Factorizations of a Coxeter element and discriminant of a reflection group

Vivien RIPOLL

LaCIM — UQÀM

Combinatorial Algebra meets Algebraic Combinatorics
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Invariant theory of W (geometry of the discriminant Δ_W) $\leftrightarrow$ Combinatorics of the noncrossing partition lattice of W (factorizations of a Coxeter element)

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$\mathrm{NC}(W) = \{w \mid \ell(w) + \ell(w^{-1}c) = \ell(c)\} = \{$"block factors" of $c\}$

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Theorem (R.)

Let Λ be a conjugacy class of elements of length 2 in W. Call submaximal factorizations of c of type Λ the block factorizations containing $n-2$ reflections and one element (of length 2) in the conjugacy class Λ. Then, their number is:

$$|\,\text{FACT}_{n-1}^{\Lambda}(c)| = \frac{(n-1)!\ h^{n-1}}{|W|} \deg D_{\Lambda}\ ,$$

where D_{Λ} is a homogeneous polynomial constructed from the geometry of the discriminant hypersurface of W.

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- We will suppose that f_n is the invariant of highest degree.

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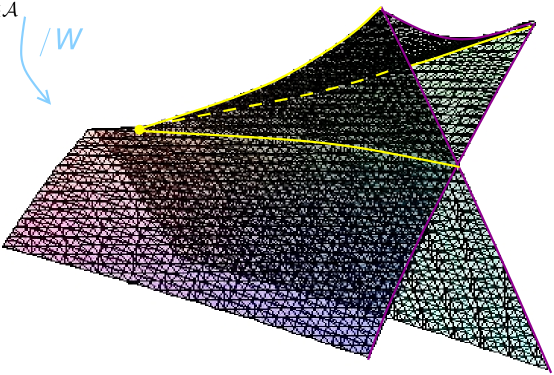

hypersurface $\mathcal{H}$ (discriminant) $\subseteq W \backslash V \simeq \mathbb{C}^3$

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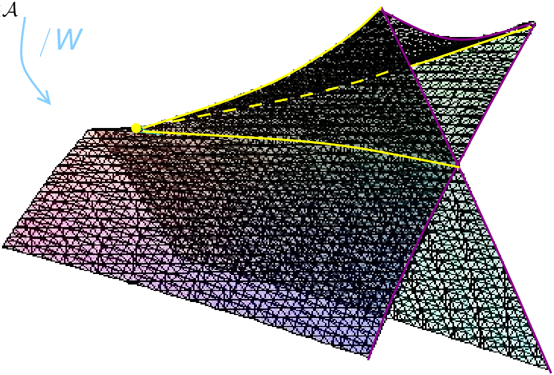

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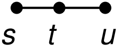

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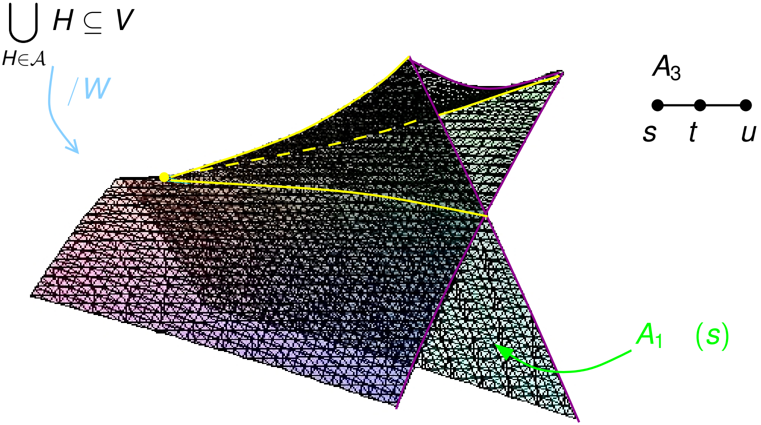

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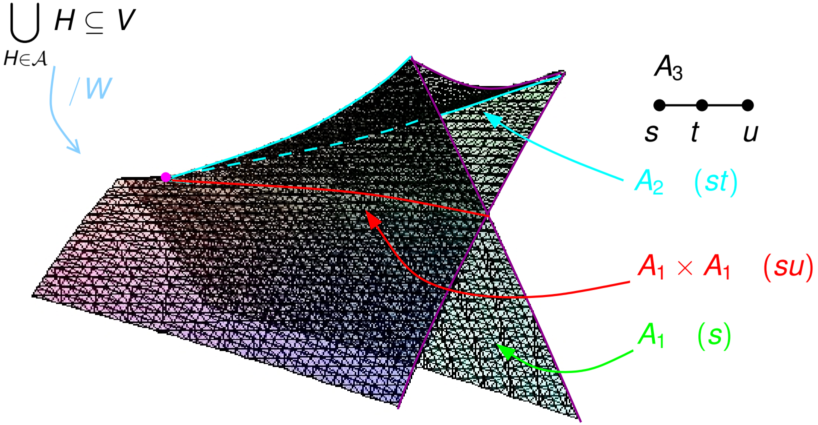

