Functors and Computations in Floer homology with Applications Part II

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Abstract

The results in this paper concern computations of Floer cohomology using generating functions. The first part proves the isomorphism between Floer cohomology and Generating function cohomology introduced by Lisa Traynor. The second part proves that the Floer cohomology of the cotangent bundle (in the sense of Part I), is isomorphic to the cohomology of the loop space of the base. This has many consequences, some of which were given in Part I, others will be given in forthcoming papers. The results in this paper had been announced (with indications of proof) in a talk at the ICM 94 in Zürich.

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1   Introduction

This paper is concerned with computations of Floer cohomology using generating functions. The first part proves the isomorphism between Floer cohomology and Generating function cohomology introduced by Lisa Traynor in [Tr]. The statement of this theorem was given in [V2] with rather precise indications of proof. However, since then, we found what we consider a simpler, even though less natural, proof. A proof along the original indications is given by Milinković and Oh in [Mi-O1], [Mi-O2]. The second part proves that the Floer cohomology of the cotangent bundle (in the sense of Part I), is isomorphic to the cohomology of the loop space of the base. This has many consequences, some of which were given in Part I, others will be given in forthcoming papers.

We would like to point out a very interesting attempt by Joachim Weber to prove the main theorem of section 3 using a different approach, namely by considering the gradient flow of the geodesic energy as a singular perturbation of the Floer flow (see [We]).

2   Floer cohomology is isomorphic to GF-homology

Let L be a Lagrange submanifold in T^*N, and assume L has a generating function quadratic at infinity, that is

$$ L = \{(x, \frac{\partial S}{\partial x}) \mid \frac{\partial S}{\partial \xi}(x, \xi) = 0\} $$

where S is a smooth function on $N \times \mathbb{R}^k$, such that $S(x, \xi)$ coincides outside a compact set with a nondegenerate quadratic form in the fibers, $Q(\xi)$. In particular if N is non-compact, L coincides with the zero section outside a compact set.

As proved by Laudenbach and Sikorav (see [LS]) this is the case for $L = \phi_1(O_N)$, where ϕ_t is a Hamiltonian flow.

Moreover S is "essentially unique" up to addition of a quadratic form in new variables and conjugation by a fiber preserving diffeomorphism (the "fibers" are those of the projection $N \times \mathbb{R}^k \to N$) see [V1] and [Th].

We may then consider, for L_0 and L_1 generated by S_0, S_1, the Floer cohomology $FH^*(L_0, L_1; a, b)$ of the chain complex $C^*(L_0, L_1; a, b)$ generated

by the points, x, in $L_0 \cap L_1$ with $A(x) \in [a, b]$ where $A(\gamma)$ is $\int_{[0,1]^2} u^*\omega$ where $u : [0,1]^2 \to T^*N$ is a map as in figure 1 such that

$$u(0, t) = \gamma_0(t) \quad u(s, 0) \in L_0$$
$$u(1, t) = \gamma(t) \quad u(s, 1) \in L_1$$

where γ_0 is a fixed path connecting L_0 to L_1, and x is identified with the constant path γ_x, at x.

Denoting by C^k the subvector space generated by the intersection points of L with the zero section having Conley-Zehnder index k, we have a differential:

$$\delta : C^k(L_0, L_1; a, b) \to C^{k+1}(L_0, L_1; a, b)$$

is obtained by counting the number of holomorphic strips, that are solutions of $\bar{\partial}u = \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = 0$ where J is an almost complex structure compatible with the symplectic form[1]

$$u : \mathbb{R} \times [0, 1] \to \qquad T^*N$$
$$u(s, 0) \in L_0 \quad u(s, 1) \in L_1$$
$$\lim_{S \to \pm\infty} u(s, t) = x_{\pm}.$$

On the other hand, we have the much simpler space $H^*(S^b, S^a)$, where $S(x, \xi, \eta) = S_1(x, \xi) - S_0(x, \eta)$, and $S^\lambda = \{(x, \xi, \eta) \mid S(x, \xi, \eta) \leq \lambda\}$. This cohomology group does not depend on the choice of S according to [V1] and [Th], up to a shift in index, and was used in [Tr] as a substitute for Floer cohomology under the name GF cohomology (denoted $GF^*(L_0, L_1; a, b)$).

Our first claim is

Theorem 2.1.

$$FH^*(L_0, L_1; a, b) \simeq GF^*(L_0, L_1; a, b)$$

The proof will take up the rest of this section.

Remark. : This was announced in [V2], together with a sketch of the proof. Our present proof is actually simpler than the one we had in mind, in particular as far as the applications to the next section are concerned.

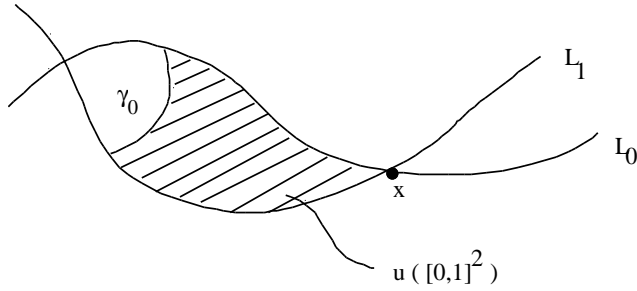[1]i.e. $\omega(\xi, J\xi)$ defines a Riemannian metric

Figure 1: The map u.

We are first going to introduce a functional interpolating between A and S.

Let $H(t, z, \xi)$ be a smooth function on $\mathbb{R} \times T^*N \times \mathbb{R}^k$, equal to some non degenerate quadratic form $Q(\xi)$ outside a compact set.

For γ a path between L_0 and L_1, set $A_H(\gamma, \xi) = A(\gamma) - \int_0^1 H(t, \gamma(t), \xi) dt$.

Set $\mathcal{P}(L_0, L_1)$ be the set of paths, γ, such that $\gamma(0) \in L_0$, $\gamma(1) \in L_1$. Then the critical points of A_H on $\mathcal{P}(L_0, L_1) \times \mathbb{R}^k$ are the pairs (γ, ξ) with

$$\begin{cases} \dot{\gamma} - X_H(\gamma) = 0 \\ \int_0^1 \frac{\partial}{\partial \xi} H(t, \gamma(t), \xi) dt = 0 \end{cases}$$

where X_H is the Hamiltonian vector field associated to the function $z \to H(t, z, \xi)$ (i.e. ξ is "frozen").

