Séries de Harish-Chandra de Modules et Correspondance de Howe Modulaire

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LMENS - 92 - 7

Mars 1992
Equivariant cohomology with generalised coefficients

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Introduction

In the text of this article, definitions, propositions, theorems, lemmas, examples, corollaries are numbered in the same sequential order. Formulas follow an independent sequential order.

Let $G$ be a real Lie group with Lie algebra $\mathfrak{g}$ and let $M$ be a smooth manifold on which $G$ acts. Let $\mathcal{A}(M) = \sum_i \mathcal{A}^i(M)$ be the space of smooth differential forms on $M$ and let $\mathcal{A}_{cpt}(M)$ be the subspace of forms with compact support. Let us recall (Cartan ;[9]) that the $G$-equivariant de Rham complex of $M$ is by definition the differential $\mathbb{Z}_+$-graded algebra

$$\mathcal{A}_G(M) := (S(\mathfrak{g}') \otimes \mathcal{A}(M))^G$$

endowed with the tensor product graded algebra structure (where elements of $\mathfrak{g}'$ are assigned degree 2) together with the equivariant de Rham differential $d_G$ of degree 1 (see section 2, Formula 4). Its cohomology denoted $H^*_G(M)$ is called the $G$-equivariant de Rham cohomology of $M$. Alternatively, an element $\alpha \in \mathcal{A}_G(M)$ can be thought of as a differential form $\alpha(X)$ on $M$ depending polynomially on $X \in \mathfrak{g}$, such that $\alpha$ is equivariant:

$$\alpha(g \cdot X) = g \cdot \alpha(X),$$

for all $g \in G$.

The complex $\mathcal{A}_G(M)$ admits a subcomplex

$$\mathcal{A}_{cpt,G}(M) := (S(\mathfrak{g}') \otimes \mathcal{A}_{cpt}(M))^G,$$
and its cohomology is called the $G$-equivariant de Rham cohomology with compact support $H_{\text{cpt}, G}^*(M)$.

Sometimes, it is natural to consider the space $A_G(M)$ of equivariant forms $\alpha(X)$ on $M$ depending smoothly on $X \in \mathfrak{g}$. The differential $d_\mathfrak{g}$ extends to this space and the cohomology of the complex $(A_G(M), d_\mathfrak{g})$ is denoted by $H_G^\infty(M)$. This cohomology $H_G^\infty(M)$ is studied in [12]. In some situations, it is also important to consider the space $A_{G/G}(M)$ of equivariant forms depending in a generalised way on the variable $X \in \mathfrak{g}$. The differential $d_\mathfrak{g}$ still has a meaning on $A_{G/G}(M)$ and the cohomology of the complex $(A_{G/G}(M), d_\mathfrak{g})$ is denoted by $H_{G/G}^\infty(M)$. (The space $A_{G/G}^\infty(M)$ and its cohomology $H_{G/G}^\infty(M)$ were introduced in [11].) One similarly defines $H_{\text{cpt}, G}^\infty(M)$. When $M$ is a point, $H_{G/G}^\infty(\text{point})$ is equal to the space $C^{-\infty}(\mathfrak{g})^G$ of $G$-invariant generalised functions on $\mathfrak{g}$. There is a natural map $H_G^\infty(M) \to H_{G/G}^\infty(M)$.

If $M$ is compact and $G$-oriented, integration over $M$ gives us a map from $H_{G/G}^\infty(M)$ to $C^{-\infty}(\mathfrak{g})^G$. More generally if $p : M \to B$ is a $G$-equivariant fibration, with $G$-oriented fibers, then there is defined an integration along the fiber map

$$p_* : H_{\text{cpt}, G}^\infty(M) \to H_{\text{cpt}, G}^\infty(B)$$

(cf. Formula 7).

