On the fundamental state energy for a Schrödinger operator with magnetic field in domains with corners

Virginie Bonnaillie
Département de Mathématiques, UMR CNRS 8628, Bât 425,
Université Paris-Sud, 91405 Orsay Cedex, France
E-mail : Virginie.Bonnaillie@math.u-psud.fr

Abstract

The superconducting properties of a sample submitted to an external magnetic field are mathematically described by the minimizers of the Ginzburg-Landau's functional. The analysis of the Hessian of the functional leads to estimate the fundamental state for the Schrödinger operator with intense magnetic field for which the superconductivity appears. So we are interested in the asymptotic behavior of the energy for the Schrödinger operator with a magnetic field. A lot of papers have been devoted to this problem, we can quote the works of Bernoff-Sternberg, Lu-Pan, Helffer-Mohamed. These papers deal with estimates of the energy in a regular domain and our goal is to establish similar results in a domain with corners. Although this problem is often mentioned in the physical literature, there are very few mathematical papers. We only know the contributions by Pan and Jadallah which deal with very particular domains like a square or a quarter plane. The physicists Brosens, Devreese, Fomin, Moshchalkov, Schweigert and Peeters propose a non optimal upper bound for the energy. Here, we present a more rigourous analysis and give an asymptotics of the smallest eigenvalue of the operator in a sector Ω_α of angle α when α is closed to 0, an estimate for the eigenfunctions and we use these results to study the fundamental state in the semi-classical case.

Acknowledgements
I am deeply grateful to B. Helffer for his help, advice and attention to this work and also to S. Fournais for his attentive reading and his suggestions. I would like to thank A. Laptev, V. Guillemin and B. Helffer for their invitation to stay at the Mittag-Leffler Institut in Stockholm and to participate to the workshop in Partial Differential Equations and Spectral Theory. Discussions with other participants, particularly with U. Smilansky who explains me his work with K. Hornberger [12], were very fruitful. I also thank H. Siedentop and the Ludwig Maximilian Universität in Munich. A part of this work was prepared at these universities. I acknowledge the support of the European Union through the IHP network of the EU N0 HPRN-CT-2002-00277. A first version of this work was diffused as a preprint of the Mittag-Leffler Institut in 2003; some points are clarified and improved here.

1 Introduction and main results

Let $\Omega \subset \mathbb{R}^2$ be an open, simply connected domain with lipschitzian boundary and let ν be the unit outer normal of the boundary $\Gamma = \partial\Omega$ when it is well defined. We define Γ' as the set of the points of Γ where the normal exists. We consider a type II cylindrical superconducting sample of cross section Ω and we apply a constant magnetic field along the cylindrical axis of intensity equal to σ. We denote by κ the characteristic of the sample, called the "Ginzburg-Landau parameter". The type I superconductor corresponds to κ small and type II to κ large. Then, up to normalization factors, the free energy writes

$$\mathcal{G}(\psi, \mathcal{A}) = \frac{1}{2}\int_\Omega \left(|(\nabla - i\kappa\mathcal{A})\psi|^2 + \frac{\kappa^2}{2}(|\psi|^2 - 1)^2 + \kappa^2|\operatorname{curl}\mathcal{A} - \sigma|^2 \right) dx. \quad (1.1)$$

The superconducting properties are described by the minimizers $(\psi, \mathcal{A})$ of this Ginzburg-Landau functional $\mathcal{G}$. The complex-valued function ψ is the order parameter ; the magnitude $|\psi|^2$ gives the density of superconducting electrons and the phase determines the current flow. The vector field $\mathcal{A}$ defined on $\mathbb{R}^2$ is the magnetic potential and $B = \operatorname{curl}\mathcal{A}$ is the induced magnetic field. To determine the apparition of the superconductivity, we linearize the Euler equation associated to (1.1) near the normal state $(\psi, \mathcal{A}) = (0, \sigma\mathcal{A}_{\mathbf{0}})$, where

$$\mathcal{A}_0 :\, = \frac{1}{2}(x_2, -x_1). \quad (1.2)$$

Therefore, defining the change of parameter $h = \frac{1}{\kappa\sigma}$, we have to determine, when $h \to 0$, the bottom of the spectrum for the Neumann realization of the operator $P_{h,\mathcal{A},\Omega}$ defined on the domain $\mathcal{D}^N(P_{h,\mathcal{A},\Omega})$ by :

$$P_{h,\mathcal{A},\Omega} = -\nabla^2_{h,\mathcal{A}}, \text{ with } \nabla_{h,\mathcal{A}} = h\nabla - i\mathcal{A}, \quad (1.3)$$

$$\mathcal{D}^N(P_{h,\mathcal{A},\Omega}) :\, = \{u \in L^2(\Omega) | \nabla_{h,\mathcal{A}}u \in L^2(\Omega), \nabla^2_{h,\mathcal{A}}u \in L^2(\Omega), \nu \cdot \nabla_{h,\mathcal{A}}u_{|\Gamma'} = 0\}.$$

We denote by $q_{h,\mathcal{A},\Omega}$ and $a_{h,\mathcal{A},\Omega}$ the quadratic and sesquilinear forms associated to the operator $P_{h,\mathcal{A},\Omega}$. These forms are defined on $H^1_{h,\mathcal{A}}(\Omega)$ by :

$$H^1_{h,\mathcal{A}}(\Omega) :\, = \{u \in L^2(\Omega) |\ \nabla_{h,\mathcal{A}}u \in L^2(\Omega)\}. \quad (1.4)$$

$$a_{h,\mathcal{A},\Omega}(u,v) = \int_\Omega \nabla_{h,\mathcal{A}}u \cdot \overline{\nabla_{h,\mathcal{A}}v}\ dx \text{ and } q_{h,\mathcal{A},\Omega}(u) = a_{h,\mathcal{A},\Omega}(u,u). \quad (1.5)$$

We omit h in the notation when $h = 1$.
It is well known that the spectrum of the operator $P_{h,\mathcal{A},\Omega}$ is invariant by gauge transformation. So, when Ω is simply connected, the spectrum of $P_{h,\mathcal{A},\Omega}$ depends only on the magnetic field and not on the choice of the corresponding magnetic potential. Then, we denote by $\mu(h, B, \Omega)$ the bottom of the spectrum of $P_{h,\mathcal{A},\Omega}$ for any $\mathcal{A}$ such that $\operatorname{curl}\mathcal{A} = B$.

Bernoff-Sternberg [3], Helffer-Mohamed [10], Helffer-Morame [11], Lu-Pan [14, 15] have already analyzed the case of regular domains and our aim is to give the proof of the results announced in [5] in order to establish similar results for domains with corners. In the previous analysis, the model operator $-\nabla^2_{\mathcal{A}_0}$ on $\mathbb{R} \times \mathbb{R}^+$ was playing an important role. Our new model to analyze the case of a non smooth domain is the operator $-\nabla^2_{\mathcal{A}_0}$ in an angular sector. We denote by Ω_α a sector with an angle equal to α and by $\mu(\alpha)$ the bottom of the spectrum for $P_{\mathcal{A}_0,\Omega_\alpha}$. If $\alpha = \pi$, we write :

$$\Theta_0 : = \mu(\pi). \tag{1.6}$$

This real Θ_0 plays an important role in the study of regular cases. It appears for the study of the bottom of the spectrum of the operator $-\nabla^2_{\mathcal{A}_0}$ on a disk (cf [2], p. 24 and [18]) and for the estimate of $\mu(h, B, \Omega)$ given by Helffer-Morame (cf [11], p.617-621). We will prove that Θ_0 is an upper bound of $\mu(\alpha)$. The main result of this paper is the construction of an asymptotics for $\mu(\alpha)$ as $\alpha \to 0$:

Theorem 1.1. *There exists a real sequence $(m_j)_{j\in\mathbb{N}}$ recursively determined with $m_0 = \frac{1}{\sqrt{3}}$ such that :*

$$\forall n \in \mathbb{N},\ \mu(\alpha) = \alpha \sum_{j=0}^{n} m_j \alpha^{2j} + \mathcal{O}_n(\alpha^{2n+3}) \ \text{as}\ \alpha \to 0. \tag{1.7}$$

To establish this result, we first determine the bottom of the essential spectrum and then construct a regular function whose Rayleigh quotient is less than the bottom of the essential spectrum.
The analysis of the model $P_{\mathcal{A}_0,\Omega_\alpha}$ is useful to approximate the case of the operator $P_{h,\mathcal{A},\Omega}$ with h a small parameter, $B : = \operatorname{curl} \mathcal{A}$ a non constant magnetic potential and Ω a domain with corners, as we see in the following theorem :

Theorem 1.2. *Let Ω be a bounded open subset of $\mathbb{R}^2$ whose boundary is a curvilinear polygon with vertices $S_1, \ldots, S_N$. Let $\Omega \ni x \mapsto B(x)$ be a positive magnetic field and let us define :*

$$b = \inf_{x\in\overline{\Omega}} B(x) \quad \text{and} \quad b' = \inf_{x\in\partial\Omega} B(x). \tag{1.8}$$

We denote by $\alpha_1, \ldots, \alpha_N$ the angles for each vertex. Then, for h small, the smallest eigenvalue $\mu(h, B, \Omega)$ for the Neumann's realization of $-(h\nabla - i\mathcal{A})^2$ admits the following asymptotics :

$$\mu(h, B, \Omega) = h\ \inf\left(b, \Theta_0 b', \inf_{j=1,\ldots,N} \mu(\alpha_j) B(S_j)\right) + \mathcal{O}(h^{5/4}). \tag{1.9}$$

We notice that for regular domains, Theorem 1.2 gives the estimate of $\mu(h, B, \Omega)$ obtained by Helffer-Morame [11], p. 617-621. The condition (1.9) takes in the case of a constant magnetic field a simpler form :

Corollary 1.3. *For B constant, we have :*

$$\mu(h, B, \Omega) = \inf\left(\inf_{j=1,\ldots,N} \mu(\alpha_j), \Theta_0\right) Bh + \mathcal{O}(h^{5/4}) \ \text{as}\ h \to 0.$$

This article is organized as follows. In Section 2, we recall some results about the Neumann realization of the Schrödinger operator with constant magnetic field in the half plane and show that the bottom of the essential spectrum for an angular sector is Θ_0, the bottom of the spectrum for the half plane. With the notation :

$$D_t = \frac{1}{i}\frac{\partial}{\partial t}, \tag{1.10}$$

we analyze, in Subsection 2.1, the operator $D_t^2 + (t-\zeta_0)^2$ to construct, in Subsection 2.3, a test function inspired by Pan [16] and propose a first upper bound of the bottom of the spectrum for an angular sector with angle close to $\frac{\pi}{2}$.
Then, from Section 3 to Section 10, we analyze the bottom of the spectrum for the Neumann realization of the Schrödinger operator with constant magnetic field in an angular sector when the angle tends to 0. As it is more easy to deal with a non variable domain, Section 3 shows how to reduce the problem to a domain independent of the angle α of the sector and to deal with a new operator P_α. This P_α is the sum of two operators with different weight according to α. Section 4 is devoted to the analysis of these two key-operators and leads in Section 5 to the construction of a formal asymptotics for the eigenvalue.
Section 6 establishes an upper bound of the bottom of the spectrum $\mu(\alpha)$ thanks to the min-max principle and the computation of the Rayleigh quotient for the previous formal solution.
Section 7 uses Agmon's techniques to prove the decay of eigenfunction which is useful to give a weak lower bound of the first eigenvalue in Section 8.
To prove that our formal construction gives further coefficients of the asymptotics of $\mu(\alpha)$ as writen in Theorem 1.1, we estimate the splitting between the two first eigenvalues and bound from below the second eigenvalue in Section 10.
In Section 11, the analysis of the Neumann realization of the Schrödinger operator with constant magnetic field in an angular sector coupled with the results about the plane $\mathbb{R}^2$ and the half plane $\mathbb{R}\times\mathbb{R}^+$ are useful to estimate the bottom of the spectrum of $-(h\nabla - i\mathcal{A})^2$ in a non smooth domain with $B = \operatorname{curl}\mathcal{A}$ non constant and h tending to 0.

2 Some remarks about Θ_0 and applications

Let us recall from [8] the link between Θ_0, introduced in (1.6), and the operators $D_t^2 + (t-\zeta)^2$ on $\mathbb{R}^+$.

2.1 Link with $D_t^2 + (t-\zeta)^2$ on $\mathbb{R}^+$

Proposition 2.1. *Let $\lambda_H(\zeta)$ be the bottom of the spectrum of the Neumann realization in $L^2(\mathbb{R}^+)$ of the operator $H(\zeta)$ defined for $\zeta \in \mathbb{R}$ by :*

$$H(\zeta) = D_t^2 + (t-\zeta)^2.$$

There exists a unique ζ_0 such that $\lambda_H(\zeta_0) = \Theta_0$, $\lambda_H(\zeta)$ is decreasing from $]-\infty, \zeta_0]$ onto $[\Theta_0, +\infty[$ and increasing from $[\zeta_0, +\infty[$ onto $[\Theta_0, 1[$. We denote by ϕ the normalized eigenvector associated to $\lambda_H(\zeta_0)$, then :

$$\int_0^\infty (t-\zeta_0)|\phi(t)|^2\, dt = 0 \quad \text{and} \quad \phi(0)^2 = \frac{\lambda_H''(\zeta_0)}{2\zeta_0}. \tag{2.1}$$

We show briefly how the operator $H(\zeta)$ appears in the analysis of $P_{\mathcal{A}_0,\mathbb{R}\times\mathbb{R}^+}$. We notice that after a gauge transform, we have to study the new operator :

$$(D_{x_1} - x_2)^2 + D_{x_2}^2 \text{ on } \mathbb{R}\times\mathbb{R}^+, \text{ with Neumann condition on } x_2 = 0.$$

Then, we make a partial Fourier transform in the first coordinate. The operator $P_{\mathcal{A}_0,\mathbb{R}\times\mathbb{R}^+}$ is also unitary equivalent to :

$$(\xi_1 - x_2)^2 + D_{x_2}^2 = H(\xi_1).$$

So the study of the operators family $H(\zeta)$ is linked to the Neumann realization of $-\nabla^2_{\mathcal{A}_0}$ on $\mathbb{R}\times\mathbb{R}^+$.
Furthermore, the bottom of the spectrum, Θ_0, is in the essential spectrum and there is no point spectrum for the realization on $\mathbb{R}\times\mathbb{R}^+$.

2.2 Bottom of the essential spectrum

Let $\Omega_\alpha\subset\mathbb{R}^2$ be an angular sector with angle α. The Persson Lemma (cf [17]) may be generalized for unbounded domains of $\mathbb{R}^2$ and Neumann realizations :

Lemma 2.2 (Persson)**.** *Let Ω be an unbounded domain of $\mathbb{R}^2$ with lipschitzian boundary and V be a semi-bounded from below regular function. We denote by $\inf\sigma_{\mathrm{ess}}(-\Delta_{\mathcal{A}}+V)$ the bottom of the essential spectrum, then :*

$$\inf\sigma_{\mathrm{ess}}(-\Delta_{\mathcal{A}}+V)=\lim_{r\to\infty}\Sigma(-\Delta_{\mathcal{A}}+V,r),\tag{2.2}$$

with, denoting $\mathcal{B}_r=\{x\in\Omega|\ |x|\le r\}$:

$$\Sigma(-\Delta_{\mathcal{A}}+V,r):=\inf_{\phi\in C_0^\infty(\overline{\Omega}\setminus\mathcal{B}_r),\phi\neq0}\frac{\displaystyle\int_\Omega\left(|\nabla_{\mathcal{A}}\phi(x)|^2+V(x)|\phi(x)|^2\right)\,dx}{\displaystyle\int_\Omega|\phi(x)|^2\,dx}.\tag{2.3}$$

We use this lemma to determine the bottom of the essential spectrum of the Schrödinger operator in an angular sector:

Proposition 2.3. *The bottom of the essential spectrum for the Neumann realization of $-\nabla^2_{\mathcal{A}_0}$ in an angular sector Ω_α, denoted by $P_{\mathcal{A}_0,\Omega_\alpha}$, is equal to Θ_0.*

Proof : We estimate $\Sigma(P_{\mathcal{A}_0,\Omega_\alpha},r)$ for $r>0$ and show that it tends to Θ_0 when r tends to infinity. We use a partition of the unity which shares the sector in three subdomains and we compare to the models $\mathbb{R}^2$ and $\mathbb{R}\times\mathbb{R}^+$ according to the support of the cut-off functions.
Let $r>0$ and $\tilde{\chi}$ be a regular function defined from $\mathbb{R}$ onto $[0,1]$ such that :

$$\tilde{\chi}(\rho)=\begin{cases}0, & \forall\rho\le0,\\ 1, & \forall\rho\ge1.\end{cases}\tag{2.4}$$

Let $\hat{\chi}_j\in C_0^\infty([-\frac12,\frac12],[0,1])$ be such that :

$$\begin{cases}\operatorname{supp}\hat{\chi}_j & \subset & \left[\frac{j-3}{4},\frac{j-1}{4}\right], & \forall j=1,2,3,\\ \displaystyle\sum_{j=1}^3\hat{\chi}_j^2(\theta) & = & 1, & \forall\theta\in\left[-\frac12,\frac12\right].\end{cases}\tag{2.5}$$

We define a cut-off function in polar coordinates :

$$\forall(\rho,\theta)\in\mathbb{R}^+\times\left]-\frac12,\frac12\right[,\ \chi_j^{r,\alpha,\mathrm{pol}}(\rho,\theta)=\tilde{\chi}\left(\frac{\rho}{r}\right)\hat{\chi}_j\left(\frac{\theta}{\alpha}\right),\ j=1,2,3,\tag{2.6}$$

and the associated functions in cartesian coordinates $\chi_j^{r,\alpha}$. We notice that :

$$\forall\phi\in C_0^\infty(\overline{\Omega_\alpha}\setminus\mathcal{B}_r),\ \sum_{j=1}^3\left|\chi_j^{r,\alpha}\right|^2\phi=\phi\text{ on }\overline{\Omega_\alpha}\setminus\mathcal{B}_r.\tag{2.7}$$

Furthermore, it is easy to prove for all $\phi \in C_0^\infty(\overline{\Omega_\alpha} \setminus \mathcal{B}_r)$ the relation :

$$||\nabla_{\mathcal{A}_0}\phi||^2_{L^2(\Omega_\alpha)} = \sum_{j=1}^{3} ||\nabla_{\mathcal{A}_0}(\chi_j^{r,\alpha}\phi)||^2_{L^2(\Omega_\alpha)} - \sum_{j=1}^{3} ||\phi\nabla\chi_j^{r,\alpha}||^2_{L^2(\Omega_\alpha)}. \tag{2.8}$$

By construction, there exists a constant C independent of α and r such that :

$$\forall \phi \in C_0^\infty(\overline{\Omega_\alpha} \setminus \mathcal{B}_r),\ ||\nabla_{\mathcal{A}_0}\phi||^2_{L^2(\Omega_\alpha)} \geq \sum_{j=1}^{3} ||\nabla_{\mathcal{A}_0}(\chi_j^{r,\alpha}\phi)||^2_{L^2(\Omega_\alpha)} - \frac{C}{\alpha^2 r^2}||\phi||^2_{L^2(\Omega_\alpha)}. \tag{2.9}$$

It is well known that the bottom of the spectrum of $P_{\mathcal{A}_0,\Omega_\alpha}$ is invariant under rotation or translation of the domain Ω_α. So, using the fact that $\Theta_0 \leq 1$ and the definition of $\Sigma(P_{\mathcal{A}_0,\Omega_\alpha}, r)$, we deduce :

$$\Sigma(P_{\mathcal{A}_0,\Omega_\alpha}, r) \geq \Theta_0 - \frac{C}{\alpha^2 r^2}. \tag{2.10}$$

This implies, taking the limit $r \to +\infty$ and using (2.2) :

$$\inf \sigma_{\mathrm{ess}}(P_{\mathcal{A}_0,\Omega_\alpha}) \geq \Theta_0.$$

Now, we establish the upper bound. Let $\epsilon > 0$ and $\psi_1 \in C_0^\infty(\overline{\mathbb{R} \times \mathbb{R}^+})$ be a function such that :

$$\Theta_0 \leq \frac{||\nabla_{\mathcal{A}_0}\psi_1||^2_{L^2(\mathbb{R}\times\mathbb{R}^+)}}{||\psi_1||^2_{L^2(\mathbb{R}\times\mathbb{R}^+)}} \leq \Theta_0 + \epsilon. \tag{2.11}$$

By translation and rotation from ψ_1, we define a function $\psi \in C_0^\infty(\overline{\Omega_\alpha} \setminus \mathcal{B}_r)$ such that :

$$\frac{||\nabla_{\mathcal{A}_0}\psi_1||^2_{L^2(\mathbb{R}\times\mathbb{R}^+)}}{||\psi_1||^2_{L^2(\mathbb{R}\times\mathbb{R}^+)}} = \frac{||\nabla_{\mathcal{A}_0}\psi||^2_{L^2(\Omega_\alpha)}}{||\psi||^2_{L^2(\Omega_\alpha)}}. \tag{2.12}$$

It is enough to take the limit as r tends to ∞ and ϵ to 0 to achieve the proof. □

Using this proposition, as soon as we find a function in $H^1_{\mathcal{A}_0}(\Omega_\alpha)$ with a Rayleigh quotient strictly less than Θ_0, then $\mu(\alpha)$ is an eigenvalue. For example, Jadallah [13] constructs a test-function for the quarter plane and we can use it to see that $\mu(\frac{\pi}{2})$ is an eigenvalue.