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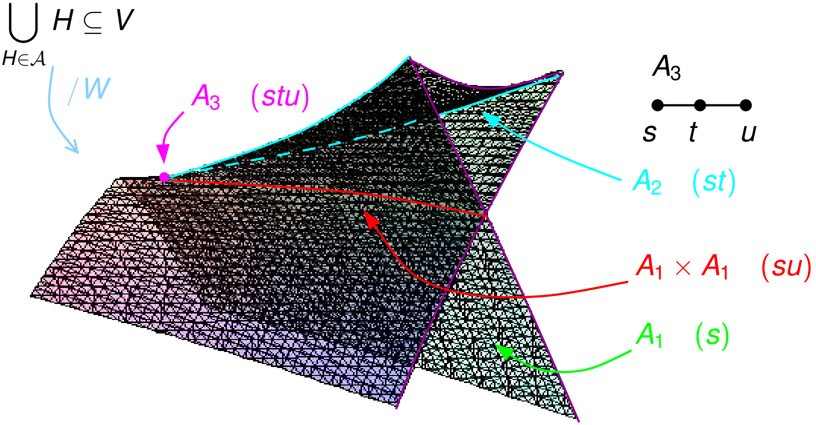

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Theorem (Orlik-Solomon, Bessis)

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Definition

The bifurcation locus of Δ_W (w.r.t. f_n) is the hypersurface of $\mathbb{C}^{n-1}$:

$$\mathcal{K} := \{D_W = 0\}$$

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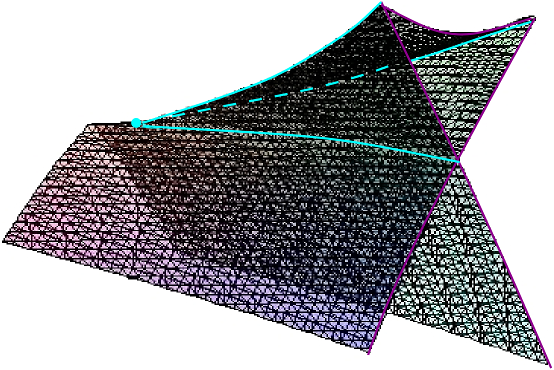

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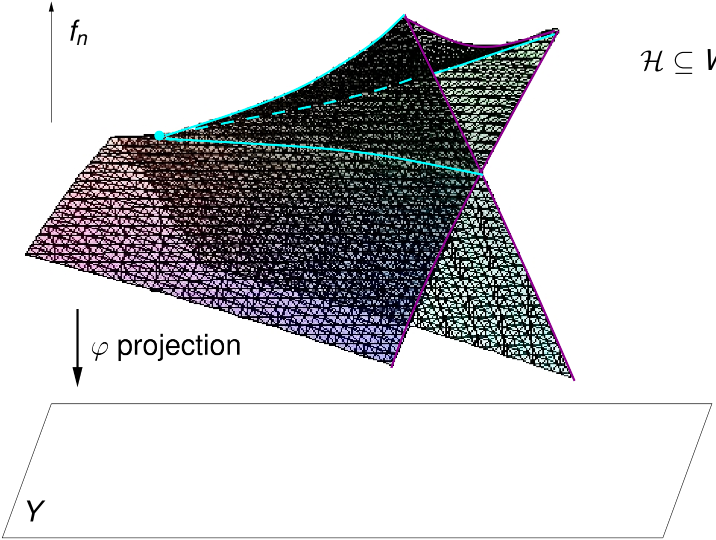