If $M$ is non compact, and if $\alpha(X)$ is an equivariant form on $M$ depending smoothly on $X \in \mathfrak{g}$, the integral of $\alpha(X)$ over $M$ may sometimes exist in a generalised sense: after integrating $\alpha(X)$ against a test function $\Phi$ on the Lie algebra $\mathfrak{g}$, the form $(\alpha, \Phi) := \int_{\mathfrak{g}} \alpha(X)\Phi(X)dX$ may become integrable over $M$ and we can define $\int_M^\infty \alpha \in C^{-\infty}(\mathfrak{g})^G$ by

$$\left( \int_M^\infty \alpha, \Phi \right) = \int_M (\alpha, \Phi).$$

Many important examples of generalised functions on $\mathfrak{g}$ arise this way. For instance, characters of representations of $G$ attached to a generic coadjoint orbit $M$ are given by the integral of an equivariant form over $M$ (see [19], [11]). If $G$ is compact, the formula of [18] for the index of a $G$-transversally elliptic operator $D$ on a compact $G$-manifold $B$ is given by the integration (in the generalised sense) over $M = T^*B$ of a $G$-equivariant form $\alpha(\sigma)(X)$ on $M$ (depending smoothly on $X \in \mathfrak{g}$) attached to the symbol $\sigma$ of $D$. 


It will thus be useful to understand the space $H^\infty_G(M)$. The aim of this article is to start a systematic study of the cohomology space $H^\infty_G(M)$.

Now we describe some of the results we prove in this article.

We first prove (in section 2) that for a $G$-equivariant real vector bundle $p : \mathcal{V} \to \mathcal{B}$, the canonical pull-back map $p^* : H^\infty_G(\mathcal{B}) \to H^\infty_G(\mathcal{V})$ is an isomorphism (cf. Proposition 6). In particular, for a real representation $V$ of $G$, $H^\infty_G(V) \cong C^\infty(G)^G$. Similarly, we prove the Thom isomorphism; asserting that if $G$ is compact and the fibers of $p$ are $G$-oriented, then the integration along the fiber map

$$p_* : H^\infty_{\text{cpt},G}(\mathcal{V}) \to H^\infty_{\text{cpt},G}(\mathcal{B})$$

is an isomorphism (cf. Proposition 9).

Let $K \subset G$ be a closed subgroup. Let $\chi = \chi_{G/K} : K \to \{\pm 1\}$ be the character of $K$ defined by $\chi(k) = \text{sign} \det_{g/k} k$, for all $k \in K$. Let $M$ be a $K$-manifold and let $H^\infty_K(M)$ be the cohomology of the complex

$$(A^\infty_K(M), d_k) := (C^\infty(k, A(M))^\chi, d_k)$$

(cf. Definition 47). Fix an orientation $o$ on $g/k$. Consider the space $G \times_K M$, fibered over $G/K$ with fiber $M$. In section 5, we define a cochain map

$$\text{Ind}_{G/K,o} : A^\infty_K(M) \to A^\infty_G(G \times_K M)$$

(cf. Proposition 48) and prove that if $K$ is compact, $\text{Ind}_{G/K,o}$ induces an isomorphism in cohomology (cf. Theorem 50). This is one of the central results of this article. The proof of this result relies on a study of the homology of the perturbed Koszul complexes defined in sections 3 and 4. This technique is already used in [12] for the study of $G$-equivariant cohomology with smooth coefficients.

Taking $M = \text{point}$, we get the isomorphism

$$C^\infty(k)^\chi \cong H^\infty_G(G/K)$$

where $C^\infty(k)^\chi := \{f \in C^\infty(k) ; k \cdot f = \chi(k)f, \text{ for all } k \in K\}$.

The explicit description of the isomorphism (cf. Proposition 41), indicates the analogy between $\int_{G/K} \text{Ind}_{G/K,o} f$ and characters of induced representations (cf. Proposition 42).
Recall [12] that $H^\infty_G(G/K)$ is canonically isomorphic to $C^\infty(k)^K$. We determine the canonical map

$$C^\infty(k)^K \cong H^\infty_G(G/K) \to C^-\infty(k)^K \cong H^-\infty_G(G/K),$$

coming from the natural map $H^\infty_G(G/K) \to H^-\infty_G(G/K)$, in Corollary 55, in the case where $G$ itself is compact.

