

2 Metric Geometry

Exercise 2.1. Give an example of two (non-compact !) metric spaces X and Y such that $d_{GH}(X, Y) = 0$ but $X \neq Y$.

Exercise 2.2 (Topology). Suppose that $(X_n)_{n \geq 0}$ is a sequence of compact metric spaces such that X_n is homeomorphic to X_0 for every $n \geq 0$. Show that $X_n \rightarrow X$ in the sense of d_{GH} does not imply X homeomorphic to X_0 .

Exercise 2.3 (Gromov's Compactness Theorem). The goal of this exercise is to characterize pre-compactness for Gromov-Hausdorff distance. A collection \mathfrak{X} of compact metric spaces is totally bounded if

- (i) There exists $C > 0$ such that for every $X \in \mathfrak{X}$, the diameter of X is bounded above by C ,
- (ii) For every $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{Z}_+$ such that every $X \in \mathfrak{X}$ admits an ε -net containing no more than $N(\varepsilon)$ points.

1. Show that every pre-compact collection is totally bounded.

Now, let $(X_n)_{n \geq 1}$ be a totally bounded sequence of compact metric spaces and denote $N(\varepsilon)$ the minimal number of balls of radius ε needed to cover any space X_i . For every n , denote the distance in the space X_n by d_n and let $(x_{k,j}^{(n)})_{k,j \geq 1}$ be a sequence of points of X_n such that for every $k \geq 1$ the points

$$x_{k,1}^{(n)}, \dots, x_{k,N(1/k)}^{(n)}, \text{ form a } 1/k\text{-net in } X_n.$$

2. Show that we can extract a subsequence $(n_i)_{i \geq 1}$ such that for every $(k, j), (k', j') \in \mathbb{Z}_+^2$

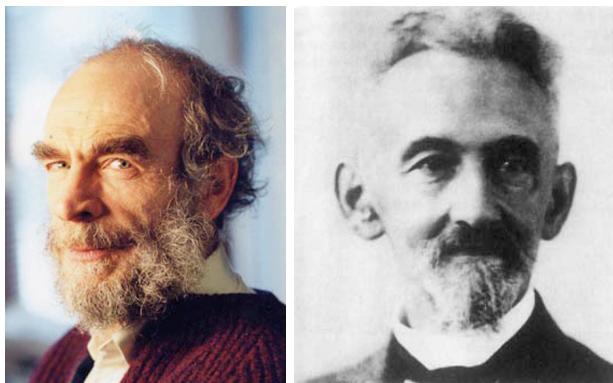
$$\lim_{n \rightarrow \infty} d_n(x_{k,j}^{(n)}, x_{k',j'}^{(n)}) \text{ exists.}$$

To simplify notation, we suppose that there is no need to take a subsequence. Consider the abstract space $X = \{(k, j)\}_{k,j \geq 1}$ endowed with $\Delta((k, j), (k', j')) = \lim d_n(x_{k,j}^{(n)}, x_{k',j'}^{(n)})$.

- 3. Show that Δ is a pseudo-metric on X and that the completion \overline{X} of X with respect to Δ is compact.
- 4. Prove that $X_n \rightarrow \overline{X}$ in the Gromov-Hausdorff sense.

Exercise 2.4. * Prove that a sequence of length spaces homeomorphic to the two-dimensional sphere \mathbb{S}_2 cannot converge to the standard two-dimensional closed ball \mathbb{B}_2 .

Exercise 2.5. *Who are these charming gentlemen ?*



References

- [BBI01] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.
- [Gro07] Misha Gromov. *Metric structures for Riemannian and non-Riemannian spaces*. Modern Birkhäuser Classics. Birkhäuser Boston Inc., Boston, MA, english edition, 2007. Based on the 1981 French original, With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.