PERIODS OF ALGEBRAIC VARIETIES

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Abstract. The *periods* of a compact complex algebraic manifold X are the integrals of its holomorphic 1-forms over paths. These integrals are in general not well-defined, but they can still be used to associate with this variety a *period point* in a suitable analytic (or sometimes quasi-projective) variety called a *period domain.* As X varies in a smooth family, this period point varies holomorphically in the period domain, defining the *period map* from the parameter space of the family to the period domain. This is a very old construction in the case when X is a complex curve. Griffiths generalized it to higher dimensions in 1968, and it became a very useful tool to study moduli spaces of varieties, in particular for K3 surfaces. I will discuss mostly examples and some recent results (cubic hypersurfaces, holomorphic symplectic varieties,...).

1. Elliptic curves

An elliptic curve is a smooth plane cubic curve E, that is, the set of points in the complex projective plane $\mathbf{P}^2$ where a homogeneous polynomial of degree 3 in 3 variables vanishes (smoothness is then equivalent to the fact that the partial derivatives of this polynomial have no common zero). After a change of variables, the equation can be written, in affine coordinates, as

$$y^2 = x(x-1)(x-\lambda),$$

where $\lambda \in \mathbf{C} - \{0,1\}$.

The differential 1-form

$$\omega := \frac{dx}{y}$$

is holomorphic everywhere on E.[1] *Elliptic integrals* of the type

$$\int_{p_0}^{p} \omega = \int_{p_0}^{p} \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$$

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[1] At points where $q(x) = x(x-1)(x-\lambda)$ (hence also y) vanishes, we may write $\omega = 2y/q'(x)$, hence ω is regular everywhere.

were considered classically. This integral is not well-defined because it depends on the choice of a path on E from the point p_0 to the point p. More precisely, if (δ, γ) is a symplectic basis for the free abelian group $H_1(E, \mathbf{Z})$, it is only well-defined up to the subgroup of $\mathbf{C}$ generated by the *periods*

$$A := \int_\delta \omega \quad \text{and} \quad B := \int_\gamma \omega.$$

One can always choose the basis (δ, γ) so that $A = 1$ and $\mathrm{Im}(B) > 0$. The point $\tau := B$ of the Siegel upper half-plane $\mathscr{H}$ is then called the *period point* of E. It is still well-defined only up to transformations of the type

$$\tau \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau := \frac{a\tau + b}{c\tau + d}, \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}).$$

All in all, we have defined a *single-valued period map*

$$\begin{array}{ccc} \mathbf{C} - \{0,1\} & \longrightarrow & \mathrm{SL}_2(\mathbf{Z}) \backslash \mathscr{H} \\ \lambda & \longmapsto & \tau. \end{array}$$

One can prove that it is *holomorphic and surjective.*[2]

2. Curves of higher genus

Let C be a (smooth projective) curve of genus g. There is in general no rational parametrization of the family of all such curves as in the case $g = 1$ (elliptic curves), but we may still consider a basis $(\delta_1, \ldots, \delta_g, \gamma_1, \ldots, \gamma_g)$ for $H_1(C, \mathbf{Z})$ which is symplectic for the intersection form on this free abelian group, and a basis $(\omega_1, \ldots, \omega_g)$ of the vector space of holomorphic differential 1-forms on C, which can be chosen so that the $g \times g$ matrix (A_{ij}) defined by

$$A_{ij} := \int_{\delta_i} \omega_j$$

[2]In fact, the correspondence $\lambda \mapsto \tau$ induces an isomorphism between $\mathbf{C} - \{0,1\}$ and the quotient of $\mathscr{H}$ by a subgroup of $\mathrm{SL}_2(\mathbf{Z})$ of index 6 classically denoted by $\Gamma(2)$. Its inverse defines a function $\tau \mapsto \lambda(\tau)$ on $\mathscr{H}$ which realizes $\mathscr{H}$ as the universal cover of $\mathbf{C} - \{0,1\}$. It is given explicitly by

$$\lambda(\tau) = \left(\sum_{n=-\infty}^{+\infty} q^{\left(n+\frac{1}{2}\right)^2} \Big/ \sum_{n=-\infty}^{+\infty} q^{n^2} \right)^4,$$

where $q = \exp(i\pi\tau)$. This function was used by Picard to prove his "little theorem" that an entire non-constant function on the complex plane cannot omit more than one value.

