)$ and is $\mathcal{C}^\infty(\partial\Omega)$. The L^2-case follows from the fact that P has compact resolvent and is selfadjoint in $\mathrm{L}^2(\partial\Omega)$. Point *5.* is a consequence of the existence of a parametrix, the fact that the symbol is real and the compactness of the resolvent (see e.g. [21, Chap. 2]). ■

We quote now without proof standard results about the Laplace-Beltrami operator and the Dirichlet-to-Neumann operator, that can be found for example in [20].

Proposition 2.4 *We have*

1. *The Laplace-Beltrami operator $-\Delta_\tau$ on $\partial\Omega$ is a semi-bounded elliptic pseudodifferential operator of real symbol of order 2;*
2. *The Dirichlet-to-Neumann operator Λ is an elliptic pseudodifferential operator of real symbol of order 1.*

Now we are able to prove a result confirming that in the general case, things look like as in the particular case of the annulus studied in the first part of this section, at least concerning the generic existence and uniqueness of solutions. Recall that the aim is to study equation (2.13) that we rewrite now:

$$\beta\Delta_\tau w + \Lambda w + \alpha w = \varphi \text{ on } \partial\Omega. \tag{2.14}$$

Theorem 2.5 *Operator $P_{\alpha,\beta} = -\beta\Delta_\tau - \Lambda - \alpha\mathrm{Id}$ is an elliptic selfadjoint semi-bounded from below pseudodifferential operator of order 2.*

Besides, for fixed $\beta > 0$, there exists series $(\alpha_n)_{n\in\mathbb{N}}$ growing to infinity such that for any $\varphi \in \mathrm{H}^s(\partial\Omega)$ with $s \in \mathbb{R}$, we have

1. *If $\alpha \notin \{\alpha_n\}$, then equation $-P_{\alpha,\beta}w = \varphi$ admits a unique solution in $\mathcal{S}'(\partial\Omega)$ which, in addition, belongs to $\mathrm{H}^{s+2}(\partial\Omega)$;*

2. *If $\alpha \in \{\alpha_n\}$, then there is either no solution or a complete affine finite dimensional space of $\mathrm{H}^{s+2}(\partial\Omega)$ solutions.*

Proof of Theorem 2.5: The first assertion is just a consequence of the fact that the symbol of $-\beta\Delta_\tau - \Lambda - \alpha\mathrm{Id}$ is real and elliptic of order 2. In particular it has the same symbol in $S^2(\partial\Omega)/S^1(\partial\Omega)$ than $-\beta\Delta_\tau$ since Λ is of order 1. We can therefore apply Proposition 2.3 to $P_{\alpha,\beta}$, and also to

$$P_\beta \stackrel{\text{def}}{=} P_{\alpha,\beta} + \alpha\mathrm{Id} = -\beta\Delta_\tau - \Lambda.$$

In particular we can introduce the series $(\alpha_n)_{n\in\mathbb{N}}$ of eigenvalues of P_β, going to infinity with n from point *3.* of Proposition 2.3. We get in particular that for any $\alpha \notin \{\alpha_n\}$, $P_\beta - \alpha\mathrm{Id}$ is invertible from $\mathrm{H}^{s+2}(\partial\Omega)$ onto $\mathrm{H}^s(\partial\Omega)$. This gives assertion *1.* of the theorem. Point *2.* is just a consequence of the Fredholm property since if $\alpha \in \{\alpha_n\}$ then $P_{\alpha,\beta}$ has a finite dimensional kernel and its image is of finite codimension. The proof is complete. ■

Remark 2.6 *The main feature of the previous study is of course that equation* (2.14) *is generically solvable for fixed β, in the following sense: for all values of $\alpha > 0$ avoiding the countable set $\{\alpha_n\}$ there exists a unique solution. This set of values $\{\alpha_n\}$ was explicitly computed in the case of a disk* (2.2) *and of a ring with either Neumann* (2.6) *or Dirichlet* (2.8) *conditions.*