From now on in the introduction, the notation $K$ will be reserved to denote a compact connected Lie group with maximal torus $T$ and Weyl group $W$. The Lie algebras of $K$, $T$ are denoted by $k$, $t$ respectively.

In section 6, we prove a Kunneth theorem: Let $D$ be a compact $K$-manifold such that $H_K(D)$ is free over $H_K(\text{point})$ (e.g. $D = K/U$, for a closed subgroup $U \subset K$ of the same rank, cf. Lemma 62). Then, for any $K$-manifold $M$, the canonical Kunneth map

$$m^-\infty : H_K(D) \otimes H_K(\text{point}) H^-K(\infty)(M) \to H^-K(\infty)(D \times M)$$

is an isomorphism (cf. Theorem 60). In particular for any such $D$, taking $M$ to be a point, we get

$$H^-K(\infty)(D) \cong C^-\infty(k)^K \otimes S(k)^K H_K(D)$$

(cf. Corollary 61).

Using Kunneth theorem and the Induction isomorphism, we obtain in Proposition 63 an extension of Chevalley’s theorem: Let $C^-\infty(t)^t$ be the space of all the generalised $W$-anti-invariant functions on $t$, then the multiplication of generalised functions on $t$ by polynomial functions induces an isomorphism

$$C^-\infty(t)^t \otimes_{S(t)^W} S(t') \cong C^-\infty(t).$$

By the same technique, we obtain the Reduction Theorem asserting that for any $K$-manifold $M$, we have a canonical isomorphism

$$H^-K(\infty)(M)^W \cong H^-K(\infty)(M),$$

where $H^-K(\infty)(M)^W$ refers to the $W$-invariants under the canonical action of $W$ on $H^-K(\infty)(M)$ (cf. Theorem 71). The proof of this reduction theorem is inspired by the proof of Theorem 4.2 in Atiyah [1].
Again combining the Kunneth theorem and the Induction isomorphism, we obtain an isomorphism of $H_L(point)$-modules
\[ H_L(point) \otimes_{H_K(point)} H_K^{-\infty}(M) \cong H_{L \times K/L}(M), \]
for any closed subgroup $L \subset K$ of the same rank and any $K$-manifold $M$ (cf. Theorem 67). In particular, taking $M = K/U$ (for any closed subgroup $U$ of $K$), we obtain
\[ H_{L \times K/L}^{-\infty}(K/U) \cong S(V)^L \otimes_{S(k^L)} (C^{-\infty}(u)^{\chi K/U}). \]

If $M$ is a $T$-manifold, we give a homology spectral sequence (in section 10) with
\[ E^2_p = Tor^S_p(C^{-\infty}(t), H_T(M)) \]
converging to the cohomology $H_T^{-\infty}(M)$, where $S(t')$ acts by multiplication on $C^{-\infty}(t)$ (cf. Theorem 99). We show that this spectral sequence degenerates at the $E^2$-term for any homogeneous space $M = K/U$ (any closed subgroup of $K$) (cf. Proposition 103).

In section 9, we study free actions. Let $P$ be a principal $G$-bundle (for any Lie group $G$) and let $q : P \to P/G$ be the quotient map. Assume that the fibers of $q$ admit a $G$-orientation $o$. Then we prove (cf. Theorem 86) that $H_G^{-\infty}(P)$ is a free module over $H_G(P) \cong H(P/G)$ with a generator $\gamma_o$. We determine this generator explicitly (cf. Proposition 77). Consider, for example, the free action of $T$ on $P = K$ by right translations. Then the space $H_T^{-\infty}(P)$ is a vector space of dimension $|W|$ over $R$. We use the description of $H_T^{-\infty}(K)$ to conclude that the canonical map $H_T^{-\infty}(K) \to H_T^{-\infty}(K)$ is identically 0 (cf. Corollary 93).

