Preface

The ternary Goldbach conjecture (or *three-prime conjecture*) states that every odd number n greater than 5 can be written as the sum of three primes. The main purpose of this book is to give the first full proof of this conjecture.

The text is intended as a definitive, peer-reviewed form of the proof, which first appeared in 2012 and 2013 as a series of preprints [Helb], [Hela], [Helc]. It is also meant to be an accessible account.

The proof builds on the great advances made in the early 20th century by Hardy and Littlewood (1922) and Vinogradov (1937). Progress since then has been gradual. In some ways, it proved necessary to proceed independently of more recent developments and work using only the main existing ideas towards the problem, together with techniques that were originally developed for many other purposes.

Part of the aim has been to keep the exposition as accessible as possible, with an emphasis on qualitative improvements and technical ideas that should be of use elsewhere. The guiding idea was to give an analytic approach that is efficient, relatively clean, and, as it must be for this problem, explicit; the focus does not lie in optimizing explicit constants, or in performing calculations, necessary as these tasks are.

The minimal background required of the reader is that typically possessed by a graduate student who has taken a course in analytic number theory and feels comfortable with basic real and complex analysis. For instance, when we define L-functions and the Fourier transform, we really do it so as to set the terminology once and for all; one would hope all readers are already acquainted with them, and need at most a quick refresher of their basic properties. However, it is not assumed that all readers are conversant with the circle method, exponential-sum estimates, the large sieve, etc. It will actually be convenient to develop some of these matters from first principles.

It should thus be feasible to use the text to teach (or teach oneself) the circle method – at least considered as a way to address additive problems involving primes, as opposed to Diophantine problems. For that matter, it is not hard to see how a proper subset of the book could be used as the basis for a more general advanced course in analytic number theory, or as a reference for foundational material in explicit analytic number theory.

Besides discussing the actual path we follow, we will sometimes mention roads not taken. We will also attempt to distinguish what is traditional and what is new – or relatively new, in the sense of being recent work done in part by people other than the author. The intent is to orient students and save the time of specialist readers.

My perspective is that of an analytic number theorist writing for others with an interest in number theory as a whole, and not solely in its computational or numerical aspects. There are doubtlessly achievements of importance in explicit analytic number theory that we will not discuss, or barely mention, simply because the bounds they give, while remarkable, are not useful here. We will have to answer the following questions, natural to an analytic number theorist: which basic tools in the field can be made explicit at present, in a way that is strong enough to be generally useful? Which standard paths need to be avoided until further progress is made? How do we proceed so that our reasoning is transparent, while our results are set out precisely and in full?

Organization. In the introduction, after a summary of the history of the problem, we will go over a detailed outline of the proof, giving, for each part, first the general strategy and main ideas, followed by a guide to the text.

Part I sets out the foundations we need for all that follows. We devote a chapter to general preliminaries – notation and basic material from analysis and analytic number theory – and another one to computational issues, with an emphasis on rigor as well as efficiency. The last chapter of Part I is dedicated to basic estimates on sums of arithmetic functions. These estimates require a mixture of analytic and computational work.

The rest of the book is divided into four parts, structured so that they can be read independently. Part II offers an explicit treatment of a natural quadratic sieve (Ch. 7), and material on the large sieve that is in part new (Ch. 8–9). The proof of the ternary Goldbach conjecture may be said to begin properly with Part III. As is the case in most proofs involving the circle method, the problem is quickly reduced to showing that a certain integral over the "circle" $\mathbb{R}/\mathbb{Z}$ is non-zero. The circle is divided into major arcs and minor arcs. In Part III – in some ways the technical heart of the proof – we will see how to give upper bounds on the integrand when α is in the minor arcs. Part IV will provide rather precise estimates for the integrand when the variable α is in the major arcs. Lastly, Part V shows how to use these inputs efficiently to estimate the integral.

Each chapter starts with a general discussion of the strategy and the main ideas involved. Some of the more technical bounds and computations are relegated to the appendices.

Göttingen and Paris, December 2019 H. A. HELFGOTT