Factorisations of the Garside element in the dual braid monoids

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Journées Garside
Caen

1. Dual braid monoids and noncrossing partition lattices
   - The dual braid monoid
   - Factorisations in the noncrossing partition lattice

2. Factorisations from the geometry of the discriminant
   - The Lyashko-Looijenga covering
   - Factorisations as fibers of LL
   - Combinatorics of the submaximal factorisations

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- $A_+(W,S)$ embeds in $B(W)$ (the braid group of W).

Dual braid monoid

Basic idea: replace S with $\mathcal{R} := \{\text{all reflections in } W\}$.
$\rightsquigarrow$ new definition of length ($\ell_{\mathcal{R}}$) and of partial order ($\preccurlyeq_{\mathcal{R}}$).

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Definition (Dual braid monoid of W)

$M(W, c)$ is the monoid with presentation

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- $M(W, c)$ embeds in $B(W)$, but is *not* isomorphic to the Artin-Tits monoid.
- the construction extends to (well-generated) complex reflection groups.

Complex reflection groups

V : complex vector space, of dim. n.

Definition

A (finite) complex reflection group is a finite subgroup of $\mathrm{GL}(V)$ generated by complex reflections.
A complex reflection is an element $s \in \mathrm{GL}(V)$ of finite order, s.t. $\mathrm{Ker}(s - \mathrm{Id}_V)$ is a hyperplane:

$$s \underset{\mathcal{B}}{\leftrightarrow} \text{matrix} \, \mathrm{Diag}(\zeta, 1, \ldots, 1) \,, \text{ with } \zeta \text{ root of unity.}$$

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Shephard-Todd's classification (1954):

- an infinite series with 3 parameters $G(de, e, r)$;
- 34 exceptional groups.

Invariant theory

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Theorem (Chevalley-Shephard-Todd)

There exist fundamental invariant polynomials $f_1, \ldots, f_n$ (homogeneous), s.t.

$$S(V^*)^W = \mathbb{C}[f_1, \ldots, f_n] .$$

Their degrees $d_1 \leq \cdots \leq d_n$ do not depend on the choice of $f_1, \ldots, f_n$ (invariant degrees of W).

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$$\rightsquigarrow \text{isomorphism} : \quad \begin{array}{ccc} W \backslash V & \xrightarrow{\sim} & \mathbb{C}^n \\ \bar{v} & \mapsto & (f_1(v), \ldots, f_n(v)). \end{array}$$

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Suppose W irreducible, well-generated.
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Very rich combinatorial object.

Multichains in NCP_W

Chapoton's formula

The number of multichains $w_1 \preccurlyeq_{\mathcal{R}} \ldots \preccurlyeq_{\mathcal{R}} w_N \preccurlyeq_{\mathcal{R}} c$ in NCP_W is :

$$Z_W(N+1) = \prod_{i=1}^{n} \frac{d_i + Nh}{d_i}.$$

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Proof (Athanasiadis, Reiner, Bessis): case-by-case using the classification... even for $N = 1$ (formula for $|\,\mathrm{NCP}_W\,|$).

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Ex. : $\mathrm{FACT}_n(c) = \mathrm{FACT}_{1^n}(c) = \mathrm{Red}_{\mathcal{R}}(c)$.
$\mathrm{FACT}_{n-1}(c) = \mathrm{FACT}_{2^1 1^{n-2}}$.

Factorisations *vs* multichains

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Conversion formulas

$$\mathrm{Cat}^{(N)}(W) = \sum_{k=1}^{n} \binom{N+1}{k} |\,\mathrm{FACT}_k(c)|$$

$$|\,\mathrm{FACT}_p(c)| = \Delta^p Z_W(0) = \sum_{k=1}^{p} (-1)^{p-k} \binom{p}{k} \mathrm{Cat}^{(k)}(W)$$

($\Delta : P \mapsto P(X+1) - P(X)$.)

Discriminant of W

$\mathcal{A} := \{\text{reflection hyperplanes of } W\}$. For $H \in \mathcal{A}$:

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Basic case in type **A** :

$$\prod_{1 \leq i < j \leq n} (x_i - x_j)^2 = \mathrm{Disc}(T^n - \sigma_1 T^{n-1} + \cdots + (-1)^n \sigma_n; T)$$

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Proposition

*If W is well-generated, the discriminant Δ_W is **monic of degree n in** f_n. The fundamental invariants $f_1, \ldots, f_n$ can be chosen s.t.:*

$$\Delta_W = f_n^n + a_2 f_n^{n-2} + a_3 f_n^{n-3} + \cdots + a_{n-1} f_n + a_n,$$

with $a_i \in \mathbb{C}[f_1, \ldots, f_{n-1}]$ (homogeneous polynomial of degree ih).

Lyashko-Looijenga morphism of type W

Definition (Lyashko-Looijenga morphism)

$$\begin{array}{cccc} \mathrm{LL}: & \mathbb{C}^{n-1} & \longrightarrow & \mathbb{C}^{n-1} \\ & (f_1, \ldots, f_{n-1}) & \longmapsto & (a_2, \ldots, a_n) \end{array}$$

It is an algebraic morphism, which is quasi-homogeneous for the weights $\deg(f_j) = d_j$, $\deg(a_i) = ih$.