is the identity matrix, and the $g \times g$ matrix $T(C) := (B_{ij})$ defined by

$$B_{ij} := \int_{\gamma_i} \omega_j$$

has symmetric positive definite imaginary part. The matrix $T(C)$ then defines a *period point* in the Siegel upper half-space $\mathscr{H}_g$. Again, this is well-defined only up to transformations of the type

$$T \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot T := (aT+b)(cT+d)^{-1}, \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbf{Z}).$$

Given a family $(C_s)_{s\in S}$ of curves of genus g parametrized by a complex variety S, this construction defines a single-valued *period map*

$$\begin{array}{rcl} S & \longrightarrow & \mathrm{Sp}_{2g}(\mathbf{Z})\backslash\mathscr{H}_g \\ s & \longmapsto & T(C_s) \end{array}$$

which is still holomorphic, but never surjective (for $g \geq 4$, it is not even dominant, i.e., its image is not dense). This leads to other questions (such as the *Schottky problem* of characterizing the possible images) which we will not discuss here.

The question of injectivity (called the *Torelli problem*) needs to be asked more carefully (injectivity will trivially be false if $(C_s)_{s\in S}$ is a constant family), but the answer is positive in the sense that a smooth projective curve of genus g is completely determined (up to isomorphism) by its period point in $\mathrm{Sp}_{2g}(\mathbf{Z})\backslash\mathscr{H}_g$.

3. Quartic surfaces

A quartic surface $X \subset \mathbf{P}^3$ is defined as the set of zeroes of a homogeneous polynomial of degree 4 in 4 variables (this is a particular case of a *K3 surface*). We ask that it be smooth (which is equivalent to saying that the partial derivatives of this polynomial have no common zeroes). This is the case for example for the Fermat quartic, defined by the equation $x_0^4 + \cdots + x_3^4 = 0$.

One can show that the vector space Ω of holomorphic differential 2-forms on X is 1-dimensional. Let ω be a generator. As in the case of elliptic curves, we choose a basis $(\gamma_1, \ldots, \gamma_{22})$ for the free abelian group $H_2(X, \mathbf{Z})$ and we want to consider the *periods* $\int_{\gamma_i} \omega$. We will present this in a slightly more conceptual fashion by considering the *Hodge decomposition*

$$H^2(X, \mathbf{C}) = H^2(X, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{C} = H^{0,2} \oplus H^{1,1} \oplus H^{2,0}$$

into vector subspaces, where $H^{2,0}$ is isomorphic to Ω, hence 1-dimensional. We will view the line $H^{2,0} = \mathbf{C}\omega \subset H^2(X, \mathbf{C})$ as a point $[\omega]$ in the complex 21-dimensional projective space $\mathbf{P}(H^2(X,\mathbf{C}))$ and call it the *period point of* X.

There are constraints on this point. To explain them, we introduce the *intersection form* Q on $H^2(X,\mathbf{Z})$ (given by cup-product), a non-degenerate quadratic form of signature $(3,19)$. Then $[\omega]$ lies in the orthogonal $H^2(X,\mathbf{C})_{\mathrm{prim}}$ (called the primitive cohomology) of the class of a plane section of X. The restriction of Q to the hyperplane $H^2(X,\mathbf{R})_{\mathrm{prim}}$ has signature $(2,19)$ and moreover,

$$Q([\omega])=0 \quad , \quad Q([\omega+\overline{\omega}])>0.$$

So we consider the space

$$\{[\alpha]\in\mathbf{P}(H^2(X,\mathbf{C})_{\mathrm{prim}}) \mid Q([\alpha])=0\ ,\ Q([\alpha+\overline{\alpha}])>0\}.$$