If we denote by ϕ_ξ^t the Hamiltonian flow of X_H (for fixed $\xi \in \mathbb{R}^k$) we have $\gamma(t) = \phi_\xi^t(\gamma(0))$, so that the critical points of A_H correspond to

$$\{z \in L_0 \cap (\phi_\xi^t)^{-1}(L_1) | \int_0^1 \frac{\partial}{\partial \xi} H(t, \phi_\xi^t(z), \xi) dt = 0\}$$

Note that

$$L_H = \{\phi_\xi^1(z) | z \in L_0, \int_0^1 \frac{\partial}{\partial \xi} H(t, \phi_\xi^t(z), \xi) dt = 0\}$$

is an immersed Lagrange submanifold, and the above critical points correspond to $L_H \cap L_0$.

Consider an almost complex structure on M compatible with the symplectic form, and denote by $\overline{\partial} = \frac{\partial}{\partial s} + J \frac{\partial}{\partial t}$.

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We now define the Floer cohomology of A_H as usual:

- C^* will be generated by the critical points of A_H on $\mathcal{P}(L_0, L_1)$
- $< d(x_-, \xi_-), (x_+, \xi_+) >$ equals the algebraic number of solutions of

$$\begin{cases} u : \mathbb{R} \times [0,1] \Leftrightarrow M \\ \overline{\partial} u(s,t) = -\nabla H(t, u(s,t), \xi(s)) \\ \frac{d}{ds}\xi(s) = -\int_0^1 \frac{\partial}{\partial \xi} H(t, u(s,t), \xi(s))dt \end{cases}$$

satisfying $\lim_{S \to \pm\infty} (u(s,t), \xi(s)) = (x\pm, \xi\pm)$.

Note that the set of such solutions has its image in a bounded subset of $T^*N \times \mathbb{R}^k$.

Indeed, for u outside a bounded subset, H vanishes, so the equation becomes

$$\begin{cases} \overline{\partial} u = 0 \\ \frac{d}{ds}\xi(s) = -\int_0^1 \frac{\partial}{\partial \xi} H(t, 0, \xi)dt \end{cases}$$

But the first equation cannot hold in the region foliated by pseudoconvex hypersurfaces.

If on the other hand $|\xi|$ is large the equation will be

$$\begin{cases} \overline{\partial} u = -\nabla H(t, u, \xi) \\ \frac{d}{ds}\xi(s) = B\xi \end{cases}$$

where $Q(\xi) = < B\xi, \xi >$.

But the second equation is such that any bounded set is contained in a set with the property that if a trajectory exits the set, it will never reenter it.

Thus, the set of bounded solutions has its image in a bounded set. Then the set of solutions satisfies the same formal properties as the set of solutions of the usual Floer equation. The proofs are just verbatim translations of those in [Fl1].

Definition. The cohomology of $(C^*(L_0, L_1, H), d)$ will be denoted by $FH^*(L_0, L_1, H)$. If we restrict the complex to those solutions with $A_H(\gamma, \xi) \in [a, b]$ the cohomology is denoted $FH^*(L_0, L_1, H; a, b)$.

A first result is

Lemma 2.2. *Let ψ_λ be a Hamiltonian flow on T^*N, and H_λ, F_λ be the Hamiltonians with flows $\psi_\lambda^{-1} \circ \phi_\xi^t \circ \psi_\lambda$ and $\phi_\xi^t \circ (\psi_\lambda^t)^{-1}$.*
    Then

$$FH^*(L_0, \psi_1(L_1), H_0) = FH^*(\psi_1^{-1}(L_0), L_1, H_1) = FH^*(L_0, L_1, F_1)$$

Proof. Consider first the chain complex $C^*(\lambda) = C^*(\psi_\lambda^{-1}(L_0), \psi_\lambda^{-1}\psi_1(L_1), H_\lambda)$ for λ in $[0, 1]$. It is generated by points in $(\psi_\lambda^{-1}(L_0))_{H_\lambda} \cap \psi_\lambda^{-1}\psi_1(L_1)$.
    But $(\psi_\lambda^{-1}(L_0))_{H_\lambda}$ is defined as

$$\{\phi_{\lambda,\xi}^1(\psi_\lambda^{-1}(z)) | z \in L_0; \int_0^1 \partial_\xi H_\lambda(t, \phi_{\lambda,\xi}^t(\psi_\lambda^{-1}(z)), \xi)dt = 0\}$$

and since $H_\lambda(t, u, \xi) = H(t, \psi_\lambda(u), \xi)$ we have $H_\lambda(t, \phi_{\lambda,\xi}^t \circ \psi_\lambda^{-1}(z), \xi) = H(t, \phi_\xi^t(z), \xi)$ and $\phi_{\lambda,\xi}^t \circ \psi_\lambda^{-1} = \psi_\lambda^{-1} \circ \phi_\xi^t$, we have that $(\psi_\lambda^{-1}(L_0))_{H_\lambda} = \psi_\lambda^{-1}((L_0)_H)$ and $\psi_\lambda^{-1}((L_0)_H) \cap \psi_\lambda^{-1}(\psi_1(L_1)) = \psi_\lambda^{-1}((L_0)_H \cap \psi_1(L_1))$
    Thus the generators of $C^*(\lambda)$ do not depend on λ.
    Note also that the intersection points of $(\psi_\lambda^{-1}(L_0))_{H_\lambda}$ and $\psi_\lambda^{-1}(\psi_1(L_1))$ stay transverse provided they are transverse for $\lambda = 0$ (that we shall always assume), and this is sufficient, together with the fact that the value of A_{H_λ} on the critical points does not depend on λ, to imply that the cohomology of $C(\lambda)$ will not depend on λ.
    Consider now the second equality. Note that we may deform H, provided its time one flow is unchanged, and the same for the flow ψ_λ.
    We may thus assume that $\partial_z H$ vanishes, (remember that H must be quadratic in ξ, hence it cannot vanish) for t in $[0, \frac{2}{3}[$, and K_λ, vanishes for t in $[\frac{1}{3}, 1]$.
    Then the flow $(\psi_\lambda^t)^{-1} \circ \phi_\xi^t$ is generated by $H(t, \psi_\lambda^t(z), \xi) + K_\lambda(t, z) = F_\lambda(t, z, \xi)$ and therefore

$$\partial_\xi F_\lambda(t, z, \xi) = \partial_\xi H(t, \psi_\lambda^t(z), \xi)$$