More generally, we consider the case of a manifold $P$ with a right action of a Lie group $G$ and where we assume that a normal closed subgroup $N$ of $G$ acts principally on $P$ (cf Definition 72). In addition, assume that the principal $N$-bundle $q_N : P \to P/N$ admits a $G$-invariant connection and that the fibers of $q_N$ admit a $G$-orientation $o$. We then construct a map
\[ m_o : H_G^{-\infty}(P/N) \to H_G^{-\infty}(P), \]
and show that $m_o$ is an isomorphism if $G$ is compact (cf. Theorem 88).

Finally, in section 11, we prove a Localisation theorem for any compact oriented $T$-manifold $M$. We first need to take a $T$-equivariant embedding
of $M$ in a representation space $V$ of $T$. This gives rise to a certain non-zero polynomial $P \in S(t')$. Now we determine

$$P(X) \int_M \alpha(X) \in C^{-\infty}(t),$$

for any $\alpha \in H_T^{-\infty}(M)$, in terms of the restriction of $\alpha$ to $M^T$ and of the equivariant Euler class of the normal bundle of the submanifold $M^T \subset M$ (cf. Theorem 104). One striking difference from the smooth case is that it is possible to have $\int_M \alpha(X) \neq 0$, for $\alpha \in H_T^{-\infty}(M)$, even though $M^T$ may be empty. In fact, we prove that $\int_K : H_T^{-\infty}(K) \to C^{-\infty}(t)$ is injective, where $T$ acts on $K$ by right multiplication (cf. Proposition 92, section 9).

**Acknowledgements**

Collaboration between the two authors on the topics of this article started when the second author visited Tata Institute of Fundamental Research in January 1991 (T.I.F.R). The second author would thus like to acknowledge the financial support from the Indian government and the hospitality of T.I.F.R. Numerous discussions with M.Duflo have influenced strongly the second author. The first author thanks M.E. Taylor for helpful conversations, and also acknowledges the hospitality of Ecole Normale Superieure during his stay in Paris in June 1991.

**1 Notation**

By a manifold $M$, we shall always mean a $C^\infty$ real manifold. We denote by $C^\infty(M)$ the space of $C^\infty$- (real-valued) functions on $M$. We denote by $C^{-\infty}(M)$ the space of generalised (real-valued) functions on $M$: By definition, $C^{-\infty}(M)$ is the continuous dual of the space of smooth compactly supported densities on $M$ under the $C^\infty$-topology. The space $C^\infty(M)$ is canonically a subspace of $C^{-\infty}(M)$ and $C^{-\infty}(M)$ is a module over $C^\infty(M)$.

We denote the space of smooth differential forms on $M$ (with real coefficients) by $\mathcal{A}^*(M)$. We denote the subspace of compactly supported differential forms by $\mathcal{A}^*_{cpt}(M)$. The exterior derivative is denoted by $d_M$ or simply by $d$. If $\xi$ is a vector field on $M$, we denote by $i(\xi) : \mathcal{A}^*(M) \to \mathcal{A}^{*-1}(M)$ the
contraction by the vector field $\xi$. We denote by $\mathcal{L}(\xi) : \mathcal{A}^*(M) \to \mathcal{A}^*(M)$ the Lie derivative action of $\xi$. The operators $d, \iota(\xi), \mathcal{L}(\xi)$ on $\mathcal{A}(M)$ satisfy the Cartan relation:

$$d\iota(\xi) + \iota(\xi)d = \mathcal{L}(\xi).$$

(1)

If $M$ is oriented, for $\alpha \in \mathcal{A}_{\text{cpt}}(M) = \bigoplus_{i=0}^{\dim M} \mathcal{A}^i_{\text{cpt}}(M)$, we note $\int_M \alpha$ the integral of the component of $\alpha$ in $\mathcal{A}^i_{\text{cpt}}(M)$.