Remark 2.4. *It would be interesting to show that $\mu(\alpha)$ is strictly bounded from above by Θ_0 for $\alpha \in]0, \pi[$ and equal to Θ_0 for $\alpha \in [\pi, 2\pi[$.*

2.3 First upper bound

The eigenfunction ϕ introduced in Proposition 2.1 will be used for the construction of a test function giving an upper bound for $\mu(\alpha)$. This first upper bound follows the idea of Pan [16] who constructs a quasi-mode in order to recover Jadallah's result (cf [13]) and show that :

$$\mu\left(\frac{\pi}{2}\right) < \Theta_0.$$

We use the same notations as in Proposition 2.1. Adapting the idea of Pan, we construct a trial function, u, in polar coordinates :

$$u(\rho,\theta) = e^{i\gamma\rho\sin\theta}\ \phi\left(\tau\rho\sin\left(\frac{\alpha}{2}+\theta\right)\right)\ \phi\left(\tau\rho\sin\left(\frac{\alpha}{2}-\theta\right)\right), \tag{2.13}$$

with :

$$\gamma = \frac{\zeta_0}{2\tau\sin\frac{\alpha}{2}} \quad \text{and} \quad \tau = \sqrt{\frac{1}{2\sin\alpha}}. \tag{2.14}$$

We obtain the following result :

Proposition 2.5. *For every $\alpha \in]0,\frac{\pi}{2}]$:*

$$\mu(\alpha) \leq \frac{\Theta_0}{\sin\alpha} - \frac{\cos\alpha}{4\sin\alpha}\phi(0)^4. \tag{2.15}$$

Furthermore, for every $\alpha \in \left]\frac{\pi}{2} - 2\arctan\left(\frac{\phi(0)^4}{4\Theta_0}\right), \frac{\pi}{2}\right[$, $\mu(\alpha)$ is an eigenvalue.

Proof : A change of coordinates sends the sector onto a quarter plane Q :

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \tau \begin{pmatrix} \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & -\cos\frac{\alpha}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{2.16}$$

The expression of u in the new coordinates is :

$$\tilde{u}(\tilde{x}_1,\tilde{x}_2) = \exp\left(i\frac{\gamma(\tilde{x}_1-\tilde{x}_2)}{2\tau\cos\frac{\alpha}{2}}\right)\phi(\tilde{x}_1)\phi(\tilde{x}_2) = e^{i\zeta_0(\tilde{x}_1-\tilde{x}_2)}\phi(\tilde{x}_1)\phi(\tilde{x}_2). \tag{2.17}$$

After computing $|(\nabla - i\mathcal{A}_0)u|^2$ with the choice (2.14) of γ and τ, we obtain :

$$\begin{aligned}
||\nabla_{\mathcal{A}_0}u||^2_{L^2(\Omega_\alpha)} = &\ \frac{\tau^2}{\tau\sin\alpha}\left(\int_0^\infty ((\tilde{x}_1-\zeta_0)^2\phi(\tilde{x}_1)^2 + \phi'(\tilde{x}_1)^2)d\tilde{x}_1 \int_0^\infty |\phi(\tilde{x}_2)|^2 d\tilde{x}_2\right. \\
&\left. + \int_0^\infty ((\tilde{x}_2-\zeta_0)^2\phi(\tilde{x}_2)^2 + \phi'(\tilde{x}_2)^2)d\tilde{x}_2 \int_0^\infty |\phi(\tilde{x}_1)|^2 d\tilde{x}_1\right) \\
&+ \frac{2\tau^2\cos\alpha}{\tau\sin\alpha}\left(\int_0^\infty (\tilde{x}_1-\zeta_0)\phi(\tilde{x}_1)^2 d\tilde{x}_1 \int_0^\infty (\tilde{x}_2-\zeta_0)\phi(\tilde{x}_2)^2 d\tilde{x}_2\right. \\
&\left. -\frac{1}{4}\int_0^\infty (\phi(\tilde{x}_1)^2)' d\tilde{x}_1 \int_0^\infty (\phi(\tilde{x}_2)^2)' d\tilde{x}_2\right).
\end{aligned}$$

We use properties of ϕ recalled in Proposition 2.1 and deduce from the min-max principle :

$$\mu(\alpha) \leq \frac{||\nabla_{\mathcal{A}_0}u||^2_{L^2(\Omega_\alpha)}}{||u||^2_{L^2(\Omega_\alpha)}} = \frac{\Theta_0}{\sin\alpha} - \frac{\cos\alpha}{4\sin\alpha}\phi(0)^4. \tag{2.18}$$

According to Proposition 2.3, $\mu(\alpha)$ is an eigenvalue as soon as $\mu(\alpha) < \Theta_0$, it is enough to solve :

$$\frac{\Theta_0}{\sin\alpha} - \frac{\cos\alpha}{4\sin\alpha}\phi(0)^4 < \Theta_0 \Longleftrightarrow \tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) < \frac{\phi(0)^4}{4\Theta_0}.$$

□

We know the estimates of Θ_0 and $\phi(0)$ according to Saint James-De Gennes [19] and computations whose details are given in [6] :

$$\Theta_0 \simeq 0.59,\ \phi(0) \simeq 0.87. \tag{2.19}$$

Remark 2.6. *Approximately, $\mu(\alpha)$ is an eigenvalue for $\alpha \in [1.09, \frac{\pi}{2}]$.*

3 Reduction to a domain independent of α

We are interested in the variation with respect to α of $\mu(\alpha)$. The shape of the domain Ω_α suggests to use polar coordinates (ρ,ϕ) and so we denote by Ω_α^{pol} the domain :

$$\Omega_\alpha^{pol}:\ =]0,+\infty[\times\left]-\frac{\alpha}{2},\frac{\alpha}{2}\right[.$$

This change of variables leads to study the new quadratic form :

$$\hat{q}(u)=\int_{\Omega_\alpha^{pol}}\left(|\partial_\rho u|^2+\frac{1}{\rho^2}\left|\left(\partial_\phi+i\ \frac{\rho^2}{2}\right)u\right|^2\right)\ \rho\ d\rho\ d\phi. \tag{3.1}$$

To have a ρ-independent Neumann condition, we make a gauge transform :

$$u_1(\rho,\phi):\ =\exp\left(i\frac{\rho^2}{2}\phi\right)u(\rho,\phi),\ \forall(\rho,\phi)\in\Omega_\alpha^{pol}.$$

We make a last change of variables and another gauge transform to obtain a quadratic form depending on α but defined on a domain independent of α, whereas at the beginning, we had a constant operator on an α-dependent domain. We use the change of variables $(\rho,\phi)\in\Omega_\alpha^{pol}\to(t,\eta)\in\Omega^0$ with :

$$(t,\eta)=\left(\alpha\frac{\rho^2}{2},\frac{\phi}{\alpha}\right),\ \Omega^0:\ =\mathbb{R}^+\times\left]-\frac{1}{2},\frac{1}{2}\right[.$$

Therefore, we have to study a family of quadratic forms q_α defined on $\mathcal{V}^N$ by :

$$q_\alpha(u)\ =\ \int_{\Omega^0}\left(2t|D_t u-\eta\ u|^2+\frac{1}{2\alpha^2 t}|\partial_\eta u|^2\right)\ dt\ d\eta, \tag{3.2}$$

$$\mathcal{V}^N:\ =\left\{u\in L^2(\Omega^0)\left|\ \frac{1}{\sqrt{t}}\partial_\eta u\in L^2(\Omega^0),\sqrt{t}(D_t-\ \eta)\in L^2(\Omega^0)\right.\right\}.$$

We define the sesquilinear form a_α associated to q_α on $\mathcal{V}^N$:

$$a_\alpha(u,v)=\int_{\Omega^0}\left(2t\,(D_t-\eta)\,u\ \overline{(D_t-\eta)\,v}+\frac{1}{2\alpha^2 t}\partial_\eta u\overline{\partial_\eta v}\right)\ dt\ d\eta. \tag{3.3}$$

So the spectrum of the operator P_α associated to the form a_α on Ω^0 satisfies :

$$\sigma(P_{\mathcal{A}_0,\Omega_\alpha})=\alpha\sigma(P_\alpha). \tag{3.4}$$

Particularly, by denoting $\lambda(\alpha)$ the bottom of $\sigma(P_\alpha)$, we observe :

$$\mu(\alpha)=\alpha\lambda(\alpha). \tag{3.5}$$

The construction of the Friedrichs extension gives the domain of P_α which respects a Neumann boundary condition in η :

$$\mathcal{D}^N(P_\alpha):\ =\{u\in\mathcal{V}^N|\quad\exists u_n\in C_0^\infty(\overline{\Omega^0})\text{ s.t. }u_n\to u\text{ in }L^2(\Omega^0)$$
$$\text{and }u_n\text{ is a Cauchy sequence for the norm }q_\alpha\}.$$

We work with the quadratic form and do not need to characterize the domain of P_α explicitly.
Furthermore, ψ is an eigenfunction associated to $\mu(\alpha)$ (if it exists) if and only if ψ_α is an eigenfunction for $\lambda(\alpha)$, where ψ and ψ_α are linked by :

$$\forall (t,\eta)\in\Omega^0,\ \psi_\alpha(t,\eta)=e^{-i\eta t}\psi\left(\sqrt{\tfrac{2t}{\alpha}}\cos(\alpha\eta),\sqrt{\tfrac{2t}{\alpha}}\sin(\alpha\eta)\right).$$

Remark 3.1. *From the expression of the form q_α, we immediately see that :*

$$\alpha\mapsto\alpha\mu(\alpha) \text{ is increasing and } \alpha\mapsto\tfrac{\mu(\alpha)}{\alpha} \text{ is decreasing.}$$

It would be interesting to show the monotonicity of μ from $]0,\pi]$ onto $]0,\Theta_0]$.

As we know from Dauge-Helffer [8], Helffer-Morame [11], Lu-Pan [15] and as we recalled in Section 2.1, then :

$$\forall\alpha\in]0,\pi],\ \frac{\mu(\alpha)}{\alpha}\geq\frac{\mu(\pi)}{\pi}=\frac{\Theta_0}{\pi}. \tag{3.6}$$

4 Analysis of the two key-operators

4.1 Presentation

In the expression of a_α in (3.3), two forms (and two associated operators) appear. The first one is ℓ (with associated L) which will be defined just below and the second is associated to the Neumann realization of $-\partial_\eta^2$ in $]-\frac{1}{2},\frac{1}{2}[$. We define the sesquilinear form ℓ, on :

$$\mathcal{V}_\ell^N:=\left\{u\in L^2(\Omega^0)\middle|\ \sqrt{t}(D_t-\eta)u\in L^2(\Omega^0)\right\},$$

by :

$$\ell(u,v)=\int_{\Omega^0}2t(D_t-\eta)u\overline{(D_t-\eta)v}\ dt\ d\eta,\ \forall u,v\in\mathcal{V}_\ell^N. \tag{4.1}$$

Let $\mathcal{P}\otimes\mathcal{S}(\overline{\mathbb{R}^+})$ be the space of polynomial functions in η whose coefficients are in $\mathcal{S}(\overline{\mathbb{R}^+})$. We define the operator L on $\mathcal{P}\otimes\mathcal{S}(\overline{\mathbb{R}^+})$ by :

$$L:=2(D_t-\eta)t(D_t-\eta).$$

Then we verify :

$$\forall u\in\mathcal{P}\otimes\mathcal{S}(\overline{\mathbb{R}^+}),\ \forall v\in\mathcal{V}_\ell^N,\ \ell(u,v)=\langle Lu,v\rangle_{L^2(\Omega^0)}. \tag{4.2}$$

The form q_α contains a term in $\frac{1}{\alpha^2}$. So when trying to minimize it, it is quite natural to begin with studying the restriction of the form to functions which are independent of η to cancel the term in $\frac{1}{\alpha^2}$.

4.2 A new key operator L^{mean}

We define the form ℓ^{mean} which appears naturally when we restrict ℓ to functions independent of η. Then the sesquilinear form ℓ^{mean} is defined on :

$$\mathcal{V}_{\mathrm{mean}}^N:=\{f\in L^2(\mathbb{R}^+)|\ \sqrt{t}f\in L^2(\mathbb{R}^+),\ \sqrt{t}D_tf\in L^2(\mathbb{R}^+)\},$$

by :

$$\ell^{\mathrm{mean}}(u,v)=\int_0^\infty 2\left(D_tu\ \overline{D_tv}+\frac{1}{12}u\overline{v}\right)\ t\ dt,\ \forall u,v\in\mathcal{V}_{\mathrm{mean}}^N. \tag{4.3}$$

The associated operator,

$$L^{\text{mean}} = 2D_t\, t\, D_t + \frac{t}{6},$$

is self-adjoint and its domain can be characterized (cf [4]) as being $\mathcal{W}_2^1(R^+) := \{u \in H^1(\mathbb{R}^+) |\ tu \in H^2(\mathbb{R}^+)\}$. Its spectrum is discrete and the eigenvalues are simple and given by $\lambda_n^{\text{mean}} = \frac{2n+1}{\sqrt{3}}$ for $n \geq 0$. Moreover each eigensubspace is included in $\mathcal{S}(\overline{\mathbb{R}^+})$. Let us give the expression of the normalized eigenvector u_1^{mean} associated to the first eigenvalue of L^{mean}, denoted by λ_1^{mean} :

$$\lambda_1^{\text{mean}} = \frac{1}{\sqrt{3}} \text{ and } u_1^{\text{mean}}(t) = \frac{1}{3^{1/4}} \exp\left(-\frac{t}{2\sqrt{3}}\right), \ \forall t \in \mathbb{R}^+. \tag{4.4}$$

We can use the function u_1^{mean} as trial function to bound from above $\mu(\alpha)$ as follows :

Proposition 4.1. *For $\alpha < \sqrt{3}\Theta_0$, the bottom of the spectrum of $P_{\mathcal{A}_0,\Omega_\alpha}$ is an eigenvalue $\mu(\alpha)$ wich satisfies :*

$$\mu(\alpha) \leq \frac{\alpha}{\sqrt{3}}. \tag{4.5}$$

Before being more precise about the operator L^{mean}, let us mention an improvment of the upper bound (4.5) due to Soeren Fournais :

Proposition 4.2. *For $\alpha < \frac{\sqrt{3}\Theta_0}{\sqrt{1-\Theta_0^2}}$, the bottom of the spectrum of $P_{\mathcal{A}_0,\Omega_\alpha}$ is an eigenvalue $\mu(\alpha)$ wich satisfies :*

$$\mu(\alpha) \leq \frac{\alpha}{\sqrt{3+\alpha^2}}. \tag{4.6}$$

Proof : The idea is to estimate (3.1) for a function like $e^{i\frac{\rho^2}{2}\phi(1-\delta)}u(\rho)$ and choose u and coefficient δ. This lead to define the function $u \in \mathcal{V}^N$ on Ω^0 by :

$$u(t,\eta) = e^{it\eta\beta^2} e^{-\frac{\beta}{\alpha}\frac{t}{2}} \text{ with } \beta = \frac{\alpha}{\sqrt{3+\alpha^2}}.$$

Then it is easy to compute :

$$\begin{aligned} q_\alpha(u) &= \int_{\Omega^0} \left(2t\left|\eta(\beta^2-1) + i\frac{\beta}{2\alpha}\right|^2 + \frac{\beta^4 t}{2\alpha^2}\right) |u(t,\eta)|^2 \ dt\ d\eta \\ &= \frac{\alpha}{2\beta}\left(\frac{(\beta^2-1)^2}{6} + \frac{\beta^2}{2\alpha^2}(1+\beta^2)\right)\int_{\Omega^0} |u(t,\eta)|^2\ dt\ d\eta \\ &= \frac{1}{\sqrt{3+\alpha^2}}\int_{\Omega^0} |u(t,\eta)|^2\ dt\ d\eta. \end{aligned} \tag{4.7}$$

Thanks to the min-max principle, we deduce that $\mu(\alpha) \leq \frac{\alpha}{\sqrt{3+\alpha^2}}$. Proposition 2.3 shows that $\mu(\alpha)$ is an eigenvalue for any angle α such that $\frac{\alpha}{\sqrt{3+\alpha^2}} < \Theta_0$.
□

Remark 4.3. *Approximately, $\mu(\alpha)$ is an eigenvalue for $\alpha \in]0, 1.26[$.*
Combining this with Remark 2.6, we conclude with a good accuracy that $\mu(\alpha)$ is an eigenvalue for $\alpha \in]0, \frac{\pi}{2}]$.

Furthermore, we deduce by standard Fredholm theory and a regularity theorem (cf [4]), the following lemma :

Lemma 4.4. *Let* $(\lambda_1^{\mathrm{mean}}, u_1^{\mathrm{mean}})$ *the fundamental state of* L^{mean}*. For every* $f \in \mathcal{S}(\overline{\mathbb{R}^+})$ *orthogonal to* u_1^{mean}*, there exists a unique* $u \in \mathcal{S}(\overline{\mathbb{R}^+})$ *such that :*

$$\left\{ \begin{array}{rcl} (L^{\mathrm{mean}} - \lambda_1^{\mathrm{mean}})u &=& f \text{ on } \mathbb{R}^+, \\ \displaystyle\int_0^\infty u(t)\ \overline{u_1^{\mathrm{mean}}(t)}\ dt &=& 0. \end{array} \right. \tag{4.8}$$

Furthermore, if u *is given by (4.8), then for all functions* $v \in \mathcal{V}^N_{\mathrm{mean}}$ *:*

$$\ell^{\mathrm{mean}}(u,v) - \lambda_1^{\mathrm{mean}} \langle u, v\rangle_{L^2(\Omega^0)} = \langle f, v\rangle_{L^2(\Omega^0)}\,. \tag{4.9}$$

In the case when the second member of (4.8) has the form Pu_1^{mean} for some polynomial P, we can explicit the solution as follows :

Lemma 4.5. *Let* P *be a polynomial of degree* n *with coefficients* p_k *such that :*

$$\sum_{k=0}^{n} (\sqrt{3})^k\ k!\ p_k = 0.$$

We define the polynomial of degree n*,* $\tilde{P} \in \mathcal{P}(\mathbb{R}^+)$*, by its coefficients* $\tilde{p}_k$ *:*

$$\left\{ \begin{array}{rcl} \tilde{p}_1 &=& -\frac{p_0}{2}, \\ \tilde{p}_k &=& \frac{1}{2k^2}\left(\frac{2}{\sqrt{3}}(k-1)\tilde{p}_{k-1} - p_{k-1}\right),\ \forall k = 2,\dots,n, \\ \tilde{p}_0 &=& -\displaystyle\sum_{k=1}^{n}(\sqrt{3})^k\ k!\ \tilde{p}_k. \end{array} \right. \tag{4.10}$$

Then $\tilde{u} = \tilde{P}u_1^{\mathrm{mean}}$ *is the unique solution for the problem :*

$$\left\{ \begin{array}{rcl} (L^{\mathrm{mean}} - \lambda_1^{\mathrm{mean}})\tilde{u} &=& Pu_1^{\mathrm{mean}}, \\ \displaystyle\int_0^\infty \tilde{u}(t)\ \overline{u_1^{\mathrm{mean}}(t)}\ dt &=& 0. \end{array} \right. \tag{4.11}$$

Proof : We first notice that :

$$\forall k \in \mathbb{N},\ \int_0^\infty t^k |u_1^{\mathrm{mean}}(t)|^2\ dt = k!\ (\sqrt{3})^k. \tag{4.12}$$

We deduce immediately that :

$$\sum_{k=0}^{n}(\sqrt{3})^k\ k!\ q_k = 0 \Longleftrightarrow \int_0^\infty \sum_{k=0}^{n} q_k t^k |u_1^{\mathrm{mean}}(t)|^2\ dt = 0. \tag{4.13}$$

Relation (4.13) applied with $q_k = \tilde{p}_k$ determined in (4.10) shows that $\tilde{u}$ is orthogonal to u_1^{mean} and so the second condition of (4.11) holds.
According to (4.13) with $q_k = p_k$, we see that Pu_1^{mean} is orthogonal to u_1^{mean}. Lemma 4.4 establishes that the problem (4.8) with $f = Pu_1^{\mathrm{mean}}$ has a unique solution. So it is enough to prove that the function $\tilde{u} = \tilde{P}u_1^{\mathrm{mean}}$ satisfies conditions (4.11). With the expression of u_1^{mean} given in (4.4), we get :

$$(L^{\mathrm{mean}} - \lambda_1^{\mathrm{mean}})(\tilde{P}u_1^{\mathrm{mean}}) = u_1^{\mathrm{mean}}\left(-2\partial_t\tilde{P} + \frac{2}{\sqrt{3}}t\partial_t\tilde{P} - 2t\partial_t^2\tilde{P}\right).$$