It has two connected components $\mathscr{D}$ and $\mathscr{D}'$, which are both 19-dimensional and isomorphic to $\mathrm{SO}(2,19)^0/\,\mathrm{SO}(2)\times\mathrm{SO}(2,19)$, a homogeneous symmetric domain of type IV in Cartan's classification. There is again some ambiguity in the definition of this point and the conclusion is that, given a family $(X_s)_{s\in S}$ of smooth quartic surfaces parametrized by a complex variety S, this construction defines a single-valued *period map*

$$S\longrightarrow\Gamma\backslash\mathscr{D},$$

where Γ is some arithmetic subgroup of the orthogonal group of the lattice $(H^2(X,\mathbf{Z})_{\mathrm{prim}},Q)$. The action of Γ on $\mathscr{D}$ is nice, and one can put a structure of quasi-projective algebraic variety on the quotient $\Gamma\backslash\mathscr{D}$. For this structure, *the period map is algebraic.*

One can take as parameter space for all smooth quartic surfaces a Zariski open subset V_{35}^0 of the 35-dimensional complex vector space V_{35} parametrizing homogeneous polynomials of degree 4 in 4 variables. Even better, one may want to use the 19-dimensional quotient $V_{35}^0/\,\mathrm{GL}_4$: using Geometric Invariant Theory (GIT), there is a way to put a structure of quasi-projective algebraic variety on $V_{35}^0/\,\mathrm{GL}_4$, so all in all, we obtain an algebraic period map

$$p: V_{35}^0/\,\mathrm{GL}_4\longrightarrow\Gamma\backslash\mathscr{D},$$

between 19-dimensional quasi-projective varieties.

Theorem 1 (Piatetski-Shapiro, Shafarevich)**.** *The map p is an open immersion whose image can be explicitly described; it is in particular the complement of a hyperplane arrangement.*

One should view the quotient $V_{35}^0/\,\mathrm{GL}_4$ as *the moduli space* for smooth quartic surfaces (its points are in one-to-one correspondance with isomorphism classes of such surfaces) and the theorem as an explicit description and uniformization of this moduli space. The description of the image of p can be used very efficiently to produce such surfaces containing certain configurations of curves or with certain automorphisms.

4. Cubic fourfolds

A cubic fourfold $X \subset \mathbf{P}^5$ is defined as the set of zeroes of a homogeneous polynomial of degree 3 in 6 variables. We ask that it be smooth (which is equivalent to saying that the partial derivatives of this polynomial have no common zeroes). This is the case for example for the Fermat cubic, defined by the equation $x_0^3 + \cdots + x_5^3 = 0$. There are no non-zero holomorphic differential forms on X, but, following Griffiths, we will still construct a period map using the Hodge decomposition

$$\begin{array}{ccccccc} H^4(X,\mathbf{C}) & = & H^{1,3}(X) & \oplus & H^{2,2}(X) & \oplus & H^{3,1}(X) \\ \text{dimensions:} & & 1 & & 21 & & 1 \end{array}$$

or rather, again, its primitive part

$$\begin{array}{ccccccc} H^4(X,\mathbf{C})_{\mathrm{prim}} & = & H^{1,3}(X) & \oplus & H^{2,2}(X)_{\mathrm{prim}} & \oplus & H^{3,1}(X) \\ \text{dimensions:} & & 1 & & 20 & & 1 \end{array}$$

(again, "primitive" means "orthogonal to the class of a hyperplane section"). Indeed, we may view the line $H^{3,1}(X) \subset H^4(X,\mathbf{C})_{\mathrm{prim}}$ as a point in the 21-dimensional projective space $\mathbf{P}(H^4(X,\mathbf{C})_{\mathrm{prim}})$, called the *period point* of X. The intersection form Q has signature $(20,2)$ on $H^4(X,\mathbf{R})_{\mathrm{prim}}$ and the period point lies in

$$\{[\alpha] \in \mathbf{P}(H^4(X,\mathbf{C})_{\mathrm{prim}}) \mid Q([\alpha]) = 0\ ,\ Q([\alpha+\overline{\alpha}]) < 0\},$$

which has again two connected components $\mathscr{D}$ and $\mathscr{D}'$, which are homogeneous symmetric domains of type IV. If we consider the quasi-projective 20-dimensional *moduli space* $V_{56}^0/\operatorname{GL}_6$ which parametrizes all smooth cubic fourfolds, this leads to an algebraic *period map*

$$p : V_{56}^0/\operatorname{GL}_6 \longrightarrow \Gamma\backslash\mathscr{D}$$

between 20-dimensional quasi-projective varieties, where Γ is some arithmetic subgroup of the orthogonal group $O(H^4(X,\mathbf{Z})_{\mathrm{prim}}, Q)$.