As a result,

$$
\begin{aligned}
(\psi_\lambda^{-1}(L_0))_{H_\lambda} &= \{\phi_{\lambda,\xi}^1(\psi_\lambda^{-1}(z)) | z \in L_0, \int_0^1 \partial_\xi H_\lambda(t, \phi_{\lambda,\xi}^t \circ \psi_\lambda^{-1}(z), \xi)dt = 0\} \\
&= \{\psi_\lambda^{-1}\phi_\xi^1(z)/z \in L_0; \int_0^1 \partial_\xi H(t, \phi_\xi^t(z), \xi)dt = 0\}
\end{aligned}
$$

$$= \{\psi_\lambda^{-1}\phi_\xi^1(z)/z \in L_0; \int_0^1 \partial_\xi F_\lambda(t,(\psi_\lambda^t)^{-1}\circ\phi_\xi^t(z),\xi)dt = 0\}$$

$$= L_{F_\lambda}$$

Thus we have

$$(\psi_\lambda^{-1}(L_0))_{H_\lambda} = \psi_\lambda^{-1}((L_0)_H) = L_{F_\lambda}$$

and $L_{F_\lambda}\cap\psi_\lambda^{-1}\psi_1(L_1)$ is independent of λ, and by the same argument as above, $FH^*(L_0,\psi_\lambda^{-1}\psi_1(L_1),F_\lambda)$ does not depend on λ. Using again the invariance of the critical levels, we proved the second equality. $\square$

In particular for $L_0 = O_N, L = \phi_1(O_N)$
we get

$$FH^*(O_N,L,0) = FH^*(O_N,O_N,H)$$

Now let $S(q,\xi)$ be a $g.f.q.i.$ for L. The flow of X_S is given by

$$\begin{cases} \dot{q} = 0 \\ \dot{p} = \frac{\partial S}{\partial q}(q,\xi) \end{cases} \text{ or } \begin{cases} q(t) \equiv q(0) \\ p(t) = t\frac{\partial}{\partial q}S(q,\xi) + p(0). \end{cases}$$

and $(O_N)_S = \{q,\frac{\partial S}{\partial q}(q,\xi))| \int_0^1 \partial_\xi S(q(t),\xi)dt = 0\}$ but since $q(t) \equiv q(0)$, we have that $\int_0^1 \partial_\xi S(q(t),\xi)dt = \partial_\xi S(q,\xi)$, hence $(O_N)_S$ is the Lagrange submanifold generated by S.

Now we claim

Lemma 2.3. *For S a g.f.q.i we have $FH^*(O_N,O_N,S;a,b) \simeq H^*(S^b,S^a)$.*

Proof. Indeed the generators on both sides are critical points of S, and connecting trajectories solve:

$$\begin{cases} \overline{\partial}u = -\nabla_q S(q(s,t),\xi(s))ds \\ \frac{d}{ds}\xi(s) = -\int_0^1 \partial_\xi S(q(s,t),\xi(s))dt. \end{cases}$$

where $u(s,t) = (q(s,t),p(s,t))$

Now the function $f(q,p) = |p|$ is pluri-subharmonic for our J, and since $\overline{\partial}u$ is tangent to $\ker df$ (note that $\nabla_q S$ is horizontal), we have that $f \circ u$ satisfies the maximum principle. Since $p(s,1) = p(s,0) = 0$, we must have $p \equiv 0$, and thus

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$$\frac{\partial}{\partial s}q(s,t) = -\nabla_q S(q(s,t),\xi(s))$$
$$\frac{\partial}{\partial t}q(s,t) = 0$$

In other words, q only depends on s, and satisfies

$$\begin{cases} \dot{q} = -\nabla_q S(q(s),\xi(s)) \\ \dot{\xi}(s) = -\partial_\xi S(q(s),\xi(s)) \end{cases}$$

and $\lim_{s\to\pm\infty}(q(s),\xi(s)) = (q_\pm,\xi_\pm)$ are critical points for S.

Thus our connecting trajectories, are just bounded gradient trajectories for S, and thus the coboundary map is the same on both complexes, hence the cohomologies are the same. □

We may finally conclude our proof. Let $L = \phi_1(0_N)$, so that it has a generating function S.

Now let S_λ be the generating function of $L_\lambda = \phi_\lambda^{-1}(L)$, so that $S_0 = 0$ and $S_1 = S$. We claim that the modules $FH^*(L_\lambda, 0_N, S_\lambda, a, b)$ are all isomorphic. Indeed, $L_\lambda \cap (0_N)_{S_\lambda} = \phi_\lambda^{-1}(0_N) \cap \phi_\lambda^{-1}(L) = \phi_\lambda^{-1}(0_N \cap L)$ is constant and the usual argument proves the constancy of the Floer cohomology ring. It remains to show that

$$FH^*(\phi_1^{-1}(0_N), 0_N, 0) = FH^*(0_N, \phi_1(0_N), 0)$$

. Again this follows from the equality

$FH^*(\phi_1^{-1}(0_N), 0_N, 0) = FH^*(\phi_t \circ \phi_1^{-1}(0_N), \phi_t(0_N), 0)$, because $\phi_t \phi_1^{-1}(0_N) \cap \phi_t(0_N) = \phi_t(\phi_1^{-1}(0_N) \cap 0_N)$.

We thus proved that

$$FH^*(O_N, L, 0; a, b) \simeq FH^*(O_N, O_N, S; a, b)$$

Since
$$FH^*(O_N, L, 0; a, b) \simeq FH^*(L; a, b)$$

and the above lemma proves states that

$$FH^*(O_N, O_N, S; a, b) \simeq H^*(S^b, S^a) = GH^*(L; a, b)$$

this concludes our proof. □

3 The isomorphism of $FH^*(DT^*N)$ and $H^*(\Lambda N)$

We will now compute $FH^*(M)$ for $M = DT^*N = \{(q,p) \in T^*N \mid |p|_g = 1\}$ for some metric g.

Let us denote by ΛN the free lopp space of N (i.e. $C^0(S^1, N)$).

We have:

Theorem 3.1.
$$FH^*(DT^*N) \simeq H^*(\Lambda N)$$

where ΛN is the free loop space of N. The same holds for S^1 equivariant cohomologies with rational coefficients.

The proof will take up the rest of this section. The next two lemmata are valid in any manifold satisfying the assumptions of section 2. First let us consider the diagonal Δ in $M \times \overline{M}$, where $\overline{M}$ is the manifold M endowed with the symplectic form $-\omega$, and φ^t is the flow of H on M.

Our first result is

Lemma 3.2.

$$FH^*(H; a, b) \simeq FH^*(\Delta, (\text{Id} \times \varphi^1)\Delta; a, b).$$

Proof. Remember that $FH^*(H, a, b) = FH^*(0, 0, H, a, b)$.

We first point out that the cochain spaces associated to both sides are the same, and generated by the fixed points of φ^1.