3 Perforated domains

3.1 Answer to Q1: Shape sensitivity analysis

We fix $d_0 > 0$ and set Ω_{d_0} the subset of Ω made of points x with $d(x, \partial\Omega) > d_0$. We consider the case where the subdomain ω is included in Ω_{d_0}. We follow the boundary variation approach of F. Murat and J. Simon: for a given vector field $\boldsymbol{h} \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$ with support in Ω_{d_0} and any nonnegative real t, we set $\Phi_t : \mathbb{R}^d \to \mathbb{R}^d$ defined by $\Phi_t(x) = x + t\boldsymbol{h}(x)$. For t small enough, Φ_t is a diffeomorphism from $\mathbb{R}^d$ into itself.

In this section, we define the Dirichlet-to-Neumann map Λ_ω, seen from $\mathrm{H}^{1/2}(\partial\Omega)$ onto $\mathrm{H}^{-1/2}(\partial\Omega)$, adapted to our problem by $\Lambda_\omega(\varphi) = \partial_n u$ where u denotes the solution of

$$\left\{\begin{array}{rcll} -\Delta u &=& 0 & \text{in } \Omega \setminus \overline{\omega}, \\ u &=& \varphi & \text{on } \partial\Omega, \\ u &=& 0 & \text{on } \partial\omega. \end{array}\right. \tag{3.1}$$

Our aim in this section is to prove the following result.

Theorem 3.1 *In the previously described geometrical setting, there are real positive numbers t_0 and C such that for all $t \le t_0$*

$$\|\Lambda_{\Phi_t(\omega)} - \Lambda_\omega\|_{\mathcal{L}(\mathrm{H}^{1/2}(\partial\Omega),\mathrm{H}^{-1/2}(\partial\Omega))} \le C\, t\, \|\boldsymbol{h}\|_{\mathrm{W}^{1,\infty}(\Omega\setminus\overline{\omega})}. \tag{3.2}$$

Proof of Theorem 3.1: There exists a bounded extension operator E from $\mathrm{H}^{1/2}(\partial\Omega)$ into $\mathrm{H}^1(\Omega)$ such that the extension $E(\varphi)$ of $\varphi \in \mathrm{H}^{1/2}(\partial\Omega)$ is supported in $\overline{\Omega \setminus \Omega_{d_0}}$. Precisely, for all $\varphi \in \mathrm{H}^{1/2}(\partial\Omega)$, we have $E(\varphi)(x) = 0$ for any $x \in \Omega_{d_0}$ and

$$\|E(\varphi)\|_{\mathrm{H}^1(\Omega)} \leq C(\Omega, d_0)\|\varphi\|_{\mathrm{H}^{1/2}(\partial\Omega)}.$$

Let us now fix $\varphi \in \mathrm{H}^{1/2}(\partial\Omega)$. In order to deal with homogeneous boundary conditions, we set $w = u - E(\varphi)$ so that w solves

$$\left\{\begin{array}{rcll} -\Delta w &=& \Delta E(\varphi) & \text{in } \Omega \setminus \overline{\omega}, \\ w &=& 0 & \text{on } \partial\Omega, \\ w &=& 0 & \text{on } \partial\omega. \end{array}\right.$$

This leads to the weak formulation: find w in $\mathrm{H}^1_0(\Omega \setminus \overline{\omega})$ such that

$$\forall v \in \mathrm{H}^1_0(\Omega \setminus \overline{\omega}), \quad \int_{\Omega\setminus\overline{\omega}} \nabla w.\nabla v = \int_{\Omega\setminus\overline{\omega}} \Delta E(\varphi) v. \tag{3.3}$$