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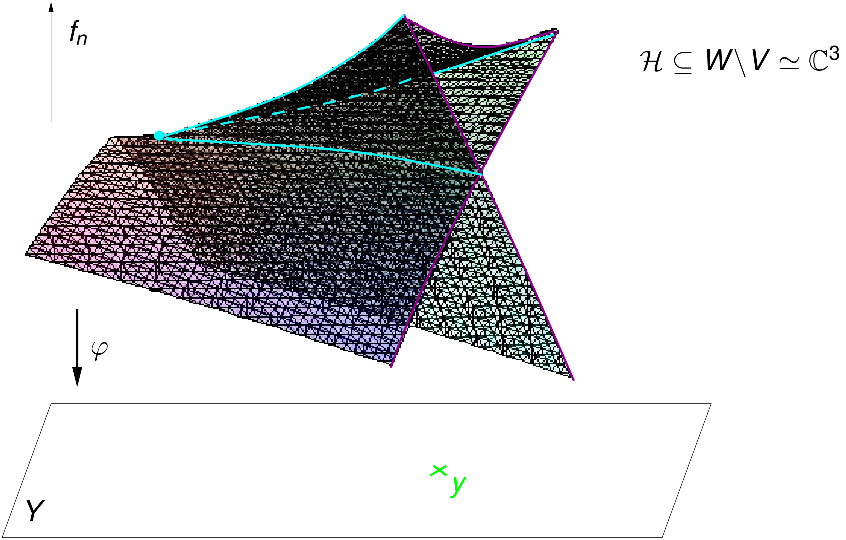

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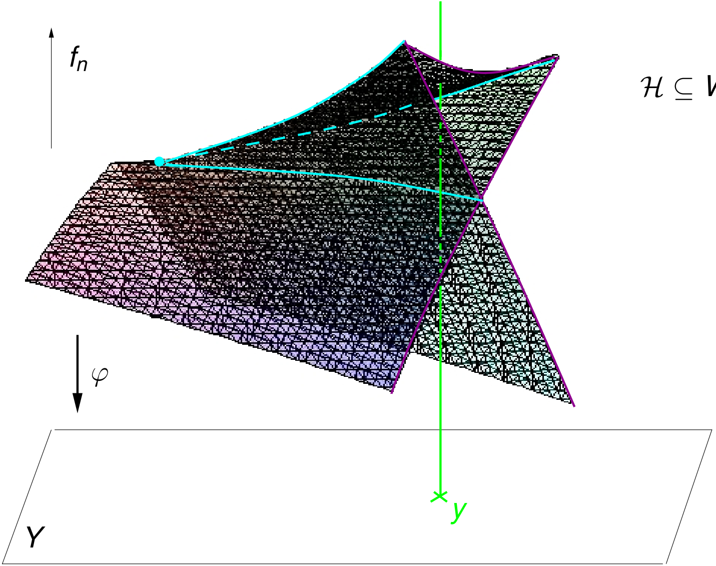

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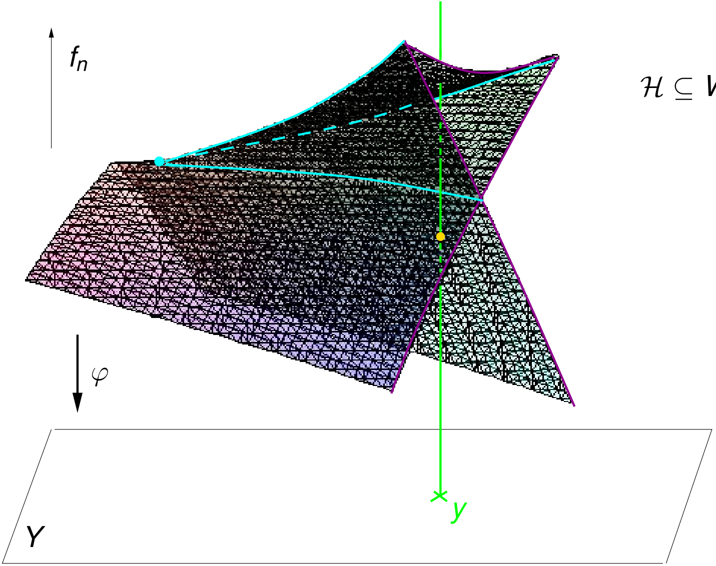

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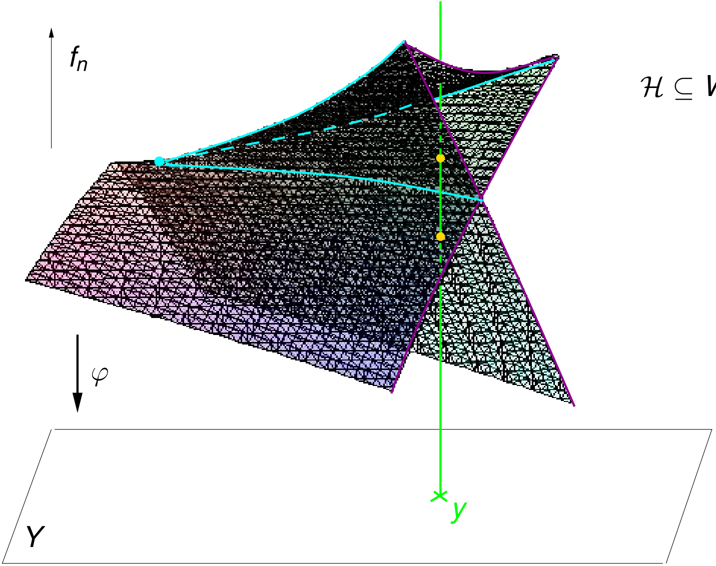

