Received: 2 July 2017 | Revised: 11 January 2018 | Accepted: 23 March 2018

DOI: 10.1002/mana.201700251

ORIGINAL PAPER

MATHEMATISCHE NACHRICHTEN

Complete families of smooth space curves and strong semistability

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Abstract

We construct the first non-trivial examples of complete families of non-degenerate smooth space curves, and show that the base of such a family cannot be a rational curve. Both results rely on the study of the strong semistability of certain vector bundles.

KEYWORDS

Hilbert scheme, Hilbert–Kunz multiplicity, space curves, strong semistability, syzygy bundles

MSC (2010)

13A35, 13D40, 14C05, 14H50, 14H60

1 | INTRODUCTION

We work over an algebraically closed field $\mathbb{k}$. A *curve* is a projective connected one-dimensional variety over $\mathbb{k}$. If B is an integral variety over $\mathbb{k}$, a *family of smooth space curves* over B is a closed subvariety $C \hookrightarrow \mathbb{P}^3_B := \mathbb{P}^3 \times B$ such that $C \to B$ is a smooth family of curves. Equivalently, it is a morphism from B to the Hilbert scheme of smooth curves in $\mathbb{P}^3$. Such a family will be said to be *trivial* if all its fibers are isomorphic as subvarieties of $\mathbb{P}^3$; in other words, if the induced morphism from B to the Hilbert scheme of $\mathbb{P}^3$ is constant. It is said to be *isotrivial* if all its fibers are isomorphic as abstract curves. We are interested in *complete* families: those whose base B is proper.

The family of lines parametrized by the Grassmannian is a non-trivial complete family of smooth space curves. It is also easy to construct (necessarily isotrivial) non-trivial complete families whose members are plane curves [2, Proposition 2.1]. For this reason, we will restrict our attention to families parametrizing *non-degenerate* space curves, that is curves whose linear span is $\mathbb{P}^3$.

Non-trivial complete families of non-degenerate smooth space curves have been studied by Chang and Ran [5,6]. They showed that the curves parametrized by the family can be neither rational nor elliptic [6, Theorem 3]. They also proved that every such family comes by base-change from a family over a curve [6, Theorem 1], so that one may restrict the study to this case.

However, they do not provide examples of such families. The existence of non-trivial complete families of non-degenerate smooth space curves is also stated as an open question in [13, p. 57]. Our main goal is to construct examples.

Theorem 1.1.

- **(i)** *There exist non-trivial complete families of non-degenerate smooth space curves over any elliptic curve.*
- **(ii)** *If $\mathbb{k}$ has characteristic p with $p \equiv \pm 1[8]$, there is a non-trivial complete family of non-degenerate smooth space curves over a smooth curve of genus ≥ 2 that does not come by base-change from a family over a curve of genus ≤ 1.*

The curves parametrized by our families have genus 2 and degree 5. As the moduli space of smooth curves of genus 2 is affine [17], such families are necessarily isotrivial. It is the degree 5 line bundle providing the embedding that varies in the family. In view of Chang and Ran's result [6, Theorem 3], those examples are minimal: they have both smallest genus and smallest degree possible.

Theorem 1.1 (i) is also optimal in the sense that the genus of the base is minimal:

Theorem 1.2. *There are no non-trivial complete families of non-degenerate smooth space curves over $\mathbb{P}^1$.*

Theorem 1.1 shows the existence of elliptic curves (and, when $\mathrm{char}(\Bbbk) \equiv \pm 1[8]$, of a curve of genus ≥ 2) in the Hilbert scheme of non-degenerate smooth space curves: it fits into the classical theme of constructing complete subvarieties of moduli spaces initiated by Oort [27].

We do not know how to remove the hypothesis on $\Bbbk$ in Theorem 1.1 (ii) (see Remark 2.12). We also leave open the question whether there exist non-isotrivial complete families of smooth space curves. Since there do not exist non-isotrivial complete families of smooth curves over curves of genus ≤ 1 [29, Théorème 4], Theorem 1.1 (ii) may be viewed as a first step towards constructing non-isotrivial complete families of smooth space curves.

In Section 2, we study when an abstract family of smooth polarized curves over a smooth projective curve gives rise to a non-trivial family of non-degenerate smooth space curves. We obtain necessary conditions in Proposition 2.7 and sufficient conditions in Proposition 2.9 that yield proofs of Theorem 1.2 and Theorem 1.1, respectively. A key role is played by the strong semistability of some vector bundles on the base, and §2.1 is devoted to recalling generalities on strong semistability.

The proof of Theorem 1.1 (ii) in Section 2 requires to verify the strong semistability of some vector bundles. We postpone this important step to Section 3. Our strategy there is to ensure that the relevant bundles are syzygy bundles (see Definition 3.2). The strong semistability of such bundles has been related by Brenner [3] and Trivedi [30] to Hilbert–Kunz multiplicities (see Definition 3.15 and Theorem 3.16). In our situation, we do not know how to compute the relevant Hilbert–Kunz multiplicities directly, as Han and Monsky did for Fermat curves [10,11,24]. Instead, we take inspiration from [4], where Brenner and Kaid obtain stronger results than strong semistability (explicit Frobenius periodicity up to a twist) for some syzygy bundles over Fermat curves. The strategy of [4] uses crucially the semistability of the syzygy bundles, that is known thanks to Han and Monsky. We need to replace these arguments by different ones: explicit syzygy computations using the strong Lefschetz property of appropriate homogeneous ideals (see §3.2). A benefit of our method is that it allows us to give new examples of how Hilbert–Kunz multiplicities vary with the characteristic of the base field in Theorem 3.17.

2 | EMBEDDING ABSTRACT FAMILIES

2.1 | Strong semistability

If $\Bbbk$ is of positive characteristic, and X is a variety over $\Bbbk$, we denote by $F : X \to X$ the absolute Frobenius morphism.

Definition 2.1. A vector bundle $\mathcal{E}$ on a smooth curve B is *strongly semistable* if $\Bbbk$ is of characteristic 0 and $\mathcal{E}$ is semistable, or if $\Bbbk$ is of positive characteristic and for every $k \geq 0$, $F^{k*}\mathcal{E}$ is semistable.

Unlike semistability, strong semistability is always preserved by finite base-change, tensor products and symmetric powers [21, 2.2.2, 2.2.3]. The following important theorem is due to Langer [22, Theorem 2.7]:

Theorem 2.2. *Let $\mathcal{E}$ be a vector bundle on a smooth curve B. Then there exists a finite morphism from a smooth curve $f : B' \to B$ such that the graded pieces of the Harder–Narasimhan filtration of $f^*\mathcal{E}$ are strongly semistable.*

Such a filtration will be called a *strong Harder–Narasimhan filtration*. In characteristic 0, the Harder–Narasimhan filtration is always strong. Over elliptic curves, the situation is very simple:

Proposition 2.3. *Let $\mathcal{E}$ be an indecomposable vector bundle over an elliptic curve.*

(i) *$\mathcal{E}$ is strongly semistable,*

(ii) *$\mathcal{E}$ is stable if and only if its degree is prime to its rank.*

Proof. In the first statement, the semistability of $\mathcal{E}$ is proved in [15, Lemma 1]. The strong semistability then follows from the more general [23, Theorem 2.1].

A semistable vector bundle whose rank and degree are prime to each other is clearly stable. Conversely, when the degree and the rank of $\mathcal{E}$ are not prime to each other, Oda has proved [25, Corollary 2.5] that $\mathcal{E}$ is not simple, hence not stable. □

We will need conditions ensuring that a vector bundle becomes isomorphic to a direct sum of isomorphic line bundles after an appropriate base-change. This is the goal of the two following propositions. The first one might be well known, but I do not know a reference for it. The second one is the Lange–Stuhler theorem.

Proposition 2.4. *Let $\mathcal{E}$ be a stable vector bundle over an elliptic curve E. Then there exists an isogeny $f : E' \to E$ such that $f^*\mathcal{E}$ is isomorphic to a direct sum of isomorphic line bundles.*

Proof. By Proposition 2.3 (i), the pull-back of $\mathcal{E}$ by any isogeny is semistable.

We first claim that there exists an isogeny $f : E' \to E$ such that $f^*\mathcal{E}$ is isomorphic to a direct sum of line bundles. To prove it, let $f : E' \to E$ be an isogeny whose degree is divisible by the rank of $\mathcal{E}$. Write $f^*\mathcal{E}$ as a direct sum of indecomposable bundles. If $\Bbbk$ is of characteristic 0, those indecomposable bundles are all stable of the same slope by [16, Lemma 3.2.3], and Proposition 2.3 (ii) implies that they are line bundles. If $\Bbbk$ is of positive characteristic p, Proposition 2.3 (ii) shows that $f^*\mathcal{E}$ cannot be stable. Considering a Jordan–Hölder filtration for $f^*\mathcal{E}$ and using induction on the rank of $\mathcal{E}$, it is possible to suppose that all the graded pieces of this filtration have rank 1. Now, extensions between line bundles of the same degree are trivial if the line bundles are not isomorphic, and parametrized by $H^1\left(E', \mathcal{O}_{E'}\right)$ otherwise. Let $[p] : E' \to E'$ denote the multiplication by p isogeny. Since the dual of $[p]$ (that is $[p]$ itself) is not separable, the pull-back map $[p]^* : H^1\left(E', \mathcal{O}_{E'}\right) \to H^1\left(E', \mathcal{O}_{E'}\right)$ vanishes. Base-changing by an appropriate power of $[p]$ thus splits all extensions appearing in the Jordan–Hölder filtration, and proves our claim.