Let w_t the solution of this problem set on the perturbed domain $\Omega \setminus \overline{\omega_t}$ with $\omega_t = \Phi_t(\omega)$. By definition, w_t belongs to $\mathrm{H}^1_0(\Omega \setminus \overline{\omega_t})$ and satisfies

$$\forall v_t \in \mathrm{H}^1_0(\Omega \setminus \overline{\omega_t}), \quad \int_{\Omega\setminus\overline{\omega_t}} \nabla w_t.\nabla v_t = \int_{\Omega\setminus\overline{\omega_t}} \Delta E(\varphi) v_t.$$

We transport this integral equation on the fixed domain setting $w^t = w_t \circ \Phi_t$ and $v^t = v_t \circ \Phi_t$. The chain rule gives $(\nabla v_t) \circ \Phi_t = (D\Phi_t^{-1})^T \nabla v^t$. Therefore by change of variables, the previous weak formulation can be rewritten under the form

$$\forall v^t \in \mathrm{H}^1_0(\Omega \setminus \overline{\omega}), \quad \int_{\Omega\setminus\overline{\omega}} A(t,x)\nabla w^t.\nabla v^t = \int_{\Omega\setminus\overline{\omega}} \Delta E(\varphi) \circ \Phi_t v^t J(t,x),$$

where $J(t,x) = \det D\Phi_t(x)$ is the Jacobian and $A(t,x) = J(t,x)(D\Phi_t(x)^{-1})(D\Phi_t(x)^{-1})^T$. The assumptions made on the extension operator E ensure that the supports of $\boldsymbol{h}$ and $E(\varphi)$ do not intersect so that the right hand side simply reads

$$\int_{\Omega\setminus\overline{\omega}} \Delta E(\varphi) \circ \Phi_t v^t J(t,x) = \int_{\Omega\setminus\overline{\omega}} \Delta E(\varphi) v^t.$$

Hence, the transported formulation is

$$\forall v^t \in \mathrm{H}^1_0(\Omega \setminus \overline{\omega}), \quad \int_{\Omega\setminus\overline{\omega}} A(t,x)\nabla w^t.\nabla v^t = \int_{\Omega\setminus\overline{\omega}} \Delta E(\varphi) v^t. \tag{3.4}$$

Now, since A is $\mathcal{C}^1$ with respect to t and continuous with respect to x in $\overline{\Omega}$, there exists t_0 such that for all $t \in [0, t_0]$,

$$\frac{1}{2}|v|^2 < v.A(t,x)v < 2|v|^2 \qquad \text{and} \qquad \|A(t,.) - \mathrm{Id}\|_{\mathrm{L}^\infty(\Omega\setminus\overline{\omega})} \leq c_1 \|\boldsymbol{h}\|_{\mathrm{W}^{1,\infty}(\Omega\setminus\overline{\omega})}\, t. \tag{3.5}$$

By classical *a priori* estimates, we obtain the existence of a constant $c > 0$ depending only on the domains Ω and ω such that

$$\|w^t\|_{\mathrm{H}^1(\Omega\setminus\overline{\omega})} \leq c\|\Delta E(\varphi)\|_{\mathrm{H}^{-1}(\Omega\setminus\overline{\omega})} \leq c\|E(\varphi)\|_{\mathrm{H}^1(\Omega\setminus\overline{\omega})} \leq c\|\varphi\|_{\mathrm{H}^{1/2}(\partial\Omega)}. \tag{3.6}$$

We form the difference between (3.3) and (3.4) to get

$$\forall v \in \mathrm{H}^1_0(\Omega \setminus \overline{\omega}), \qquad \int_{\Omega\setminus\overline{\omega}} \left(A(t,x)\nabla w^t - \nabla w\right).\nabla v = 0.$$

Therefore, $A(t,x)\nabla w^t - \nabla w = 0$ in $\mathrm{L}^2(\Omega \setminus \overline{\omega})$. But the decomposition

$$A(t,x)\nabla w^t - \nabla w = [A(t,x) - \mathrm{Id}]\,\nabla w^t + \nabla w^t - \nabla w$$

provides

$$\|\nabla w^t - \nabla w\|_{\mathrm{L}^2(\Omega\setminus\overline{\omega})} \leq c_2 \|A(t,.) - \mathrm{Id}\|_{\mathrm{L}^\infty(\Omega\setminus\overline{\omega})} \|\nabla w^t\|_{\mathrm{L}^2(\Omega\setminus\overline{\omega})}.$$