Theorem 2 (Voisin, Looijenga, Laza)**.** *The map p is an open immersion whose image can be explicitly described; it is the complement of a hyperplane arrangement.*

Moreover, there are canonical compactifications on both sides: the GIT compactification of $V_{56}^0/\operatorname{GL}_6$ and the Baily-Borel compactification of $\Gamma\backslash\mathscr{D}$. What Laza proves is that the period map extends to an isomorphism between the blow-up of the GIT compactification at one (explicit) boundary point[3] and the Baily-Borel compactification.

5. Irreducible holomorphic symplectic varieties (IHS)

For us, an irreducible holomorphic symplectic variety is a smooth projective simply-connected variety X such that the vector space of holomorphic differential 2-forms on X is generated by a symplectic (i.e., non-degenerate at each point of X) 2-form ω. The dimension of X is necessarily even and when X is a surface, it is simply a K3 surface. Examples were constructed by Beauville and O'Grady in every even dimension (but there are not many).

Example 3 (Beauville-Donagi)**.** Let $Y \subset \mathbf{P}^5$ be a smooth cubic fourfold. Let $L(Y) \subset G(1, \mathbf{P}^5)$ be the (smooth projective) variety that parametrizes lines contained in Y. It is an irreducible symplectic fourfold.

Contrary to what we did earlier, where we always looked at the middle cohomology, we concentrate here on

$$\begin{array}{cccccccc} H^2(X, \mathbf{C})_{\text{prim}} & = & H^{0,2}(X) & \oplus & H^{1,1}(X)_{\text{prim}} & \oplus & H^{2,0}(X) \\ \text{dimensions:} & & 1 & & b_2 - 3 & & 1 \end{array}$$

(here, "primitive" means "orthogonal to the class of a given hyperplane class"). Beauville constructed a quadratic form Q_{BB} on $H^2(X, \mathbf{R})_{\text{prim}}$. It has signature $(2, b_2 - 3)$ and the corresponding period domain is

$$\mathscr{D} := \{[\alpha] \in \mathbf{P}(H^2(X, \mathbf{C})_{\text{prim}}) \mid Q_{BB}([\alpha]) = 0\ ,\ Q_{BB}([\alpha + \overline{\alpha}]) > 0\}.$$

Verbitski recently proved that the period map

$$\mathscr{M} \longrightarrow \Gamma \backslash \mathscr{D}$$

from the connected component $\mathscr{M}$ of the moduli space of polarized IHS corresponding to deformations of X to the quotient of $\mathscr{D}$ by a suitable arithmetic group Γ, is generically injective.

Example 4 (Beauville-Donagi)**.** Let $Y \subset \mathbf{P}^5$ be a smooth cubic fourfold. Consider the incidence variety

$$I := \{(y, \ell) \in Y \times L(Y) \mid y \in \ell\}$$

[3]It is the point corresponding to the (singular) determinantal cubic $\begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{vmatrix} = 0.$

with its projections $p : I \to Y$ and $q : I \to L(Y)$. Beauville & Donagi proved that the *Abel-Jacobi map*

$$q_*p^* : (H^4(Y, \mathbf{Z})_{\mathrm{prim}}, Q) \to (H^2(L(Y), \mathbf{Z})_{\mathrm{prim}}, Q_{BB})$$

is an anti-isometry of polarized Hodge structures. In other words, the period maps are compatible: the following diagram is commutative

$$\begin{tikzcd}
\left\{\begin{matrix}\text{moduli space of}\\ \text{cubic fourfolds}\end{matrix}\right\} \arrow[rr, "L"] \arrow[dr, "p"'] & & \left\{\begin{matrix}\text{moduli space of}\\ \text{polarized IHS}\end{matrix}\right\} \arrow[dl, "p"] \\
& \Gamma\backslash\mathscr{D}. &
\end{tikzcd}$$

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