It remains to prove that, up to another base-change by an isogeny, all these line bundles are isomorphic. Let us write $f^*\mathcal{E} \simeq \bigoplus_i \mathcal{F}_i$, where the $\mathcal{F}_i$ are the isotypical factors: each $\mathcal{F}_i$ is the direct sum of isomorphic line bundles. Write $f = g \circ h$, where $h : E' \to F$ is separable of Galois group G and $g : F \to E$ is purely inseparable. If the group G did not act transitively on the isotypical factors, a non-trivial direct sum $\mathcal{G}$ of some of them would descend to F by Galois descent. Since, $\mathrm{Hom}\left(\mathcal{G}, f^*\mathcal{E}/\mathcal{G} \otimes \Omega^1_{E'}\right) = \mathrm{Hom}(\mathcal{G}, f^*\mathcal{E}/\mathcal{G}) = 0$, inseparable descent [18, Theorem 5.1] shows that this sheaf descends even to E, contradicting the stability of $\mathcal{E}$. Hence G permutes transitively the isotypical components. But since G acts on E' as a finite subgroup of translations, it follows that the line bundles appearing in $\mathcal{E}$ differ from each other by torsion line bundles. Hence all the line bundles appearing become isomorphic after further pull-back by a well-chosen isogeny. □

Proposition 2.5. *Let $\mathcal{E}$ be a vector bundle on a smooth curve B over the algebraic closure of a finite field. Then the following conditions are equivalent:*

- **(i)** *$\mathcal{E}$ is strongly semistable.*
- **(ii)** *There exists a finite morphism from a smooth curve $f : B' \to B$ such that $f^*\mathcal{E}$ is isomorphic to a direct sum of isomorphic line bundles.*

Proof. If (ii) holds, the vector bundle $f^*F^{k*}\mathcal{E} = F^{k*}f^*\mathcal{E}$ is semistable as a direct sum of isomorphic line bundles. This implies that $F^{k*}\mathcal{E}$ is semistable, proving (i).

Let us explain the other implication, due to Lange and Stuhler [20]. First, it is easy to find a finite morphism from a smooth curve $g : B'' \to B$ and a line bundle $\mathcal{N}$ on B'' such that $g^*\mathcal{E} \otimes \mathcal{N}$ has degree 0. By our hypothesis on the base field, the strongly semistable vector bundle $g^*\mathcal{E} \otimes \mathcal{N}$ is trivialized by a finite surjective morphism $h : B' \to B''$ by [20, Satz 1.9]. Setting $f = g \circ h$, one sees that $f^*\mathcal{E}$ is a direct sum of line bundles isomorphic to $h^*\mathcal{N}^{-1}$. □

2.2 | The Harder–Narasimhan filtration

We start with a lemma:

Lemma 2.6. *Let B be a smooth curve, let $\pi : C \to B$ be a smooth projective family of curves over B and let $\mathcal{L}$ be a line bundle on C. Then $\mathcal{E} := \pi_*\mathcal{L}$ is locally free and its formation commutes with base-change by any finite map from a smooth curve $B' \to B$. Moreover, for every $b \in B$, the natural map $\mathcal{E}|_b \to H^0(C_b, \mathcal{L}_b)$ is injective.*

Proof. The sheaf $\mathcal{E}$ is locally free as a torsion-free coherent sheaf over a smooth curve. The second statement is a consequence of flat base-change [14, III Proposition 9.3]. As for the third statement, consider the exact sequence $0 \to \mathcal{O}_B(-b) \to \mathcal{O}_B \to \mathcal{O}_B|_b \to 0$. Pull it back to C, tensor with $\mathcal{L}$ and push it forward to B to get an exact sequence $0 \to \mathcal{E}(-b) \to \mathcal{E} \to \mathrm{Im}\left(\mathcal{E} \to H^0(C_b, \mathcal{L}_b)\right) \to 0$. Restricting it to b using right-exactness of tensor product, and noticing that the morphism $\mathcal{E}(-b)|_b \to \mathcal{E}|_b$ vanishes, one sees that $\mathcal{E}|_b \to H^0(C_b, \mathcal{L}_b)$ is indeed injective. □

In the following proposition, we make use of the *secant variety* $S \subset \mathbb{P}^4$ of a smooth curve $C \subset \mathbb{P}^4$, which is the union of all lines in $\mathbb{P}^4$ that meet C with multiplicity ≥ 2.

Proposition 2.7. *Let $\pi : C \to B$, $\phi : C \hookrightarrow \mathbb{P}^3_B$ be a non-trivial complete family of non-degenerate smooth space curves over a smooth curve B, and $\mathcal{E} := \pi_*\phi^*\mathcal{O}_{\mathbb{P}^3}(1)$. Then the constant subbundle $\mathcal{O}_B^{\oplus 4} \subset \mathcal{E}$ with fibers $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$ is the first step of the Harder–Narasimhan filtration of $\mathcal{E}$.*

Proof. We argue by contradiction and suppose that the conclusion does not hold. The idea of the proof is to use the hypothesis that $\mathcal{O}_B^{\oplus 4} \subset \mathcal{E}$ is not the first step of the Harder–Narasimhan filtration of $\mathcal{E}$ to produce an embedding of C in a four-dimensional projective bundle over B, and to derive a contradiction by studying geometrically this embedding. At any point of the proof, we may replace B by a finite cover by a smooth curve B' because the formation of $\mathcal{E}$ commutes with this base-change by Lemma 2.6.

Let $\mathcal{Q}$ be the quotient of $\mathcal{E}$ by $\mathcal{O}_B^{\oplus 4}$. Using Theorem 2.2, perform a base-change to ensure that the strong Harder–Narasimhan filtration of $\mathcal{Q}$ is defined over B. Since $\mathcal{O}_B^{\oplus 4}$ is not the first step of the Harder–Narasimhan filtration of $\mathcal{E}$, the first step of the Harder–Narasimhan filtration of $\mathcal{Q}$ has nonnegative degree.

Choosing a field of definition of finite type, spreading out and specializing to a general closed point, we get data defined over a finite field. It still contradicts the proposition, as $\mathcal{Q}$ still has a subbundle of nonnegative degree after such a general specialization. Consequently, we may suppose that $\mathbb{k}$ is the algebraic closure of a finite field. As above, we may assume that the strong Harder–Narasimhan filtration of $\mathcal{Q}$ is defined over B and that its first step has nonnegative degree.

Base-changing again using Proposition 2.5, we may assume that this subbundle is a direct sum of line bundles of nonnegative degree. In particular, $\mathcal{Q}$ contains a subbundle $\mathcal{M}$ of rank 1 of nonnegative degree. Consequently, there exists a subbundle $\mathcal{F}$ of $\mathcal{E}$ that is an extension of a line bundle of nonnegative degree $\mathcal{M}$ by $\mathcal{O}_B^{\oplus 4}$. Base-changing using Theorem 2.2, the strong Harder–Narasimhan filtration of $\mathcal{F}$ is defined over B. Let $\mathcal{G} \subset \mathcal{F}$ be the first step of this filtration.

Now, let us use $\mathcal{F}$ to embed C in a relative projective bundle over B: we get an immersion $\psi : C \to \mathbb{P}_B\mathcal{F}$. Moreover, one recovers the original embedding ϕ by projecting away from $\mathbb{P}_B\mathcal{M}$. Note that, as $\mathcal{F}|_b \to H^0(C_b, \mathcal{L}_b)$ is injective by Lemma 2.6, ψ embeds all fibers of π in a non-degenerate way in $\mathbb{P}^4$. Let us introduce the relative secant variety $\mathcal{S} \hookrightarrow \mathbb{P}_B\mathcal{F}$ that is the union of the secant varieties of the embedded curves $C_b \hookrightarrow \mathbb{P}\mathcal{F}_b$. It is a hypersurface of $\mathbb{P}_B\mathcal{F}$ because secant varieties of non-degenerate curves in $\mathbb{P}^4$ are of dimension 3. It does not meet $\mathbb{P}_B\mathcal{M}$ because, for every $b \in B$, the linear system $H^0\big(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)\big)$ induced an embedding of C_b.