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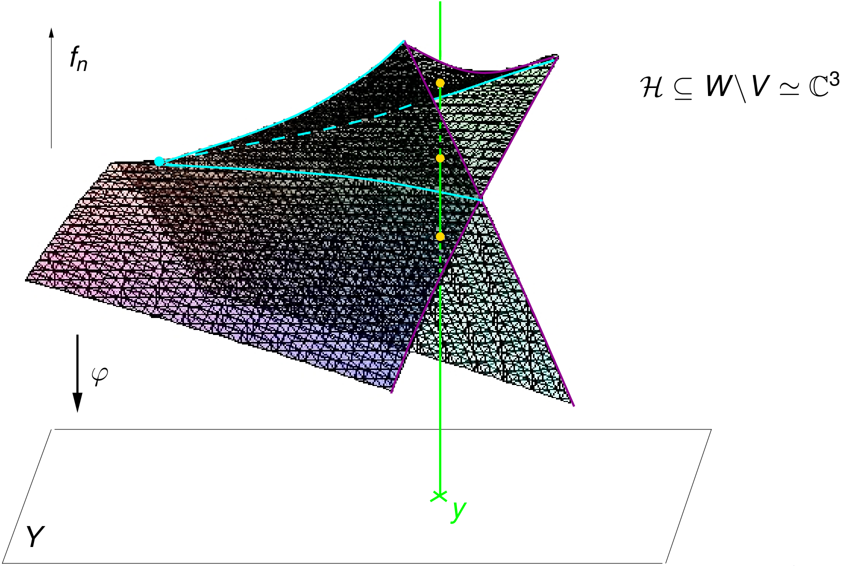

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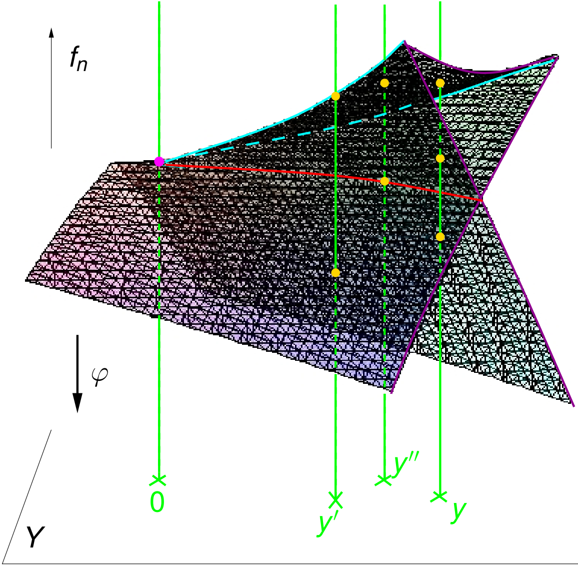

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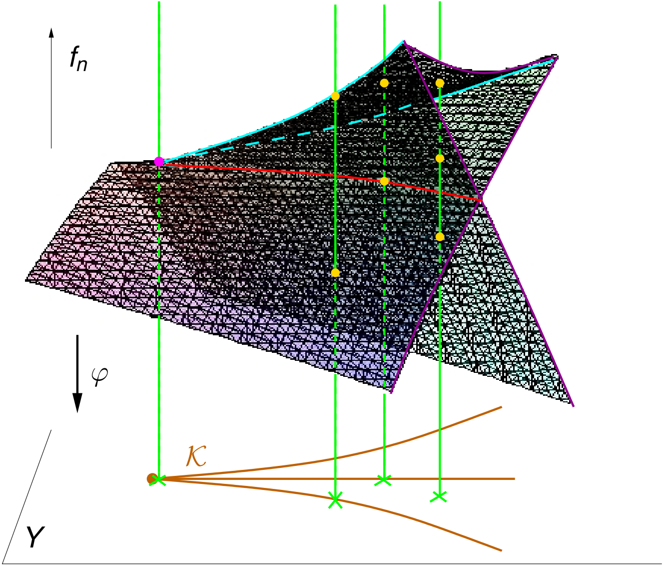