Applying (3.6) and (3.5), we get

$$\|w^t - w\|_{\mathrm{H}^1(\Omega\setminus\overline{\omega})} \leq c_3 \|\boldsymbol{h}\|_{\mathrm{W}^{1,\infty}(\Omega\setminus\overline{\omega})}\, t\, \|\varphi\|_{\mathrm{H}^{1/2}(\partial\Omega)}.$$

By restriction to $\Omega\setminus\overline{\Omega_{d_0}}$, we also have $\|w^t - w\|_{\mathrm{H}^1(\Omega\setminus\overline{\Omega_{d_0}})} \leq Ct\|\varphi\|_{\mathrm{H}^{1/2}(\partial\Omega)}$. On this tubular neighborhood of $\partial\Omega$, $\boldsymbol{h} = \boldsymbol{0}$, then $\Phi_t = \mathrm{Id}$ and $w^t = w_t$. Now, for any $\psi \in \mathrm{H}^{1/2}(\partial\Omega)$, Green's formula yields

$$\langle \psi, \partial_n w - \partial_n w_t\rangle_{\mathrm{H}^{1/2}(\partial\Omega)\times\mathrm{H}^{-1/2}(\partial\Omega)} = \int_{\Omega\setminus\overline{\omega}} (\Delta w - \Delta w_t)\, E(\psi) + (\nabla w - \nabla w_t)\nabla E(\psi).$$

Since $\Delta w = -\Delta E(\varphi) = \Delta w_t$ by construction, we obtain the estimate

$$\begin{aligned}\left|\langle \psi, \partial_n w - \partial_n w_t\rangle_{\mathrm{H}^{1/2}(\partial\Omega)\times\mathrm{H}^{-1/2}(\partial\Omega)}\right| &\leq \|\nabla w_t - \nabla w\|_{\mathrm{L}^2(\Omega\setminus\overline{\omega})} \|\nabla E(\psi)\|_{\mathrm{L}^2(\Omega\setminus\overline{\omega})} \\ &\leq C\|\boldsymbol{h}\|_{\mathrm{W}^{1,\infty}(\Omega\setminus\overline{\omega})}\, t\, \|\varphi\|_{\mathrm{H}^{1/2}(\partial\Omega)} \|\psi\|_{\mathrm{H}^{1/2}(\partial\Omega)}.\end{aligned}$$

The proof is complete. ■

3.2 Answer to Q2: asymptotic analysis

The boundary value problems we have in mind for the applications are invariant by dilatation: see the scaling included in the boundary condition of (1.2). Therefore, instead of working with a large domain surrounding a fixed domain, we will work in this section with a fixed domain and a small inclusion inside. For consistency with the motivation of (1.2), we restrict ourselves to dimension 2 and the outer domain can be thought as a disk.

Let Ω and ω be two bounded domains in $\mathbb{R}^2$. We fix x_0 in Ω; the set $x_0 + \varepsilon\omega$ is denoted by $\omega_\varepsilon(x_0)$. For ε small enough, $\omega_\varepsilon(x_0) \subset \Omega$ and we set $\Omega_\varepsilon = \Omega \setminus \overline{\omega_\varepsilon(x_0)}$. For a subdomain D of Ω (D will takes the values $\emptyset$, $\omega_\varepsilon(x_0)$), we consider the differential operator L_D on $\mathrm{H}^1(\partial\Omega)$ with values in $\mathrm{H}^{-1}(\partial\Omega)$ defined by

$$L_D(u) = \alpha u + \Lambda_D(u) + \beta\Delta_\tau u, \tag{3.7}$$

where Λ_D is the Dirichlet-to-Neumann map defined as $\Lambda_D(\varphi) = \partial_n U$ and U denotes the solution of

$$\left\{\begin{array}{rcll} -\Delta U &=& 0 & \text{in } \Omega\setminus\overline{D}, \\ U &=& \varphi & \text{on } \partial\Omega, \\ U &=& 0 & \text{on } \partial D. \end{array}\right.$$

We address the question of the invertibility of $L_{\omega_\varepsilon(x_0)}$. Our first result is the following.

