Problem set 1

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Problem 1. Recall that a topological space X is *irreducible* if it is non-empty and is not the union of two strict closed subsets. In other words, if X_1 and X_2 are closed subsets of X and $X = X_1 \cup X_2$, then $X = X_1$ or $X = X_2$.

a) Let X be a topological space and let $V \subset X$ be a subset (endowed with the induced topology). Prove that V is irreducible if and only if its closure $\overline{V}$ is irreducible.

Proof. Assume that V is irreducible. If $\overline{V} = W_1 \cup W_2$, where W_1 and W_2 are closed in $\overline{V}$ (hence also in X), we have $V = (V \cap W_1) \cup (V \cap W_2)$, where $V \cap W_1$ and $V \cap W_2$ are closed in V. Since V is irreducible, we obtain $V = V \cap W_1$ or $V = V \cap W_2$, hence $V \subset W_1$ or $V \subset W_2$. Since W_1 and W_2 are closed in X, this implies $\overline{V} \subset W_1$ or $\overline{V} \subset W_2$, hence $\overline{V} = W_1$ or $\overline{V} = W_2$. This proves that $\overline{V}$ is irreducible.

For the converse, assume that $\overline{V}$ is irreducible. If $V = W_1 \cup W_2$, where W_1 and W_2 are closed in V, we can write $W_i = V \cap F_i$, where F_i is closed in X (we could take $F_i = \overline{W}_i$). We then have $V \subset F_1 \cup F_2$, hence $\overline{V} \subset F_1 \cup F_2$ and $\overline{V} = (\overline{V} \cap F_1) \cup (\overline{V} \cap F_2)$. Since $\overline{V}$ is irreducible, we obtain $\overline{V} = \overline{V} \cap F_1$ or $\overline{V} = \overline{V} \cap F_2$, hence $V \subset F_1$ or $V \subset F_2$ and $V = W_1$ or $V = W_2$. This proves that V is irreducible.

b) Let X and Y be topological spaces and let $u : X \to Y$ be a continuous map. If X is irreducible, prove that $u(X)$ is irreducible.

Proof. Assume $u(X) = W_1 \cup W_2$, where W_1 and W_2 are closed in $u(X)$. We can write $W_i = u(X) \cap F_i$, where F_i is closed in Y. We have then $u(X) \subset F_1 \cup F_2$, hence $X = u^{-1}(F_1) \cup u^{-1}(F_2)$. Since X is irreducible and $u^{-1}(F_i)$ is closed in X, we get $X = u^{-1}(F_1)$ or $X = u^{-1}(F_2)$, hence $u(X) \subset F_1$ or $u(X) \subset F_2$. This implies $u(X) = W_1$ or $u(X) = W_2$ and proves that $u(X)$ is irreducible.

Problem 2. Let $\mathbf{k}$ be an *infinite* (not necessarily algebraically closed) field. Let $C \subset \mathbf{k}^2$ be the vanishing set $V(X^2 - Y^3)$.

a) Prove that the ideal of C is the ideal in $\mathbf{k}[X, Y]$ generated by $X^2 - Y^3$ and that C is irreducible (*Hint*: use the "parametrization" $\mathbf{k} \to C$ given by $t \mapsto (t^3, t^2)$ and express $A(C) = \mathbf{k}[X, Y]/I(C)$ as a subring of $\mathbf{k}[T]$).

Proof. Obviously, $X^2 - Y^3$ is in $I(C)$. Assume that $P \in \mathbf{k}[X, Y]$ vanishes on C. We have $P(t^3, t^2)$ for all $t \in \mathbf{k}$. Since the field $\mathbf{k}$ is infinite, this implies that the *polynomial* $P(T^3, T^2) \in \mathbf{k}[T]$ vanishes. Modulo the ideal I generated by $X^2 - Y^3$, one can write $P \equiv A(Y) + XB(Y)$. We then have $A(T^2) + T^3 B(T^2) = 0$ in $\mathbf{k}[T]$. Since only even powers of T appear in $A(T^2)$ and only odd powers of T appear in $T^3 B(T^2)$, we obtain $A = B = 0$ and $P \in I$. This proves the opposite inclusion $I(C) \subset I$.