Let $G$ be a real Lie group. By a $G$-manifold, we mean a manifold on which $G$ acts smoothly. Let $\mathfrak{g}$ be the Lie algebra of $G$. If $X \in \mathfrak{g}$, we denote by $X_M$ (or simply $X$, if no confusion is likely) the vector field on $M$ such that

$$(X_M \cdot \varphi)(x) = \frac{d}{d\varepsilon}\varphi((\exp -\varepsilon X)x)|_{\varepsilon=0}$$

for $\varphi \in C^\infty(M)$, $x \in M$.

Unless otherwise stated, vector spaces are over $\mathbb{R}$, and the dual space $\text{Hom}_{\mathbb{R}}(V, \mathbb{R})$ of a vector space $V$ is denoted by $V^\ast$. If $E^i, 1 \leq i \leq n$ is a basis of a $n$-dimensional vector space $V$, $E^i$ denotes the dual basis of $V^\ast$.

Tensor products without subscripts will mean over $\mathbb{R}$. Unless otherwise indicated, cohomology of a manifold is taken to be the de Rham cohomology (with real coefficients).

In this article, $\mathbb{Z}/2$-graded objects will carry a superscript $\bullet$, while $\mathbb{Z}$-graded objects will carry a superscript $\ast$. A vector space with a $\mathbb{Z}/2$-grading will often be called a superspace. In defining actions on the tensor product $V \otimes W$ of two $\mathbb{Z}/2$-graded vector spaces $V, W$, we will respect usual rules of signs. For example, an odd endomorphism $A$ of the $\mathbb{Z}/2$-graded vector space $W$ is extended to an endomorphism still denoted by $A$ of $V \otimes W$ by defining

$$A(v \otimes w) = v \otimes Aw \quad \text{if } v \in V^{\text{even}}, w \in W.$$  

(2)

$$A(v \otimes w) = -v \otimes Aw \quad \text{if } v \in V^{\text{odd}}, w \in W.$$  

(3)

Any $\mathbb{Z}$-graded object $C^\ast$ can of course be thought of as a $\mathbb{Z}/2$-graded object $C^\bullet$ by defining

$$C^{\text{even}} = \sum_{n \in \mathbb{Z}} C^{2n}, \quad C^{\text{odd}} = \sum_{n \in \mathbb{Z}} C^{2n+1}.$$  

Lie algebra of any real Lie group will be denoted by the same lower case German letter.
2 G-equivariant cohomology with generalised coefficients - Basic definitions

Let $G$ be a Lie group and let $M$ be a $G$-manifold. Let us recall the definition of the $G$-equivariant de Rham cohomology $H^*_G(M)$ of $M$.

Let $S(g')$ be the symmetric algebra of $g'$. Consider the $\mathbb{Z}_+$-graded space $S(g') \otimes \mathcal{A}(M)$, where the degree of an element $P \otimes \alpha, P \in S^p(g'), \alpha \in \mathcal{A}^q(M)$ is defined to be equal to $2p+q$. Let $E^i$ be a basis of $g$ and let $E_i \in \mathfrak{g}'$ be the dual basis. Define the operator $d_g$ of degree 1 on $S(g') \otimes \mathcal{A}(M)$ by:

$$d_g(P \otimes \alpha) = P \otimes d_M \alpha - \sum_i E_i P \otimes \iota(E_M^i) \alpha$$

for $P \in S(g'), \alpha \in \mathcal{A}(M)$. This expression is independent of the choice of the basis $E^i$, as the element $\sum_i E_i \otimes E^i \in \mathfrak{g}' \otimes g$ is the canonical element $I \in \text{End}(g)$, where $I$ is the identity element of $\text{End}(g)$.