Let $q : \mathbb{P}_B\mathcal{F} \to B$ be the projection and $\mathcal{O}_q(1)$ be the relative tautological bundle. By description of the Picard group of a projective bundle, there exist $\mathcal{A} \in \mathrm{Pic}(B)$ and $l \in \mathbb{Z}$ such that $\mathcal{S}$ is the zero-locus of a section $\sigma \in H^0\big(\mathbb{P}_B\mathcal{F}, \mathcal{O}_q(l) \otimes q^*\mathcal{A}\big) = H^0\big(B, \mathrm{Sym}^l \mathcal{F} \otimes \mathcal{A}\big)$. That $\mathcal{S}$ does not meet $\mathbb{P}_B\mathcal{M}$ means exactly that σ induces a nowhere vanishing section of $\mathcal{M}^{\otimes l} \otimes \mathcal{A}$ on B. In particular, $\mathcal{A} \simeq \mathcal{M}^{\otimes -l}$.

We distinguish three cases. Suppose first that $\mu(\mathcal{G}) < \mu(\mathcal{M})$, so that the graded pieces $\mathcal{G}_i$ of the strong Harder–Narasimhan filtration of $\mathcal{F}$ all have slope $< \mu(\mathcal{M})$. This filtration induces a filtration of $\mathrm{Sym}^l \mathcal{F}$ whose graded pieces are tensor products of symmetric powers of the $\mathcal{G}_i$: these are strongly semistable of slope $< \mu\big(\mathcal{M}^{\otimes l}\big)$. Consequently, $H^0\big(B, \mathrm{Sym}^l \mathcal{F} \otimes \mathcal{M}^{\otimes -l}\big) = 0$, which is a contradiction.

Next, suppose that $\mu(\mathcal{G}) \geq \mu(\mathcal{M}) > 0$. The morphism $\mathcal{G} \to \mathcal{M}$ cannot be zero as there are no non-zero morphisms $\mathcal{G} \to \mathcal{O}_B^{\oplus 4}$ by semistability of $\mathcal{G}$. Again by semistability of $\mathcal{G}$, this morphism has to be surjective. Then $\mathcal{G}$ is an extension of $\mathcal{M}$ by a subbundle of $\mathcal{O}_B^{\oplus 4}$, and the inequality $\mu(\mathcal{G}) \geq \mu(\mathcal{M})$ implies that $\mathcal{G} \to \mathcal{M}$ is an isomorphism. Hence $\mathcal{F}$ splits as a direct sum $\mathcal{O}_B^{\oplus 4} \oplus \mathcal{M}$. The space $H^0\big(B, \mathrm{Sym}^l \mathcal{F} \otimes \mathcal{M}^{\otimes -l}\big)$ is one-dimensional because $\mu(\mathcal{M}) > 0$, and the zero locus of one of its sections on a fiber of q is a hyperplane with multiplicity l. This contradicts the fact that, the curve C_b being embedded in a non-degenerate way in $\mathbb{P}^4$, its secant variety is also non-degenerate.

Finally, suppose that $\mu(\mathcal{M}) = 0$. Then $\mathcal{F}$ is strongly semistable as an extension of strongly semistable bundles of the same degree. Applying Proposition 2.5, we may assume that $\mathcal{F}$ is a direct sum of isomorphic line bundles, so that $\mathbb{P}_B\mathcal{F} \simeq \mathbb{P}^4_B$. The relative secant variety $\mathcal{S}$ is then a hypersurface of $\mathbb{P}^4_B$ avoiding a constant section. It follows that $\mathcal{S}$ is a product hypersurface, isomorphic to $S \times B$ where $S \subset \mathbb{P}^4$ is a hypersurface. Consequently, S is the secant variety of all curves $C_b \hookrightarrow \mathbb{P}^4$. Recall that Chang and Ran [6, Theorem 3] proved that the curves C_b have genus ≥ 2. Hence, by Lemma 2.8 below, there are only finitely many possibilities for the curves $C_b \subset \mathbb{P}^4$, and the subvariety $\psi : C \hookrightarrow \mathbb{P}^4_B$ has to be a product itself. Since the original family ϕ is obtained by projecting away from a constant section, it follows that our original family was a product, contradicting its non-triviality. □

We have used the following lemma:

Lemma 2.8. *Let $C \subset \mathbb{P}^4$ be a smooth non-degenerate curve of genus at least 2, and let S be its secant variety. Then there is a unique family of lines that covers S, namely the 2-dimensional family of secants of C. Moreover, C is an irreducible component of the set of points included in infinitely many of these lines.*

Proof. Let P be the $\mathbb{P}^1$-bundle over the two-fold symmetric product $C^{(2)}$ of C whose fiber over $(x, y) \in C^{(2)}$ is the line through x and y if $x \neq y$ (resp. the tangent at x if $x = y$). The natural surjective morphism $p : P \to S$ is birational by [7]. As this claim is not explicitly stated by Dale, we explain how to deduce it from [7].

To do so, we introduce a few notation. Let Q be the pull-back of P by the degree 2 morphism $C^2 \to C^{(2)}$: it is a $\mathbb{P}^1$-bundle over C^2. Define $M(C)$ to be the set of triples (x, z, l), where $x \in C$, $z \in \mathbb{P}^4$, and l is a line containing x and z that is secant to C. The morphism $r : Q \to M(C)$ sending a point z on the line l over (x, y) to (x, z, l) is birational by [7, Theorem 1.8]. Define $SB(C)$ to be the set of pairs (z, l) where l is a line secant to C and $z \in l$. The morphism $s : M(C) \to SB(C)$ defined by $(x, z, l) \mapsto (z, l)$ is separable of degree 2 by [7, Theorem 1.8, Lemma 3.5], and the morphism $t : SB(C) \to S$ defined by $t(z, l) = z$ is birational by [7, Theorem 4.1, Theorem 1.10]. The composition $t \circ s \circ r : Q \to S$ then has degree 2. Since it factors as the composition of the natural degree 2 morphism $Q \to P$ and of $p : P \to S$, it follows that p is indeed birational.

Since C has genus ≥ 2, the Abel–Jacobi map shows that $C^{(2)}$ contains only finitely many rational curves. Hence, the only family of rational curves that covers S is the one induced by the fibers of the $\mathbb{P}^1$-bundle structure, that is the family of secants of C. The subset of S consisting of points included in infinitely many of these secants is the image by t of the union of the positive-dimensional fibers of t. A dimension count shows that it is an algebraic variety of dimension at most 1. Since C is obviously contained in it, C has to be an irreducible component of this locus. □

Proposition 2.7 gives necessary conditions for a polarized family $(\pi : \mathcal{C} \to B, \mathcal{L})$ to induce a non-trivial family of non-degenerate smooth space curves $\phi : \mathcal{C} \hookrightarrow \mathbb{P}^3_B$ with $\mathcal{L} \simeq \phi^*\mathcal{O}_{\mathbb{P}^3}(1)$: the first graded piece of the Harder–Narasimhan filtration of $\mathcal{E} := \pi_*\mathcal{L}$ has to be of rank 4, and the corresponding sections have to induce embeddings of the fibers of π in $\mathbb{P}^3$. The proof of Theorem 1.2 follows:

Proof of Theorem 1.2. Let $\pi : \mathcal{C} \to \mathbb{P}^1$, $\phi : \mathcal{C} \hookrightarrow \mathbb{P}^3_{\mathbb{P}^1}$ be a complete family of non-degenerate smooth space curves over $\mathbb{P}^1$. It is isotrivial by [29, Théorème 4]: all the fibers of π are isomorphic to a fixed curve C. By Chang and Ran [6, Theorem 3], C has genus ≥ 2. Since the automorphism group of C is finite [28] and $\mathbb{P}^1$ is simply connected, the family has to be a product: $\mathcal{C} \simeq C \times \mathbb{P}^1$.

Since the Picard scheme $\mathrm{Pic}(C)$ does not contain non-trivial rational curves, all the fibers are even isomorphic as polarized curves and $\phi^*\mathcal{O}_{\mathbb{P}^3}(1) \simeq \mathcal{M} \boxtimes \mathcal{N}$ for some line bundles $\mathcal{M}$ (resp. $\mathcal{N}$) on C (resp. $\mathbb{P}^1$). Consequently, $\mathcal{E} := \pi_*\phi^*\mathcal{O}_{\mathbb{P}^3}(1)$ is isomorphic to a direct sum of isomorphic line bundles, hence is strongly semistable. It follows from Proposition 2.7 that the subbundle of $\mathcal{E}$ used to construct the embedding ϕ is $\mathcal{E}$ itself, so that our family is trivial. □

2.3 | Constructing embeddings

We now provide a sufficient condition for an abstract family of curves to give rise to a complete family of non-degenerate smooth space curves, up to maybe replacing the base by a finite surjective cover.

Proposition 2.9. *Let $\pi : \mathcal{C} \to B$ be a smooth projective family of curves over a smooth projective curve. Let $\mathcal{L}$ be a line bundle on $\mathcal{C}$ and $\mathcal{E} := \pi_*\mathcal{L}$. Let $\mathcal{F} \subset \mathcal{E}$ be a subbundle of rank 4 such that for every $b \in B$, $\mathcal{F}|_b \subset H^0(\mathcal{C}_b, \mathcal{L}_b)$ embeds $\mathcal{C}_b$ in $\mathbb{P}^3$.*

Suppose that one of the following conditions is satisfied:

- **(i)** *$\mathcal{F}$ is stable and B is an elliptic curve,*
- **(ii)** *$\mathcal{F}$ is strongly semistable and $\Bbbk$ is the algebraic closure of a finite field.*

Then there exist a finite morphism from a smooth curve $f : B' \to B$ and, denoting by $(\pi' : \mathcal{C}' \to B', \mathcal{L}')$ the base-change, a complete family of non-degenerate smooth space curves $\phi' : \mathcal{C}' \hookrightarrow \mathbb{P}^3_{B'}$ such that $\mathcal{L}'|_{\mathcal{C}'_b} \simeq \phi'^*\mathcal{O}(1)|_{\mathcal{C}'_b}$ for every $b \in B'$.