So we have $A(C) = \mathbf{k}[X,Y]/I$ and the parametrization $\mathbf{k} \to C$ induces an isomorphism between $A(C)$ and the subring $\mathbf{k}[T^2, T^3]$ of $\mathbf{k}[T]$. The latter is obviously an integral domain, hence so is $A(C)$ and C is irreducible.

b) Prove that C is not isomorphic to $\mathbf{k}$ (*Hint*: prove that $A(C)$ is not a principal ideal domain).

Proof. We saw in a) that $A(C)$ is isomorphic to the subring $\mathbf{k}[T^2, T^3]$ of $\mathbf{k}[T]$. Let J be the ideal of $A(C)$ generated by T^2 and T^3. If it is generated by one element $P(T) \in J \subset \mathbf{k}[T^2, T^3]$, we can write $T^2 = A(T)P(T)$ and $T^3 = B(T)P(T)$, hence $2 = \deg(A) + \deg(P)$ and $3 = \deg(B) + \deg(P)$, with $\deg(P) \geq 2$ (because $P \in J$). Since each of these degrees is different from 1, we get a contradiction.

c) How do these these results generalize to the vanishing set $V(X^r - Y^s)$, where r and s are relatively prime positive integers?

Proof. The conclusions are the same but the arguments are slightly more complicated. For a), a polynomial $P \in \mathbf{k}[X,Y]$ that vanishes on C is such that $P(T^s, T^r)$ vanishes in $\mathbf{k}[T]$. Write

$$P(X,Y) = P_0(Y) + XP_1(Y) + \cdots + X^{r-1}P_{r-1}(Y)$$

modulo the ideal I generated by $X^r - Y^s$. We then have

$$0 = P(T^s, T^r) = P_0(T^r) + T^s P_1(T^r) + \cdots + T^{s(r-1)} P_{r-1}(T^r) \in \mathbf{k}[T].$$

The powers of T that appear in the term $T^{si}P_i(T^r)$ are congruent to si modulo r. Since r and s are relatively prime, these numbers are all different modulo r for $i \in \{0, \ldots, r-1\}$. It follows that $P_i = 0$ for all i and $P \in I$. This proves $I(C) = I$ and $A(C) \simeq \mathbf{k}[T^s, T^r]$.

To prove that $A(C)$ is not a principal ideal domain, we may assume $r < s$. As a $\mathbf{k}$-vector space, $A(C)$ is generated by all monomial T^{mr+ns}, with m, n non-negative integers. When of degree $< s$, these monomials are of the type T^{mr}. If $T^r = A(T)P(T)$ and $T^s = B(T)P(T)$, we have $r = \deg(A) + \deg(P)$ and $s = \deg(B) + \deg(P)$, with $\deg(P) \geq r$, hence we may assume $P(T) = T^r$. This implies $B(T) = T^{s-r} \in \mathbf{k}[T^s, T^r]$. Since $s - r < s$, it must be a multiple of r, which is absurd.

Problem 3. Let $\mathbf{k}$ be an *infinite* (not necessarily algebraically closed) field, let $u \colon \mathbf{P}^1_{\mathbf{k}} \to \mathbf{P}^3_{\mathbf{k}}$ be the regular map defined by $u(s,t) = (s^3, s^2t, st^2, t^3)$, and set $C := u(\mathbf{P}^1_{\mathbf{k}})$.

a) Prove that no 4 distinct points of C are contained in a hyperplane in $\mathbf{P}^3_{\mathbf{k}}$.