The connecting trajectories are, for the left hand side given by

$$\overline{\partial} v = -\nabla H(t, v)$$
$$v : \mathbb{R} \times S^1 \to M$$

while for the right hand side, we first notice that $FH^*(\Delta, (\text{id} \times \varphi^1)\Delta; a, b) \simeq FH^*(\Delta, (\text{Id} \times \varphi^1)\Delta, 0; a, b) \simeq FH^*(\Delta, \Delta, 0 \oplus H; a, b)$ (Note: by $K = 0 \oplus H$ we mean the Hamiltonian on $M \times \overline{M}$ defined by $K(t, z_1, z_2) = H(t, z_2)$).

The second isomorphism follows from lemma 2.2.

Now the coboundary map for this cohomology is obtained by counting solutions of

$$\overline{\partial} u = -\nabla K$$

where $u = (u_1, u_2) : \mathbb{R} \times S^1 \Leftrightarrow M \times \overline{M}$ and $\overline{\partial} u_1 = 0; \quad \overline{\partial} u_2 = -\nabla H(t, u_2)$.

Moreover, since $t \to u(s,t)$ connects Δ to itself, we have $u_1(s,0) = u_2(s,0)$ and $u_1(s,1) = u_2(s,1)$.

Therefore we may glue together u_1 and u_2 to obtain a map $\widetilde{v} : S^1 \times \mathbb{R} \to M$ such that

$$\overline{\partial}\widetilde{v} = -\nabla \widetilde{H}(t,\widetilde{v})$$

$$\text{where} \quad \widetilde{H}(t,u) \;=\; 0 \quad \text{for} \quad t \quad \text{in} \quad [0,1]$$
$$\;=\; H(t,u) \quad \text{for} \quad t \quad \text{in} \quad [1,2]$$

(the circle being identified with $\mathbb{R}/2\mathbb{Z}$). We shall make the simplifying assumption that $H(t,z) = 0$ for t close to 0 or 1, so that $\tilde{H}$ is continuous in t. This is of course not a restriction on the time one map.

But the time 2 map of $\widetilde{H}$ coincides with the time 1 map for H, hence we may continuously deform one equation into the other, and the two cohomologies are isomorphic. (Rememeber that if we have a family of Hamiltonians depending continuously on some parameter, but having the same time 1 map, then the corresponding Floer cohomologies are alos independnt from the parameter). $\qquad\qquad\square$

Assume φ^1 is equal to ψ^r. We denote by Γ_φ the graph of φ (i.e. $\Gamma_\phi = (\mathrm{id} \times \varphi)\Delta$), and $\Gamma_\psi^{\rho,r} = \{(z_1, \psi(z_2), z_2, \psi(z_3) \ldots, z_{k-1}, \psi(z_r), z_r, \psi(z_1))\}$ in $(M \times \overline{M})^r$
Set $\Delta^r = \Delta \times \quad \times \Delta$ (r times)

Lemma 3.3. . *For $\varphi = \psi^r$ we have*

$$FH^*(\Delta, \Gamma_\varphi; a, b) \simeq FH^*(\Delta^r, \Gamma_\psi^{\rho,r}; a, b)$$

Proof. Clearly, we may identify $\Gamma_\varphi \cap \Delta$ and $\Delta^r \cap \Gamma_\psi^{\rho,r}$, since a point in this intersection is given by $(z_1, \psi(z_2), \ldots z_r, \psi(z_1)) = (z_1, z_1, \ldots z_r, z_r)$ that is $z_1 = \psi(z_2); z_2 = \psi(z_3), \ldots, z_r = \psi(z_1)$ or else

$$z_1 = \psi^r(z_1), z_i = \psi^{r+1-i}(z_1) \quad \text{for} \quad i \geq 2$$

Note that there is a $\mathbb{Z}/r$ action on the set of such points, induced by $z \to \psi(z)$ on $\Gamma_\varphi \cap \Delta$ and by the shift $(z_j) \to (z_{j-1})$ on $\Gamma_\psi^{\rho,r} \cap \Delta^r$, and these two actions obviously coincide. We shall not mention this point in the proof, but all our results hold for $\mathbb{Z}/r$ equivariant cohomology with any coefficient

ring, and this eventually allows us to recover the S^1 equivariant cohomology (with rational coefficients), due to the Lemma in appendix 2 of [V3].

Now let us compare the trajectories.

For the second cohomology $(u_j, v_j) : \mathbb{R} \times [0, 1] \to M$ satisfying

$$\begin{cases} \overline{\partial} u_j = \overline{\partial} v_j = 0 \\ u_j(s, 0) = v_j(s, 0) = 0 \\ u_j(s, 1) = \psi(v_j(s, 1)) \end{cases}$$

If we set $w_j(s, t) = \psi_t(v_j(s, t))$ so that

$$\begin{cases} \overline{\partial} w_j = \nabla K(s, t) \quad \overline{\partial} u_j = 0 \\ u_j(s, 0) = w_j(s, 0) \\ u_j(s, 1) = w_{j+1}(s, 1) \end{cases}$$

We may now glue together the u_j and w_j as in figure 2 to get a map

$$u : \mathbb{R} \times \mathbb{R}/2r\mathbb{Z} \to M$$

such that

$$\begin{array}{llll} u(s, t) & = & u_j(s, t - 2j) & \text{for} \quad 2j \leq t \leq 2j + 1 \\ u(s, t) & = & w_j(s, t - 2j - 1) & \text{for} \quad 2j + 1 \leq t < 2j + 2 \end{array}$$

It satisfies $\overline{\partial} u = \nabla F(u)$ where

$$\begin{array}{llll} F(t, u) & = & K(t - 2j, u) & \text{for} \quad 2j \leq t \leq 2j + 1 \\ & = & 0 & \text{otherwise} \end{array}$$

Thus the time $2k$ map of X_F is equal to $\psi^k = \varphi$.

We thus identified the Floer trajectories defining $FH^*(\Delta, \Gamma_\psi^{\rho,r}; a, b)$ with those defining

$$FH^*(H; a, b) \simeq FH^*(\Delta, \Gamma_\varphi; a, b)$$

This concludes our proof.

$\square$

Let us now prove theorem 3.1. Let H be a Hamiltonian equal to $H(t, q, p) = c|p|$ for $|p|$ large, where c is some constant (that differs from the length of a closed geodesic, but will eventually become large), and φ^t its flow.