According to (4.10), the constant coefficient of $-2\partial_t\tilde{P}+\frac{2}{\sqrt{3}}t\partial_t\tilde{P}-2t\partial_t^2\tilde{P}$ is equal to $-2\tilde{p}_1=p_0$ and the coefficient of t^k for $k=1,\dots,n$ is equal to :

$$-2(k+1)\tilde{p}_{k+1}+\frac{2}{\sqrt{3}}k\tilde{p}_k-2k(k+1)\tilde{p}_{k+1}=-2(k+1)^2\tilde{p}_{k+1}+\frac{2}{\sqrt{3}}k\tilde{p}_k=p_k.$$

So, the first equation of (4.11) holds. □

4.3 Study of $-\partial_\eta^2$

An elementary study of the Neumann realization of $-\partial_\eta^2$ leads to the following lemma :

Lemma 4.6. *Let $f\in\mathcal{P}\otimes\mathcal{S}(\overline{\mathbb{R}^+})$ such that for all $t\in\overline{\mathbb{R}^+}$, $\displaystyle\int_{-\frac12}^{\frac12}f(t,\eta)\,d\eta=0$. Then there exists a unique $\tilde{u}\in\mathcal{P}\otimes\mathcal{S}(\overline{\mathbb{R}^+})$ such that :*

$$\begin{cases}-\partial_\eta^2\tilde{u} &= 2tf \text{ on } \Omega^0,\\ \partial_\eta\tilde{u}_{|_{\eta=-\frac12,\frac12}}=0 &\text{and } \displaystyle\int_{-\frac12}^{\frac12}\tilde{u}(t,\eta)\,d\eta=0.\end{cases}\tag{4.14}$$

Then, for all $v\in\mathcal{V}^N$, we have :

$$\frac12\int_{\Omega^0}\frac1t\partial_\eta\tilde{u}\;\partial_\eta\overline{v}\;d\eta\;dt=\int_{\Omega^0}f\;\overline{v}\;d\eta\;dt.\tag{4.15}$$

As above, we have an explicit solution of problem (4.14) :

Lemma 4.7. *Let $P(t,\eta)=\displaystyle\sum_{k=0}^{n}\left(p_k^e(t)(i\eta)^{2k}+p_k^o(t)(i\eta)^{2k+1}\right)$ be polynomial in η such that $p_k^e\in\mathcal{S}(\mathbb{R}^+)$, $p_k^o\in\mathcal{S}(\mathbb{R}^+)$ and :*

$$\forall t\in\mathbb{R}^+,\;\sum_{k=0}^{n}\frac{p_k^e(t)}{(2k+1)}\left(\frac{i}{2}\right)^{2k+1}=0.\tag{4.16}$$

Then the polynomial S defined by :

$$\begin{aligned}S(t,\eta) &= 2t\sum_{k=0}^{n}\frac{p_k^e(t)}{(2k+1)(2k+2)}\left((i\eta)^{2k+2}-\frac{i^{2k+2}}{(2k+3)2^{2k+2}}\right)\\ &\quad+2t\sum_{k=0}^{n}\frac{p_k^o(t)}{2k+2}\left(\frac{(i\eta)^{2k+3}}{2k+3}-\frac{i^{2k+3}\eta}{2^{2k+2}}\right),\end{aligned}\tag{4.17}$$

is the unique solution for the problem :

$$\begin{cases}-\partial_\eta^2 S &= 2tP \text{ on } \Omega^0,\\ \partial_\eta S_{|_{\eta=-\frac12,\frac12}}=0 &\text{and } \displaystyle\int_{-\frac12}^{\frac12}S(t,\eta)\,d\eta=0.\end{cases}\tag{4.18}$$

Remark 4.8. *The expression (4.17) shows that S has the same parity as P. Furthermore, if p_k^e and p_k^o are real, then S can be written as a polynomial in the variable $(i\eta)$ with real coefficents in $\mathcal{S}(\mathbb{R}^+)$.*

Proof : We first notice that the condition (4.16) means exactly $\int_{-\frac{1}{2}}^{\frac{1}{2}} P(t,\eta) d\eta = 0$.
According to Lemma 4.6, we know that the problem (4.18) has a unique solution denoted by S. It is enough to prove that the explicit polynomial S given by (4.17) fills conditions (4.18).
We remark that $\int_{-\frac{1}{2}}^{\frac{1}{2}} S(t,\eta)\, d\eta = 0$ for every $t \in \mathbb{R}^+$. It is easy to estimate $\partial_\eta S$, to verify the Neumann condition and to compute the second derivative of S. □

5 Construction of a formal solution

5.1 First terms of the construction

We look for two sequences $(m_k)_{k\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ and $(u_k)_{k\in\mathbb{N}} \in (\mathcal{P} \otimes \mathcal{S}(\overline{\mathbb{R}^+}))^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, if we introduce $U^{(n)} = \sum_{k=0}^{n} \alpha^{2k} u_k$ and $\lambda^{(n)}(\alpha) = \sum_{k=0}^{n} \alpha^{2k} m_k$, then, modulo $\mathcal{O}_n(\alpha^{2n+2})$, we have :

$$a_\alpha(U^{(n)}, v) \equiv \lambda^{(n)}(\alpha) \left\langle U^{(n)}, v \right\rangle_{L^2(\Omega^0)}, \ \forall v \in \mathcal{V}^N.$$

We look for a formal solution $U^{(\infty)}$ and $\lambda^{(\infty)}$. We expand the equation in powers of α and express that the coefficients of α^{2k} $(k \geq -1)$ should cancel.
The cancellation of the coefficient of $\frac{1}{\alpha^2}$ gives :

$$\forall v \in \mathcal{V}^N, \ \int_{\Omega^0} \frac{1}{t} \partial_\eta u_0 \ \partial_\eta \overline{v} \ dt \ d\eta = 0. \tag{5.1}$$

The unique condition coming from this relation is that u_0 depends only on t.
The vanishing of the coefficient of α^{2k}, for $k \geq 0$, gives :

$$\ell(u_k, v) + \frac{1}{2} \int_{\Omega^0} \frac{1}{t} \partial_\eta u_{k+1} \partial_\eta \overline{v} \ dt \ d\eta = \int_{\Omega^0} \sum_{j=0}^{k} m_j u_{k-j} \ \overline{v} \ dt \ d\eta, \ \forall v \in \mathcal{V}^N. \quad (5.2)_k$$

Let us show that this determines u_k and m_k recursively.

5.2 Algorithm for the determination of the coefficients

This method can be applied at every order to determine the other coefficients and we will indeed prove the following proposition :

Proposition 5.1. *We determine recursively the coefficients $u_j \in \mathcal{P} \otimes \mathcal{S}(\overline{\mathbb{R}^+})$ and $m_j \in \mathbb{R}$ such that, formally :*

$$P_\alpha \left(\sum_{j=0}^{\infty} \alpha^{2j} u_j \right) \equiv \left(\sum_{j=0}^{\infty} \alpha^{2j} m_j \right) \left(\sum_{j=0}^{\infty} \alpha^{2j} u_j \right). \tag{5.3}$$

For all $j \geq 0$, we can choose m_j and $u_j = u_j^0 + \tilde{u}_j$, with $u_j^0 \in \mathcal{S}(\overline{\mathbb{R}^+})$ and $\tilde{u}_j \in \mathcal{P} \otimes \mathcal{S}(\overline{\mathbb{R}^+})$ uniquely determined by the relations :

$$m_0 = \lambda_1^{\text{mean}}, \quad (5.4)_0$$

$$m_j = \langle \tilde u_j, L u_1^{\rm mean}\rangle_{L^2(\Omega^0)}. \qquad (5.4)_j$$

$$u_0^0 = u_1^{\rm mean}, \qquad (5.5)_0$$

$$\left\{\begin{array}{rcl} (L^{\rm mean}-\lambda_1^{\rm mean})u_j^0 &=& -\displaystyle\int_{-\frac12}^{\frac12} L\tilde u_j \; d\eta + \sum_{i=1}^{j} m_i u_{j-i}^0, \\ \displaystyle\int_0^\infty u_j^0(t)\; \overline{u_1^{\rm mean}(t)}\; dt &=& 0. \end{array}\right. \qquad (5.5)_j$$

$$\tilde u_0 = 0, \qquad (5.6)_0$$

$$\left\{\begin{array}{rcl} -\partial_\eta^2 \tilde u_j &=& 2t\left(\displaystyle\sum_{i=0}^{j-1} m_{j-1-i}u_i - Lu_{j-1}\right), \\ \partial_\eta \tilde u_{j|_{\eta=-\frac12,\frac12}} = 0 & \mathit{and} & \displaystyle\int_{-\frac12}^{\frac12} \tilde u_j \; d\eta = 0. \end{array}\right. \qquad (5.6)_j$$

Proof : To prove Proposition 5.1, we expand in power of α^2 Relation (5.3), cancel the coefficients of α^{2k} for $k \geq -1$. We have already studied the case $k=-1$ and deduce $\tilde u_0 = 0$. We now look at Relations $(5.2)_k$ for $k \geq 0$. We denote by $\mathcal{P}(k)$, for $k \in \mathbb{N}$, the property :

$\mathcal{P}(k)$: "the cancellation of the coefficient α^{2k}, given by Relation $(5.2)_k$, determines the real m_k and the functions u_k^0, $\tilde u_{k+1}$ which are given by solving $(5.4)_k$, $(5.5)_k$ and $(5.6)_{k+1}$".

We now prove that $\mathcal{P}(k)$ holds for every $k \in \mathbb{N}$.

We look at the relation coming from the vanishing of the constant coefficient given by Relation $(5.2)_0$. We restrict Relation $(5.2)_0$ to functions only depending on t and obtain a new relation :

$$\forall v \in \mathcal{V}^N_{\rm mean},\ \ \ell^{\rm mean}(u_0, v) = m_0 \langle u_0, v\rangle_{L^2(\Omega^0)}. \qquad (5.7)$$

So we determine u_0 and m_0 by solving the spectral problem $L^{\rm mean}u_0 = m_0 u_0$ and we choose the fundamental state $(\lambda_1^{\rm mean}, u_1^{\rm mean})$.

We return to the initial Relation $(5.2)_0$ where the only unknown is u_1 :

$$\forall v \in \mathcal{V}^N,\ \frac12 \int_{\Omega^0} \partial_\eta u_1 \partial_\eta \overline{v}\ \frac1t \ dt\ d\eta = \langle \lambda_1^{\rm mean} u_1^{\rm mean}, v\rangle_{L^2(\Omega^0)} - \ell(u_1^{\rm mean}, v). \qquad (5.8)$$

Since $u_1^{\rm mean} \in \mathcal{S}(\overline{\mathbb{R}^+})$, the integration by parts $\ell(u_1^{\rm mean}, v) = \langle L u_1^{\rm mean}, v\rangle_{L^2(\Omega^0)}$ holds for any $v \in \mathcal{V}_\ell^N$ and so we define the function $v_0 :\ = ((\lambda_1^{\rm mean} - L)u_1^{\rm mean})$. We can choose a function u_1 such that $u_1 = u_1^0 + \tilde u_1$ with $u_1^0 \in \mathcal{S}(\overline{\mathbb{R}^+})$ free and $\tilde u_1$ only determined by Lemma 4.6 with $f = v_0$. So $\mathcal{P}(0)$ holds.

Let $k \in \mathbb{N}$. We assume that property $\mathcal{P}(j)$ holds for any $j \leq k$. So, by induction, we have determined the reals m_j, the functions u_j^0, $\tilde u_j$ satisfying Relations $(5.4)_j$, $(5.5)_j$ and $(5.6)_j$ for $j = 0, \ldots, k$ and the function $\tilde u_{k+1}$ solving $(5.6)_{k+1}$. We vanish the coefficient of α^{2k+2} given by $(5.2)_{k+1}$.

We choose $v = u_1^{\rm mean}$ in $(5.2)_{k+1}$. By assumptions on m_j, u_j^0, $\tilde u_j$ and particularly due to the expression of $\tilde u_{k+1}$ given by $(5.6)_{k+1}$, we determine m_{k+1} exactly by $(5.4)_{k+1}$:

$$m_{k+1} = \langle \tilde u_{k+1}, L u_1^{\rm mean}\rangle_{L^2(\Omega^0)}. \qquad (5.9)$$

We now restrict Relation $(5.2)_{k+1}$ to functions $v \in \mathcal{V}^N_{\text{mean}}$. Since $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{u}_j \, d\eta = 0$ for any $j = 1, \ldots, k+1$, we have to find u^0_{k+1} independent of η such that :

$$\forall v \in \mathcal{V}^N_{\text{mean}},\ \ell^{\text{mean}}(u^0_{k+1}, v) = -\ell(\tilde{u}_{k+1}, v) + \int_0^\infty \left(\sum_{j=0}^{k+1} m_j u^0_{k+1-j} \right) \overline{v} \, dt. \quad (5.10)$$

We define the function $f\colon = -\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} L\tilde{u}_{k+1} \, d\eta + \sum_{j=0}^{k} m_{k+1-j} u^0_j$. By regularity of $\tilde{u}_{k+1}$ and assumptions on u^0_j for $j = 0, \ldots, k$, it is easy to prove that $f \in \mathcal{S}(\overline{\mathbb{R}^+})$ and f is orthogonal to u^{mean}_1. We apply Lemma 4.4 and so obtain a unique u^0_{k+1} satisfying the spectral problem (4.8) and so Relation (5.10). We see that u^0_{k+1} solves $(5.5)_{k+1}$.
We now come back to Relation $(5.2)_{k+1}$ to determine $\tilde{u}_{k+2}$ such that :

$$\forall v \in \mathcal{V}^N,\ \int_{\Omega^0} \frac{1}{t} \partial_\eta u_{k+2} \partial_\eta \overline{v} \, dt \, d\eta = 2 \left\langle \sum_{j=0}^{k+1} m_j u_{k+1-j} - Lu_{k+1}, v \right\rangle_{L^2(\Omega^0)}.$$

We apply Lemma 4.6 with $f = \displaystyle\sum_{j=0}^{k+1} m_j u_{k+1-j} - Lu_{k+1}$. Then $f \in \mathcal{P} \otimes \mathcal{S}(\overline{\mathbb{R}^+})$ and it is easy to show that $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} f(t,\eta) \, d\eta = 0$, $\forall t \in \mathbb{R}^+$ (using $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{u}_j d\eta = 0$ for $j = 0, \ldots, k+1$ and the expression of $(L^{\text{mean}} - \lambda^{\text{mean}}_1) u^0_{k+1}$). Then we choose $\tilde{u}_{k+2}$ uniquely determined by Relation $(5.6)_{k+2}$.
So we have established that $\mathcal{P}(k+1)$ holds.
Thus $\mathcal{P}(k)$ holds for every integer $k \geq 0$ and Proposition 5.1 is established. □

5.3 Particular form of the coefficients in the asymptotics

We can determine an explicit expression for every coefficient given by Proposition 5.1 and $(5.4)_j$, $(5.5)_j$, $(5.6)_j$. Just before, we make an easy remark about u^{mean}_1.

Remark 5.2. *The application :*

$$\Phi\colon P \mapsto (u^{\text{mean}}_1)^{-1} L(P u^{\text{mean}}_1)$$

is well defined from $\mathcal{P}\left(\mathbb{R}^+ \times \left]-\frac{1}{2}, \frac{1}{2}\right[\right)$ *onto* $\mathcal{P}\left(\mathbb{R}^+ \times \left]-\frac{1}{2}, \frac{1}{2}\right[\right)$.
If $P(t,\eta) = \displaystyle\sum_{k=0}^{n} p_k(t)(i\eta)^k$ *with* $p_k \in \mathcal{P}(\mathbb{R}^+)$, *then* $\Phi(P)(t,\eta) = \displaystyle\sum_{k=0}^{n} \tilde{p}_k(t)(i\eta)^k$ *with* $\tilde{p}_k \in \mathcal{P}(\mathbb{R}^+)$ *such that for* $k = 0, \ldots, n$ *:*

$$\tilde{p}_k = \left(\frac{1}{\sqrt{3}} - \frac{t}{6} \right) p_k + \frac{2t}{\sqrt{3}} p'_k - 2tp''_k + 2\left(1 - \frac{t}{\sqrt{3}} p_{k-1} \right) + 4tp'_{k-1} - 2tp_{k-2},$$

with the convention $p_{-1} = 0$ *and* $p_{-2} = 0$.
So we deduce immediately that if p_k *are real for every* $k \leq n$, *then* $\tilde{p}_k$ *are real.*

By induction, Remark 5.2, Lemmas 4.5 and 4.7 easily lead to the proposition :

Proposition 5.3. *For $k \in \mathbb{N}$, we consider u_k^0 and $\tilde{u}_k$ determined by Relations $(5.5)_k$ and $(5.6)_k$. Then there exist two polynomials $P_k^0 \in \mathcal{P}\left(\left]-\frac{1}{2},\frac{1}{2}\right[\right)$ and $\tilde{P}_k \in \mathcal{P}\left(\mathbb{R}^+ \times \left]-\frac{1}{2},\frac{1}{2}\right[\right)$ such that :*

$$u_k^0 = P_k^0 u_1^{\text{mean}} \text{ and } \tilde{u}_k = \tilde{P}_k u_1^{\text{mean}}.$$

Furthermore $\tilde{P}_k$ has the form $\tilde{P}_k(t,\eta) = \sum_{j\geq 0} \tilde{p}_j(t)(i\eta)^j$ with $\tilde{p}_j \in \mathcal{P}(\mathbb{R}^+)$ real.

Proposition 5.1 gives an expression of all the coefficients m_j, u_j^0 and $\tilde{u}_j$. According to Proposition 5.3, we know that there exist polynomials P_j^0 and $\tilde{P}_j$ such that $u_j^0 = P_j^0 u_1^{\text{mean}}$ and $\tilde{u}_j = \tilde{P}_j u_1^{\text{mean}}$. So we deduce an algorithm to determine m_j, P_j^0 and $\tilde{P}_j$ recursively with formal computations.
At the first order, we have :

$$m_0 = \tfrac{1}{\sqrt{3}}, \quad P_0^0(t) = 1, \quad \tilde{P}_0(t,\eta) = 0.$$

For the second term :

$$\begin{aligned} m_1 &= -\frac{23}{35\sqrt{3}}, \\ P_1^0(t) &= -\frac{19}{42} + \frac{23}{70\sqrt{3}}t + \frac{11}{210}t^2 - \frac{2}{189\sqrt{3}}t^3, \\ \tilde{P}_1(t,\eta) &= \frac{7}{720}t^2 + i\eta\frac{t}{2}\left(-1+\frac{t}{\sqrt{3}}\right) - \eta^2\frac{t^2}{6} + i\eta^3\left(\frac{2}{3}t - \frac{2}{3\sqrt{3}}t^2\right) + \eta^4\frac{t^2}{3}. \end{aligned}$$

The same algorithm works for any finite expansion.