Moreover, in case (i), B' may be chosen isomorphic to B.

Proof. By Propositions 2.4 and 2.5, there exists a finite morphism from a smooth curve $f : B' \to B$ such that $f^*\mathcal{F}$ is isomorphic to a direct sum of isomorphic line bundles. By Lemma 2.6, $f^*\mathcal{F}$ is a subbundle of $\pi'_*\mathcal{L}'$, and for every $b \in B$, $(f^*\mathcal{F})|_b \subset H^0(\mathcal{C}'_b, \mathcal{L}'_b)$ embeds $\mathcal{C}'_b$ in a non-degenerate way in $\mathbb{P}^3$. Consequently, $f^*\mathcal{F}$ induces an embedding $\phi' : \mathcal{C}' \hookrightarrow \mathbb{P}_{B'}(f^*\mathcal{F})$ that is non-degenerate over every $b \in B'$. Since this projective bundle is trivial by our choice of f, we are done.

In case (i), one may choose f to be an isogeny by Proposition 2.4. The isogeny f factors some multiplication isogeny $[N] : B \to B$, allowing us to assume $B' = B$. □

Remark 2.10. A family constructed by Proposition 2.9 is non-trivial if the $(\mathcal{C}_b, \mathcal{L}_b)$ are not all isomorphic as polarized curves. In this case, Proposition 2.7 shows that $\mathcal{F}$ has to be the first graded piece of the Harder–Narasimhan filtration of $\mathcal{E}$.

Remark 2.11. In Proposition 2.9 (ii), the genus of the curve B' is not explicit as the construction of B' relies on the Lange–Stuhler theorem (Proposition 2.5). However, it follows from the proof of this theorem [20, Satz 1.4 b)] that we can give bounds for the genus of B' if we know an explicit Frobenius periodicity property for the strongly semistable vector bundle $\mathcal{F}$ (that is a relation of the form $F^{r*}\mathcal{F} \simeq F^{s*}\mathcal{F} \otimes \mathcal{N}$ for some $r \neq s$ and some line bundle $\mathcal{N}$).

Fortunately, in our applications to Theorem 1.1 (ii), we prove the strong semistability of the relevant vector bundle precisely by exhibiting such a relation (see the proof of Corollary 3.12). Consequently, Lange–Stuhler's proof provides bounds for the genus of the base of the families constructed in Theorem 1.1 (ii).

2.4 | Curves of genus 2 and degree 5

We may now give the:

Proof of Theorem 1.1. Let C be a smooth curve of genus 2 and $\mathcal{L}$ be a degree 5 line bundle on C. The Riemann–Roch theorem shows that $h^1(C, \mathcal{L}) = 0$, $h^0(C, \mathcal{L}) = 4$, and that these four sections embed C in $\mathbb{P}^3$. Let $A := \mathrm{Pic}^5(C)$ be the variety parametrizing degree 5 line bundles on C and $\mathcal{P}$ be a Poincaré bundle on $C \times A$. By [14, III Theorem 12.11], the sheaf $\mathcal{E} := p_{2*}\mathcal{P}$ is a rank 4 vector bundle on A whose formation commutes with base-change.

Let B be a smooth projective curve and $i : B \to A$ be a non-constant morphism. Consider the constant family $\pi : \mathcal{C} := C \times B \to B$ polarized by $\mathcal{L} := (\mathrm{Id}, i)^*\mathcal{P}$. By base-change, $\pi_*\mathcal{L} = i^*\mathcal{E}$. To prove Theorem 1.1, we apply Proposition 2.9 to polarized families $(\mathcal{C} \to B, \mathcal{L})$ as above, with $\mathcal{F} := i^*\mathcal{E}$.

Let us show that it is possible to choose C, B and i carefully so that the stability hypotheses (i) or (ii) in Proposition 2.9 are satisfied. In the setting of Theorem 1.1 (i), the curve B be an elliptic curve, and Proposition 2.13 below produces a genus 2 curve C and a non-constant morphism $i : B \to A$ such that $i^*\mathcal{E}$ is stable. In the setting of Theorem 1.1 (ii), the field $\Bbbk$ is of characteristic $p \equiv \pm 1[8]$, and Proposition 3.1 proven in Section 3 produces a genus 2 curve C and, setting $B := C$, an immersion $i : B \to A$, both defined over $\bar{\mathbb{F}}_p$, such that $i^*\mathcal{E}$ is strongly semistable.

To conclude the proof, it remains to verify that the families of smooth space curves constructed by applying Proposition 2.9 are non-trivial. To do so, fix $b \in B$, and consider the polarized variety $(\mathcal{C}_b, \mathcal{L}_b)$. Since i is non-constant and $\mathrm{Aut}(\mathcal{C}_b)$ is finite [28], there are at most finitely many $b' \in B$ such that $(\mathcal{C}_b, \mathcal{L}_b) \simeq (\mathcal{C}_{b'}, \mathcal{L}_{b'})$, allowing to apply Remark 2.10. □

Remark 2.12. The difficulty of removing the assumption that $p \equiv \pm 1[8]$ in the statement of Theorem 1.1 (ii) lies in the verification of the strong semistability assumption in Proposition 2.9 (ii), for an appropriate choice of C, B and i.

Proposition 2.13 relies on a construction of curves of genus 2 whose jacobian is not simple, that is very well explained in the first section of [9]. We keep the notations A and $\mathcal{E}$ of the proof of Theorem 1.1 above.

Proposition 2.13. *Let B be an elliptic curve over $\Bbbk$. Then there exist a genus 2 curve C and an immersion $i : B \to A := \mathrm{Pic}^5(C)$ such that $i^*\mathcal{E}$ is stable.*

Proof. Let E be an elliptic curve over $\Bbbk$ not isogenous to B. Let n be an odd integer invertible in $\Bbbk$. Choose an isomorphism $E[n] \xrightarrow{\sim} B[n]$ whose graph Γ is isotropic with respect to the Weil pairings on $E[n]$ and $B[n]$. Let $A := (E \times B)/\Gamma$. The quotient of A by the image B of $\{0\} \times B$ in A is $(E \times B)/\langle \{0\} \times B, \Gamma\rangle \simeq E/E[n] \simeq E$, yielding an exact sequence $0 \to B \to A \xrightarrow{q} E \to 0$ of abelian varieties. By [9, Propositions 1.1 and 1.4], A is isomorphic to the Jacobian of a smooth curve C of genus 2 and the theta divisor of A has degree n on B.

Choose an isomorphism $A \simeq \mathrm{Pic}^5(C)$ and let $i : B \to A$ be the inclusion of a general fiber of q. Suppose for contradiction that $i^*\mathcal{E}$ is not stable. As $\det(\mathcal{E})$ is numerically equivalent to the opposite of the theta divisor [1, VII.4], the rank 4 of $i^*\mathcal{E}$ is prime with its degree $-n$, showing that $i^*\mathcal{E}$ is not semistable.

By the existence of a relative Harder–Narasimhan filtration with respect to q [16, Theorem 2.3.2], there exists a saturated subsheaf $\mathcal{F} \subset \mathcal{E}$ whose restriction to a general fiber of q destabilizes $\mathcal{E}$. Outside of a finite number of points of A, $\mathcal{F}$ is a vector bundle. Its determinant $\det(\mathcal{F})$ extends uniquely as a line bundle $\mathcal{N}$ on A by smoothness of A. By construction, $\mathcal{N}$ has degree greater than $-n$ on the fibers of $\mathcal{F}$.

Consider the projection $u : E \times B \to A$. The isomorphism class of the line bundle $u^*\mathcal{N}$ is Γ-invariant. Since E and B are not isogenous, $\mathrm{Pic}(E \times B) \simeq \mathrm{Pic}(E) \oplus \mathrm{Pic}(B)$. The action of Γ on $\mathrm{Pic}(E \times B)$ is easy to describe, and one sees that $\mathrm{Pic}(E \times B)^\Gamma$ consists of line bundles of the form $\mathcal{N}_E \boxtimes \mathcal{N}_B$, where $\mathcal{N}_E$ (resp. $\mathcal{N}_B$) have degree divisible by n on E (resp. B). Hence $\mathcal{N} \cdot B = u^*\mathcal{N} \cdot (\{0\} \times B)$ is a multiple of n.