$v_j(s,1) = u_{j-1}(s,1)$ $v_{j-1}(s,0) = u_{j-1}(s,0)$
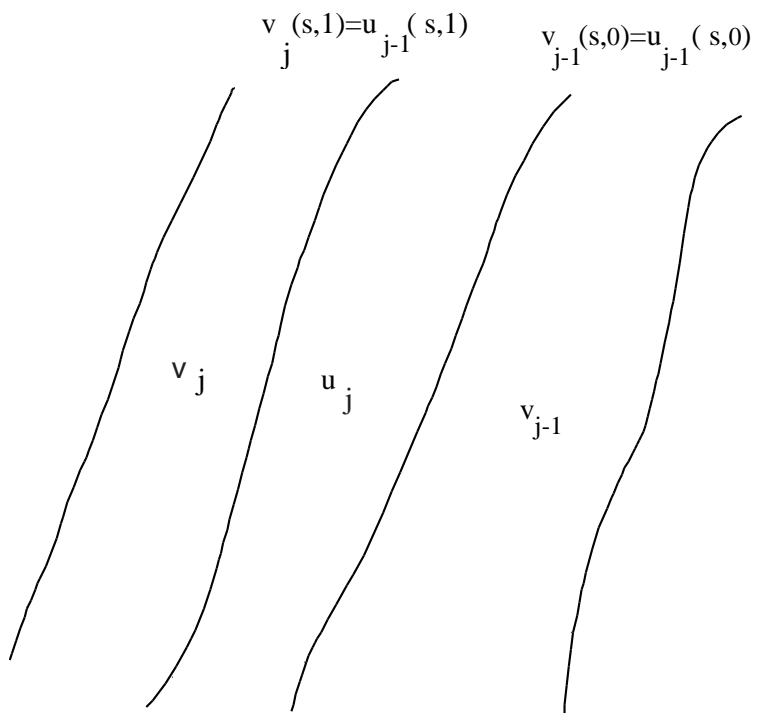
v_j u_j v_{j-1}

j-1

Figure 2: The glueing of the maps u_j and v_j

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For some integer r (that will eventually become large) set $\psi = \varphi^{1/r}$, so that $\psi^r = \varphi^1 = \varphi$.

Then according to lemmata 3.2 and 3.3, we have that

$$FH^*(H; a, b) \simeq FH^*(\Delta^r, \Gamma_\psi^{\rho, r}; a, b)$$

Now we may assume that $H(t, q, p) = h(|p|)$ with h increasing and convex, and k is the Legendre dual of h. Let $q(t)$ be a loop in M, and $E(q) = \int_{S^1} k(|\dot{q}|) dt$. Then we claim

$$FH^*(H; a, b) \simeq H^*(E^b, E^a)$$

We know from [V3] that there is a subset $\mathcal{U}_{r,\varepsilon}$ in Δ^r, defined by

$$\mathcal{U}_{r,\varepsilon} = \{(q_j, P_j)_{j \in \mathbb{Z}/r\mathbb{Z}} | d(q_j, q_{j+1}) \leq \varepsilon/2\}$$

for ε small enough, independent from r, such that $\Gamma_\psi^{\rho, r}$ (that is Γ_Φ in the notation of [V3]) is the graph of dS_Φ over $\mathcal{U}_{r,\varepsilon}$.

From proposition 1.8 in [V3], we have that there is a pseudogradient vector field ξ_Φ of S_Φ such that, denoting by HI^* the cohomological Conley index (see [Co]), we have, for some integer d,

$$HI^*(S_\Phi^b, S_\Phi^a; \xi_\Phi) \simeq H^{*-d}(E_r^b, E_r^a)$$

where

$$E_r^b = \{q = (q_j) \in N^r | d(q_j, q_{j+1}) \leq \varepsilon/2 \quad \text{and} \quad E_r(q) < b\}$$

(Note: in [V3], E_r^b is denoted $\Lambda_{r,\varepsilon}^b$)

$(E_r(q) = \sup_P S_\Phi(q, P)$ and d is some normalizing constant (equal in fact to rn).

On the other hand $E_r^b \simeq E^b$ where $E^b = \{q \in \Lambda M | E(q) \leq b\}$ (see e.g. [Mi]), so we only need:

Lemma 3.4.

$$FH^*(\Delta^r, \Gamma_\psi^{\rho, r}; a, b) \simeq HI^{*+d}(S_\Phi^b, S_\Phi^a, \xi_\Phi)$$

Proof. We would like to find an almost complex structure $J = J_0$, on $(T^*N \times \overline{T^*N})^r$ such that the holomorphic maps corresponding to Floer trajectories are also in one to one correspondence with bounded trajectories of ξ_Φ.
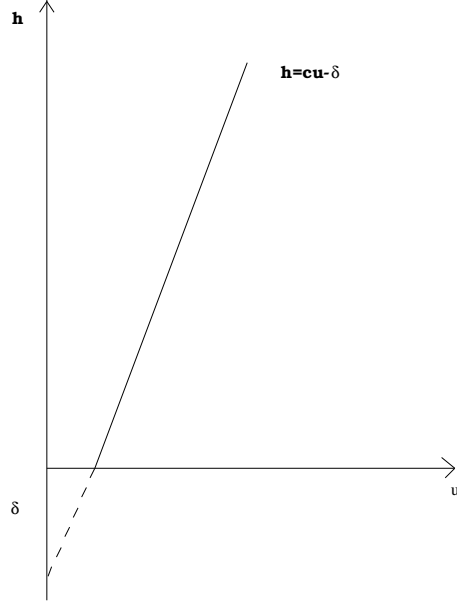
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Figure 3: The hamiltonian h

We first choose h so that the critical values of S_Φ are in $[-\delta, 0]$. Indeed, we may impose that $h'(u)u - h(u)$ is in $[-\delta, 0]$ (see Figure 3).

Then the Floer trajectories used to define the left hand side will have area less than 2δ.

Now, let us show that for δ small enough, such a Floer trajectory must stay inside $T^*\mathcal{U}_{r,\varepsilon_0}$.

Indeed, since all critical points are inside $T^*(\mathcal{U}_{r,\varepsilon_0/2})$, we have, for a Floer trajectory exiting from $T^*\mathcal{U}_{r,\varepsilon_0/2}$, that it defines a J-holomorphic curve Σ such that

(i)
$$\partial\Sigma \subset \Delta^r \cup \Gamma_\Phi^{\rho,r}.$$

(ii)
$$\Sigma \quad \text{is in} \quad T^*(\mathcal{U}_{r,\varepsilon_0} - \mathcal{U}_{r,\varepsilon_0/2}).$$

Now we are in the following abstract situation:

Let $V_0 \subset V_1$ be convex sets, L_0, L_1 be disjoint Lagrange submanifolds in $T^*(V_1 - V_0)$. We set $S_t = \partial V_t = \{x \in V_1 \mid (1-t)d(x, V_0) = td(x, V_1)\}$, and $B_t = \partial T^*V_t = T^*_{S_t}V$.