6 Upper bound for the asymptotics of $\mu(\alpha)$

Proposition 6.1. *Let n be a positive integer and m_k be the reals determined by Proposition 5.1, for $k \leq n$. We define :*

$$\lambda^{(n)} = \sum_{k=0}^{n} \alpha^{2k} m_k. \tag{6.1}$$

Then there exist $\alpha_0 > 0$ and a positive constant c such that :

$$\forall \alpha \in]0, \alpha_0[, \; \mu(\alpha) \leq \alpha\lambda^{(n)} + c\alpha^{2n+3}.$$

Furthermore, there exists an eigenvalue $\tilde{\mu}(\alpha)$ of $P_{\mathcal{A}_0,\Omega_\alpha}$ such that :

$$\tilde{\mu}(\alpha) = \alpha\lambda^{(n)} + \mathcal{O}(\alpha^{2n+1}) \text{ when } \alpha \to 0.$$

Proof : We consider the functions u_k and the reals m_k determined by Proposition 5.1 and define for every integer $n \geq 1$:

$$U^{(n)} = \sum_{k=0}^{n} \alpha^{2k} u_k, \tag{6.2}$$

We want to estimate the Rayleigh quotient for $U^{(n)}$. Let $v \in \mathcal{V}^N$, then, by using Proposition 5.1, the vanishing of the coefficient of α^{2k} for $k \leq n$, given in $(5.2)_k$, and the definition of $u_0 = u_1^{\text{mean}}$, we have :

$$a_\alpha(U^{(n)}, v) = \sum_{k=0}^{n-1} \alpha^{2k} \int_{\Omega^0} \left(\sum_{j=0}^{k} m_j u_{k-j} \right) \overline{v} \; dt \; d\eta + \alpha^{2n} \ell(u_n, v). \tag{6.3}$$

In (6.3), we rewrite the sum $\sum_{j=0}^{k} m_j u_{k-j}$:

$$\sum_{k=0}^{n} \alpha^{2k} \sum_{j=0}^{k} m_j u_{k-j} = \lambda^{(n)} U^{(n)} - \alpha^{2n+2} \sum_{k=0}^{n-1} \alpha^{2k} \sum_{j=k+1}^{n} m_{k+n+1-j} u_j. \tag{6.4}$$

Thus, there exists a constant C such that for $\alpha < 1$ and for every $v \in \mathcal{V}^N$:

$$\begin{aligned} \left| a_\alpha(U^{(n)}, v) - \lambda^{(n)} \left\langle U^{(n)}, v \right\rangle_{L^2(\Omega^0)} \right| \quad \leq \quad & \alpha^{2n} \left| \ell(u_n, v) - \sum_{j=0}^{n} m_j \left\langle u_{n-j}, v \right\rangle_{L^2(\Omega^0)} \right| \\ & + C \alpha^{2n+2} ||v||_{L^2(\Omega^0)}. \end{aligned} \tag{6.5}$$

We bound from below the norm $||U^{(n)}||_{L^2(\Omega^0)}$ as follows :

$$||U^{(n)}||_{L^2(\Omega^0)} \geq 1 - \sup_{k=0,\ldots,n} ||u_k||_{L^2(\Omega^0)} \frac{\alpha^2}{1-\alpha^2}.$$

So there exist $\alpha_0 < 1$ and a constant $C_0 > 0$ such that for every $\alpha \in]0, \alpha_0[$:

$$||U^{(n)}||_{L^2(\Omega^0)} \geq C_0 > 0. \tag{6.6}$$

Due to the Cauchy-Schwarz inequality and the lower bound (6.6), a classical spectral theorem shows that :

$$d(\lambda^{(n)}, \sigma(P_\alpha)) \leq \mathcal{O}(\alpha^{2n}).$$

By change of variables, there exists an eigenvalue $\tilde{\mu}(\alpha)$ of $P_{\mathcal{A}_0, \Omega_\alpha}$ with :

$$|\alpha \lambda^{(n)} - \tilde{\mu}(\alpha)| \leq \mathcal{O}(\alpha^{2n+1}).$$

For using the min-max principle, we choose $v = U^{(n)}$ in (6.5) and estimate $\ell(u_n, U^{(n)})$ using $(5.4)_n$, the orthogonality between u_n^0 and u_1^{mean} and the relation $\ell(\tilde{u}_n, u_1^{\text{mean}}) = \langle \tilde{u}_n, L u_1^{\text{mean}} \rangle_{L^2(\Omega^0)}$, then :

$$\ell(u_n, U^{(n)}) = \ell(u_n, u_1^{\text{mean}}) + \sum_{k=1}^{n} \alpha^{2k} \ell(u_n, u_k) = m_n + \sum_{k=1}^{n} \alpha^{2k} \ell(u_n, u_k),$$

Now, since u_1^{mean} is normalized, we deduce :

$$\begin{aligned} & \ell(u_n, U^{(n)}) - \sum_{j=0}^{n} m_j \left\langle u_{n-j}, U^{(n)} \right\rangle_{L^2(\Omega^0)} \\ = \quad & \sum_{k=1}^{n} \alpha^{2k} \left(\ell(u_n, u_k) - \sum_{j=0}^{n} m_j \left\langle u_{n-j}, u_k \right\rangle_{L^2(\Omega^0)} \right). \end{aligned}$$

According to the expression of u_k given in Proposition 5.3 and Remarks 4.12 and 5.2, there exists a constant c_n only dependent on n such that :

$$\left| \ell(u_n, U^{(n)}) - \sum_{j=0}^{n} m_j \left\langle u_{n-j}, U^{(n)} \right\rangle_{L^2(\Omega^0)} \right| \leq c_n \alpha^2. \tag{6.7}$$

Using the upper bound (6.7), Inequality (6.5) and estimate of the norm (6.6), we get a constant $c > 0$ such that :

$$\frac{a_\alpha(U^{(n)}, U^{(n)})}{||U^{(n)}||^2_{L^2(\Omega^0)}} \leq \lambda^{(n)} + c\alpha^{2n+2}, \forall \alpha \in]0, \alpha_0[. \tag{6.8}$$

The min-max principle achieves the proof of Proposition 6.1. □

7 Some estimates of eigenfunctions

7.1 Agmon's techniques

We give a priori estimates on the decay of the eigenfunctions. By using Agmon's paper [1], we propose some estimates for the localization of the eigenfunctions. Let us recall principles of the Agmon's estimates.
Let α_1 small enough such that $\mu_k(\alpha) < \Theta_0$ for every $\alpha \in]0, \alpha_1]$ where $\mu_k(\alpha)$ is the k-th element of the spectrum of $P_{\mathcal{A}_0,\Omega_\alpha}$. We denote by $u_{k,\alpha}$ a normalized eigenfunction associated to $\mu_k(\alpha)$.
Let ϕ be uniformaly lipschitzian on Ω_α, then, by assumptions on $u_{k,\alpha}$:

$$\int_{\Omega_\alpha} \left(\mu_k(\alpha) + |\nabla\phi|^2\right) \ e^{2\phi}|u_{k,\alpha}|^2 \ dx = \int_{\Omega_\alpha} |(\nabla - i\mathcal{A}_0)(e^{\phi}u_{k,\alpha})|^2 \ dx. \tag{7.1}$$

Let χ_1 and χ_2 be real, positive, regular functions on $\overline{\Omega_\alpha}$, with support respectively included in $\overline{\Omega_\alpha} \cap \mathcal{B}_2$ and $\overline{\Omega_\alpha} \setminus \mathcal{B}_1$ (where $\mathcal{B}_R$ denotes $\{x \in \overline{\Omega_\alpha} | \ |x| < R\}$) such that $|\chi_1|^2 + |\chi_2|^2 = 1$ on $\overline{\Omega_\alpha}$. We define $\chi_j^R : \ = \chi_j\left(\frac{\cdot}{R}\right)$ on Ω_α, then :

$$q_{\mathcal{A}_0,\Omega_\alpha}(e^{\phi}u_{k,\alpha}) = \sum_{j=1}^{2} q_{\mathcal{A}_0,\Omega_\alpha}(\chi_j^R e^{\phi}u_{k,\alpha}) - \frac{1}{R^2}\sum_{j=1}^{2} ||e^{\phi}u_{k,\alpha} \ |\nabla\chi_j^R| \ ||^2_{L^2(\Omega_\alpha)}. \tag{7.2}$$

We use the assumptions on χ_j^R and report (7.2) on (7.1), then the positivity of $q_{\mathcal{A}_0,\Omega_\alpha}(\chi_1^R e^{\phi}u_{k,\alpha})$ leads to the upper bound :

$$q_{\mathcal{A}_0,\Omega_\alpha}(\chi_2^R e^{\phi}u_{k,\alpha}) \quad \leq \quad \int_{\Omega_\alpha} \left(\mu_k(\alpha) + |\nabla\phi|^2 + \frac{C}{R^2}\right) e^{2\phi}|u_{k,\alpha}|^2 dx. \tag{7.3}$$

Let us assume that there exists $\mu(\alpha, R) > 0$ such that :

$$q_{\mathcal{A}_0,\Omega_\alpha}(\chi_2^R e^{\phi}u_{k,\alpha}) \geq \mu(\alpha, R)||\chi_2^R e^{\phi}u_{k,\alpha}||^2_{L^2(\Omega_\alpha)}. \tag{7.4}$$

By assumption on the support of χ_2^R, we can bound from below $||\chi_2^R e^{\phi}u_{k,\alpha}||^2_{L^2(\Omega_\alpha)}$ by $||e^{\phi}u_{k,\alpha}||^2_{L^2(\Omega_\alpha \setminus \mathcal{B}_{2R})}$, and so deduce from (7.4) and (7.3) that :

$$\begin{aligned} &\int_{\Omega_\alpha \setminus \mathcal{B}_{2R}} \left(\mu(\alpha, R) - \mu_k(\alpha) - |\nabla\phi|^2 - \frac{C}{R^2}\right) e^{2\phi}|u_{k,\alpha}|^2 \ dx \\ \leq \ &\int_{\Omega_\alpha \cap \mathcal{B}_{2R}} \left(\mu_k(\alpha) + |\nabla\phi|^2 + \frac{C}{R^2}\right) e^{2\phi}|u_{k,\alpha}|^2 \ dx. \end{aligned} \tag{7.5}$$

If we are able to bound respectively $\mu(\alpha,R)-\mu_k(\alpha)-|\nabla\phi|^2-\frac{C}{R^2}$ from below on $\Omega_\alpha\setminus\mathcal{B}_{2R}$ and $\mu_k(\alpha)+|\nabla\phi|^2+\frac{C}{R^2}$ from above on $\Omega_\alpha\cap\mathcal{B}_{2R}$ by respective positive constants C_1 and C_2 for α small enough and R large enough, then we deduce from (7.5) :

$$\int_{\Omega_\alpha} e^{2\phi}|u_{k,\alpha}|^2\,dx \le \left(\frac{C_2}{C_1}+1\right)\exp\left(2\sup_{x\in\Omega_\alpha\cap\mathcal{B}_{2R}}\phi(x)\right). \tag{7.6}$$

7.2 Decay of eigenfunction

Theorem 7.1. *Let α_1 small enough such that the k-th smallest element of the spectrum of $P_{\mathcal{A}_0,\Omega_\alpha}$ verifies $\mu_k(\alpha)<\Theta_0$ for every $\alpha\in]0,\alpha_1]$. Let $u_{k,\alpha}$ be a normalized eigenfunction associated to $\mu_k(\alpha)$. For $\alpha\le\alpha_1$ and for all $\epsilon\in\left]0,\Theta_0-\sup_{\alpha\in]0,\alpha_1]}\mu_k(\alpha)\right[$, there exists a constant $C_{\epsilon,\alpha}$ such that :*

$$\int_{\Omega_\alpha} e^{2\sqrt{\Theta_0-\mu_k(\alpha)-\epsilon}\;|x|}\;|u_{k,\alpha}(x)|^2\,dx \le C_{\epsilon,\alpha}. \tag{7.7}$$

Proof : By definition (2.3) of $\Sigma(P_{\mathcal{A}_0,\Omega_\alpha},r)$, the assumption (7.4) holds with $\mu(\alpha,r)=\Sigma(P_{\mathcal{A}_0,\Omega_\alpha},r)$. Furthermore, (2.10) gives a lower bound of $\Sigma(P_{\mathcal{A}_0,\Omega_\alpha},r)$. We define ϕ on Ω_α by $\phi(x):=\sqrt{\Theta_0-\mu_k(\alpha)-\epsilon}\;|x|$, then (7.5) combined with (2.10) leads to :

$$\left(\epsilon-\frac{c}{\alpha^2R^2}-\frac{C}{R^2}\right)\int_{\Omega_\alpha\setminus\mathcal{B}_{2R}} e^{2\phi}|u_{k,\alpha}|^2dx \le \left(\Theta_0-\epsilon+\frac{C}{R^2}\right)\int_{\Omega_\alpha\cap\mathcal{B}_{2R}} e^{2\phi}|u_{k,\alpha}|^2dx. \tag{7.8}$$

If we choose $R=\frac{C_1}{\alpha}$ big enough for that $\epsilon-\frac{c}{\alpha^2R^2}-\frac{C}{R^2}\ge\frac{\epsilon}{2}$ and $\Theta_0-\epsilon+\frac{C}{R^2}\le\Theta_0-\frac{\epsilon}{2}$, then (7.8) allows us to conclude that Relation (7.7) is true with $C_{\epsilon,\alpha}=c_\epsilon\exp\left(4C_1\frac{\sqrt{\Theta_0-\mu_k(\alpha)-\epsilon}}{\alpha}\right)$. □

Theorem 7.2. *Let $\epsilon_0>0$. We define by $\mu^{NDN}(\alpha,\epsilon_0)$ the bottom of the spectrum of $-\nabla^2_{\mathcal{A}_0}$ on $\Omega_\alpha\setminus\mathcal{B}_R$ with a Neumman boundary condition in $\partial\Omega_\alpha$ and a Dirichlet boundary condition in $|x|=R$, with $R=\frac{\epsilon_0}{\alpha}$. Then, there exist $\mu^{NDN}(\epsilon_0)>0$ and $\alpha_0>0$ such that :*

$$\forall\alpha\in]0,\alpha_0],\ \mu^{NDN}(\alpha,\epsilon_0)\ge\mu^{NDN}(\epsilon_0). \tag{7.9}$$

Let $u_{k,\alpha}$ be a normalized eigenvector associated to the k-th eigenvalue $\mu_k(\alpha)$ for $P_{\mathcal{A}_0,\Omega_\alpha}$. Then, there exists $\alpha_1\le\alpha_0$ such that for every $\alpha\in]0,\alpha_1]$:

$$\int_{\Omega_\alpha} e^{\sqrt{2\mu^{NDN}(\epsilon_0)}|x|}|u_{k,\alpha}(x)|^2\,dx \le 4e^{2\frac{\epsilon_0}{\alpha}\sqrt{2\mu^{NDN}(\epsilon_0)}}. \tag{7.10}$$

Proof : If (7.9) holds, we choose $\phi(x)=\sqrt{\frac{\mu^{NDN}(\epsilon_0)}{2}}|x|$ and look at (7.5) with $\mu(\alpha,R)=\mu^{NDN}(\epsilon_0)$, then the conclusion of Theorem 7.2 follows.
So, we have just to establish (7.9). As illustrated in Figure 1, we define :

$$\Omega^{NDN}_{\alpha,\epsilon_0} := \{x\in\mathbb{R}^2|\ (x_1+R,x_2)\in\Omega_\alpha\setminus\mathcal{B}_R\}\text{ with }R=\frac{\epsilon_0}{\alpha}, \tag{7.11}$$

$$\Omega(\epsilon_0) :=]0,+\infty[\times\left]-\frac{\epsilon_0}{4},\frac{\epsilon_0}{4}\right[. \tag{7.12}$$

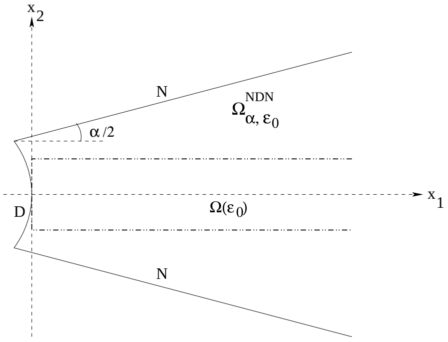

Figure 1: Domain $\Omega^{NDN}_{\alpha,\epsilon_0}$.

Then, by gauge invariance and invariance under translation of the domain, $\mu^{NDN}(\alpha,\epsilon_0)$ is the bottom of the spectrum of $P\colon = D_{x_1}^2 + (D_{x_2} - x_1)^2$ on $\Omega^{NDN}_{\alpha,\epsilon_0}$ with a Neumann condition on the two half-lines of the boundary and a Dirichlet condition on the curve. We denote by q the form associated to P. Before proving (7.9), we give a lemma :

Lemma 7.3. *The bottom of the spectrum of $P_{\tilde{\mathcal{A}},\Omega(\epsilon_0)}$ with $\tilde{\mathcal{A}} = (0, x_1)$, denoting by $\lambda(\Omega(\epsilon_0))$, is strictly positive.*

Proof : Using Persson Lemma 2.2 and the notation $\Omega_r(\epsilon_0)\colon =]r, +\infty[\times\,]-\frac{\epsilon_0}{4}, \frac{\epsilon_0}{4}[$, the bottom of the essential spectrum of $P_{\tilde{\mathcal{A}},\Omega(\epsilon_0)}$ can be expressed by :

$$\begin{aligned} \inf \sigma_{ess}(P_{\tilde{\mathcal{A}},\Omega(\epsilon_0)}) &= \sup_{r>0} \inf_{||u||_{L^2(\Omega(\epsilon_0))}=1} \left\{ \int_{\Omega(\epsilon_0)} |\nabla_{\tilde{\mathcal{A}}} u|^2 dx,\ u \in C_0^\infty(\Omega_r(\epsilon_0)) \right\} \\ &\geq \inf_{u \in H^1_{\tilde{\mathcal{A}}}(\Omega(\epsilon_0)), u\neq 0} \frac{\displaystyle\int_{\Omega(\epsilon_0)} |\nabla_{\tilde{\mathcal{A}}} u|^2 dx}{||u||^2_{L^2(\Omega(\epsilon_0))}}. \qquad (7.13) \end{aligned}$$

So we have :

$$\inf \sigma_{ess}(P_{\tilde{\mathcal{A}},\Omega(\epsilon_0)}) \geq \mu(\Omega(\epsilon_0)). \qquad (7.14)$$

Let us define $\psi(x) = x_1 x_2$ for $x \in \Omega(\epsilon_0)$, then, by gauge transform :

$$\mu(\Omega(\epsilon_0)) = \inf_{||u||_{L^2(\Omega(\epsilon_0))}=1} \left\{ \int_{\Omega(\epsilon_0)} |\nabla_{\tilde{\mathcal{A}}+\nabla\psi} u|^2 dx,\ u \in H^1_{\tilde{\mathcal{A}}+\nabla\psi}(\Omega(\epsilon_0)) \right\}. \qquad (7.15)$$

A Fourier transform according to the first coordinate x_1 leads to define the real $\mu(\xi_1)$ for any $\xi_1 \in \mathbb{R}$ by :

$$\mu(\xi_1)\colon = \inf_{u \in C_0^\infty([-\frac{\epsilon_0}{4}, \frac{\epsilon_0}{4}]), u\neq 0} \frac{\displaystyle\int_{-\frac{\epsilon_0}{4}}^{\frac{\epsilon_0}{4}} (\xi_1 - x_2)^2 |u(x_2)|^2 + |D_{x_2} u(x_2)|^2 \ dx_2}{\displaystyle\int_{-\frac{\epsilon_0}{4}}^{\frac{\epsilon_0}{4}} |u(x_2)|^2 \ dx_2}. \qquad (7.16)$$

Easily $\lim_{\xi_1 \to \pm\infty} \mu(\xi_1) = +\infty$. Furthermore, μ is positive and continuous on $\mathbb{R}$, so there exists a lower bound $\underline{\mu} > 0$ of $\mu(\xi_1)$ for every $\xi_1 \in \mathbb{R}$. Consequently, $\mu(\Omega(\epsilon_0)) \geq \underline{\mu} > 0$ and a fortiori, recalling (7.14), $\inf \sigma_{ess} P_{\tilde{\mathcal{A}},\Omega(\epsilon_0)} \geq \underline{\mu} > 0$.
Assuming that the smallest eigenvalue of $P_{\tilde{\mathcal{A}},\Omega(\epsilon_0)}$ is equal to 0, then there exists a normalized function $u \in \mathcal{D}^N(P_{\tilde{\mathcal{A}},\Omega(\epsilon_0)})$ such that :

$$D_{x_1} u = 0,\ (D_{x_2} - x_1) u = 0 \text{ in } \mathcal{D}'(\Omega(\epsilon_0)).$$

We deduce by these relations that $u = 0$ in $\mathcal{D}'(\Omega(\epsilon_0))$, which is not possible. So the smallest eigenvalue of $P_{\tilde{\mathcal{A}},\Omega(\epsilon_0)}$ is strictly positive, thus $\lambda(\Omega(\epsilon_0)) > 0$. □
Let us use Lemma 7.3 to prove (7.9).
We assume (7.9) does not hold. So there exists a sequence $v_{\alpha_n} \in H^1_{\tilde{\mathcal{A}}}(\Omega^{NDN}_{\alpha_n,\epsilon_0})$ such that $||v_{\alpha_n}||_{L^2(\Omega^{NDN}_{\alpha_n,\epsilon_0})} = 1$ and $\lim_{\alpha_n \to 0} q(v_{\alpha_n}) = 0$. The inclusion $\Omega(\epsilon_0) \subset \Omega^{NDN}_{\alpha_n,\epsilon_0}$, Lemma 7.3 and the min-max principle bound from below $q(v_{\alpha_n})$ by :

$$q(v_{\alpha_n}) \geq \int_{\Omega(\epsilon_0)} |D_{x_1} v_{\alpha_n}|^2 + |(D_{x_2} - x_1) v_{\alpha_n}|^2 dx \geq \lambda(\Omega(\epsilon_0)) ||v_{\alpha_n}||^2_{L^2(\Omega(\epsilon_0))}. \quad (7.17)$$