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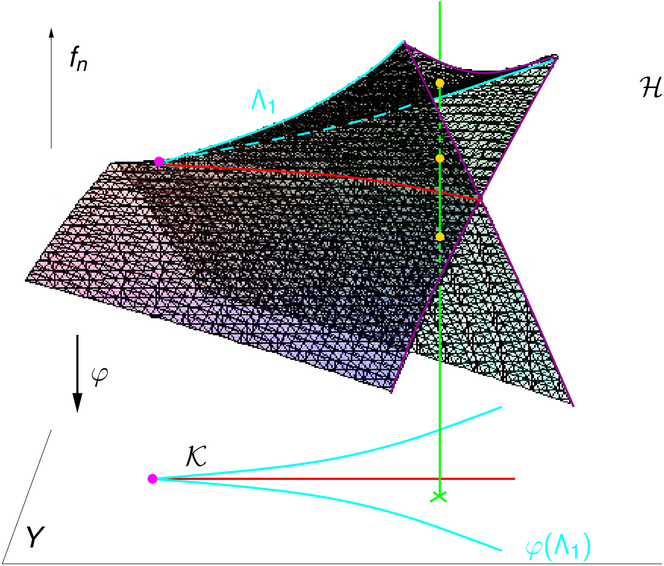

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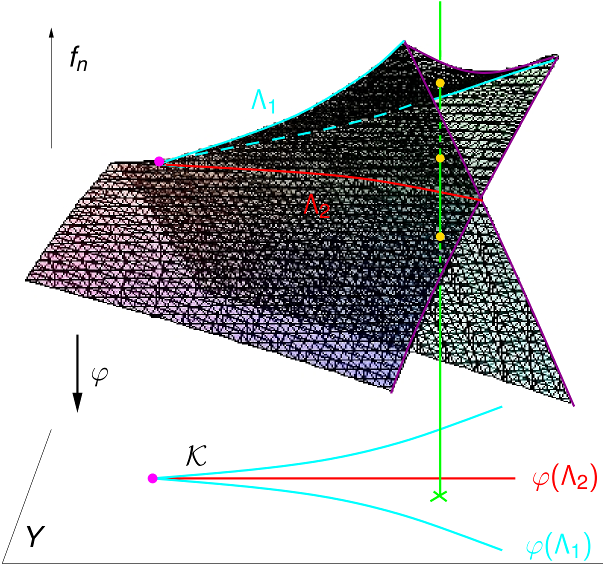

Submaximal factorizations of type Λ

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For $\Lambda \in \bar{\mathcal{L}}_2$, the number of submaximal factorizations of c of type Λ (i.e. , whose unique length 2 element lies in the conjugacy class Λ) is:

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Corollary

*The number of **block factorisations of a Coxeter element** c **in** $n-1$ **factors** is:*

$$|\operatorname{FACT}_{n-1}(c)| = \frac{(n-1)!\, h^{n-1}}{|W|}\left(\frac{(n-1)(n-2)}{2}h + \sum_{i=1}^{n-1} d_i\right) ,$$

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Chapoton's formula for multichains in NC(W)

Suppose W irreducible of rank n, and let c be a Coxeter element.
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Return to thm End

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- Our corollary is also a consequence of Chapoton's formula;
- but the proof is geometric and more enlightening: we travelled from the numerology of $\mathrm{FACT}_n(c)$ to that of $\mathrm{FACT}_{n-1}(c)$, without adding any case-by-case analysis.

Some ingredients of the proof

▸ Return to thm ▸ End

- Lyashko-Looijenga morphism LL: ▸ Picture

  $y \in Y = \operatorname{Spec} \mathbb{C}[f_1, \ldots, f_{n-1}] \mapsto$ multiset of roots of $\Delta_W(y, f_n)$.

- Construction of topological factorisations: [Bessis, R.]

$$\text{facto} : Y \to \text{FACT}(c) \ .$$

- Fundamental property that the product map:

$$Y \xrightarrow{\text{LL} \times \text{facto}} E_n \times \text{FACT}(c)$$

  is injective, and its image is the set of "compatible" pairs. In other words, the map facto induces a bijection between any fiber $\text{LL}^{-1}(\omega)$ and the set of factorisations of same "composition" as ω.

- Consequently, we can use some algebraic properties of LL to obtain cardinalities of certain fibers, and deduce enumeration of certain factorisations.

Lyashko-Looijenga morphism and topological factorisations

Return to proof