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Let Σ be a holomorphic curve with boundary in $L_0 \cup L_1$. Then we claim that the area of Σ is bounded from below.

First let us consider the case of a pseudo-holomorphic curve, Σ closed in $T^*(V_1 - V_0)$ (and in particular with boundary in $T^*_{S_0}V \cup T^*_{S_1}V$), such that $\partial\Sigma \cap T^*_{S_0}V = \partial\Sigma \cap B_0$ is non-empty.

Then $B_{1/2}$ must intersect Σ, since otherwise, as $\partial\Sigma \cap B_0 \neq \emptyset$, there would be an interior tangency between some B_t (for $0 < t < 1/2$) and Σ, and this is impossible by the pseudoconvexity of B_t.

Thus for $\alpha = \frac{1}{2}d(V_0, V_1)$, and $x_0 \in B_{1/2} \cap \Sigma$ we have that $B(x_0, \alpha)$ is in $T^*(V_1 - V_0)$, and thus, we have

$$\text{area}(\Sigma) \geq \text{area}(\Sigma \cap B(x_0, \alpha) \geq \pi\alpha^2 \exp(\eta(c\alpha))$$

where c is an upper bound for the sectional curvatures of the metric g_0 associated to ω and J (see Appendix).

Now consider the case where Σ has a boundary contained in the union of the two Lagrange submanifolds L_0, L_1.

Let U_0, U_1 be tubular neighbourhoods of L_0, L_1 respectively, and assume that they are disjoint, symmetric (i.e. there is an anti-holomorphic diffeomorphism of U_i, fixing L_i), and pseudoconvex. This can be easily achieved, through a perturbation of J near the L_i.

Consider now $\partial\Sigma \cap L_i = \gamma_i$. Then either γ_0 and γ_1 are both contained inside $B_{1/2}$ or one of them is not.

In the first case, consider $\Sigma \cap T^*(V_1 - V_{1/2})$. Then this intersection does not have a boundary in $T^*(V_1 - V_{1/2})$, except on $B_{1/2} \cup B_1$, and we thus have again a lower bound on the area of Σ as in the case of a closed curve, except that α is to be replaced by $\alpha/2$.

In the second case, assume for instance that $B_{1/2}$ intersects γ_0.

Let us then consider the symetrization of Σ inside U_0, where J is integrable, (see for instance [Si]). This will be a closed curve $\widehat{\Sigma}$ in U_0, that has a point x_0 in $B_{1/2}$, and we also have a ball $B(x_0, \alpha)$ in $U_0 \cap T^*(V_1 - V_0)$ for $\alpha \leq \inf\{\frac{1}{2}d(V_0, V_1), \frac{1}{2}d(L_0, \partial U_0)\}$, and again, we get a lower estimate of the area of $\widehat{\Sigma}$. Since $\text{area}(\Sigma) \geq 1/2\,\text{area}(\widehat{\Sigma})$, we also get an estimate on the area of Σ.

This proves our abstract statement. We now claim that we are in the above framework, with $\alpha \simeq \varepsilon_0/4$.

Indeed, the diameter of $T^*(\mathcal{U}_{r,\varepsilon} - \mathcal{U}_{r,\varepsilon/2})$ is $\varepsilon/2$, and we have to show that Δ^r has a pseudoconvex symmetric neighbourhood of radius $\varepsilon/4$.

But we will show that

$$(ii) \qquad \Gamma_\Phi^{\rho,r} \cap T^*(\mathcal{U}_{r,\varepsilon_0} - \mathcal{U}_{r,\varepsilon_0/2}) \cap \{|X_j| \le \varepsilon_0/4|Y_j| \le \varepsilon_0/4\} = \emptyset.$$

This implies that $\{|X_j| \le \varepsilon/4|Y_j| \le \varepsilon/4\}$ is a tubular neighbourhood of the zero section, is disjoint from $\Gamma_\Phi^{\rho,r}$ in the region we are considering. Thus we have our U_0, with radius $\varepsilon/4$. A similar fact would hold for $\Gamma_\Phi^{\rho,r}$.

Let us now prove our last claim.

Indeed we only have to show that $|\frac{\partial S_\Phi}{\partial P_j}| \ge \varepsilon/4$ for (q, P) in $\mathcal{U}_{r,\varepsilon_0} - \mathcal{U}_{r,\varepsilon_0/2}$.

But $\frac{\partial S_\Phi}{\partial P_j} \simeq q_{j+1} - q_j - \frac{1}{r}\frac{\partial H}{\partial p} + \eta(q_{j+1} - q_j, P_j)$ (see [V3], proof of 1.2) where $\eta(0, P) = d\eta(0, P) = 0$, and since for some j, $d(q_{j+1}, q_j) \ge \frac{\varepsilon}{2}$, we have $|X_j| = |\frac{\partial S}{\partial P_j}| \ge \frac{\varepsilon}{2} - \frac{C}{r} \ge \frac{\varepsilon_0}{4}$ for r large enough.

It is particularly important to notice that our lower bounds are independent from r, since they only depend on an upper bound for the sectional curvatures of the metric, and this quantity stays bounded as r goes to infinity. $\qquad\square$

If we choose $\delta < \frac{1}{4}\varepsilon$ we get that all Floer trajectories for J_0 must stay in $T^*\mathcal{U}_{r,\varepsilon_0}$.

We claim that

(i) for a suitable choice of J_1, the J_1 holomorphic curves defining the Floer cohomology are in one to one correspondence with trajectories of S_Φ.

(ii) there is a family J_λ of almost complex structures connecting J_0 to J_1, taming ω and making $T^*_{\partial\mathcal{U}_{r,\varepsilon_0}}\mathcal{U}_{r,\varepsilon_0}$ pseudo convex. Note that our almost complex structures will be time dependent, but this is not important.