By assumptions, $\lim_{\alpha_n \to 0} q(v_{\alpha_n}) = 0$ so $\lim_{\alpha_n \to 0} ||v_{\alpha_n}||_{L^2(\Omega(\epsilon_0))} = 0$ since $\lambda(\Omega(\epsilon_0)) > 0$.
Let χ_1 be a regular cut-off function defined on $\mathbb{R}$ with $0 \leq \chi_1 \leq 1$ and :

$$\chi_1(x_2) = \begin{cases} 1 & \text{if} \quad x_2 \geq \frac{\epsilon_0}{4}, \\ 0 & \text{if} \quad x_2 \leq -\frac{\epsilon_0}{4}. \end{cases} \quad (7.18)$$

We also define $\chi_2 = \sqrt{1 - \chi_1^2}$ and easily deduce :

$$q(v_{\alpha_n}) = q(\chi_1 v_{\alpha_n}) + q(\chi_2 v_{\alpha_n}) - ||D_{x_2}\chi_1\ v_{\alpha_n}||^2_{L^2(\Omega^{NDN}_{\alpha_n,\epsilon_0})} - ||D_{x_2}\chi_2\ v_{\alpha_n}||^2_{L^2(\Omega^{NDN}_{\alpha_n,\epsilon_0})}. \quad (7.19)$$

Looking at the support of $(D_{x_2}\chi_j)_{j=1,2}$, we define $a :=\frac{\epsilon_0}{4} \tan \frac{\alpha_n}{2}$ and $\Omega_2(\epsilon_0) := \Omega^{NDN}_{\alpha_n,\epsilon_0} \cap]-a, 0] \times]-\frac{\epsilon_0}{4}, \frac{\epsilon_0}{4}[$ and so there exists a constant $c > 0$ such that :

$$||D_{x_2}\chi_j\ v_{\alpha_n}||^2_{L^2(\Omega^{NDN}_{\alpha_n,\epsilon_0})} \leq \frac{c}{\epsilon_0^2} ||v_{\alpha_n}||^2_{L^2(\Omega(\epsilon_0) \cup \Omega_2(\epsilon_0))},\ j = 1, 2. \quad (7.20)$$

We know that $\lim_{\alpha_n \to 0} ||v_{\alpha_n}||_{L^2(\Omega(\epsilon_0))} = 0$, so it is enough to estimate $||v_{\alpha_n}||^2_{L^2(\Omega_2(\epsilon_0))}$.
Since v_{α_n} vanishes on the curve of the boundary, we can use some Poincaré techniques to establish :

$$\int_{\Omega_2(\epsilon_0)} |v_{\alpha_n}(x_1, x_2)|^2\ dx_1\ dx_2 \quad \leq \quad a^2 q(v_{\alpha_n}). \quad (7.21)$$

Relations (7.17) and (7.21) prove the existence of $\tilde{\alpha}$ and $C(\epsilon_0) > 0$ such that :

$$||v_{\alpha_n}||^2_{L^2(\Omega(\epsilon_0) \cup \Omega_2(\epsilon_0))} \leq C(\epsilon_0) q(v_{\alpha_n}), \forall \alpha_n \leq \tilde{\alpha}. \quad (7.22)$$

Let $\epsilon > 0$, (7.19), (7.20) and (7.22) show that there exists $\tilde{\alpha}_0 \leq \tilde{\alpha}$ such that :

$$\forall \alpha_n \leq \tilde{\alpha}_0,\ q(v_{\alpha_n}) \geq q(\chi_1 v_{\alpha_n}) + q(\chi_2 v_{\alpha_n}) - \epsilon. \quad (7.23)$$

By comparison with the operator $P_{\tilde{\mathcal{A}},\mathbb{R}\times\mathbb{R}^+}$, using gauge transform and invariance under domain's translation, we deduce :

$$q(\chi_1 v_{\alpha_n}) \geq \Theta_0 ||\chi_1 v_{\alpha_n}||^2_{L^2(\Omega^{NDN}_{\alpha_n,\epsilon_0})} \text{ and } q(\chi_2 v_{\alpha_n}) \geq \Theta_0 ||\chi_2 v_{\alpha_n}||^2_{L^2(\Omega^{NDN}_{\alpha_n,\epsilon_0})}.$$

Thus :

$$q(v_{\alpha_n}) \geq \Theta_0 ||v_{\alpha_n}||^2_{L^2(\Omega^{NDN}_{\alpha_n,\epsilon_0})} - \epsilon. \tag{7.24}$$

This is impossible since v_{α_n} is normalized. □

Corollary 7.4. *We denote by $u_{k,\alpha}$ the k-th normalized eigenfunction for P_α. Then, for all $\epsilon > 0$, there exist $\delta > 0$ and $\alpha_0 > 0$ such that :*

$$\forall \alpha \leq \alpha_0,\ \forall T \geq \frac{\epsilon}{\alpha}, \int_{]T,+\infty[\times]-\frac{1}{2},\frac{1}{2}[} |u_{k,\alpha}|^2\ dt\ d\eta \leq e^{-\frac{\delta}{\alpha}}. \tag{7.25}$$

8 First term of the asymptotic expansion

8.1 Weak lower bound

Theorem 8.1. *For every $\epsilon > 0$, there exists $\alpha_0 > 0$ such that :*

$$\mu(\alpha) \geq \frac{\alpha}{\sqrt{3}} - \epsilon\alpha,\ \forall \alpha \leq \alpha_0.$$

The proof of Theorem 8.1 consists of comparing the eigenvalue of the operator P_α with the first eigenvalue of L^{mean} by using min-max principle and estimating the decay of the first eigenvector of P_α.
Looking at the operator P_α, we see that the term $\frac{1}{t}$ is difficult to analyze when t becomes very large. So, we avoid this problem by dealing with new operators defined on a bounded domain :

$$\Omega_T :\ =]0,T[\times \left]-\frac{1}{2},\frac{1}{2}\right[,$$

we estimate the bottom of the spectrum for these new operators and compare with the old ones. We begin with some lemmas preparing the proof of the lower bound in Theorem 8.1. As we will see in the following, remainders coming from the cut-off are very small and we will estimate the error.

8.2 The mean operator $L^{\text{mean},T}$

We show that the bottom of the spectrum of L^{mean} is very close to the bottom obtained from the realization of L^{mean} on a bounded domain. We define :

$$\begin{aligned} \mathcal{V}^N_{\text{mean},T} &:= \{u \in L^2(0,T)|\ \sqrt{t}D_t u \in L^2(0,T),\ u_{|t=T} = 0\}, \\ \ell^{\text{mean},T}(u,v) &:= \ell^{\text{mean}}(u,v),\ \forall u,v \in \mathcal{V}^N_{\text{mean,T}}, \\ \lambda_1^{\text{mean},T} &:= \inf_{u \in \mathcal{V}^N_{\text{mean,T}}, u \neq 0} \frac{\ell^{\text{mean},T}(u,u)}{||u||^2_{L^2(0,T)}}. \end{aligned} \tag{8.1}$$

Lemma 8.2. *There exists a positive constant $\tilde{c}$ such that :*

$$\lambda_1^{\text{mean}} \leq \lambda_1^{\text{mean},T} \leq \lambda_1^{\text{mean}} + \tilde{c}e^{-\frac{T}{2\sqrt{3}}},\ \forall T \geq 1. \tag{8.2}$$

Proof : The proof is very easy. The inclusion $\mathcal{V}^N_{\text{mean,T}} \subset \mathcal{V}^N_{\text{mean}}$ gives immediately $\lambda_1^{\text{mean}} \leq \lambda_1^{\text{mean},T}$. For the second inequality, we use the eigenvector u_1^{mean} of L^{mean} which is exponentially decreasing and a cut-off function χ with support in Ω_T. We estimate the Rayleigh quotient $\frac{\ell^{\text{mean},T}(\chi u_1^{\text{mean}}, \chi u_1^{\text{mean}})}{||\chi u_1^{\text{mean}}||^2_{L^2(\Omega_T)}}$ to achieve the proof. □

8.3 The operator P_α^T on a truncated sector

In order to compare $\lambda(\alpha)$ and $\lambda_1^{\text{mean},T}$, we introduce the new operator P_α^T associated to the form a_α, restricted to Ω_T, with a Dirichlet boundary condition at $t = T$ and a Neumann boundary condition at $\eta = \pm\frac{1}{2}$ defined on :

$$\mathcal{V}_{T,0}^N : = \left\{ u \in L^2(\Omega_T) \middle| \frac{1}{\sqrt{t}}\partial_\eta u \in L^2(\Omega_T),\ \sqrt{t} D_t u \in L^2(\Omega_T),\ u_{|t=T} = 0 \right\}.$$

The domain of P_α^T is given by the construction of the Friedrichs extension. We do not need to characterize it explicitly because we work with the quadratic form q_α^T. We denote by $\lambda^T(\alpha)$ the bottom of the spectrum of P_α^T :

$$\lambda^T(\alpha) = \inf_{u \in \mathcal{V}_{T,0}^N, u \neq 0} \frac{q_\alpha^T(u)}{||u||^2_{L^2(\Omega_T)}} \text{ with } q_\alpha^T(u) = \int_{\Omega_T} 2t|(D_t - \eta)u|^2 + \frac{1}{2\alpha^2 t}|D_\eta u|^2 \, dt\, d\eta. \tag{8.3}$$

Lemma 8.3. *For every $\epsilon > 0$, there exist $\delta > 0$ and $\alpha_0 > 0$ such that :*

$$\forall \alpha \leq \alpha_0,\ \forall T \geq \frac{\epsilon}{\alpha},\ \lambda(\alpha) \leq \lambda^T(\alpha) \leq \lambda(\alpha) + e^{-\frac{\delta}{\alpha}}. \tag{8.4}$$

Proof : By inclusion of the form domain, we deduce the lower bound :

$$\lambda^T(\alpha) \geq \lambda(\alpha). \tag{8.5}$$

To establish an upper bound, we multiply the functions in the domain $\mathcal{V}^N$ by a cut-off function, so they belong to the form domain $\mathcal{V}_{T,0}^N$ and we estimate the error coming from a cutting. We consider a regular real cut-off function χ, defined on $\mathbb{R}^+$, with support in $[0,1]$ and equal to 1 on $[0, \frac{1}{2}]$. For $T > 0$, we define $\chi_T : = \chi\left(\frac{\cdot}{T}\right)$ on $\mathbb{R}^+$ and u_α a normalized eigenvector for $\lambda(\alpha)$. We remark that $\chi_T u_\alpha \in \mathcal{V}_{T,0}^N$ and easily prove that :

$$q_\alpha^T(\chi_T u_\alpha) \quad = \quad \text{Re}\left\langle \chi_T^2\ P_\alpha u_\alpha, u_\alpha \right\rangle_{L^2(\Omega_T)} + ||\ |\nabla \chi_T| u_\alpha||^2_{L^2(\Omega_T)}. \tag{8.6}$$

By assumption on u_α and construction of χ_T, there exists a positive constant c independent of α and T such that :

$$q_\alpha^T(\chi_T u_\alpha) \quad \leq \quad \lambda(\alpha)||\chi_T u_\alpha||^2_{L^2(\Omega_T)} + \frac{c}{T}||u_\alpha||^2_{L^2(]\frac{T}{2},T[\times]-\frac{1}{2},\frac{1}{2}[)}. \tag{8.7}$$

According to Corollary 7.4, there exist $\delta > 0$ and $\alpha_0 > 0$ such that :

$$1 - e^{-\frac{\delta}{\alpha}} \leq ||\chi_T u_\alpha||^2_{L^2(\Omega_T)} \leq 1. \tag{8.8}$$

The min-max principle concludes to (8.4) (with a smaller δ). □

8.4 The regularized operator $P^{T,\rho}$

To avoid the singularity on P_α with the term $\frac{1}{t}$ which tends to 0 when t tends to infinity, we deal with the operator P_α^T and we can bound from below $\frac{1}{t}$ by $\frac{1}{T}$ which is a constant. So it is quite natural to introduce the self-adjoint extension $P^{T,\rho}$ to the form $a^{T,\rho}$ defined on $\mathcal{V}_T^{N,D}$ as follows :

$$\mathcal{V}_T^{N,D} : = \left\{ u \in L^2(\Omega_T) |\ (D_t - \eta)u \in L^2(\Omega_T),\ D_\eta u \in L^2(\Omega_T),\ u_{|t=T} = 0 \right\},$$

$$a^{T,\rho}(u,v) : = \int_{\Omega_T} \left(2t(D_t - \eta)u\overline{(D_t - \eta)v} + \frac{\rho}{2} D_\eta u \overline{D_\eta v} \right) \, dt\, d\eta. \tag{8.9}$$

We denote by $\nu(T,\rho)$ the bottom of the spectrum of $P^{T,\rho}$ defined on $\mathcal{D}^{N,D}(P^{T,\rho})$:

$$\begin{aligned}\mathcal{D}^{N,D}(P^{T,\rho}) \quad &:= \quad \Big\{u\in\mathcal{V}_T^{N,D} \,|\, (D_t-\eta)t(D_t-\eta)u\in L^2(\Omega_T),\ D_\eta^2 u\in L^2(\Omega_T),\\ &\qquad tD_t u_{|t=0}=0,\ D_\eta u_{|\eta=\pm\frac{1}{2}}=0\Big\},\\ &\qquad P^{T,\rho} := 2(D_t-\eta)t(D_t-\eta)+\frac{\rho}{2}D_\eta^2. \end{aligned} \tag{8.10}$$

Let us define the continuous, bounded and self-adjoint projector Π_0 by :

$$\Pi_0 : L^2(\Omega_T)\to L^2(\Omega_T),\ f\mapsto \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t,\eta)\ d\eta. \tag{8.11}$$

Lemma 8.4. *We assume that $T>1$ and $\frac{T}{\rho}$ is uniformly bounded from above. Then there exists $\hat{C}$ such that :*

$$\nu(T,\rho)\le\lambda_1^{\mathrm{mean},T}\le\nu(T,\rho)+\hat{C}\sqrt{\frac{T}{\rho}}. \tag{8.12}$$

Furthermore, if we denote by $u_1^{T,\rho}$ a normalized eigenvector associated to $\nu(T,\rho)$, there exists a constant C such that :

$$||(Id-\Pi_0)u_1^{T,\rho}||_{L^2(\Omega_T)}\le\frac{C}{\sqrt{\rho}}, \tag{8.13}$$

$$1-\frac{C}{\rho}\le||\Pi_0 u_1^{T,\rho}||_{L^2(\Omega_T)}\le 1. \tag{8.14}$$

Proof : By the inclusion $\mathcal{V}^N_{\mathrm{mean},T}\subset\mathcal{V}_T^{N,D}$, we immediately see that :

$$\nu(T,\rho)\le\lambda_1^{\mathrm{mean},T},\ \forall\rho\ge 0. \tag{8.15}$$

Combining (8.2) and (8.15), we bound from above $\nu(T,\rho)$ by $\lambda_1^{\mathrm{mean}}+\tilde{c}=: C'$. By definition of $\nu(T,\rho)$ and $u_1^{T,\rho}$, we have $\nu(T,\rho)=a^{T,\rho}(u_1^{T,\rho},u_1^{T,\rho})$ and deduce :

$$||D_\eta u_1^{T,\rho}||_{L^2(\Omega_T)}\le\sqrt{\frac{2}{\rho}}C'. \tag{8.16}$$

Using the projector Π_0, we write $u_1^{T,\rho}$ as a sum of two functions :

$$u_1^{T,\rho}=\Pi_0(u_1^{T,\rho})+\tilde{u}^{T,\rho}. \tag{8.17}$$

We identify the range of Π_0 with $L^2(0,T)$: $L^2(0,T)$ is injected in $L^2(\Omega_T)$ by defining $g\mapsto i(g)=g$. Now, we omit take i.
By definition of $\Pi_0(u_1^{T,\rho})$ and $\tilde{u}^{T,\rho}$, it is easy to see that :

$$||D_\eta\tilde{u}^{T,\rho}||_{L^2(\Omega_T)}\le\frac{\sqrt{2}C'}{\sqrt{\rho}},\ \int_{-1/2}^{1/2}\tilde{u}^{T,\rho}(t,\eta)\ d\eta=0,\ D_\eta\tilde{u}^{T,\rho}{}_{|_{\eta=\pm\frac{1}{2}}}=0. \tag{8.18}$$

Properties (8.18) show that $\tilde{u}^{T,\rho}$ is orthogonal to constant functions and since π^2 is the smallest positive eigenvalue for $-\partial_\eta^2$ on $]-\frac{1}{2},\frac{1}{2}[$, we deduce that :

$$||D_\eta\tilde{u}^{T,\rho}||_{L^2(\Omega_T)}\quad\ge\quad\pi||\tilde{u}^{T,\rho}||_{L^2(\Omega_T)}. \tag{8.19}$$

Relation (8.19) and the lower bound (8.18) achieve the proof of (8.13) with $C = \frac{\sqrt{2}C'}{\pi}$. Since $||\Pi_0(u_1^{T,\rho})||^2_{L^2(\Omega_T)} = 1 - ||\tilde{u}^{T,\rho}||^2_{L^2(\Omega_T)}$, Relation (8.13) leads to (8.14). We know that :

$$2(D_t - \eta)t(D_t - \eta)u_{T,\rho} + \frac{\rho}{2}D_\eta^2 u_{T,\rho} = \nu(T,\rho)u_{T,\rho}. \tag{8.20}$$

We apply the projector Π_0 to Relation (8.20) and so :

$$L^{\mathrm{mean,T}}\Pi_0(u_1^{T,\rho}) + \int_{-\frac{1}{2}}^{\frac{1}{2}} 2(D_t - \eta)t(D_t - \eta)\tilde{u}^{T,\rho}\ d\eta = \nu(T,\rho)\Pi_0(u_1^{T,\rho}).$$

We take the scalar product with $\Pi_0(u_1^{T,\rho})$ and obtain, after simplifications :

$$\begin{aligned}\left\langle L^{\mathrm{mean},T}\Pi_0 u_1^{T,\rho}, \Pi_0 u_1^{T,\rho}\right\rangle_{L^2(0,T)} &= \nu(T,\rho)||\Pi_0 u_1^{T,\rho}||^2_{L^2(0,T)} - 2i\int_{\Omega_T} \eta\tilde{u}^{T,\rho}\ \overline{\Pi_0 u_1^{T,\rho}} dt d\eta \\ &+2\int_{\Omega_T} \eta t D_t\tilde{u}^{T,\rho}\overline{\Pi_0 u_1^{T,\rho}} dt d\eta - 2\int_{\Omega_T} t\eta^2\tilde{u}^{T,\rho}\ \overline{\Pi_0 u_1^{T,\rho}} dt d\eta. \end{aligned} \tag{8.21}$$

We now estimate each term of (8.21) :

- $\left|\int_{\Omega_T} \eta\tilde{u}^{T,\rho}\ \overline{\Pi_0 u_1^{T,\rho}}\ dt\ d\eta\right| \leq \frac{1}{2}\frac{C''}{\sqrt{\rho}}$ according to (8.13) and (8.14).
- $\left|\int_{\Omega_T} \eta^2 t\tilde{u}^{T,\rho}\ \overline{\Pi_0 u_1^{T,\rho}}\ dt\ d\eta\right| \leq \frac{C''}{4}\sqrt{\frac{T}{\rho}}||\sqrt{t}\Pi_0 u_1^{T,\rho}||_{L^2(\Omega_T)}$.
- For the third term, we use an integration by parts and then :

$$\left|\int_{\Omega_T} \eta t D_t\tilde{u}^{T,\rho}\ \overline{\Pi_0 u_1^{T,\rho}}\ dt\ d\eta\right| \leq \frac{C''}{2\sqrt{\rho}} + \frac{C''\sqrt{T}}{2\sqrt{\rho}}||\sqrt{t}D_t\Pi_0 u_1^{T,\rho}||_{L^2(0,T)}.$$

We define $C(T,\rho) := \left\langle L^{\mathrm{mean},T}\Pi_0 u_1^{T,\rho}, \Pi_0 u_1^{T,\rho}\right\rangle_{L^2(0,T)}$ and remark that :

$$||\sqrt{t}\Pi_0 u_1^{T,\rho}||_{L^2(0,T)} \leq \sqrt{6}\sqrt{C(T,\rho)} \text{ and } ||\sqrt{t}D_t\Pi_0 u_1^{T,\rho}||_{L^2(0,T)} \leq \sqrt{C(T,\rho)}.$$

Then an upper bound coming from Relation (8.21) and previous estimates of each term lead to the bound :

$$C(T,\rho) \leq \nu(T,\rho) + \frac{3C''}{\sqrt{\rho}} + 4C''\sqrt{\frac{T}{\rho}}\sqrt{C(T,\rho)}.$$

We assume that $T \geq 1$ and that there exists a constant M such that $\frac{T}{\rho} \leq M$. Then $C(T,\rho)$ is finite and there exists a constant $C(\infty) > 0$ independent of T and ρ such that : $C(T,\rho) \leq C(\infty) < +\infty$. This combined with (8.14) leads to the bound (8.12) of $\lambda_1^{\mathrm{mean},T}$. □

8.5 Proof of the lower bound on Theorem 8.1

Let $\eta > 0$. We apply Lemmas 8.2 and 8.4 with $\epsilon = \frac{\eta}{2\hat{C}}$ and $T = \frac{\epsilon}{\alpha}$, then :

$$\nu\left(T, \frac{1}{\alpha^2 T}\right) \geq \lambda_1^{\mathrm{mean}} - \frac{\eta}{2}. \tag{8.22}$$

Since $\lambda^T(\alpha) \geq \nu\left(T, \frac{1}{\alpha^2 T}\right)$, Lemma 8.3 shows that with the previous choice of ϵ, there exist $\alpha_0 > 0$ and $\delta > 0$ such that :

$$\lambda(\alpha) \geq \nu\left(T, \frac{1}{\alpha^2 T}\right) - e^{-\frac{\delta}{\alpha}},\ \forall \alpha \in]0, \alpha_0]. \tag{8.23}$$

So there exists $\alpha_1 \leq \alpha_0$ such that for any $\alpha \in]0, \alpha_1]$, $e^{-\frac{\delta}{\alpha}} \leq \frac{\eta}{2}$; then (8.22) and (8.23) lead to :

$$\lambda(\alpha) \geq \lambda_1^{\mathrm{mean}} - \eta. \tag{8.24}$$

Since $\mu(\alpha) = \alpha\lambda(\alpha)$, we can also achieve the proof of Theorem 8.1. □

9 Behavior of eigenvectors

It is easy to deduce from the proof of Theorem 8.1 the behavior of the eigenvectors of $P^{T,\rho}$ and $L^{\mathrm{mean,T}}$ as follows.