Theorem 3.2 *If $L_\emptyset$ is invertible, then there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the operator $L_{\omega_\varepsilon(x_0)}$ is invertible.*

As a direct consequence of this theorem and of the study of Section 2.1, we obtain the following statement, related to the fact that $L_\emptyset$ is generically invertible:

Corollary 3.3 *Assume that Ω is the unit disk and that conditions* (2.2) *hold, or more generally that Ω is smooth and compact and that conditions of point 1. of Theorem 2.5 hold. Then there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the operator $L_{\omega_\varepsilon(x_0)}$ is invertible, and the norm of the inverse is uniformly bounded with respect to ε.*

In the case of the disk we therefore have the following : for any bounded open ω containing 0 and any α, $\beta \neq 0$, the boundary value problem

$$\left\{ \begin{array}{rcll} -\Delta u &=& 0 & \text{in } B(0,R) \setminus \overline{\omega}, \\ \partial_n u + \frac{\alpha}{R} u + \frac{\beta}{R} \Delta_\tau u &=& 0 & \text{on } \partial B(0,R), \\ u &=& g & \text{on } \partial\omega, \end{array} \right. \tag{3.8}$$

where g is a fixed right hand side in $\mathrm{H}^{1/2}(\partial\omega)$, is wellposed if R is large enough.

Proof of Theorem 3.2: The first step is to prove that

$$\|\Lambda_{\omega_\varepsilon(x_0)} - \Lambda_\emptyset\|_{\mathcal{L}(\mathrm{H}^{1/2}(\partial\Omega), \mathrm{H}^{-1/2}(\partial\Omega))} \leq \frac{C}{|\ln \varepsilon|}. \tag{3.9}$$

To that end, fix $\varphi \in \mathrm{H}^{1/2}(\partial\Omega)$ and let u_0 and u_ε be solutions of

$$\left\{ \begin{array}{rcll} -\Delta u_0 &=& 0 & \text{in } \Omega, \\ u_0 &=& \varphi & \text{on } \partial\Omega; \end{array} \right. \qquad \text{and} \qquad \left\{ \begin{array}{rcll} -\Delta u_\varepsilon &=& 0 & \text{in } \Omega_\varepsilon, \\ u_\varepsilon &=& \varphi & \text{on } \partial\Omega, \\ u_\varepsilon &=& 0 & \text{on } \partial\omega_\varepsilon(x_0). \end{array} \right.$$

Let d denote the distance to x_0 and w the solution of

$$\left\{ \begin{array}{rcll} -\Delta w &=& 0 & \text{in } \Omega, \\ w &=& \ln d & \text{on } \partial\Omega. \end{array} \right.$$

Following the ideas of Maz′ya, Nazarov and Plamenevskij exposed in [18], we formulate the ansatz

$$u_\varepsilon = u_0 + a_\varepsilon(\ln d - w) + r, \quad \text{with } a_\varepsilon = o(1) \text{ and } r = o(a_\varepsilon). \tag{3.10}$$