We often will identify $S(g')$ with the space of polynomial functions on $g$. Writing $X \in g$ as $X = \sum x_i E^i$, we identify $E_i$ with the linear coordinate function $x_i$. An element $\alpha$ of the space $S(g') \otimes \mathcal{A}(M)$ can be viewed as a polynomial map $X \rightarrow \alpha(X)$ from $g$ to $\mathcal{A}(M)$ and then the operator $d_g$ is given by the formula:

$$(d_g \alpha)(X) = d_M(\alpha(X)) - \iota(X_M)(\alpha(X))$$

or by

$$(d_g \alpha)(X) = d_M(\alpha(X)) - \sum x_i \iota(E_M^i)(\alpha(X)).$$

Consider the action of $G$ on $S(g')$ induced from the adjoint representation of $G$ on $g$ and the action of $G$ on $\mathcal{A}(M)$ induced from the action of $G$ on $M$. Let

$$\mathcal{A}_G(M) := (S(g') \otimes \mathcal{A}(M))^G$$

be the space of $G$-invariants in $S(g') \otimes \mathcal{A}(M)$. In other words, an element $\alpha$ of $\mathcal{A}_G(M)$ is an equivariant polynomial map (i.e. $\alpha(g \cdot X) = g \cdot (\alpha(X))$) from $g$ to $\mathcal{A}(M)$. The operator $d_g$ commutes with the tensor product action of $G$ on $S(g') \otimes \mathcal{A}(M)$, thus $d_g$ preserves $\mathcal{A}_G(M)$. The Cartan relation (1)

$$\mathcal{L}(X_M) = d\iota(X_M) + \iota(X_M)d$$

is satisfied for $\alpha \in \mathcal{A}_G(M)$.
implies \((d^2_\mathfrak{g}\alpha)(X) = -\mathcal{L}(X_M)(\alpha(X))\). Thus \((d^2_\mathfrak{g}\alpha)(X) = 0\) for \(\alpha \in \mathcal{A}_G(M)\) and thus \((\mathcal{A}_G(M), d_\mathfrak{g})\) is a complex.

**Definition 1** Define:

\[
Z_G(M) = \{\alpha \in \mathcal{A}_G(M), d_\mathfrak{g}\alpha = 0\},
\]

\[
B_G(M) = \{\alpha \in \mathcal{A}_G(M), \alpha = d_\mathfrak{g}\beta, \text{for some } \beta \in \mathcal{A}_G(M)\}
\]

and

\[
H_G(M) = Z_G(M)/B_G(M).
\]

The space \(H_G(M)\) is called the \(G\)-equivariant de Rham cohomology of \(M\). The cohomology \(H_G(M)\) inherits the \(\mathbb{Z}_+\)-grading from \(\mathcal{A}_G^*(M)\). The graded algebra \(S(\mathfrak{g}^\ast)\) of invariant polynomial functions on \(\mathfrak{g}\) acts by multiplication on \(\mathcal{A}_G(M)\). This action commutes with the differential \(d_\mathfrak{g}\). Thus \(H_G^*(M)\) is a \(\mathbb{Z}\)-graded \(S(\mathfrak{g}^\ast)\)-module.

If \(K\) is a closed subgroup of \(G\), the restriction to \(k\) of a function defined on \(\mathfrak{g}\) induces a map from \(H_G^*(M)\) to \(H_K^*(M)\). In particular, evaluation at \(0 \in \mathfrak{g}\); \(\alpha \to \alpha(0)\) induces a map from \(\tilde{H}_G^*(M)\) to \(H^*(M)\), where \(H^*(M)\) is the usual de Rham cohomology of \(M\) with coefficients in \(\mathbb{R}\).

The complex \(\mathcal{A}_G(M)\) has a subcomplex

\[
\mathcal{A}_{\text{cpt},G}(M) := (S(\mathfrak{g}^\ast) \otimes \mathcal{A}_{\text{cpt}}(M))^G.
\]

The compactly supported \(G\)-equivariant de Rham cohomology \(H_{\text{cpt},G}^*(M)\) of \(M\) is defined as the cohomology of the complex \((\mathcal{A}_{\text{cpt},G}^*, d_\mathfrak{g})\).