Proposition 9.1. *There exist $c > 0$ and $T_0 > 0$ such that the first eigenvalue of $P^{T,\rho}$ is simple for any ρ and $T \geq T_0$ such that $\frac{1}{\rho} + \sqrt{\frac{T}{\rho}} \leq c$. Denoting by $u_1^{T,\rho}$ and $u_1^{\mathrm{mean},T}$ the first normalized eigenvectors for $P^{T,\rho}$ and $L^{\mathrm{mean},T}$ respectively, there exists C such that for $T \geq T_0$ and $\rho > 0$ with $\frac{1}{\rho} + \sqrt{\frac{T}{\rho}} \leq c$:*

$$||\Pi_0(u_1^{T,\rho}) - u_1^{\mathrm{mean},T}||^2_{L^2(0,T)} \leq C\left(\frac{1}{\rho} + \sqrt{\frac{T}{\rho}}\right), \tag{9.1}$$

$$||u_1^{T,\rho} - u_1^{\mathrm{mean},T}||^2_{L^2(\Omega_T)} \leq C\left(\frac{1}{\rho} + \sqrt{\frac{T}{\rho}}\right). \tag{9.2}$$

Proof : Let $u_1^{T,\rho}$ be a normalized eigenfunction associated to the first eigenvalue for $P^{T,\rho}$ and we define the function $u^0_{T,\rho} :\ = \Pi_0(u_1^{T,\rho})$ on $]0, T[$. After a possible multiplication by a complex number, we assume that $u^0_{T,\rho}$ is decomposed according to a basis of eigenvectors for $L^{\mathrm{mean},T}$ with $a_0 \geq 0$:

$$u^0_{T,\rho} = a_0 u_1^{\mathrm{mean},T} + \tilde{u}^0_{T,\rho}, \tag{9.3}$$

with $\tilde{u}^0_{T,\rho}$ orthogonal to $u_1^{\mathrm{mean},T}$. We estimate $\left\langle L^{\mathrm{mean},T} u^0_{T,\rho}, u^0_{T,\rho}\right\rangle_{L^2(0,T)}$ with (8.21), Relation (8.15) and the expression (9.3). Then there exists a constant A independent of T and ρ such that if $T \geq 1$ and $\frac{T}{\rho}$ is uniformly bounded, then :

$$\left\langle L^{\mathrm{mean},T} u^0_{T,\rho}, u^0_{T,\rho}\right\rangle_{L^2(0,T)} \leq \lambda_1^{\mathrm{mean},T}(a_0^2 + ||\tilde{u}^0_{T,\rho}||^2_{L^2(0,T)}) + A\sqrt{\frac{T}{\rho}}. \tag{9.4}$$

We denote by $\lambda_2^{\text{mean},T}$ the second eigenvalue for $L^{\text{mean},T}$ and apply the min-max principle, then :

$$\left\langle L^{\text{mean},T} u^0_{T,\rho}, u^0_{T,\rho} \right\rangle_{L^2(0,T)} \geq \lambda_1^{\text{mean},T} a_0^2 + \lambda_2^{\text{mean},T} ||\tilde{u}^0_{T,\rho}||^2_{L^2(0,T)}. \tag{9.5}$$

From (9.4) and (9.5), we deduce that :

$$\left(\lambda_2^{\text{mean},T} - \lambda_1^{\text{mean},T}\right) ||\tilde{u}^0_{T,\rho}||^2_{L^2(0,T)} \quad \leq \quad A\sqrt{\frac{T}{\rho}}. \tag{9.6}$$

We can bound from below $\lambda_2^{\text{mean},T}$ by λ_2^{mean} and according to (8.2), we have :

$$\frac{1}{\lambda_2^{\text{mean},T} - \lambda_1^{\text{mean},T}} \quad \leq \quad \frac{1}{\lambda_2^{\text{mean}} - \lambda_1^{\text{mean}} - \tilde{c}e^{-\frac{T}{2\sqrt{3}}}}. \tag{9.7}$$

Due to (9.7), there exist T_0 and $\tilde{A}$ independent of ρ such that for all $T \geq T_0$:

$$||\tilde{u}^0_{T,\rho}||^2_{L^2(0,T)} \quad \leq \quad \tilde{A}\sqrt{\frac{T}{\rho}}. \tag{9.8}$$

This shows that $||\tilde{u}^0_{T,\rho}||^2_{L^2(0,T)}$ tends to 0 when ρ tends to infinity.
We use the decomposition of $u^0_{T,\rho}$, Relations (8.14) and (9.8) to obtain :

$$1 - \frac{\tilde{C}''}{\rho} - \tilde{A}\sqrt{\frac{T}{\rho}} \leq a_0^2 \leq 1. \tag{9.9}$$

This leads to Relation (9.1) since :

$$||u^0_{T,\rho} - u_1^{\text{mean},T}||^2_{L^2(0,T)} = (1-a_0)^2 + ||\tilde{u}^0_{T,\rho}||^2_{L^2(0,T)} \leq \frac{C}{\rho} + C\sqrt{\frac{T}{\rho}}. \tag{9.10}$$

Due to the decomposition of $u_1^{T,\rho} = \Pi_0(u_1^{T,\rho}) + \tilde{u}^{T,\rho}$, we have :

$$||u_1^{T,\rho} - u_1^{\text{mean},T}||^2_{L^2(\Omega_T)} \quad \leq \quad 2||\Pi_0(u_1^{T,\rho}) - u_1^{\text{mean},T}||^2_{L^2(\Omega_T)} + 2||\tilde{u}^{T,\rho}||^2_{L^2(0,T)}.$$

This last relation coupled with estimates (9.1) and (8.13) achieve the proof of (9.2).
Let us assume by contradiction that the first eigenvalue of $P^{T,\rho}$ is not simple, then there exists a normalized eigenvector denoted by $v_1^{T,\rho}$, orthogonal to $u_1^{T,\rho}$ for the smallest eigenvalue. Using Proposition 9.1 with $u_1^{T,\rho}$ and $v_1^{T,\rho}$, we have :

$$\begin{aligned} ||v_1^{T,\rho} - u_1^{T,\rho}||^2_{L^2(\Omega_T)} &\leq 2||v_1^{T,\rho} - u_1^{\text{mean},T}||^2_{L^2(\Omega_T)} + 2||u_1^{T,\rho} - u_1^{\text{mean},T}||^2_{L^2(\Omega_T)} \\ &\leq C'\left(\frac{1}{\rho} + \sqrt{\frac{T}{\rho}}\right). \end{aligned} \tag{9.11}$$

This is in contradiction with the fact that $||v_1^{T,\rho} - u_1^{T,\rho}||^2_{L^2(\Omega_T)} = 2$ as soon as $C'\left(\frac{1}{\rho} + \sqrt{\frac{T}{\rho}}\right) < 2$. Therefore $\nu(T,\rho)$ is simple in this case. □
We also can compare the first eigenvectors for L^{mean} and $L^{\text{mean},T}$:

Proposition 9.2. *Let u_1^{mean} be the normalized positive eigenvector associated to λ_1^{mean}, $u_1^{\text{mean},T}$ be the normalized positive eigenvector associated to $\lambda_1^{\text{mean},T}$ and $\tilde{u}_1^{\text{mean},T}$ its extension by 0 on $[T,+\infty[$. Then :*

$$||u_1^{\text{mean}} - \tilde{u}_1^{\text{mean},T}||_{L^2(\mathbb{R}^+)} = \mathcal{O}\left(e^{-\frac{T}{4\sqrt{3}}}\right) \text{ as } T \to +\infty. \tag{9.12}$$

Proof : We decompose $\tilde{u}_1^{\text{mean},T} = a_0 u_1^{\text{mean}} + v_1$, with $a_0 \geq 0$ (after a possible multiplication of $\tilde{u}_1^{\text{mean},T}$ by a complex number) and v_1 orthogonal to u_1^{mean}, then :

$$||\tilde{u}_1^{\text{mean},T}||^2_{L^2(\mathbb{R}^+)} = 1 = a_0^2 + ||v_1||^2_{L^2(\mathbb{R}^+)}. \tag{9.13}$$

Furthermore :

$$\lambda_1^{\text{mean},T} = \left\langle L^{\text{mean}}\tilde{u}_1^{\text{mean},T}, \tilde{u}_1^{\text{mean},T}\right\rangle_{L^2(\mathbb{R}^+)} \geq a_0^2\lambda_1^{\text{mean}} + \lambda_2^{\text{mean}}||v_1||^2_{L^2(\mathbb{R}^+)}. \tag{9.14}$$

Relations (9.13) and (9.14) lead to :

$$(\lambda_2^{\text{mean}} - \lambda_1^{\text{mean},T})||v_1||^2_{L^2(\mathbb{R}^+)} \leq (\lambda_1^{\text{mean},T} - \lambda_1^{\text{mean}})a_0^2 \leq \lambda_1^{\text{mean},T} - \lambda_1^{\text{mean}}.$$

From Lemma 8.2, we deduce :

$$\left(\lambda_2^{\text{mean}} - \lambda_1^{\text{mean}} - \tilde{c}e^{-\frac{T}{2\sqrt{3}}}\right)||v_1||^2_{L^2(\mathbb{R}^+)} \quad \leq \quad \tilde{c}e^{-\frac{T}{2\sqrt{3}}}.$$

But observing that $\lambda_2^{\text{mean}} - \lambda_1^{\text{mean}} = \frac{2}{\sqrt{3}}$, we get $||v_1||^2_{L^2(\mathbb{R}^+)} = \mathcal{O}\left(e^{-\frac{T}{2\sqrt{3}}}\right)$ and $a_0 = 1 + \mathcal{O}\left(e^{-\frac{T}{2\sqrt{3}}}\right)$ as T tends to infinity. To achieve the proof, we just write :

$$||u_1^{\text{mean}} - \tilde{u}_1^{\text{mean},T}||^2_{L^2(\mathbb{R}^+)} \quad = \quad (1-a_0)^2||u_1^{\text{mean}}||^2_{L^2(\mathbb{R}^+)} + ||v_1||^2_{L^2(\mathbb{R}^+)}.$$

So $||u_1^{\text{mean}} - \tilde{u}_1^{\text{mean},T}||^2_{L^2(\mathbb{R}^+)} = \mathcal{O}\left(e^{-\frac{T}{2\sqrt{3}}}\right)$. □

10 Splitting between the two first eigenvalues

10.1 Main proposition

The upper bound of the bottom of the spectrum given in Section 6 proves the existence of an eigenvalue with the asymptotics $\alpha\lambda^{(n)} + \mathcal{O}(\alpha^{2n+3})$. In the previous section, we have established the lower bound $\mu(\alpha) \geq \frac{\alpha}{\sqrt{3}} - \epsilon\alpha$. If the splitting between the two first eigenvalues of $P_{\mathcal{A}_0,\Omega_\alpha}$ is big enough or if we obtain a "good" lower bound of the second eigenvalue, then spectral theorem will give the asymptotics of $\mu(\alpha)$.

Proposition 10.1. *As α tends to 0, the number of eigenvalues of $P_{\mathcal{A}_0,\Omega_\alpha}$ below the essential spectrum tends to infinity. If we denote by $\mu(\alpha)$, $\mu_2(\alpha)$ the two first eigenvalues of $P_{\mathcal{A}_0,\Omega_\alpha}$ and by λ_1^{mean}, λ_2^{mean} the smallest eigenvalues for L^{mean}, then for every $\epsilon > 0$, there exist α_0 and $C > 0$ such that for all $\alpha \in]0, \alpha_0]$:*

$$\mu_2(\alpha) \quad \geq \quad \alpha\lambda_2^{\text{mean}} - \epsilon\alpha, \tag{10.1}$$

$$\frac{\mu_2(\alpha) - \mu(\alpha)}{\alpha} \quad \geq \quad (\lambda_2^{\text{mean}} - \lambda_1^{\text{mean}}) - \epsilon. \tag{10.2}$$

To prove Proposition 10.1, we will use the same operators as in Section 8, that is to say, the operators $L^{\text{mean},T}$, P_α^T and $P^{T,\rho}$ defined respectively in (8.1), (8.3), (8.10) and studied respectively in Subsections 8.2, 8.3 and 8.4.

10.2 Estimates of the second eigenvalue

We use the index 2 to indicate the second eigenvalue (when it exists). We have an expression of $\lambda_2(\alpha)$ according to the min-max principle :

$$\lambda_2(\alpha) \quad = \quad \sup_{v\in L^2(\Omega^0)} \inf_{u\in\mathcal{V}^N, u\perp v} \frac{q_\alpha(u)}{||u||^2_{L^2(\Omega^0)}}. \tag{10.3}$$

By the same principle, we have expressions for $\lambda_2^T(\alpha)$, λ_2^{mean}, $\lambda_2^{\text{mean},T}$ and $\nu_2(T,\rho)$. So we deduce that :

$$\lambda_2^{\text{mean}} \quad \geq \quad \lambda_2(\alpha), \tag{10.4}$$

$$\lambda_2^T(\alpha) \quad \geq \quad \lambda_2(\alpha), \tag{10.5}$$

$$\lambda_2^{\text{mean},T} \quad \geq \quad \lambda_2^{\text{mean}}, \tag{10.6}$$

$$\lambda_2^{\text{mean},T} \quad \geq \quad \nu_2(T,\rho), \tag{10.7}$$

$$\lambda_2^T(\alpha) \quad \geq \quad \nu_2\left(T, \frac{1}{\alpha^2 T}\right). \tag{10.8}$$

Due to Relation (10.4), we can bound from above $\lambda_2(\alpha)$ by a positive constant and so $\mu_2(\alpha) = \alpha\lambda_2(\alpha)$ tends to 0 with α. Then there exists α_1 such that for all $\alpha \leq \alpha_1$, $\mu_2(\alpha) < \Theta_0$. By Lemma 2.3, we have established that the bottom of the essential spectrum of $P_{\mathcal{A}_0,\Omega_\alpha}$ is equal to Θ_0 and so, $\mu_2(\alpha)$ is an eigenvalue as soon as $\mu_2(\alpha) < \Theta_0$.

10.3 Comparison between ν_2 and λ_2^{mean}

Lemma 10.2. *There exist ϵ_0, $\alpha_0 > 0$ and $c > 0$ such that for $\epsilon \leq \epsilon_0$, $\alpha \leq \alpha_0$ and $T = \frac{\epsilon}{\alpha}$:*

$$\nu_2\left(T, \frac{1}{\alpha^2 T}\right) \geq \lambda_2^{\text{mean}} - c\epsilon. \tag{10.9}$$

Proof : According to Proposition 9.1 with $\rho = \frac{1}{\alpha^2 T}$ and $T = \frac{\epsilon}{\alpha}$, there exist ϵ_0 and α_0 such that for all $\epsilon \leq \epsilon_0$ and $\alpha \leq \alpha_0$, the smallest eigenvalue of $P^{T,\rho}$ is simple. Let u_1^T and u_2^T the normalized eigenvectors associated respectively to $\nu\left(T, \frac{1}{\alpha^2 T}\right)$ and $\nu_2\left(T, \frac{1}{\alpha^2 T}\right)$. We define $u_1^{T,0} := \Pi_0 u_1^T$, $u_2^{T,0} := \Pi_0 u_2^T$ and $\tilde{u}_1^T = u_1^T - u_1^{T,0}$, $\tilde{u}_2^T = u_2^T - u_2^{T,0}$. According to Lemma 8.4 and Relations (8.13) and (8.14), there exists C' independent of α and T such that :

$$||\tilde{u}_1^T||_{L^2(0,T)} \leq C'\sqrt{\epsilon}\sqrt{\alpha}, \qquad ||\tilde{u}_2^T||_{L^2(0,T)} \leq C'\sqrt{\epsilon}\sqrt{\alpha}, \tag{10.10}$$

$$1 - C'\epsilon\alpha \leq ||u_1^{T,0}||_{L^2(0,T)} \leq 1, \qquad 1 - C'\epsilon\alpha \leq ||u_2^{T,0}||_{L^2(0,T)} \leq 1. \tag{10.11}$$

We decompose $u_2^{T,0} = a_1 u_1^{\text{mean},T} + \tilde{u}_2^{T,0}$ with $\tilde{u}_2^{T,0}$ orthogonal to $u_1^{\text{mean},T}$, then $a_1^2 + ||\tilde{u}_2^{T,0}||^2_{L^2(0,T)} \leq 1$, so $|a_1| \leq 1$. Since $\left\langle u_2^T, u_1^T \right\rangle_{L^2(\Omega_T)} = 0$, we estimate :

$$\left\langle u_2^{T,0}, u_1^{T,0} \right\rangle_{L^2(0,T)} = \left\langle \tilde{u}_2^T, \tilde{u}_1^T \right\rangle_{L^2(\Omega_T)} - \left\langle \tilde{u}_2^T, u_1^T \right\rangle_{L^2(\Omega_T)} - \left\langle u_2^T, \tilde{u}_1^T \right\rangle_{L^2(\Omega_T)}. \tag{10.12}$$

The decomposition of $u_2^{T,0}$, the normalization of $u_1^{\text{mean},T}$ and the orthogonality between $\tilde{u}_2^{T,0}$ and $u_1^{\text{mean},T}$ give an other expression for the scalar product :

$$\left\langle u_2^{T,0}, u_1^{T,0} \right\rangle_{L^2(0,T)} = a_1 + \left\langle a_1 u_1^{\text{mean},T} + \tilde{u}_2^{T,0}, u_1^{T,0} - u_1^{\text{mean},T} \right\rangle_{L^2(0,T)}. \tag{10.13}$$

Relation (9.1) shows that $||u_1^{T,0} - u_1^{\text{mean},T}||_{L^2(0,T)} \leq \tilde{C}\epsilon$. Furthermore, Relations (10.10) bound from above $\left|\langle \tilde{u}_2^T, u_1^T \rangle_{L^2(\Omega_T)}\right|$ and $\left|\langle u_2^T, \tilde{u}_1^T \rangle_{L^2(\Omega_T)}\right|$ by $C'\sqrt{\epsilon}\sqrt{\alpha}$ and also $\left|\langle \tilde{u}_2^T, \tilde{u}_1^T \rangle_{L^2(\Omega_T)}\right|$ by $C'^2\epsilon\alpha$. We report these estimates in (10.12) and (10.13) knowing that $|a_1| \leq 1$, then :

$$|a_1| \leq 2\tilde{C}\epsilon + 2C'\sqrt{\epsilon}\sqrt{\alpha} + C'^2\epsilon\alpha. \tag{10.14}$$

So there exist $\alpha_1 \leq \alpha_0$ and C_1 such that for $\alpha \leq \alpha_1$:

$$|a_1| \quad \leq \quad C_1\sqrt{\epsilon}. \tag{10.15}$$

Taking again (10.11) and (10.15), there exists $\hat{C}$ with :

$$1 - \hat{C}\epsilon \leq ||\tilde{u}_2^{T,0}||^2_{L^2(0,T)} = ||u_2^{T,0}||^2_{L^2(0,T)} - a_1^2 \leq 1. \tag{10.16}$$

As we made for $\left\langle L^{\text{mean},T} u_1^{T,0}, u_1^{T,0} \right\rangle_{L^2(0,T)}$, we bound $\left\langle L^{\text{mean},T} u_2^{T,0}, u_2^{T,0} \right\rangle_{L^2(0,T)}$:

$$\left\langle L^{\text{mean},T} u_2^{T,0}, u_2^{T,0} \right\rangle_{L^2(0,T)} \leq \nu_2\left(T, \frac{1}{\alpha^2 T}\right) ||u_2^{T,0}||^2_{L^2(0,T)} + C\epsilon, \tag{10.17}$$

and also :

$$\left\langle L^{\text{mean},T} u_2^{T,0}, u_2^{T,0} \right\rangle_{L^2(0,T)} \geq a_1^2 \lambda_1^{\text{mean},T} + \lambda_2^{\text{mean},T} ||\tilde{u}_2^{T,0}||^2_{L^2(0,T)}. \tag{10.18}$$