Hence, the restriction of $\mathcal{F}$ to a general fiber of q has nonnegative degree. Equivalently, the restriction of $\mathcal{F}^\vee$ to a general fiber of q has non-positive degree. Consequently, the vector bundle $\mathcal{E}^\vee$ is not ample. This contradicts [1, VII 2.2]. □

3 | CONSTRUCTING STRONGLY SEMISTABLE VECTOR BUNDLES

In this section, $\Bbbk$ is assumed to be of positive characteristic p.

Let C be a smooth curve of genus 2, $c \in C$ a point and $\mathcal{M}$ a degree 6 line bundle on C. Let $A := \mathrm{Pic}^5(C)$, and $\mathcal{P}$ be the Poincaré bundle on $C \times A$ normalized so that $\mathcal{P}|_{\{c\}\times A} \simeq \mathcal{O}_A$, and $\mathcal{E} := p_{2*}\mathcal{P}$. Let $i : C \to A$ be defined by $i(P) := \mathcal{M} \otimes \mathcal{O}_C(-P)$.

The main goal of this section is to prove the following proposition, thus completing the proof of Theorem 1.1 (ii) given in § 2.4. More precisely, Proposition 3.1 follows from Lemma 3.3, Corollary 3.12 and Proposition 3.13.

Proposition 3.1. *Suppose that C has hyperelliptic equation $Z^2 = X^6 + Y^6$, and that $\mathcal{M} = \omega_C^{\otimes 3}$. Then $i^*\mathcal{E}$ is strongly semistable if and only if $p \equiv \pm 1[8]$.*

The restrictive assumptions on the curve C and the line bundle $\mathcal{M}$ in the hypotheses of this proposition will be made explicitely later, when they become useful.

3.1 | Syzygy bundles

Let us first recall what a syzygy bundle is.

Definition 3.2. Let X be a variety, let $(\mathcal{L}_i)_{1\le i\le n}$ be line bundles on X and let $\sigma_i \in H^0(X, \mathcal{L}_i)$ be sections with no common zero. The *syzygy bundle* associated to these sections is the vector bundle of rank $n-1$ on X defined by the exact sequence:

$$0 \to \mathrm{Syz}_X(\sigma_i) \to \bigoplus_i \mathcal{L}_i^{-1} \xrightarrow{\oplus_i \sigma_i} \mathcal{O}_X \to 0. \tag{3.1}$$

If $\mathcal{N}$ is a line bundle on X, one can compute $H^0\big(X, \mathrm{Syz}_X(\sigma_i) \otimes \mathcal{N}\big)$ using (3.1): it consists of sections $\tau_i \in H^0\big(X, \mathcal{L}_i^{-1} \otimes \mathcal{N}\big)$ such that $\sum_i \tau_i \sigma_i = 0$.

If $\mathcal{L}$ is a base-point free line bundle on X and the σ_i form a base of $H^0(X, \mathcal{L})$, we set $\mathrm{Syz}_X(\mathcal{L}) := \mathrm{Syz}_X(\sigma_i)$. Let $\mathcal{S} := \mathrm{Syz}_C(\mathcal{M})$.

Lemma 3.3. *There is an isomorphism $i^*\mathcal{E} \simeq \mathcal{S} \otimes \mathcal{M}(c)$.*

Proof. Consider the pull-back $(\mathrm{Id}, i)^*\mathcal{P}$ of the Poincaré bundle on $C \times C$. Its restriction to $\{c\} \times C$ is trivial and its restriction to $C \times \{x\}$ is isomorphic to $\mathcal{M}(-x)$ for every $x \in C(\Bbbk)$. It follows that $(\mathrm{Id}, i)^*\mathcal{P} \simeq p_1^*\mathcal{M} \otimes p_2^*\mathcal{O}(c)(-\Delta)$, where $\Delta \subset C \times C$ is the diagonal. As a consequence, there is a short exact sequence on $C \times C$:

$$0 \to (\mathrm{Id}, i)^*\mathcal{P} \to p_1^*\mathcal{M} \otimes p_2^*\mathcal{O}(c) \to \big(p_1^*\mathcal{M} \otimes p_2^*\mathcal{O}(c)\big)|_\Delta \to 0.$$

Pushing it forward by p_2 and using the vanishing of the appropriate H^1, one gets:

$$0 \to i^*\mathcal{E} \to H^0(C, \mathcal{M}) \otimes \mathcal{O}(c) \to \mathcal{M}(c) \to 0,$$

where the arrow $H^0(C, \mathcal{M}) \otimes \mathcal{O}(c) \to \mathcal{M}(c)$ is the evaluation. One recognizes the definition of a syzygy bundle, up to a twist. □

From now on, we restrict to the case where $\mathcal{M}$ is the tricanonical line bundle $\omega_C^{\otimes 3}$. Since $\omega_C \simeq f^*\mathcal{O}(1)$, where $f : C \to \mathbb{P}^1$ is the hyperelliptic double cover, this will allow us to compare $F^*\mathcal{S}$ with bundles on $\mathbb{P}^1$, that are easier to describe.

Let X, Y be homogeneous coordinates on $\mathbb{P}^1$ and $P(X, Y)$ be a degree 6 homogeneous polynomial defining the ramification locus of f: the curve C is defined by $Z^2 = P(X, Y)$. The canonical ring $\bigoplus_{i\ge 0} H^0\big(C, \omega_C^{\otimes i}\big)$ of C is then isomorphic to $\Bbbk[X, Y, Z]/\big\langle Z^2 - P(X, Y)\big\rangle$, where the generators X, Y are of degree 1 and Z is of degree 3. In particular, it is isomorphic to $\Bbbk[X, Y] \oplus \Bbbk[X, Y] \cdot Z$ as a $\Bbbk[X, Y]$-module.

Let us introduce the two following syzygy bundles on $\mathbb{P}^1$:

$$\begin{cases} \mathcal{S}_+ := \mathrm{Syz}_{\mathbb{P}^1}\Big(X^{3p}, X^{2p}Y^p, X^pY^{2p}, Y^{3p}, P(X, Y)^{\frac{p+1}{2}}\Big), \\ \mathcal{S}_- := \mathrm{Syz}_{\mathbb{P}^1}\Big(X^{3p}, X^{2p}Y^p, X^pY^{2p}, Y^{3p}, P(X, Y)^{\frac{p-1}{2}}\Big). \end{cases}$$

Lemma 3.4. *There is an exact sequence:*

$$0 \to F^*\mathcal{S}\left(\omega_C^{\otimes -3}\right) \to f^*\mathcal{S}_+ \oplus f^*\mathcal{S}_-\left(\omega_C^{\otimes -3}\right) \to F^*\mathcal{S} \to 0. \tag{3.2}$$

Moreover, if $m \geq 0$*, the complex obtained by tensoring* (3.2) *by* $\omega_C^{\otimes m}$ *and taking global sections is exact.*

Proof. From the definition of a syzygy bundle, one sees that:

$$F^*\mathcal{S} \simeq \mathrm{Syz}_C\left(X^{3p}, X^{2p}Y^p, X^pY^{2p}, Y^{3p}, Z^p\right).$$

It is easy to describe the morphisms in (3.2) at the level of local sections. The morphism $f^*\mathcal{S}_+ \to F^*\mathcal{S}$ is $(A,B,C,D,E) \mapsto (A,B,C,D,ZE)$, and the morphism $f^*\mathcal{S}_-\left(\omega_C^{\otimes -3}\right) \to F^*\mathcal{S}$ is $(A,B,C,D,E) \mapsto (ZA,ZB,ZC,ZD,E)$. Similarly, $F^*\mathcal{S}\left(\omega_C^{\otimes -3}\right) \to f^*\mathcal{S}_+$ is $(A,B,C,D,E) \mapsto (ZA,ZB,ZC,ZD,E)$ and $F^*\mathcal{S}\left(\omega_C^{\otimes -3}\right) \to f^*\mathcal{S}_-\left(\omega_C^{\otimes -3}\right)$ is $(A,B,C,D,E) \mapsto -(A,B,C,D,ZE)$.

To prove the exactness of (3.2), it suffices to prove the second statement. This is easy using the description of the canonical ring as $\Bbbk[X,Y] \oplus \Bbbk[X,Y]\cdot Z$. □

3.2 | The strong Lefschetz property and syzygy computations

To compute the syzygy bundles $\mathcal{S}_+$ and $\mathcal{S}_-$, we need to restrict the situation again, by choosing carefully the polynomial P. We will take $P(X,Y) = X^6 + Y^6$, so that C is the curve of equation $Z^2 = X^6 + Y^6$. In particular, from now on, we suppose that $p \neq 2, 3$ so that C is indeed smooth.

Our main tool will be the strong Lefschetz property for homogeneous ideals.

Definition 3.5. Let $R := \Bbbk[x_1, \dots, x_n]$. A homogeneous Artinian ideal $I \subset R$ satisfies the *strong Lefschetz property* if there is a linear form $l \in R_1$ such that for every $r, d \geq 0$, the multiplication map $(R/I)_r \xrightarrow{\cdot l^d} (R/I)_{r+d}$ is of maximal rank.