Considering the traces of r on both $\partial\Omega$ and $\partial\omega_\varepsilon(x_0)$, we get

$$\begin{array}{ll} \text{on } \partial\Omega: & -r = 0, \\ \text{on } \partial\omega_\varepsilon(x_0): & -r = [u_0(x_0) + o(1)] + a_\varepsilon\left[\ln \varepsilon - w(x_0) + \mathcal{O}(1)\right], \end{array}$$

whence the expression for a_ε:

$$a_\varepsilon = \frac{u_0(x_0)}{w(x_0) - \ln \varepsilon}. \tag{3.11}$$

Applying the mean value theorem to u_0 in a small ball B around x_0, contained in Ω (choosing ε small enough, we also assume $\omega_\varepsilon(x_0) \subset B$, see (3.13) below), and the standard *a priori* estimate for u_0, we obtain

$$|u_0(x_0)| \leq \frac{|B|^{1/2}}{2\pi} \|u_0\|_{\mathrm{L}^2(B)} \leq c_1 \|\varphi\|_{\mathrm{H}^{1/2}(\partial\Omega)}.$$

Combined with equation (3.11), we immediately get

$$a_\varepsilon \leq \frac{c_2}{|\ln \varepsilon|} \|\varphi\|_{\mathrm{H}^{1/2}(\partial\Omega)}. \tag{3.12}$$

To prove (3.9), we need to estimate the H^1-norm of $u_\varepsilon - u_0$ over Ω_ε. Using the expression (3.10), it is enough to show $r = \mathcal{O}(a_\varepsilon)$ (even if a sharper estimate may be obtained to validate (3.10) as an asymptotic formula).

We first check that r is an harmonic function in Ω_ε vanishing on $\partial\Omega$. The problem then reduces to compute the norm of its trace on $\partial\omega_\varepsilon(x_0)$. Setting $x = x_0 + \varepsilon X$ with $X \in \partial\omega$, this trace takes the form

$$-r(x) = [u_0(x) - u_0(x_0)] + a_\varepsilon[\ln|X| - w(x) + w(x_0)].$$

For the first part of r, we use an L^∞-estimate in the ball B for the harmonic function u_0:

$$\|u_0 - u_0(x_0)\|_{\mathrm{H}^{1/2}(\partial\omega_\varepsilon(x_0))} \leq c_3\varepsilon\|\nabla u_0\|_{\mathrm{L}^\infty(B)}. \tag{3.13}$$

Using the mean value theorem for ∇u_0 we get

$$\|\nabla u_0\|_{\mathrm{L}^\infty(B)} \leq c_4\|\nabla u_0\|_{\mathrm{L}^2(\Omega)} \leq c_5\|\varphi\|_{\mathrm{H}^{1/2}(\partial\Omega)}.$$

We combine the previous estimates with (3.12):

$$\|r\|_{\mathrm{H}^{1/2}(\partial\omega_\varepsilon(x_0))} \leq \left(c_3c_5\varepsilon + \frac{c_2}{|\ln\varepsilon|}\right)\|\varphi\|_{\mathrm{H}^{1/2}(\partial\Omega)} \leq \frac{c_6}{|\ln\varepsilon|}\|\varphi\|_{\mathrm{H}^{1/2}(\partial\Omega)} = \mathcal{O}(a_\varepsilon). \tag{3.14}$$

This achieves to prove the estimate

$$\|u_\varepsilon - u_0\|_{\mathrm{H}^1(\Omega_\varepsilon)} \leq \frac{C}{|\ln\varepsilon|}\|\varphi\|_{\mathrm{H}^{1/2}(\partial\Omega)}.$$

By a standard *a priori* estimate, this yields (3.9).

A similar estimate is easily obtained for $L_{\omega_\varepsilon(x_0)} - L_\emptyset$ by linearity. To conclude, we write

$$L_{\omega_\varepsilon(x_0)} = L_\emptyset\left[I + L_\emptyset^{-1}(L_{\omega_\varepsilon(x_0)} - L_\emptyset)\right].$$

Then, for ε small enough, the operator $L_{\omega_\varepsilon(x_0)}$ is invertible and its inverse can be written in terms of the Neumann's series

$$L_{\omega_\varepsilon(x_0)}^{-1} = \sum_{n=0}^{\infty}(I - L_\emptyset^{-1}L_{\omega_\varepsilon(x_0)})^n L_\emptyset^{-1}.$$