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Define $Y := \operatorname{Spec} \mathbb{C}[f_1, \ldots, f_{n-1}]$.
$\rightsquigarrow W \backslash V \simeq Y \times \mathbb{C}$.

$$\mathrm{LL}: \begin{array}{ccc} Y & \to & E_n = \{\text{configurations of } n \text{ points in } \mathbb{C}\} \\ y & \mapsto & \{\text{roots, with multiplicities, of } \Delta_W(y, f_n) \text{ in } f_n\} \end{array}$$

Lyashko-Looijenga covering

$$
\begin{aligned}
E_n^{\mathrm{reg}} &:= \{\text{configurations of } n \text{ \textbf{distincts} points}\} \subseteq E_n \\
\mathcal{K} &:= \mathsf{LL}^{-1}(E_n - E_n^{\mathrm{reg}}) \\
&= \{y \in Y \mid \Delta_W(y, f_n) \text{ has multiple roots in } f_n\} \\
&= \{y \in Y \mid D_{\mathsf{LL}}(y) = 0\}\ ,
\end{aligned}
$$

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E_n^{\mathrm{reg}} &:= \{\text{configurations of } n \textbf{ distincts} \text{ points}\} \subseteq E_n \\
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Theorem (Looijenga, Lyashko, Bessis)

- *The extension* $\mathbb{C}[a_2, \ldots, a_n] \subseteq \mathbb{C}[f_1, \ldots, f_{n-1}]$ *is free, with rank* $n!h^n/|W|$.
- LL *is a finite morphism.*
- *its restriction* $Y - \mathcal{K} \twoheadrightarrow E_n^{\mathrm{reg}}$ *is an unramified covering of degree* $n!h^n/|W|$.

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Hypersurface $\mathcal{H} \subseteq W \backslash V \simeq Y \times \mathbb{C}$.

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$\mathcal{R}$-length and conjugacy class of $c_{y,x}$ depend on the position of (y, x) in $\mathcal{H}$ ...

Stratification of $W\backslash V$ and parabolic Coxeter elements

Stratification of V with the *flats* (**intersection lattice**) :
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Factorisations and LL

Proposition

*Fix $y \in Y$. For all $x \in \mathrm{LL}(y)$, $c_{y,x}$ is a parabolic Coxeter element of W. Its **length** is the multiplicity of x in $\mathrm{LL}(y)$; its **conjugacy class** corresponds to the unique stratum Λ in $\bar{\mathcal{L}}$ s.t. $(y, x) \in \Lambda^0$.*

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$\forall \omega \in E_n$, $\underline{\mathrm{fact}}$ induces $\mathrm{LL}^{-1}(\omega) \xrightarrow{\sim} \mathrm{FACT}_\mu(c)$, where μ is the composition of ω.

Reduced decompositions of c

$$|\,\mathrm{Red}_{\mathcal{R}}(c)| \quad = \quad |\,\mathrm{FACT}_{\mu}(c)| \quad \text{for } \mu = (1, \ldots, 1)$$

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Can we compute in the same way
$|\,\mathrm{FACT}_{n-1}(c)| = |\,\mathrm{FACT}_{2^1 1^{n-2}}(c)|$?

Irreducible components of $\mathcal{K}$

Ramified part of LL : $\mathcal{K} \twoheadrightarrow E_n - E_n^{\mathrm{reg}}$.
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Proposition

The irreducible components of $\mathcal{K}$ are the $\varphi(\Lambda)$, for $\Lambda \in \bar{\mathcal{L}}_2$.

Restriction of LL to a component of $\mathcal{K}$

Write $D_{\mathrm{LL}} = \prod_{\Lambda \in \bar{\mathcal{L}}_2} D_\Lambda^{r_\Lambda}$, with D_Λ irreducibles in $\mathbb{C}[f_1, \ldots, f_{n-1}]$ s.t. $\varphi(\Lambda) = \{D_\Lambda = 0\}$.

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$$\frac{\prod \deg(a_i)}{\deg(D_{\mathrm{LL}})} \Big/ \frac{\prod \deg(f_j)}{\deg(D_\Lambda)} = \frac{(n-2)!h^{n-2}}{|W|} \deg D_\Lambda$$

Factorisations of type Λ

Theorem (R.)

For any stratum Λ in $\bar{\mathcal{L}}_2$:

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$$|\operatorname{FACT}_{n-1}^{\Lambda}(c)| = \frac{(n-1)!\ h^{n-1}}{|W|} \deg D_\Lambda \ .$$

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To obtain $|\textsc{Fact}_{n-1}(c)|$, we need to compute $\sum_{\Lambda \in \bar{\mathcal{L}}_2} \deg(D_\Lambda)$.

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Conclusion, prospects

- We travelled from the numerology of $\mathrm{Red}_{\mathcal{R}}(c)$ (non-ramified part of LL) to that of $\mathrm{FACT}_{n-1}(c)$, without adding any case-by-case analysis.

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Thank you !

(Merci, gracias...)