Proof. A hyperplane has equation $L(x_0, x_1, x_2, x_3) = 0$, where L is a non-zero linear form. If L vanishes at 4 points of C, we have $L(s^3, s^2t, st^2, t^3) = 0$ for 4 distinct points (s,t) of $\mathbf{P}^1_{\mathbf{k}}$. But this is a non-zero homogeneous polynomial of degree 3 in two variables, hence it cannot have 4 distinct zeroes in $\mathbf{P}^1_{\mathbf{k}}$.

b) Prove that any quadric in $\mathbf{P}^3_{\mathbf{k}}$ (i.e., any subset of $\mathbf{P}^3_{\mathbf{k}}$ defined by a non-zero homogoneous polynomial of degree 2) that contains 7 distinct points of C contains C.

Proof. A quadric has equation $Q(x_0, x_1, x_2, x_3) = 0$, where Q is a non-zero quadratic form. If Q vanishes at 7 points of C, we have $Q(s^3, s^2t, st^2, t^3) = 0$ for 7 distinct points (s,t) of $\mathbf{P}^1_{\mathbf{k}}$. But this is a non-zero homogeneous polynomial of degree 6 in two variables, hence it cannot have 7 distinct zeroes in $\mathbf{P}^1_{\mathbf{k}}$.

c) Prove that C is the vanishing set in $\mathbf{P}^3_{\mathbf{k}}$ of the (homogeneous) ideal I in $\mathbf{k}[T_0, T_1, T_2, T_3]$ generated by the homogeneous polynomials $T_0T_2 - T_1^2, T_2^2 - T_1T_3, T_1T_2 - T_0T_3$, which can be neatly expressed as the 2×2-minors of the matrix

$$\begin{pmatrix} T_0 & T_1 & T_2 \\ T_1 & T_2 & T_3 \end{pmatrix}.$$

Proof. The set C is contained in $V(I)$. Conversely, assume $x := (x_0, x_1, x_2, x_3) \in V(I)$. If $x_0 \neq 0$, we may take $x_0 = 1$ and we have $x_2 = x_1^2$ and $x_3 = x_1x_2 = x_1^3$, hence $x = u(1, x_1) \in C$. If $x_0 = 0$, we have $x_1 = 0$, $x_2 = 0$, hence $x = u(0, 1) \in C$.

d) Prove that the ideal of C is I (*Hint*: prove that any polynomial $P \in \mathbf{k}[T_0, T_1, T_2, T_3]$ is congruent modulo I to a polynomial of the type $A(T_0, T_1, T_3) + T_2B(T_3)$ and that if P vanishes on C, one has $B = 0$; then, use a similar method to show that A is divisible by $T_1^3 - T_0^2T_3$).

Proof. The inclusion $I \subset I(C)$ is clear. Using the fact that I contains $T_0T_2 - T_1^2, T_1T_2 - T_0T_3, T_2^2 - T_1T_3$, we reduce modulo I any polynomial P to the form $A(T_0, T_1, T_3) + T_2B(T_3)$. If P vanishes on C, we obtain as in Problem 2 (using the fact that $\mathbf{k}$ is infinite) $A(S^3, S^2T, T^3) + ST^2B(T^3) = 0$ in $\mathbf{k}[S, T]$. Monomials of the type ST^m only appear in $ST^2B(T^3)$, hence $B = 0$ and $A(S^3, S^2T, T^3) = 0$. The polynomial $T_1^3 - T_0^2T_3 = -T_1(T_0T_2 - T_1^2) + T_0(T_1T_2 - T_0T_3)$ is in I, hence one can write $A(T_0, T_1, T_3) \equiv A_0(T_0, T_3) + T_1A_1(T_0, T_3) + T_1^2A_2(T_0, T_3) \pmod I$, with $A_0(S^3, T^3) + S^2TA_1(S^3, T^3) + S^4T^2A_2(S^3, T^3) = 0$. Looking at the exponents of T that appear in this polynomial modulo 3, we obtain $A_0 = A_1 = A_2$, hence $A \in I$. This proves the opposite inclusion $I(C) \subset I$.

e) **(Extra credit)** How do these results generalize to the regular map $u\colon \mathbf{P}^1_{\mathbf{k}} \to \mathbf{P}^n_{\mathbf{k}}$ ($n \geq 3$) defined by $u(s, t) = (s^n, s^{n-1}t, \ldots, st^{n-1}, t^n)$?