We look at Relations (10.17), (10.18); we use (10.11), (10.15), (10.16) and (10.6) to achieve the proof of (10.9). □

10.4 Comparison between $\lambda_2(\alpha)$ and $\lambda_2^T(\alpha)$

Lemma 10.3. *For every $\epsilon \in \left]0, \frac{2}{\sqrt{3}}\right[$, there exist α_0, $\delta > 0$ and $C > 0$ such that for $\alpha \in]0, \alpha_0[$ and $T = \frac{\epsilon}{\alpha}$:*

$$0 \leq \lambda_2^T(\alpha) - \lambda_2(\alpha) \leq Ce^{-\frac{\delta}{\alpha}}. \tag{10.19}$$

Proof : As for the first eigenvalue, we show that $\lambda_2(\alpha)$ is closed to $\lambda_2^T(\alpha)$ by using spectral theorem. We define $v_\alpha^T = \chi_T u_{2,\alpha}$ and estimate $||(P_\alpha^T - \lambda_2(\alpha))v_\alpha^T||_{L^2(\Omega_T)}$. For $(t,\eta) \in \Omega_T$:

$$P_\alpha^T v_\alpha^T(t,\eta) = P_\alpha v_\alpha^T(t,\eta) = \chi_T P_\alpha u_{2,\alpha}(t,\eta) + [P_\alpha, \chi_T] u_{2,\alpha}(t,\eta).$$

By assumption, there exists C such that $|D_t\chi_T| \leq \frac{C}{T}$ and $|D_t^2\chi_T| \leq \frac{C}{T^2}$. Corollary 7.4 proves the existence of $\delta > 0$ and $\alpha_0 > 0$ such that for $\alpha \leq \alpha_0$ and $T = \frac{\epsilon}{\alpha}$:

$$||u_{2,\alpha}||_{L^2(]\frac{T}{2},T[\times]-\frac{1}{2},\frac{1}{2}[)} \leq e^{-\frac{\delta}{\alpha}}. \tag{10.20}$$

So, with an integration by parts for the term $||tD_t u_{2,\alpha} D_t \chi_T||_{L^2(\Omega_T)}$, we deduce :

$$||(P_\alpha^T - \lambda_2(\alpha))v_\alpha^T||_{L^2(\Omega_T)} \leq Ce^{-\frac{\delta}{\alpha}}. \tag{10.21}$$

With $||v_\alpha^T||^2_{L^2(\Omega_T)} \geq ||u^T_{2,\alpha}||^2_{L^2(\Omega_{\frac{T}{2}})} = 1 - ||u^T_{2,\alpha}||^2_{L^2(]\frac{T}{2},+\infty[\times]-\frac{1}{2},\frac{1}{2}[)}$ and (10.20), there is c such that :

$$||v_\alpha^T||_{L^2(\Omega_T)} \geq 1 - ce^{-\frac{\delta}{\alpha}}. \tag{10.22}$$

We apply a classical spectral theorem and Relations (10.21) and (10.22). So there exist $\alpha_1 \leq \alpha_0$, $\tilde{c}$ and for all $T = \frac{\epsilon}{\alpha}$, $\lambda_k^T(\alpha) \in \sigma(P_\alpha^T)$ such that for $\alpha \leq \alpha_1$:

$$|\lambda_k^T(\alpha) - \lambda_2(\alpha)| \leq \tilde{c}e^{-\frac{\delta}{\alpha}}. \tag{10.23}$$

We show that $k \geq 2$ by contradiction. We assume $k = 1$. It is easy to see that :

$$\lambda_2(\alpha) - \lambda_1^{\text{mean},T} \leq \tilde{c}e^{-\frac{\delta}{\alpha}}.$$

But, due to (10.8), we deduce :

$$\nu_2\left(T, \frac{1}{\alpha^2 T}\right) - \lambda_1^{\text{mean},T} \leq \tilde{c}e^{-\frac{\delta}{\alpha}}.$$

We apply Lemmas 10.2 and 8.2 to establish the upper bound :

$$\lambda_2^{\text{mean}} - \lambda_1^{\text{mean}} \leq \tilde{c}e^{-\frac{\delta}{\alpha}} + Ce^{-\frac{T}{2\sqrt{3}}} + c\epsilon = \tilde{c}e^{-\frac{\delta}{\alpha}} + Ce^{-\frac{\epsilon}{2\sqrt{3}\alpha}} + c\epsilon. \tag{10.24}$$

Since $\lambda_2^{\text{mean}} - \lambda_1^{\text{mean}} = \frac{2}{\sqrt{3}}$, (10.24) is impossible as soon as $\epsilon < \frac{2}{\sqrt{3}}$. Thus $k \geq 2$ and then $\lambda_k^T(\alpha) \geq \lambda_2^T(\alpha)$. We deduce (10.19) by using (10.5). □

10.5 Proof of Proposition 10.1

Let $\eta > 0$. We apply Lemma 10.2 with $\epsilon = \frac{\eta}{2c}$. Then, there exists α_0 such that for $\alpha \in]0, \alpha_0]$ and $T = \frac{\epsilon}{\alpha}$:

$$\nu_2\left(T, \frac{1}{\alpha^2 T}\right) \geq \lambda_2^{\text{mean}} - \frac{\eta}{2}. \tag{10.25}$$

With this choice of ϵ, Lemma 10.3 and Relation (10.8) give the existence of $\alpha_1 \leq \alpha_0$ such that for $\alpha \in]0, \alpha_1]$:

$$\lambda_2(\alpha) \geq \nu_2\left(T, \frac{1}{\alpha^2 T}\right) - \frac{\eta}{2}. \tag{10.26}$$

We take the lower bounds (10.25) and (10.26), then, for $\alpha \in]0, \alpha_1]$:

$$\lambda_2(\alpha) \geq \lambda_2^{\text{mean}} - \eta. \tag{10.27}$$

Since $\lambda(\alpha) \leq \lambda_1^{\text{mean}}$, we deduce :

$$\lambda_2(\alpha) - \lambda(\alpha) \geq (\lambda_2^{\text{mean}} - \lambda_1^{\text{mean}}) - \eta. \tag{10.28}$$

□

We now use Proposition 10.1 to justify the asymptotics of $\mu(\alpha)$.

10.6 Asymptotics of $\mu(\alpha)$

Theorem 10.4. *Let n a positive integer. We consider $(m_j)_{j\leq n}$ given recursively by Proposition 5.1, then $\mu(\alpha)$ has the asymptotics :*

$$\mu(\alpha) \quad = \quad \alpha\sum_{j=0}^{n} m_j\alpha^{2j} + \mathcal{O}_n(\alpha^{2n+3}) \text{ as } \alpha \to 0. \tag{10.29}$$

Proof : We take again $U^{(n)}$ and $\lambda^{(n)}$ defined in (6.2) and (6.1). According to Proposition 6.1, there is α_0 and for $\alpha \leq \alpha_0$, $\mu_N(\alpha) \in \sigma(P_{\mathcal{A}_0,\Omega_\alpha})$ such that :

$$|\mu_N(\alpha) - \alpha\lambda^{(n)}| \leq C\alpha^{2n+1}. \tag{10.30}$$

Assume that $N \geq 2$, then :

$$\mu_N(\alpha) - \alpha\lambda^{(n)} \quad \geq \quad \mu_2(\alpha) - \alpha\lambda^{(n)}. \tag{10.31}$$

We choose $\epsilon \leq \frac{1}{\sqrt{3}}$, then due to Proposition 10.1 and construction of $\lambda^{(n)}$, there exists $\alpha_1 \leq \alpha_0$ such that for $\alpha \in]0, \alpha_1]$:

$$\mu_2(\alpha) \geq \alpha\lambda_2^{\mathrm{mean}} - \alpha\frac{\epsilon}{2}, \qquad \alpha\lambda^{(n)} \leq \alpha\lambda_1^{\mathrm{mean}} + \alpha\frac{\epsilon}{2}. \tag{10.32}$$

We report (10.32) in (10.31), then, since $\lambda_2^{\mathrm{mean}} - \lambda_1^{\mathrm{mean}} = \frac{2}{\sqrt{3}}$:

$$\mu_N(\alpha) - \alpha\lambda^{(n)} \quad \geq \quad \alpha(\lambda_2^{\mathrm{mean}} - \lambda_1^{\mathrm{mean}}) - \alpha\epsilon \geq \frac{\alpha}{\sqrt{3}}. \tag{10.33}$$

This is impossible with (10.30), so $N = 1$ and $\mu_N(\alpha) = \mu(\alpha)$. □

11 Estimates in the semi-classical case

11.1 Localization techniques

11.1.1 Localization with a partition of unity

As in [11], p. 617-621, we can give a lower bound and an upper bound for the fundamental state of the Schrödinger operator with a non constant magnetic field and a bounded open set of $\mathbb{R}^2$ whose boundary is a curvilinear polygon. Our goal is now to prove Theorem 1.2. The partition of unity plays an important role and we recall its construction. The idea of the localization to compare the model cases $\mathbb{R}^2$, $\mathbb{R}\times\mathbb{R}^+$ and Ω_α comes from the following proposition :

Proposition 11.1. *Let $0 \leq \rho \leq 1$. There exist a constant C and a partition of unity χ_j^h of Ω satisfying :*

$$\chi_j^h(x) \;=\; \chi_j\left(\frac{x}{h^\rho}\right), \text{ with } \chi_j \text{ a partition of unity of } \mathbb{R}^2,$$

$$\sum_j |\chi_j^h|^2 \;=\; 1, \tag{11.1}$$

$$\sum_j |\nabla\chi_j^h|^2 \;\leq\; Ch^{-2\rho}, \tag{11.2}$$

$$\mathrm{supp}(\chi_j^h) \;\subset\; \mathcal{B}(z_j, h^\rho) \text{ s.t. } \begin{cases} \text{either } \mathrm{supp}(\chi_j^h)\cap\partial\Omega = \emptyset, \\ \text{either } z_j \in \partial\Omega \text{ and} \\ \quad \mathrm{supp}(\chi_j^h)\cap\{S_k, k = 1,\dots,N\} = \emptyset, \\ \text{or } z_j = S_j. \end{cases} \tag{11.3}$$

(with the choice of index j such that $z_j = S_j$ for $j = 1, \dots, N$).

We notice immediately :

$$\sum_j \chi_j^h (\nabla \chi_j^h) = 0. \tag{11.4}$$

To compare with models $\mathbb{R}^2$, $\mathbb{R} \times \mathbb{R}^+$ and Ω_α, we share indices in three parts :

$$cor\colon = \{j|\ z_j = S_j\},\ bd\colon = \{j|\ z_j \in \partial\Omega \setminus \{S_1, \dots, S_N\}\},\ int\colon = \{j|\ z_j \in \Omega\},$$

corresponding respectively to the corners, the regular points of the boundary and the points of the interior. We deduce for every $u \in H^1_{h,\mathcal{A}}(\Omega)$ that :

$$\begin{aligned} q_{h,\mathcal{A},\Omega}(u) &= \sum_{int} q_{h,\mathcal{A},\Omega}(\chi_j^h u) + \sum_{bd} q_{h,\mathcal{A},\Omega}(\chi_j^h u) + \sum_{cor} q_{h,\mathcal{A},\Omega}(\chi_j^h u) \\ &\quad -h^2 \sum_j ||\ |\nabla \chi_j^h| u||^2_{L^2(\Omega)}. \end{aligned} \tag{11.5}$$

We see two kinds of errors : errors coming from approximation which suggest the choice ρ large and errors coming from localization (last term of (11.5)) which suggest to choose ρ small. So we will try to optimize between these two constraints.

11.1.2 Change of variables

To compare with the models $\mathbb{R} \times \mathbb{R}^+$ and Ω_α, we make a local change of variables.

Lemma 11.2. *There exist positive constants h_0 and C_1 such that for any regular point $z \in \partial\Omega \setminus \{S_1, \dots, S_N\}$, for any $\epsilon < d(z, \{S_1, \dots, S_N\})$, we can write for $u \in H^1_{h,\mathcal{A}}(\Omega)$ with support in $\overline{\Omega} \cap \mathcal{B}(z, \epsilon)$ the form in the new domain $\mathbb{R} \times \mathbb{R}^+$ by :*

$$q_{h,\mathcal{A},\Omega}(u) = \int_{\mathbb{R}\times\mathbb{R}^+} \sum_{1\leq k,l\leq 2} g_{k,l}(\tilde{x}) \left(h\frac{\partial \tilde{u}}{\partial \tilde{x}_k} - i\tilde{A}_k \tilde{u} \right) \overline{\left(h\frac{\partial \tilde{u}}{\partial \tilde{x}_l} - i\tilde{A}_l \tilde{u} \right)} \sqrt{\det\, g}\; d\tilde{x}, \tag{11.6}$$

with $\tilde{u}$ and $\tilde{\mathcal{A}}$ deduced from u and $\mathcal{A}$ by change of variables.
Furthermore, for $h \in]0, h_0]$ and u with support in $\mathcal{B}(z, h^\rho) \cap (\overline{\Omega} \setminus \{S_1, \dots, S_N\})$:

$$(1 - C_1 h^\rho) q_{h,\tilde{\mathcal{A}},\mathbb{R}\times\mathbb{R}^+}(\tilde{u}) \leq q_{h,\mathcal{A},\Omega}(u) \leq (1 + C_1 h^\rho) q_{h,\tilde{\mathcal{A}},\mathbb{R}\times\mathbb{R}^+}(\tilde{u}). \tag{11.7}$$

We obtain a similar result for corners.

Lemma 11.3. *There exist positive constants h_0 and C_1 such that for any corners S_j, for any $u \in H^1_{h,\mathcal{A}}(\Omega)$ with support in $\overline{\Omega} \cap \mathcal{B}(S_j, \epsilon)$ (with $\epsilon < d(S_j, S_k)$ for every $k \neq j$), we can write the form in the new domain Ω_{α_j} by :*

$$q_{h,\mathcal{A},\Omega}(u) = \int_{\Omega_{\alpha_j}} \sum_{1\leq k,l\leq 2} g_{k,l}(\tilde{x}) \left(h\frac{\partial \tilde{u}}{\partial \tilde{x}_k} - i\tilde{A}_k \tilde{u} \right) \overline{\left(h\frac{\partial \tilde{u}}{\partial \tilde{x}_l} - i\tilde{A}_l \tilde{u} \right)} \sqrt{\det\, g}\; d\tilde{x}, \tag{11.8}$$

with $\tilde{u}$ and $\tilde{\mathcal{A}}$ deduced from u and $\mathcal{A}$ by change of variables.
Furthermore, for any $h \in]0, h_0]$ and u with support in $\overline{\Omega} \cap \mathcal{B}(S_j, h^\rho)$:

$$(1 - C_1 h^\rho) q_{h,\tilde{\mathcal{A}},\Omega_{\alpha_j}}(\tilde{u}) \leq q_{h,\mathcal{A},\Omega}(u) \leq (1 + C_1 h^\rho) q_{h,\tilde{\mathcal{A}},\Omega_{\alpha_j}}(\tilde{u}). \tag{11.9}$$

The main difficulty in Lemma 11.2 is to control the uniformity with respect to z. Details on the proofs of Lemmas 11.2 and 11.3 are given in [6].

11.1.3 Gauge transform

We assume that we have chosen the coordinates such that $z=(0,0)\in\partial\Omega$. By gauge transform, we can assume that $\mathcal{A}(0)$ and the linear part of the potential $\mathcal{A}$ are respectively equal to 0 and $\mathcal{A}^{\ell in}\colon = B(0)\mathcal{A}_0$ (with $\mathcal{A}_0$ defined on (1.2)). We define $\mathcal{A}'\colon = \mathcal{A}-\mathcal{A}^{\ell in}$, then there exists a positive constant $\tilde{C}$ such that :

$$|\mathcal{A}-\mathcal{A}^{\ell in}| = |\mathcal{A}'| \leq \tilde{C}|x|^2. \tag{11.10}$$

By the decomposition of $\mathcal{A}$ as the sum $\mathcal{A}^{\ell in}+\mathcal{A}'$, we write :

$$\begin{aligned} q_{h,\mathcal{A},\Omega}(v) &= \int_\Omega \left(\left|(h\partial_{x_1}-iA_1^{\ell in})v\right|^2 + \left|(h\partial_{x_2}-iA_2^{\ell in})v\right|^2 \right)\, dx \\ &\quad +2\,\mathrm{Re}\, i\int_\Omega \left((h\partial_{x_1}-iA_1^{\ell in})v\overline{A_1'v} + (h\partial_{x_2}-iA_2^{\ell in})v\overline{A_2'v} \right)\, dx \\ &\quad + \int_\Omega \left(|A_1'v|^2+|A_2'v|^2\right)\, dx. \end{aligned} \tag{11.11}$$

There exists a constant C such that if v has a support in $\overline{\Omega}\cap\mathcal{B}(0,h^\rho)$, then :

$$q_{h,\mathcal{A},\Omega}(v) \quad\leq\quad q_{h,\mathcal{A}^{\ell in},\Omega}(v) + Ch^{2\rho}||v||^2_{L^2(\Omega)}. \tag{11.12}$$

11.2 Lower bound

Proposition 11.4. *Under the assumptions of Theorem 1.2, there exist h_0 and a constant C such that for any $h\in]0,h_0[$:*

$$\mu(h,B,\Omega) \geq h\min\left(b, \Theta_0 b', \inf_{j=1,\dots,N}\mu(\alpha_j)B(S_j)\right) - Ch^{5/4}. \tag{11.13}$$

Proof : We have to estimate each term of (11.5). From (11.2), we deduce :

$$h^2\sum_j ||\ |\nabla\chi_j^h|\ u||^2_{L^2(\Omega)} \leq Ch^{2-2\rho}||u||^2_{L^2(\Omega)}. \tag{11.14}$$

This estimate shows that, for $\rho=\frac{3}{8}$, the error is in $\mathcal{O}\left(h^{\frac{5}{4}}\right)$.
Observing that $[hD_{x_1}-A_1, hD_{x_2}-A_2] = i\ h\ B$, we deduce the estimate :

$$\sum_{int} q_{h,\mathcal{A},\Omega}(\chi_j^h u) \geq h\sum_{int}\int_\Omega B(x)|\chi_j^h u(x)|^2dx, \tag{11.15}$$

where we have used that $\operatorname{supp}\chi_j^h\subset\Omega$, for $j\in int$.