We may also consider the space \(\mathcal{C}^\infty(\mathfrak{g}, \mathcal{A}(M))\) of \(\mathcal{C}^\infty\)-maps from \(\mathfrak{g}\) to \(\mathcal{A}(M)\). The group \(G\) acts naturally on \(\mathcal{C}^\infty(\mathfrak{g}, \mathcal{A}(M))\). The operator \(d_\mathfrak{g}\) is defined by the same formula (4) on \(\mathcal{C}^\infty(\mathfrak{g}, \mathcal{A}(M))\). The space \(\mathcal{C}^\infty(\mathfrak{g}, \mathcal{A}(M))\) has a \(\mathbb{Z}/2\)-grading given by parity of differential forms. The operator \(d_\mathfrak{g}\) is an odd operator on this superspace. However it is impossible to define a \(\mathbb{Z}_+\)-grading on \(\mathcal{C}^\infty(\mathfrak{g}, \mathcal{A}(M))\), such that \(d_\mathfrak{g}\) would be of degree 1.

We denote by

\[
\mathcal{A}_G^\infty(M) = (\mathcal{C}^\infty(\mathfrak{g}, \mathcal{A}(M)))^G,
\]

the space of \(G\)-equivariant \(\mathcal{C}^\infty\) maps from \(\mathfrak{g}\) to \(\mathcal{A}(M)\). The Cartan relation implies again \(d^2_\mathfrak{g} = 0\) on \(\mathcal{A}_G^\infty(M)\).
Definition 2 Define:

\[ Z_G^\infty (M) = \{ \alpha \in A_G^\infty (M), d_g \alpha = 0 \}, \]
\[ B_G^\infty (M) = \{ \alpha \in A_G^\infty (M), \alpha = d_g \beta \text{ for some } \beta \in A_G^\infty (M) \} \]

and

\[ H_G^\infty (M) = Z_G^\infty (M)/B_G^\infty (M). \]

Introduce (as in [11]) the space \( C^{-\infty} (g, A(M)) \) of generalised functions on \( g \) with values in the space \( A(M) \). This is, by definition, the space of continuous \( \mathbb{R} \)-linear maps \( \text{Hom}(\mathcal{D}(g), A(M)) \) from the space of smooth compactly supported densities \( \mathcal{D}(g) \) on \( g \) to the space \( A(M) \), where \( \mathcal{D}(g) \) and \( A(M) \) are both endowed with the \( C^\infty \)-topologies. Thus, if \( \alpha \) is an element of \( C^{-\infty} (g, A(M)) \) and if \( \Phi \) is a smooth compactly supported density on \( g \), then \( (\alpha, \Phi) \) is a differential form on \( M \) denoted by \( \int_g \alpha(X) d\Phi(X) \).

A compactly supported \( C^\infty \) density on \( g \) will be called a test density (on \( g \)). A compactly supported \( C^\infty \) function on \( g \) will be called a test function.

We write \( d_M \) for the operator on \( C^{-\infty} (g, A(M)) \) defined by

\[ (d_M \alpha, \Phi) = d_M (\alpha, \Phi), \quad \text{for } \Phi \text{ a test density}, \]

and \( \iota \) for the operator defined by

\[ (\iota \alpha, \Phi) = \sum_i \iota (E^i_M)(\alpha, x_i \Phi). \]

Then define the operator \( d_g \) on \( C^{-\infty} (g, A(M)) \) by

\[ d_g \alpha = d_M \alpha - \iota \alpha. \]