Lemma 3.6. *Let* $I \subset \Bbbk[x,y]$ *be a homogeneous Artinian ideal. Suppose that* $(R/I)_r = 0$ *for* $r \geq p$*. Then* I *satisfies the strong Lefschetz property.*

Proof. In characteristic 0, this is [12, Proposition 4.4]. In this proof, the characteristic 0 hypothesis is only used for the explicit description of Borel-fixed ideals, applied to the generic initial ideal of I. The description of Borel-fixed ideals in positive characteristic p [8, Theorem 15.23] is more complicated in general, but coincides with the simple one in characteristic 0 when the condition that $(R/I)_r = 0$ for $r \geq p$ is satisfied. Consequently, under this hypothesis, the proof goes through. □

It is now possible to prove:

Proposition 3.7.

- **(i)** *If* $p \equiv 1[8]$, $\mathcal{S}_+ \simeq \mathcal{O}\left(\frac{-15p-9}{4}\right) \oplus \mathcal{O}\left(\frac{-15p-1}{4}\right)^{\oplus 3}$ *and* $\mathcal{S}_- \simeq \mathcal{O}\left(\frac{-15p+3}{4}\right)^{\oplus 4}$.
- **(ii)** *If* $p \equiv -1[8]$, $\mathcal{S}_+ \simeq \mathcal{O}\left(\frac{-15p-3}{4}\right)^{\oplus 4}$ *and* $\mathcal{S}_- \simeq \mathcal{O}\left(\frac{-15p+1}{4}\right)^{\oplus 3} \oplus \mathcal{O}\left(\frac{-15p+9}{4}\right)$.
- **(iii)** *If* $p \equiv 3[8]$, $\mathcal{S}_+ \simeq \mathcal{O}\left(\frac{-15p-7}{4}\right)^{\oplus 2} \oplus \mathcal{O}\left(\frac{-15p+1}{4}\right)^{\oplus 2}$ *and* $\mathcal{S}_- \simeq \mathcal{O}\left(\frac{-15p-3}{4}\right) \oplus \mathcal{O}\left(\frac{-15p+5}{4}\right)^{\oplus 3}$.
- **(iv)** *If* $p \equiv -3[8]$, $\mathcal{S}_+ \simeq \mathcal{O}\left(\frac{-15p-5}{4}\right)^{\oplus 3} \oplus \mathcal{O}\left(\frac{-15p+3}{4}\right)$ *and* $\mathcal{S}_- \simeq \mathcal{O}\left(\frac{-15p-1}{4}\right)^{\oplus 2} \oplus \mathcal{O}\left(\frac{-15p+7}{4}\right)^{\oplus 2}$.

Proof. We know the degrees of $\mathcal{S}_+$ and $\mathcal{S}_-$ from their definition. Moreover, by Grothendieck's theorem, a vector bundle on $\mathbb{P}^1$ splits as a direct sum of line bundles. As a consequence, to prove the proposition, it is enough to compute the global sections of some twists of $\mathcal{S}_+$ and $\mathcal{S}_-$. For instance, to prove that $\mathcal{S}_+ \simeq \mathcal{O}\left(\frac{-15p-9}{4}\right) \oplus \mathcal{O}\left(\frac{-15p-1}{4}\right)^{\oplus 3}$ if $p \equiv 1[8]$, it is sufficient to show that $h^0\left(\mathbb{P}^1, \mathcal{S}_+\left(\frac{15p-3}{4}\right)\right) = 0$ and that $h^0\left(\mathbb{P}^1, \mathcal{S}_+\left(\frac{15p+1}{4}\right)\right) = 3$.

Even if the result depends only on p modulo 8, we distinguish between different values of p modulo 24. As all the global section computations needed are similar, we only carry out one: assuming that $p \equiv 1[24]$, we prove that $h^0\left(\mathbb{P}^1, \mathcal{S}_+\left(\frac{15p+1}{4}\right)\right) = 3$.

Applying the global sections functor to an appropriate twist of the exact sequence defining $\mathcal{S}_+$, we see that $H^0\left(\mathbb{P}^1, \mathcal{S}_+\left(\frac{15p+1}{4}\right)\right)$ is the vector space of solutions of the equation:

$$AX^{3p} + BX^{2p}Y^p + CX^pY^{2p} + DY^{3p} + E\left(X^6 + Y^6\right)^{\frac{p+1}{2}} = 0, \tag{3.3}$$

where the unknowns A, B, C, D, E are homogeneous polynomials in X and Y, the first four being of degree $\frac{3p+1}{4}$ and E being of degree $\frac{3p-11}{4}$. Equation (3.3) is a linear system in the coefficients of A, B, C, D, E.

The matrix of this linear system in the monomial bases has six rectangular blocks, as one sees by separating the monomials according to the value modulo 6 of the exponent of X. Consequently, the solution space of (3.3) is the direct sum of the solution spaces of six smaller systems, that we may solve independently.

Let us look at the first one, obtained by keping in (3.3) only monomials in which the exponent of X is a multiple of 6. Then, setting $x := X^6$ and $y := Y^6$, it is possible to write $A = X^3Y^4a(x, y)$, $B = X^4Y^3b(x, y)$, $C = X^5Y^2c(x, y)$, $D = Yd(x, y)$ and $E = Y^4e(x, y)$. Dividing by Y^4, we get the new equation:

$$ax^{\frac{p+1}{2}} + bx^{\frac{p+2}{3}}y^{\frac{p-1}{6}} + cx^{\frac{p+5}{6}}y^{\frac{p-1}{3}} + dy^{\frac{p-1}{2}} + e(x + y)^{\frac{p+1}{2}} = 0, \tag{3.4}$$

where the unknowns a, b, c, d, e are homogeneous polynomials in x and y of respective degrees $\frac{p-9}{8}$, $\frac{p-9}{8}$, $\frac{p-9}{8}$, $\frac{p-1}{8}$ and $\frac{p-9}{8}$: it is a linear system in $\frac{5p+3}{8}$ unknowns and as many equations.

Introduce the ideal $I := \left\langle x^{\frac{p+1}{2}}, x^{\frac{p+2}{3}}y^{\frac{p-1}{6}}, x^{\frac{p+5}{6}}y^{\frac{p-1}{3}}, y^{\frac{p-1}{2}}\right\rangle$ of $R := \Bbbk[x, y]$. The linear system (3.4) has maximal rank exactly when $\cdot(x + y)^{\frac{p+1}{2}} : (R/I)_{\frac{p-9}{8}} \to (R/I)_{\frac{5p-5}{8}}$ has maximal rank. If $\alpha, \beta \in \Bbbk^*$, this rank is equal to the rank of multiplication by $(\alpha x + \beta y)^{\frac{p+1}{2}}$, as one sees by performing the change of variables $x' = \alpha x$, $y' = \beta y$, hence of the multiplication by a power of a general linear form. By Lemma 3.6, I satisfies the strong Lefschetz property and such multiplication maps have maximal rank. We have proven that (3.4) has maximal rank. Since it has as many unknowns as equations, it has no nontrivial solution.

The same argument using the strong Lefschetz property shows that the five other sub-linear systems have maximal rank. Three of them (corresponding to exponents of X congruent to 1, 2 and 3 modulo 6) have exactly one more unknown than equations. Another has as many unknowns as equations (the one corresponding to exponents of X congruent to 4 modulo 6), and the last one has more equations than unknowns. Consequently, only three have non-trivial solutions, and moreover a one-dimensional solution space. It follows, as wanted, that $h^0\left(\mathbb{P}^1, \mathcal{S}_+\left(\frac{15p+1}{4}\right)\right) = 3$. □

Remark 3.8. The matrices of the linear systems in the proof of Proposition 3.7 are complicated matrices of binomial coefficients, very similar to those appearing in Han's thesis [10]. It seems difficult to check directly that they are of maximal rank.

Remark 3.9. Proposition 3.7 and Lemma 3.4 show at once that $F^*\mathcal{S}$ is unstable when $p \equiv \pm 3[8]$. We will obtain more precise information in Paragraph 3.4.

3.3 | Frobenius periodicity

We are ready to prove the strong semistability of $\mathcal{S}$ when $p \equiv \pm 1[8]$. Denote by R the ramification locus of f: it consists of the points $P_i = [\zeta_i : 1]$, where the ζ_i are the sixth roots of -1. We view R either as a subset of $\mathbb{P}^1$ or as a subset of C. Note that these ramification points are transitively permuted by the natural action of the group μ_6 of sixth roots of unity on $\mathbb{P}^1$.

Proposition 3.10. *There are exact sequences:*

$$0 \to F^*\mathcal{S} \to \left(\omega_C^{\otimes\frac{-15p+3}{4}}\right)^{\oplus 5} \to \omega_C^{\otimes\frac{-15p+15}{4}} \to 0, \ \textit{if } p \equiv 1[8], \tag{3.5}$$

$$0 \to \omega_C^{\otimes\frac{-15p-15}{4}} \to \left(\omega_C^{\otimes\frac{-15p-3}{4}}\right)^{\oplus 5} \to F^*\mathcal{S} \to 0, \ \textit{if } p \equiv -1[8]. \tag{3.6}$$

Proof. We first construct (3.5). By Proposition 3.7, the injective morphism in the exact sequence (3.2) writes: $F^*\mathcal{S} \to \left(\omega_C^{\otimes\frac{-15p+11}{4}}\right)^{\oplus 3} \oplus \left(\omega_C^{\otimes\frac{-15p+3}{4}}\right) \oplus \left(\omega_C^{\otimes\frac{-15p+3}{4}}\right)^{\oplus 4}$. We will prove that the induced morphism $F^*\mathcal{S} \to \left(\omega_C^{\otimes\frac{-15p+3}{4}}\right)^{\oplus 5}$

is injective in restriction to every point $P \in C$. This concludes because its quotient is then a line bundle, isomorphic to $\omega_C^{\otimes\frac{-15p+15}{4}}$ for degree reasons.