We have now to estimate the terms which are localized at the boundary. Let $j\in cor\cup bd$. By change of variables, we send locally the domain onto $\mathbb{R}\times\mathbb{R}^+$ or Ω_α. We apply Lemmas 11.2 and 11.3, so there exists a constant C_1 independent of h such that, for every v with support in $\overline{\Omega}\cap\mathcal{B}(z_j,h^\rho)$,

$$(1-C_1h^\rho)q_{h,\tilde{\mathcal{A}},\Omega_\alpha}(\tilde{v}) \leq q_{h,\mathcal{A},\Omega}(v) \leq (1+C_1h^\rho)q_{h,\tilde{\mathcal{A}},\Omega_\alpha}(\tilde{v}). \tag{11.16}$$

So it is enough to analyze $q_{h,\tilde{\mathcal{A}},\Omega_\alpha}(\tilde{v})$. We now omit the tilda due do the change of variables. We make a gauge transform such that the linear term of the magnetic potential is equal to $\mathcal{A}^{\ell in} : = B(z_j)\mathcal{A}_0$. We denote by $\mathcal{A}'$ the remainder : $\mathcal{A}' = \mathcal{A} - \mathcal{A}^{\ell in}$. Relation (11.11) gives :

$$q_{h,\mathcal{A},\Omega}(v) \geq q_{h,\mathcal{A}^{\ell in},\Omega}(v) - 2\,\mathrm{Im}\int_\Omega \Big((h\partial_{x_1} - iA_1^{\ell in})v\,\overline{A_1'v} + (h\partial_{x_2} - iA_2^{\ell in})v\,\overline{A_2'v} \Big)\,dx.$$

A Cauchy-Schwarz inequality leads to :

$$\left| \int_\Omega (h\partial_{x_k} - iA_k^{\ell in})v\overline{A_k'v}dx \right| \leq \frac{h^{2\theta}}{2}\int_\Omega \left|(h\partial_{x_k} - iA_k^{\ell in})v\right|^2 dx + \frac{h^{-2\theta}}{2}\int_\Omega |A_k'v|^2\,dx. \tag{11.17}$$

But, by a Taylor expansion, we see that $|A_k'| \leq C|x - z_j| \leq Ch^\rho$ where C is a constant independent of h and z_j. Using this estimate, taking account of the error due to the change of variables (11.16) and choosing $v = \chi_j^h u$, we get :

$$q_{h,\mathcal{A},\Omega}(\chi_j^h u) \geq (1 - h^{2\theta} - C_1 h^\rho)h\mu(\alpha_j)B(z_j)\int_\Omega |\chi_j^h u|^2\,dx - \tilde{C}^2 h^{4\rho-2\theta}||\chi_j^h u||^2. \tag{11.18}$$

The estimates (11.14), (11.15) and (11.18) give the following lower bound for $q_{h,\mathcal{A},\Omega}(u)$:

$$\begin{aligned} q_{h,\mathcal{A},\Omega}(u) \geq\ & h\sum_{int}\int B(x)|\chi_j^h u|^2 dx - Ch^{2-2\rho}||u||^2 \\ & + (1 - h^{2\theta} - Ch^\rho)h\Theta_0\sum_{bd}B(z_j)\int_\Omega |\chi_j^h u|^2 dx - C^2h^{4\rho-2\theta}\sum_{bd}||\chi_j^h u||^2 \\ & + (1 - h^{2\theta} - Ch^\rho)h\sum_{cor}\mu(\alpha_j)B(S_j)\int_\Omega |\chi_j^h u|^2 dx - C^2 h^{4\rho-2\theta}\sum_{cor}||\chi_j^h u||^2. \end{aligned}$$

We choose $\rho = \frac{3}{8}$ and $\theta = \frac{1}{8}$, so there exist a positive constant C and $h_0 > 0$ such that for every $h \in]0, h_0]$, we have the lower bound for any $u \in H^1_{h,\mathcal{A}}(\Omega)$:

$$q_{h,\mathcal{A},\Omega}(u) \geq \left(h\min\left(b, \Theta_0 b', \inf_{j=1,\dots,N}\mu(\alpha_j)B(S_j) \right) - C_0 h^{5/4} \right)\ ||u||^2_{L^2(\Omega)}.$$

We apply the min-max principle and get (11.13). □

11.3 Upper bound

Proposition 11.5. *Under the assumptions of Theorem 1.2 and assuming that $\mu(\alpha_j) < \Theta_0$ for any $j = 1, \dots, N$, then there exist h_0 and a constant C such that for any $h \in]0, h_0[$:*

$$\mu(h, B, \Omega) \leq h\min\left(b, \Theta_0 b', \inf_{j=1,\dots,N}\mu(\alpha_j)B(S_j) \right) + Ch^{4/3}. \tag{11.19}$$

Proof : We establish three upper bounds :

1. $\mu(h, B, \Omega) \leq h\inf_{j=1,\dots,N}\mu(\alpha_j)B(S_j) + C_0h^{4/3}$.
2. $\mu(h, B, \Omega) \leq hb'\Theta_0 + C_0h^{4/3}$.
3. $\mu(h, B, \Omega) \leq hb + C_0h^{4/3}$.

• We begin with looking at the vertices. Let S_j be a vertex. By change of variables, we send locally Ω onto Ω_{α_j}. We will consider a function u with a support in a ball $\overline{\Omega}\cap\mathcal{B}(S_j,h^\rho)$, so according to (11.9), it is enough to estimate $q_{h,\mathcal{A},\Omega_{\alpha_j}}(u)$ with the new coordinates and the error is in h^ρ.
By assumption $\mu(\alpha_j)<\Theta_0$, then $\mu(\alpha_j)$ is an eigenvalue. We use the normalized eigenfunction v_0 associated to the Schrödinger operator $P_{\mathcal{A}_0,\Omega_{\alpha_j}}$ and we define :

$$\forall x\in\Omega_{\alpha_j},\ v_h(x)=\sqrt{\frac{B(S_j)}{h}}\ v_0\left(\sqrt{\frac{B(S_j)}{h}}x\right).$$

With a gauge transform, we assume that $\mathcal{A}=\mathcal{A}^{\ell in}+\mathcal{A}'$ with the same notations as in Subsection 11.1.3.
We consider χ a smooth function defined on $\mathbb{R}^2$ such that :

$$\operatorname{supp}\chi\subset\mathcal{B}_1,\quad \chi=1 \text{ on } \mathcal{B}_{1/2},\quad 0\leq\chi\leq 1. \tag{11.20}$$

We denote by $\chi_{h^\rho}:\ =\chi\left(\frac{\cdot-x_0}{h^\rho}\right)$ where x_0 will be chosen later. We now compute $q_{h,\mathcal{A},\Omega_{\alpha_j}}(\chi_{h^\rho}v_h)$:

$$\begin{aligned} q_{h,\mathcal{A},\Omega_{\alpha_j}}(\chi_{h^\rho}v_h) &= q_{h,\mathcal{A}^{\ell in},\Omega_{\alpha_j}}(\chi_{h^\rho}v_h)+\int_{\Omega_{\alpha_j}}|\mathcal{A}'|^2|\chi_{h^\rho}v_h|^2\ dx\\ &\quad +2\ \mathrm{Re}\ i\int_{\Omega_{\alpha_j}}\sum_{j=1}^{2}(h\partial_{x_j}-iA_j^{\ell in})(\chi_{h^\rho}v_h)\overline{A'_j\chi_{h^\rho}v_h}\ dx.\end{aligned}$$

By construction of v_h and assumptions about χ_{h^ρ} and $\mathcal{A}'$, we deduce that there exists a positive constant C such that :

$$q_{h,\mathcal{A},\Omega_{\alpha_j}}(\chi_{h^\rho}v_\phi)\leq hB(S_j)\mu(\alpha_j)||\chi_{h^\rho}v_h||^2_{L^2(\Omega_{\alpha_j})}+Ch^{2-2\rho}+Ch^2+Ch^{\frac{3}{2}}. \tag{11.21}$$

We now bound from below the norm of $\chi_{h^\rho}v_h$:

$$||\chi_{h^\rho}v_h||^2_{L^2(\Omega_{\alpha_j})}\geq 1-||v_h||^2_{L^2(\Omega_{\alpha_j})\cap\complement\mathcal{B}_{h^\rho/2})}\geq 1-C_1\ e^{-C_2h^{\rho-\frac{1}{2}}}, \tag{11.22}$$

by using the behavior of the eigenfuction mentioned in Section 7 and particularly in Theorem 7.1. Recalling (11.9), we deduce :

$$\begin{aligned}\frac{q_{h,\tilde{\mathcal{A}},\Omega}(\chi_{h^\rho}v_h)}{||\chi_{h^\rho}v_h||^2_{L^2(\Omega)}} &\leq (1+C_1h^\rho)\left(hB(S_j)\mu(\alpha_j)+C\frac{h^{2-2\rho}+h^2+h^{\frac{3}{2}}}{1-C_1\ e^{-C_2h^{\rho-\frac{1}{2}}}}\right)\\ &\leq hB(S_j)\mu(\alpha_j)+C(h^{\rho+1}+h^{\frac{3}{2}}+h^{2-2\rho}). \end{aligned}\tag{11.23}$$

Choosing $\rho=\frac{1}{3}$, we get :

$$\mu(h,B,\Omega)\quad\leq\quad hB(S_j)\mu(\alpha_j)+Ch^{\frac{4}{3}}. \tag{11.24}$$

• Now, we prove the upper bound 2. Let x_0 a point of the boundary such that $B(x_0)=b'$. Either x_0 is a regular point, either x_0 is a vertex. If x_0 is a vertex, we take account of $B(x_0)=b'$, (11.24), the assumption $\mu(\alpha)<\Theta_0$ and deduce :

$$\mu(h,B,\Omega)\quad\leq\quad hb'\Theta_0+ch^{\frac{4}{3}}. \tag{11.25}$$

Let us now assume that x_0 is not a vertex. We use the same techniques as Helffer-Morame [11], p. 648. By change of variables, we have a small perturbation of the sesquilinear form on the half-plane and we use the following function as a trial function :

$$\forall x \in \mathbb{R} \times \mathbb{R}^+,\ v(x) = h^{-\frac{1}{8}} e^{i\sqrt{\frac{b'}{h}}\zeta_0 x_1} \left(\frac{b'}{h}\right)^{\frac{1}{4}} \phi\left(\sqrt{\frac{b'}{h}} x_2\right) \chi(x_2) g\left(h^{-\frac{1}{4}} x_1\right), \tag{11.26}$$

with $g \in C_0^\infty(\mathbb{R})$ L^2-normalized supported by $]-\frac{1}{2}, \frac{1}{2}[$ and $\chi \in C_0^\infty(\mathbb{R}^+)$ such that :

$$\chi(x_2) = \begin{cases} 1 & \text{if} \quad 0 \leq x_2 \leq \frac{\epsilon_0}{2}, \\ 0 & \text{if} \quad x_2 \geq \epsilon_0, \end{cases}$$

and ϕ, ζ_0 defined by Proposition 2.1. We obtain the upper bound :

$$\mu(h, B, \Omega) \leq h\Theta_0 B(x_0) + Ch^{3/2}. \tag{11.27}$$

Combining (11.25) and (11.27), we deduce :

$$\mu(h, B, \Omega) \leq h\Theta_0 b' + Ch^{4/3}. \tag{11.28}$$

• Let us analyze the upper bound 3. Let $x_0 \in \overline{\Omega}$ such that $B(x_0) = b$. Either $x_0 \in \Omega$, either $x_0 \in \partial\Omega$. If $x_0 \in \partial\Omega$, then $b = b'$. Since we have proved that $\mu(h, B, \Omega) \leq hb'\Theta_0 + Ch^{4/3}$ and since $\Theta_0 < 1$, we deduce that :

$$\mu(h, B, \Omega) \leq hb + Ch^{4/3}. \tag{11.29}$$

Let us now assume $x_0 \in \Omega$, then there exists $h_0 > 0$ such that $\mathcal{B}\left(x_0, h_0^{\frac{1}{2}}\right) \subset \Omega$. Let $h \in]0, h_0]$. We consider the function u defined on Ω by :

$$u(x) = \sqrt{\frac{B(x_0)}{4\pi h}} \exp\left(\frac{-B(x_0)|x - x_0|^2}{4h}\right),\ \ \tilde{u} = \exp\left(i\frac{\phi}{h}\right) u, \tag{11.30}$$

with ϕ which realizes a gauge transform such that the linear expansion of $\mathcal{A} + \nabla\phi$ is equal to $\mathcal{A}_0$.
We use a cut-off function to localize $\tilde{u}$ in $\mathcal{B}(x_0, h^{\frac{1}{2}})$ and we make a Taylor expansion of $\mathcal{A}$ at the first order as mentioned in (11.12). We also get :

$$\mu(h, B, \Omega) \leq hB(x_0) + C_0 h^{3/2} = hb + C_0 h^{3/2}, \tag{11.31}$$

with C_0 a positive constant independent of h. Thanks to (11.29) and (11.31), we obtain :

$$\mu(h, B, \Omega) \leq hb + C_0 h^{3/2}, \tag{11.32}$$

Taking account of Relations (11.24), (11.28) and (11.32), we get the upper bound (11.19). □

Remark 11.6. *It seems that the assumption $\mu(\alpha) < \Theta_0$ holds for any angular $\alpha \in]0, \pi[$ but it is not proved for the moment. If we do not make this assumption, the result still holds with a worse remainder but we have a fourth case to study : the case of a vertex S_j with angle $\alpha_j < \pi$ and $\mu(\alpha_j) = \Theta_0$ as $\inf\limits_{x \in \partial\Omega} B(x) = B(S_j)$. Let us shortly explain how we can treat this case. Let us consider T_j a point of* *the regular boundary such that* $d(S_j,T_j)=h^{\frac14}$*, we localize in a small ball around* T_j *which do not meet corner or other piece of the boundary. After change of variables near* T_j *to send locally* Ω *onto* $\mathbb{R}\times\mathbb{R}^+$*, we consider instead of* v *defined by (11.26) the new function for* $x\in\mathbb{R}\times\mathbb{R}^+$ *:*

$$v(x)=e^{i\sqrt{\frac{B(T_j)}{h}}\zeta_0 x_1}\left(\frac{B(T_j)}{h}\right)^{\frac14}\phi\left(\sqrt{\frac{B(T_j)}{h}}x_2\right)\ g_2\left(\frac{x_2}{h^{\frac38}}\right)\ h^{-\frac{3}{16}}g_1\left(h^{-\frac38}x_1\right), \tag{11.33}$$

with $g_1\in C_0^\infty(]-\frac12,\frac12[,\mathbb{R})$ L^2*-normalized and* $g_2\in C_0^\infty(\overline{\mathbb{R}}^+)$ *such that :*

$$g_2(x_2)=\left\{\begin{array}{lll}1 & if & 0\le x\le\frac12\\ 0 & if & x_2\ge 1\end{array}\right.,$$

and ϕ,ζ_0 *defined by Proposition 2.1. Computations lead to the estimate :*

$$\mu(h,B,\Omega)\le h\Theta_0B(T_j)+Ch^{\frac98}. \tag{11.34}$$

Using the Taylor formula, there exists a positive constant β *such that :*

$$B(T_j)\le B(S_j)+\beta|T_j-S_j|\le B(S_j)+\beta h^{\frac14}. \tag{11.35}$$

We report (11.35) in (11.34) and so :

$$\mu(h,B,\Omega)\le h\Theta_0(B(S_j)+\beta h^{\frac14})+Ch^{\frac98}=h\Theta_0B(S_j)+\mathcal{O}(h^{\frac98}). \tag{11.36}$$

This achieves the demonstration in the general case for the upper bound.

12 Conclusion

Some physicists [7, 20] were already interested in the smallest eigenvalue for the Neumann realization of the Schrödinger operator with a constant magnetic field in an angular sector Ω_α and in its dependence on α. They gave already estimates but these results are without rigorous proof. Relation (1.7) gives an expansion at any order and goes far beyond the work of Brosens-Devreese-Fomin-Moshchalkov [7] who mention only the first term $\frac{\alpha}{\sqrt3}$ and a paper of Schweigert-Peeters [20] who propose on the basis of numerical computations a two-terms formula. As usual in the physical literature, the best one can hope through their techniques is an upper bound of $\mu(\alpha)$ because they only construct quasi-modes. We emphasize that we have obtained here the asymptotics and a control of the splitting between the first and the second eigenvalue. We have also given some upper bounds of $\mu(\alpha)$ and showed that the bottom of the spectrum is an eigenvalue for any angular in $]0,\pi/2]$. Let us recall these estimates and give the localization of $\mu(\alpha)$ on Figure 12.

Proposition 2.3 :	$\forall\alpha\in]0,2\pi[,$	$\mu(\alpha)\le\Theta_0,$	(1)
Proposition 4.1 :	$\forall\alpha\in]0,2\pi[,$	$\mu(\alpha)\le\frac{\alpha}{\sqrt3},$	(2)
Relation (2.15) :	$\forall\alpha\in]0,\frac{\pi}{2}[,$	$\mu(\alpha)\le\frac{\Theta_0}{\sin\alpha}-\frac{\cos\alpha}{4\sin\alpha}\phi(0)^4,$	(3)
Relation (3.6) :	$\forall\alpha\in]0,\pi[,$	$\mu(\alpha)\ge\Theta_0\frac{\alpha}{\pi},$	(4)
Proposition 4.2 :	$\forall\alpha\in]0,2\pi[,$	$\mu(\alpha)\le\frac{\alpha}{\sqrt{3+\alpha^2}}.$	(5)

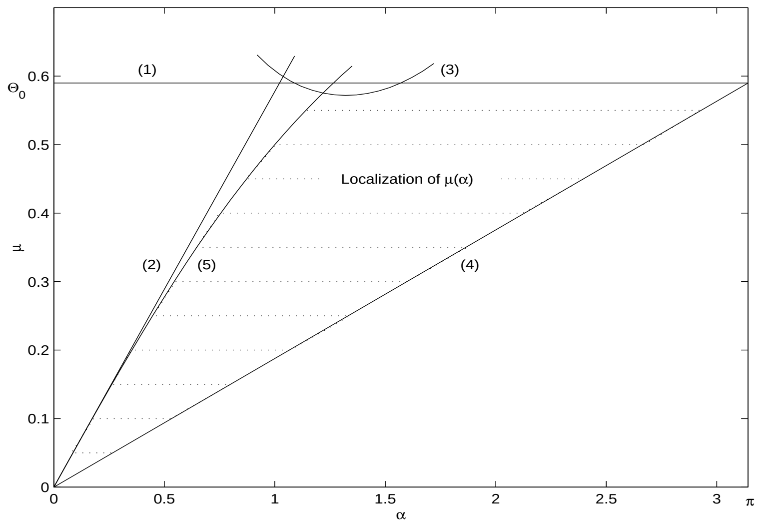

Figure 2: Localization of $\mu(\alpha)$

An open problem is to prove the monotony of $\mu(\alpha)$. Computations by physicists [20] confirmed by our own computations based on a finite elements method (cf [6]) indeed suggest that $\mu(\alpha)$ is increasing with α.
This paper also completes the results of Helffer-Morame [11], Jadallah [13], Pan [16] by dealing with the case of the Schrödinger operator with non constant magnetic field in a bounded open set with a curvilinear boundary.
Another point to establish is the localization of the ground state in the semi-classical case. The aim is to prove that this state is localized at the corners where the eigenvalue is the smallest. We hope to come back to this point in a future paper.

References

[1] S. Agmon : Lectures on exponential decay of solutions of second order elliptic equations. *Math. Notes* **29**, Princeton University Press (1982).

[2] P. Bauman, D. Phillips and Q. Tang : Stable nucleation for the Ginzburg-Landau system with an applied magnetic field. *Arch. Rational Mech. Anal.* **142**, p. 1-43 (1998).

[3] A. Bernoff and P. Sternberg : Onset of superconductivity in decreasing fields for general domains. *J. Math. Phys.* **39** (3), p. 1272-1284 (1998).

[4] P. Bolley et J. Camus : Sur une classe d'opérateurs elliptiques et dégénérés à une variable. *J. Math. Pures et Appl.* **51**, p. 429-463 (1972).

[5] V. Bonnaillie : On the fundamental state for a Schrödinger operator with magnetic field in a domain with corners. *C. R. Acad. Sci. Paris, Ser. I* **336** (2), p. 135-140 (2003).

[6] V. Bonnaillie : Analyse mathématique de la supraconductivité dans un domaine à coins : méthodes semi-classiques et numériques. *Thèse de Doctorat, Université Paris XI - Orsay* (2003).

[7] F. Brosens, J. T. Devreese, V. M. Fomin and V. V. Moshchalkov : Superconductivity in a wedge : analytical variational results. *Solid State Comm.* **111** (12), p. 565-569 (1999).

[8] M. Dauge and B. Helffer : Eigenvalues variation I, Neumann problem for Sturm-Liouville operators. *J. Differential Equations* **104** (2), p. 243-262 (1993).

[9] V. M. Fomin, J. T. Devreese and V. V. Moshchalkov : Surface superconductivity in a wedge. *Europhys. Lett.* **42** (5), p. 553-558 (1998).

[10] B. Helffer and A. Mohamed : Semiclassical analysis for the ground state energy of a Schrödinger operator with magnetic wells. *J. Funct. Anal.* **138** (1), p. 40-81 (1996).

[11] B. Helffer and A. Morame : Magnetic bottles in connection with superconductivity. *J. Funct. Anal.* **185**, p. 604-680 (2001).

[12] K. Hornberger and U. Smilansky : The boundary integral method for magnetic billiards. *J. Phys. A : Math. Gen.* **33**, p. 2829-2855 (2000).

[13] H-T. Jadallah : The onset of superconductivity in a domain with a corner. *J. Math. Phys.* **42** (9), p. 4101-4121 (2001).

[14] K. Lu and X. B. Pan : Estimates of the upper critical field for the Ginzburg-Landau equations of superconductivity. *Physica D* **127**, p. 73-104 (1999).

[15] K. Lu and X. B. Pan : Gauge invariant eigenvalue problems on $\mathbb{R}^2$ and $\mathbb{R}^2_+$. *Trans. A. M. S.* **352**, p. 1247-1276 (2000).

[16] X. B. Pan : Upper critical field for superconductors with edges and corners. *Calc. Var. Partial Differential Equations* **14**, p. 447-482 (2002).

[17] A. Persson : Bounds for the discrete part of the spectrum of a semi-bounded Schrödinger operator. *Math. Scand.* **8**, p. 143-153 (1960).

[18] M. del Pino, P. L. Felmer and P. Sternberg : Boundary concentration for eigenvalue problems related to the onset of superconductivity. *Comm. Math. Physics* **210**, p. 413-446 (2000).

[19] D. Saint-James and P. G. de Gennes : Onset of superconductivity in decreasing fields. *Phys. Let.* **7** (5), p. 306-308 (1963).

[20] V. A. Schweigert and F. M. Peeters : Influence of the confinement geometry on surface superconductivity. *Phys. Rev. B* **60** (5), p. 3084-3087 (1999).