Lemma 3.5. *Let U be a manifold with boundary, and $f : U \to \mathbb{R}$ be a smooth function. Then for $L = graph(df)$, ϕ_t the Hamiltonian flow $(q, p) \to (q, p + tdf(q))$, set $J(t, u) = (\phi_t)_* J_0(u)$.*

Then, there is a one to one correspondence between solutions of

$$\begin{cases} \dot{q}(t) = -\nabla f(q(t)) \\ \lim_{s \to \pm\infty} q(t) = x_\pm \end{cases}$$

and

$$\begin{cases} \overline{\partial}_J v = 0 \\ v(s, 0) \in 0_N \\ \lim_{s \to \pm\infty} v(s, t) = x_\pm \end{cases} \qquad , \qquad v(s, 1) \in L$$

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Proof. We have, setting $v(s,t) = \phi_t u(s,t)$

$$\overline{\partial}_J v = d\phi_t(u)\frac{\partial}{\partial s}u(s,t)$$
$$+ J(\frac{\partial}{\partial t}\phi_t)(u)$$
$$= d\phi_t(u)[\frac{\partial}{\partial t} + d\phi_t(u)[\frac{\partial}{\partial S} + d\phi_t^{-1}(\phi_t(u))Jd\phi_t(u)\frac{\partial}{\partial t}u$$
$$+ \nabla H(u)]$$
$$= d\phi_t(u)[\overline{\partial}_{J_0}u + \nabla H(u)$$
$$= d\phi_t(u)[\overline{\partial}_{J_0}u + \nabla H(u)].$$

Thus

$$\begin{cases} \overline{\partial}_{J_0}u = -\nabla H(u) \\ u(S,0) \in O_U \qquad u(s,1) \in O_U. \end{cases}$$

Now since $H(q,p) = f(q)$, $\nabla H(q,p) = \nabla f(q)$, and in local coordinates, we have $dp_j \cdot \nabla f(q) = 0$.

Hence $d(|p|) \cdot \overline{\partial}_{J_0}u = \frac{1}{|p|}\sum_{j=1}^{n} dp_j \cdot \nabla f(q) = 0$ and $|p \circ u|$ satisfies the maximum principle. But since $u(s,0) \in O_U$, $u(s,1) \in O_U$, we have $p \circ u \equiv 0$, hence $u(s,t) = q(s,t),0)$. Now $\frac{\partial q}{\partial t} + \frac{\partial p}{\partial S} = 0$ is the second half of $\overline{\partial}_{J_0}u = -\nabla H(u)$, hence $q(s,t) = q(t)$, and the first half becomes $\dot{q}(t) = -\nabla f(q)$. $\square$

Note that if J_0 makes $T^*_{\partial U}U$ pseudoconvex, the same holds for J_λ the linear interpolation between J_0 and J_1.

From this lemma, and the previous arguments, we may conclude that $FH^*(\Delta^k, \Gamma_\psi^{\sigma,k}; a, b)$ is isomorphic to the cohomology of the Thom-Smale-Witten complex of ∇S_Φ restricted to $S_\Phi^b - S_\Phi^a$.

According to [Fl3] this last cohomology equals $HI^*(S_\Phi^b, S_\Phi^a; \nabla S_\Phi)$.

Our proof will be complete if we are able to show that

$$HI^*(S_\Phi^b, S_\Phi^a; \nabla S_\Phi) \simeq HI^*(S_\Phi^b, S_\Phi^a; \xi_\Phi)$$

where ξ_Φ is as in [V3]. This may be explained by looking at figure 4.

We represented there a pseudogradient vector field η for S_Φ, equal to ∇S_Φ in a neighbourhood of $\mathcal{U}_{r,\varepsilon} - \mathcal{U}_{r,2\varepsilon/3}$ and to ξ_Φ in a neighbourhood of $\mathcal{U}_{r,\varepsilon/2}$.
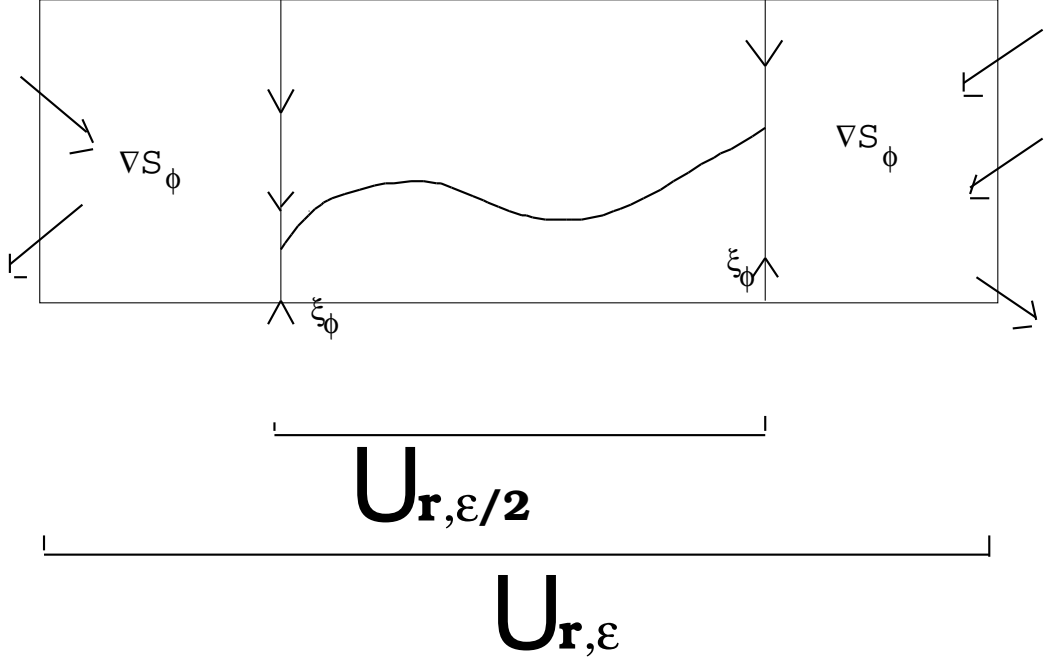
Figure 4: The phase portrait of the pseudo-gradient ξ_Φ of S_Φ

Now, all critical points of S_Φ are inside $\mathcal{U}_{r,\varepsilon/2}$, and a heteroclinic trajectory for η stays inside $\mathcal{U}_{r,\varepsilon/2}$ since $\eta = \xi_\Phi$ on $\partial \mathcal{U}_{r,\varepsilon/2}$ is tangent to $\partial \mathcal{U}_{r,\varepsilon/2}$.

Thus $I^*(\mathcal{U}_{r,\varepsilon}, \eta) \simeq I^*(\mathcal{U}_{r,\varepsilon/2}, \eta)$.