From its description, one sees that the morphism $F^*\mathcal{S} \to f^*\mathcal{S}_- \simeq \left(\omega_C^{\otimes\frac{-15p+3}{4}}\right)^{\oplus 4}$ is an isomorphism on the fibers outside R, and that if $P \in R$, the kernel of $F^*\mathcal{S}|_P \to f^*\mathcal{S}_-|_P$ consists of syzygies (A, B, C, D, E) such that A, B, C, D vanish at P. It remains to see that this kernel is not killed by the composition $F^*\mathcal{S}|_P \to f^*\mathcal{S}_+\left(\omega_C^{\otimes 3}\right)|_P \to \omega_C^{\otimes\frac{-15p+3}{4}}|_P$, i.e. that its image in $f^*\mathcal{S}_+\left(\omega_C^{\otimes 3}\right)|_P$ does not belong to $\left(\omega_C^{\otimes\frac{-15p+11}{4}}\right)^{\oplus 3}|_P$.

If it failed for $P = P_1$, there would exist a non-zero section $(A, B, C, D, E) \in H^0\left(C, f^*\mathcal{S}_+\left(\omega_C^{\otimes\frac{15p+1}{4}}\right)\right) = H^0\left(\mathbb{P}^1, \mathcal{S}_+\left(\frac{15p+1}{4}\right)\right)$ such that A, B, C, D vanish at P_1. Writing $A = (X - \zeta_1 Y)\tilde{A}_1$, $B = (X - \zeta_1 Y)\tilde{B}_1$, $C = (X - \zeta_1 Y)\tilde{C}_1$, $D = (X - \zeta_1 Y)\tilde{D}_1$, $\tilde{E}_1 = \prod_{i=2}^6 (X - \zeta_i Y)E$, one gets a section $\sigma_1 = (\tilde{A}_1, \tilde{B}_1, \tilde{C}_1, \tilde{D}_1, \tilde{E}_1) \in H^0\left(\mathbb{P}^1, \mathcal{S}_-\left(\frac{15p-3}{4}\right)\right)$ such that $\tilde{E}_1$ vanishes at $P_2, \dots, P_6$. For symmetry reasons, using the μ_6-action, there exists, for every $1 \leq i \leq 6$ a non-zero section $\sigma_i = (\tilde{A}_i, \tilde{B}_i, \tilde{C}_i, \tilde{D}_i, \tilde{E}_i) \in H^0\left(\mathbb{P}^1, \mathcal{S}_-\left(\frac{15p-3}{4}\right)\right)$ such that $\tilde{E}_i$ vanishes at P_j for $j \neq i$. Since $H^0\left(\mathbb{P}^1, \mathcal{S}_-\left(\frac{15p-3}{4}\right)\right)$ is 4-dimensional by Proposition 3.7, these six sections cannot be linearly independent: for instance, $\sigma_1 \in \langle \sigma_2, \dots, \sigma_6 \rangle$. It follows that $\tilde{E}_1$ vanishes at all P_i. Then $\left(\tilde{A}_1, \tilde{B}_1, \tilde{C}_1, \tilde{D}_1, \tilde{E}_1/(X^6 + Y^6)\right) \in H^0\left(\mathbb{P}^1, \mathcal{S}_+\left(\frac{15p-3}{4}\right)\right)$ is non-zero, contradicting Proposition 3.7.

Let us explain how to obtain (3.6) by a similar argument. By Lemma 3.4 and Proposition 3.7, there is a morphism $\left(\omega_C^{\otimes\frac{-15p-3}{4}}\right)^{\oplus 5} \to F^*\mathcal{S}$, and it suffices to prove its surjectivity. Using only the four factors coming from $\mathcal{S}_+$, one gets surjectivity at points $P \notin R$, and the fact that, if $P \in R$, all $(A, B, C, D, E) \in F^*\mathcal{S}|_P$ such that $E = 0$ are in the image. Hence, it suffices to prove that the unique section $(A, B, C, D, E) \in H^0\left(C, f^*\mathcal{S}_-\left(\omega_C^{\otimes\frac{15p-9}{4}}\right)\right) = H^0\left(\mathbb{P}^1, \mathcal{S}_-\left(\frac{15p-9}{4}\right)\right)$ satisfies $E(P) \neq 0$. If it didn't, E_R would vanish by symmetry, and $\left(A, B, C, D, E/(X^6 + Y^6)\right) \in H^0\left(\mathbb{P}^1, \mathcal{S}_+\left(\frac{15p-9}{4}\right)\right)$ would be a non-zero section contradicting Proposition 3.7. □

Proposition 3.11. *There are isomorphisms:*

(i) $F^*\mathcal{S} \simeq \mathcal{S}\left(\omega_C^{\otimes\frac{15-15p}{4}}\right)$, *if* $p \equiv 1[8]$,

(ii) $F^*\mathcal{S} \simeq \mathcal{S}^\vee\left(\omega_C^{\otimes\frac{-15-15p}{4}}\right)$, *if* $p \equiv -1[8]$.

Proof. Denote by $\sigma_i \in H^0\left(C, \omega_C^{\otimes 3}\right)$ the sections appearing in the last arrow of (3.5). Tensoring (3.5) by $\omega_C^{\otimes\frac{15p-3}{4}}$ and taking cohomology, one gets:

$$0 \to H^0\left(C, F^*\mathcal{S}\left(\omega_C^{\otimes\frac{15p-3}{4}}\right)\right) \to \Bbbk^{\oplus 5} \xrightarrow{\sigma_i} H^0\left(C, \omega_C^{\otimes 3}\right).$$

But $H^0\left(C, F^*\mathcal{S}\left(\omega_C^{\otimes\frac{15p-3}{4}}\right)\right) = 0$ by the second part of Lemma 3.4 applied with $m = \frac{15p-3}{4}$ and Proposition 3.7. Thus, the σ_i are linearly independant and (3.5) is, up to a twist, the exact sequence defining the syzygy bundle $\mathcal{S}$, proving (i).

We prove (ii) in a similar way. Denote by $\tau_i \in H^0\left(C, \omega_C^{\otimes 3}\right)$ the sections appearing in the first arrow of (3.6). Tensoring it by $\omega_C^{\otimes\frac{15p+7}{4}}$ and taking cohomology, one gets:

$$0 \to H^0(C, \omega_C)^{\oplus 5} \to H^0\left(C, F^*\mathcal{S}\left(\omega_C^{\otimes\frac{15p+7}{4}}\right)\right) \to H^1\left(C, \omega_C^{\otimes -2}\right) \xrightarrow{\tau_i} H^1(C, \omega_C)^{\oplus 5}.$$

The vector space $H^0(C,\omega_C)^{\oplus 5}$ is 10-dimensional. By the second part of Lemma 3.4 and Proposition 3.7, one sees that $H^0\left(C, F^*S\left(\omega_C^{\otimes\frac{15p+7}{4}}\right)\right)$ has dimension ≤ 10. It follows that $H^1\left(C,\omega_C^{\otimes -2}\right)\to H^1(C,\omega_C)^{\oplus 5}$ is injective. This map being Serre-dual to $\Bbbk^{\oplus 5}\xrightarrow{\tau_i} H^0\left(C,\omega_C^{\otimes 3}\right)$, the τ_i generate $H^0\left(C,\omega_C^{\otimes 3}\right)$. Hence, the dual of (3.6) is, up to a twist, the exact sequence defining the syzygy bundle $\mathcal{S}$, proving (ii). □

Corollary 3.12. *If $p\equiv \pm 1[8]$, $\mathcal{S}$ is strongly semistable.*

Proof. Proposition 3.11 shows that when $p\equiv\pm 1[8]$, $\mathcal{S}$ is Frobenius periodic up to a twist: $F^{2*}\mathcal{S}\simeq \mathcal{S}\left(\omega_C^{\otimes\frac{15p^2-15}{4}}\right)$. It is classical that such bundles are strongly semistable. We recall the argument. Suppose that $\mathcal{S}$ is not semistable, and let $\mathcal{F}\subset\mathcal{S}$ be the first graded piece of its Harder–Narasimhan filtration. Then $F^{2*}\mathcal{F}\left(\omega_C^{\otimes\frac{-15p^2+15}{4}}\right)\subset\mathcal{S}$ has greater slope than $\mathcal{F}$, a contradiction. Hence $\mathcal{S}$ is semistable. By the periodicity property, so are all its Frobenius pull-backs. □

3.4 | Unstability

Let us now describe what happens when $p\equiv\pm 3[8]$.