This expression gives a fortiori a unform bound for the norm of the inverse. ■

Remark 3.4 *Theorem* 3.2 *remains valid in higher dimensions. Nethertheless, the proof has to be adapted: the right hand side of* (3.9) *has to be modified. The upper-bound in* ε *depends on the dimension and is closely linked to the fundamental solution of Laplace equation. Namely, in dimension* 3*, the logarithmic potential is replaced by* $1/|x|$ *so that the bound in* (3.9) *becomes* $C\varepsilon$*.*

3.3 Some numerical experiments

The previous result states that, for R large enough, problem (3.8) has a unique solution. From a practical point of view, it would be useful to quantify the expression “large enough”. In the particular example of a ring, see Section 2.1.2, this can be done thanks to analytic computations, see formula (2.10). However, this can not be achieved in the general case but numerical simulations can help to have an idea on this point.

We denote by $\mathfrak{L}_R$ the operator associated to problem (3.8). In Figures 2 and 3, we plot the norm of the inverse of $\mathfrak{L}_R$ (actually the infinity norm of the corresponding discrete matrix) with respect to R. In the case of a ring, we naturally recover the forbidden radii – in red dashed lines on the figure – characterized by (2.10). Figure 3 shows a more general situation ; the geometries are represented in Figure 1.

All results shown have been obtained with a high degree finite element method implemented with the library MÉLINA, see [17], with the following values

$$\alpha = \beta = 1.$$

Note that the most difficult computations concern very thin rings, which explain that the forbidden ratios close to 1 are not well captured by the simulations. However only large R have to be considered in the absorbing conditions framework (the larger R, the better the approximation). Besides, the computations show that the radius R has not to be very large to get a wellposed problem. For large R, Figure 4 does not show any forbidden radii beyond $R = 1.1$. The drift in R^2 only reflects the asymptotic behavior of the inverse $\mathfrak{L}_R^{-1}$.

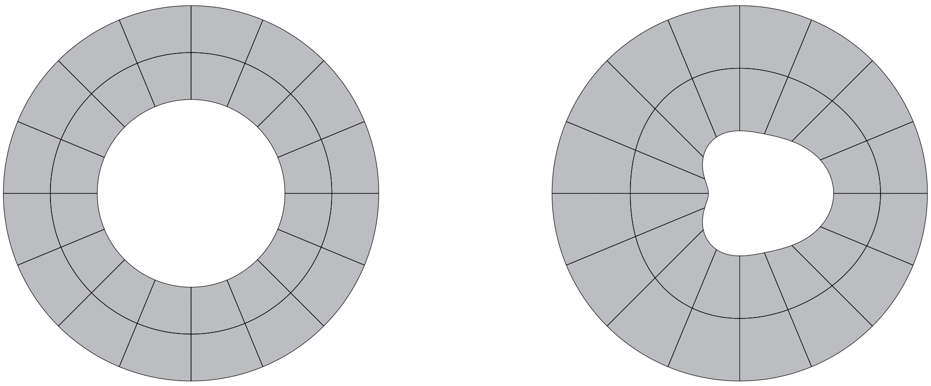

Figure 1: The geometries and meshes used ($R = 2$).

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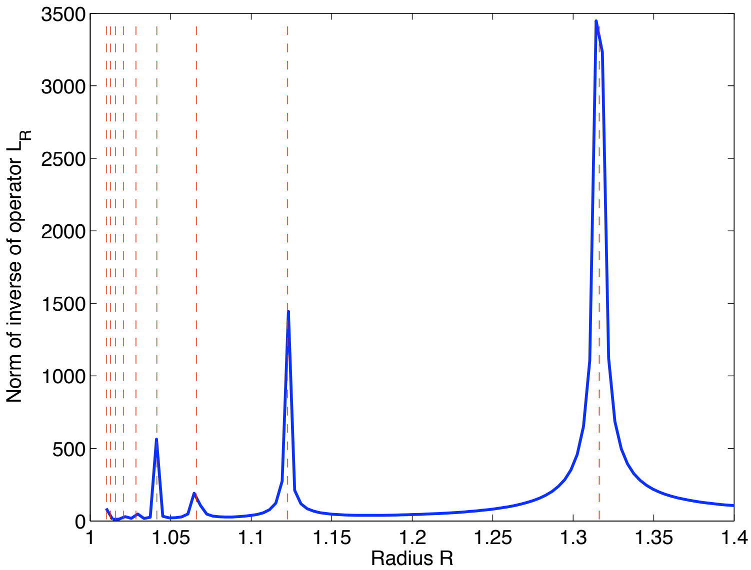