But $I^*(\mathcal{U}_{r,\varepsilon}, \eta) \simeq I^*(\mathcal{U}_{r,\varepsilon}, \nabla S_\Phi)$ since $\eta = \nabla S_\Phi$ near $\partial \mathcal{U}_{r,\varepsilon}$, and

$$I^*(\mathcal{U}_{r,\varepsilon/2}, \eta) \simeq I^*(\mathcal{U}_{r,\varepsilon/2}, \xi_\Phi).$$

Therefore

$$I^*(\mathcal{U}_{r,\varepsilon}, \nabla S_\Phi) \simeq I^*(\mathcal{U}_{r,\varepsilon/2}, \xi_\Phi)$$

and our argument obviously extends if we restrict ourselves to $S_\Phi^b - S_\Phi^a$. Thus

$$I^*(S_\Phi^b, S_\Phi^a, \nabla S_\Phi) \simeq I^*(S_\Phi^b, S_\Phi^a, \xi_\Phi).$$

Note that these indices do not depend on ε, for r large enough, hence it is not important that the previous inequality relates the index of $\mathcal{U}_{r,\varepsilon}$ to the index of $U_{r,\varepsilon/2}$.

4 Appendix

Let (M, ω) be a symplectic manifold with contact type boundary, and J_0 be an admissible almost complex structure on M.

Given U a domain in M, and x_0 in U, we consider

$$w_{J_0,x_0}(U) = \inf\{\int_C \omega \quad | \, C \quad \text{is } J_0\text{- holomorphic,} \quad x_0 \in C\}$$

and $w(U)$, the usual Gromov width, is then given by

$$w(U) = \sup\{w_{J_0,x_0}(U) \mid x_0 \in U, J_0 \quad \text{is admissible} \quad\}$$

A natural question is to compute the limits

$$\lim_{k \to +\infty} w(U^k) = \overline{w}(U)$$

$$\lim_{k \to +\infty} w_{J_0,x_0}(U^k) = \overline{w}_{J_0,x_0}(U)$$

Here we still denote by J_0 the almost complex structure $J_0 \times \ldots \times J_0$ on M^k, and by x_0 the point $(x_0, \ldots, x_0)$ in M^k.

Note that the sequence $w_{J_0,x_0}(U)$ is obviously decreasing.

On the other hand, $\overline{w}(U)$ is not, a priori, equal to $\sup \{\overline{w}_{J_0,x_0}(U) | x_0 \in U, J_0 \text{ is admissible}\} = \widetilde{w}(U)$ since there are many more almost complex structures on M^k than those of the type $J_0 \times \ldots \times J_0$. Clearly, we have $\widetilde{w}(U) \leq \overline{w}(U)$.

While it is clear, if U contains a symplectic ball of radius r, that $\overline{w}(U) \geq \pi r^2$, no such lower bound holds for $\overline{w}_{J_0,x_0}(U)$.

In fact, it is not a priori obvious that $\overline{w}_{J_0,x_0}(U)$ is non zero. This is what we prove in this appendix.

Proposition 4.1. *Let g_0 be the metric associated to (ω, J_0). Then, if the sectional curvature of g_0 is bounded by c, and injectivity radius at x_0 bounded by ρ, we have*

$$\overline{w}_{J_0,x_0}(U) \geq \pi \rho^2 \delta(c, \rho)$$

where $\delta(c, \rho)$ is a continuous positive function, such that $\delta(c, 0) = 1$.

Lemma 4.2. *Let $B_{g_0}(r)$ be the ball of radius r, centered at x_0, for the metric g_0. Let C be a minimal surface for g_0 through x_0. Then*

$$area\,(C \cap B_{g_0}(r)) \geq \pi r^2 \exp(\eta(cr))$$

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where η is continuous, $\eta(0) = 0$, and c is the upper bound of the sectional curvature of g_0.

Proof. It is similar to the case where g_0 is the euclidean metric. Set

$$a(r) = \text{area}\,(C \cap B_{g_0}(r))$$

then $a'(r) = \text{length}(C \cap \partial B_{g_0}(r))$ and since C is minimal, $a(r)$ must be less than the area of the cone through x_0, spanned by $C \cap \partial B_{g_0}(r)$.

Let us then compute the area of such a cone. It is clearly given by $\int_0^r \text{length}(\gamma_s)ds$ where $\gamma_s(t) = \exp(\frac{s}{r} \cdot \exp^{-1}(c_r(t)))$ and $c_r(t)$ is the curve $C \cap \partial B_{g_0}(r)$, the exponential being taken at x_0.

Let M be a bound on the sectional curvature of g_0. Then we have, by classical comparison theorems ([Pan] p.117, remark 8.14b)

$$\|D\exp_{x_0}(u)\| \leq \frac{\sinh(Mr)}{Mr} \text{ for } \|u\| \leq r$$

$$\|D\exp_{x_0}^{-1}(y)\| \leq \frac{Mr}{\sin(Mr)} \text{ for } y \in B(x_0, r)$$

Thus

$$\text{length}\,(\gamma_s) \leq \frac{s}{r}\frac{\sinh(Mr)}{\sin(Mr)} \cdot \text{length}(Cr)$$

and

$$\int_0^r \text{length}\,(\gamma_s)ds \leq \frac{r}{2}\frac{\sinh(Mr)}{\sin(Mr)} \cdot \text{length}(Cr)$$

and we have

$$a(r) \leq \frac{r}{2}\varphi(Mr)a'(r)$$

so that

$$\frac{a'(r)}{a(r)} \geq \frac{2}{r\varphi(Mr)}$$

$$\log(\frac{a(r)}{a(\varepsilon)}) \geq \log\frac{r^2}{\varepsilon^2} + \int_\varepsilon^r \frac{2}{u\varphi(Mu)}(1 - \varphi(Mu))du$$

$$\geq \log(\frac{r^2}{\varepsilon^2}) + \int_{M\varepsilon}^{Mr} 2\frac{1 - \varphi(v)}{v\varphi(v)}dr$$

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Since $1 - \varphi(u) \sim u$ as u goes to zero, the quantity $\int_{M\varepsilon}^{Mr} \frac{2}{u(\varphi(u))}(1 - \varphi(u))du$ converges to $\eta(Mr)$ as ε goes to zero, with η continuous and $\eta(0) = 0$.

Then, since $\lim\limits_{\varepsilon \to 0} \dfrac{a(\varepsilon)}{\varepsilon^2} \geq \pi$, we have $a(r) \geq \pi r^2 \exp(\eta(Mr))$. $\qquad\square$

Now, replacing U by U^k, the sectional curvature of the induced metrics stays bounded (even though the bound may change as we go from $k = 1$ to $k = 2$), and since U^k contains $(B_{g_0}(\rho))^k$, we get

$$\omega_{J_0, x_0}(U^k) \geq \pi \rho^2 \delta(c, \rho).$$

$\qquad\square$

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