Proposition 3.13. *If $p\equiv\pm 3[8]$, then $F^*\mathcal{S}$ is not semistable and its Harder–Narasimhan filtration is strong. This filtration is of the form:*

(i) $0\to\mathcal{T}\to F^*\mathcal{S}\to\omega_C^{\otimes\frac{-15p-3}{4}}\to 0$ *if $p\equiv 3[8]$,*

(ii) $0\to\omega_C^{\otimes\frac{-15p+3}{4}}\to F^*\mathcal{S}\to\mathcal{T}\to 0$ *if $p\equiv -3[8]$.*

Proof. We will only prove (i), as the second statement is similar. From Lemma 3.4 and Proposition 3.7, we get a morphism $F^*\mathcal{S}\to f^*\mathcal{S}_-\to\omega_C^{\otimes\frac{-15p-3}{4}}$. Let us prove that it is surjective. Since $F^*\mathcal{S}\to f^*\mathcal{S}_-$ is surjective at all points $P\notin R$, and since if $P\in R$, the image of $F^*\mathcal{S}|_P\to f^*\mathcal{S}_-|_P$ consists of syzygies (A,B,C,D,E) such that $E(P)=0$, we need to show that not all sections $(A,B,C,D,E)\in H^0\left(C, f^*\mathcal{S}_-\left(\omega_C^{\otimes\frac{15p-5}{4}}\right)\right)=H^0\left(\mathbb{P}^1,\mathcal{S}_-\left(\frac{15p-5}{4}\right)\right)$ satisfy $E(P)=0$. Suppose it is not the case: then, by symmetry using the μ_6-action, for all sections $(A,B,C,D,E)\in H^0\left(\mathbb{P}^1,\mathcal{S}_-\left(\frac{15p-5}{4}\right)\right)$, E would vanish on R. Dividing E by X^6+Y^6, we would get a non-sero section in $H^0\left(\mathbb{P}^1,\mathcal{S}_+\left(\frac{15p-5}{4}\right)\right)$, contradicting Proposition 3.7. Hence our morphism was surjective, and we denote its kernel by $\mathcal{T}$.

From Lemma 3.4 and Proposition 3.7 again, we get a morphism $\left(\omega_C^{\otimes\frac{-15p+1}{4}}\right)^{\oplus 2}\to f^*\mathcal{S}_+\to F^*\mathcal{S}$. Let us prove that it is injective on every fiber. Since $f^*\mathcal{S}_+\to F^*\mathcal{S}$ is injective on the fibers at $P\notin R$, and since, if $P\in R$, the kernel of $f^*\mathcal{S}_+|_P\to F^*\mathcal{S}|_P$ consists of syzygies (A,B,C,D,E) such that A,B,C,D all vanish at P, it suffices to rule out the existence of a section $(A,B,C,D,E)\in H^0\left(C,f^*\mathcal{S}_+\left(\omega_C^{\otimes\frac{15p-1}{4}}\right)\right)=H^0\left(\mathbb{P}^1,\mathcal{S}_+\left(\frac{15p-1}{4}\right)\right)$ such that A,B,C,D all vanish at P. We proceed by contradiction. Then, for symmetry reasons, there exist for $1\leq i\leq 6$ a section $(A_i,B_i,C_i,D_i,E_i)\in H^0\left(\mathbb{P}^1,\mathcal{S}_+\left(\frac{15p-1}{4}\right)\right)$ such that A_i,B_i,C_i,D_i all vanish at P_i. Dividing A_i, B_i, C_i, D_i by $X-\zeta_iY$ and multiplying E by $\prod_{j\neq i}(X-\zeta_jY)$, we get non-zero sections $\sigma_i=(\tilde{A}_i,\tilde{B}_i,\tilde{C}_i,\tilde{D}_i,\tilde{E}_i)\in H^0\left(\mathbb{P}^1,\mathcal{S}_-\left(\frac{15p-5}{4}\right)\right)$ such that $\tilde{E}_i$ vanishes at P_j for $j\neq i$. By Proposition 3.7, $H^0\left(\mathbb{P}^1,\mathcal{S}_-\left(\frac{15p-5}{4}\right)\right)$ is 3-dimensional, hence the σ_i cannot be linearly independent, say $\sigma_1\in\langle\sigma_2,\dots,\sigma_6\rangle$. Then $\tilde{E}_1$ vanishes at all the P_i and $\left(\tilde{A}_1,\tilde{B}_1,\tilde{C}_1,\tilde{D}_1,\tilde{E}_1/(X^6+Y^6)\right)\in H^0\left(\mathbb{P}^1,\mathcal{S}_+\left(\frac{15p-5}{4}\right)\right)$ is a non-zero section contradicting Proposition 3.7.

Since there are obviously no non-zero morphisms $\omega_C^{\otimes\frac{-15p+1}{4}}\to\omega_C^{\otimes\frac{-15p-3}{4}}$, the subbundle $\left(\omega_C^{\otimes\frac{-15p+1}{4}}\right)^{\oplus 2}$ factors through $\mathcal{T}$, and a degree computation shows that this realizes $\mathcal{T}$ as an extension:

$$0\to\left(\omega_C^{\otimes\frac{-15p+1}{4}}\right)^{\oplus 2}\to\mathcal{T}\to\omega_C^{\otimes\frac{-15p+1}{4}}\to 0. \tag{3.7}$$

Now $\mathcal{T}$ is strongly semistable as an extension of strongly semistable bundles of the same slope, and writing $F^*\mathcal{S}$ as an extension of $\omega_C^{\otimes\frac{-15p-3}{4}}$ by $\mathcal{T}$ indeed realizes the Harder–Narasimhan filtration of $F^*\mathcal{S}$. □

3.5 | Hilbert–Kunz multiplicities

We now apply our results to the computation of Hilbert–Kunz multiplicities. Let us first recall the definition.

Definition 3.14. Let A be a noetherian n-dimensional ring of characteristic p and let $\mathfrak{m}$ be a maximal ideal of A. Let $\mathfrak{m}^{[e]}$ be the ideal of A generated by p^e-th powers of elements of $\mathfrak{m}$. The *Hilbert–Kunz multiplicity* of $(A, \mathfrak{m})$ is defined to be:

$$e_{\mathrm{HK}}(A,\mathfrak{m}) := \lim_{e\to\infty} \frac{\mathrm{length}\left(A/\mathfrak{m}^{[e]}\right)}{p^{ne}}.$$

This invariant was first considered by Kunz [19], and the limit was shown to exist and to be finite by Monsky [26]. It is difficult to compute in general.

We will be interested in the following geometric case:

Definition 3.15. Let C be a smooth curve endowed with a line bundle $\mathcal{L}$ whose sections embed C as a projectively normal curve. Consider the section ring $A := \bigoplus_{l\geq 0} H^0\left(C, \mathcal{L}^{\otimes l}\right)$ with its maximal ideal $\mathfrak{m} := \bigoplus_{l>0} H^0\left(C, \mathcal{L}^{\otimes l}\right)$. Define:

$$e_{\mathrm{HK}}(C,\mathcal{L}) := e_{\mathrm{HK}}(A,\mathfrak{m}).$$

In this particular case, Brenner [3, Theorem 1] and Trivedi [30, Theorem 4.12] have related the Hilbert–Kunz multiplicity to properties of a syzygy bundle:

Theorem 3.16. (Brenner, Trivedi). *Let C be a smooth curve endowed with a degree d line bundle $\mathcal{L}$ whose sections embed C in $\mathbb{P}^{k-1}$ as a projectively normal curve. Using Theorem 2.2, choose a finite morphism of degree e from a smooth curve $f : C' \to C$ such that the Harder–Narasimhan filtration of $f^*\,\mathrm{Syz}_C(\mathcal{L})$ is strong. Let r_i and $e\delta_i$ be the ranks and degrees of the graded pieces of this filtration (so that r_i and δ_i are independent of f). Then:*

$$e_{\mathrm{HK}}(C,\mathcal{L}) = \frac{1}{2d}\sum_i \frac{\delta_i^2}{r_i} - \frac{kd}{2}.$$

Applying this theorem using Corollary 3.12 and Proposition 3.13, we get:

Theorem 3.17. *Let C be the curve of genus 2 with equation $Z^2 = X^6 + Y^6$. Then:*

(i) $e_{\mathrm{HK}}\left(C, \omega_C^{\otimes 3}\right) = \frac{15}{4}$ *if* $p \equiv \pm 1[8]$,

(ii) $e_{\mathrm{HK}}\left(C, \omega_C^{\otimes 3}\right) = \frac{15}{4} + \frac{1}{4p^2}$ *if* $p \equiv \pm 3[8]$.

ACKNOWLEDGEMENT

I would like to thank Chungsim Han for having kindly made available to me a copy of her thesis [10].

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How to cite this article: Benoist O. Complete families of smooth space curves and strong semistability. *Mathematische Nachrichten*. 2018;291:2354–2366. https://doi.org/10.1002/mana.201700251