Figure 2: Norm of inverse with respect to R for the ring.

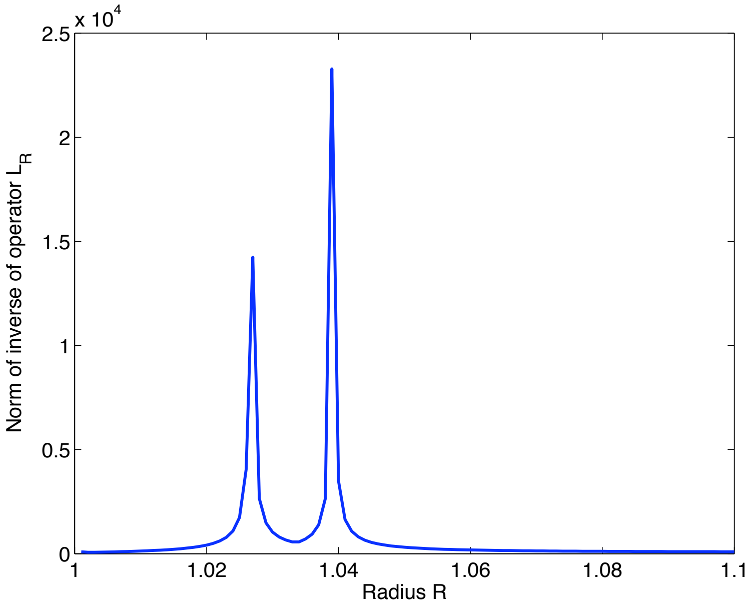

Figure 3: Norm of inverse with respect to R for the more general case.

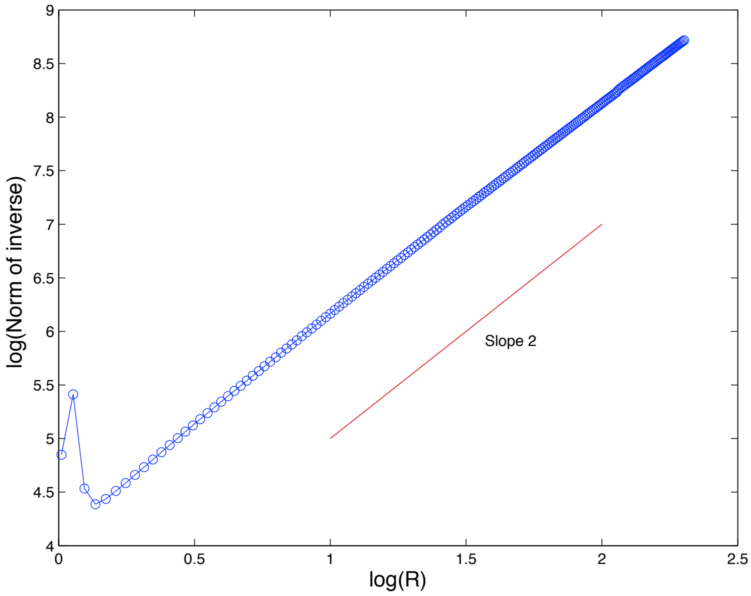

Figure 4: Norm of inverse with respect to R for large radii (loglog scale).

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