Observe that for \( \alpha \in C^\infty (g, A(M)) \subset C^{-\infty} (g, A(M)) \), the operator \( d_g \) coincides with the operator \( d_g \) introduced above. We thus will also write informally

\[ (d_g \alpha)(X) = d_M (\alpha(X)) - \sum_i x_i \iota (E^i_M)|\alpha(X)| \]

for \( \alpha \in C^{-\infty} (g, A(M)) \).
The group $G$ acts naturally on $C^{-\infty}(g, \mathcal{A}(M))$:

$$(g\alpha, \Phi) = g \cdot (\alpha, g^{-1} \cdot \Phi).$$

It can be easily seen that the operators $d$ and $\iota$ commute with the action of $G$. Define

$${\mathcal{A}}^{-\infty}_G(M) = C^{-\infty}(g, \mathcal{A}(M))^G$$

as the space of $G$-equivariant $C^{-\infty}$-maps from $g$ to $\mathcal{A}(M)$. An element of the space $\mathcal{A}^{-\infty}_G(M)$ will be called a $G$-equivariant form with generalised coefficients, or simply an equivariant form. If $\Phi$ is a test function on $g$, we denote by $\Phi^g$ the function $\Phi^g(X) = \Phi(gX)$. Let $dX$ be an Euclidean measure on $g$. For $\alpha \in \mathcal{A}^{-\infty}_G(M)$ and $g \in G$, we have

$$|\det_g(g)|(\int_g \alpha(X)\Phi^g(X)dX) = g^{-1} \cdot (\int_g \alpha(X)\Phi(X)dX).$$

The operator $d_g$ preserves $\mathcal{A}^{-\infty}_G(M)$ and the Cartan relation (1) implies again $d_g^2 = 0$ on $\mathcal{A}^{-\infty}_G(M)$.

**Definition 3** Define:

$$Z^{-\infty}_G(M) = \{ \alpha \in \mathcal{A}^{-\infty}_G(M), d_g\alpha = 0 \},$$

$$B^{-\infty}_G(M) = \{ \alpha \in \mathcal{A}^{-\infty}_G(M), \alpha = d_g\beta \text{ for some } \beta \in \mathcal{A}^{-\infty}_G(M) \}$$

and

$$H^{-\infty}_G(M) = Z^{-\infty}_G(M)/B^{-\infty}_G(M).$$

An equivariant form in $Z^{-\infty}_G(M)$ (resp. $B^{-\infty}_G(M)$) is said to be **closed** (resp. **exact**).

Observe that the parity of the exterior degree on $\mathcal{A}(M)$ induces a $\mathbb{Z}/2$-degree on the preceding spaces. We denote them by

$$Z^{-\infty}_G(M)^\bullet, B^{-\infty}_G(M)^\bullet, H^{-\infty}_G(M)^\bullet.$$.

The ring $S(g')^G$ of invariant polynomial functions on $g$ acts by multiplication on $\mathcal{A}^{-\infty}_G(M)$. This action commutes with the differential $d_g$. Thus $H^{-\infty}_G(M)^\bullet$ is a $S(g')^G$-module. In fact $H^{-\infty}_G(M)$ is a module for $H^{-\infty}_G(M)$ under left multiplication.
If $M$ is a point, then, from the definition, it is clear that $H_G(\text{point}) = S(\mathfrak{g})^G$, $H_G^\infty(\text{point}) = C^\infty(\mathfrak{g})^G$ and $H_G^{-\infty}(\text{point}) = C^{-\infty}(\mathfrak{g})^G$.

There is a natural map

$$H_G^{-\infty}(M) \to H_G^{-\infty}(N).$$

Define similarly $(\mathcal{A}_{\text{cpt},G}^{-\infty}(M), d_{\mathfrak{g}})$ as the subcomplex of $(\mathcal{A}_G^{-\infty}(M), d_{\mathfrak{g}})$, consisting of all $\alpha \in \mathcal{A}_G^{-\infty}(M)$ such that $(\alpha, \Phi) \in \mathcal{A}_{\text{cpt}}(M)$, for all test densities $\Phi$, and define $H_{\text{cpt},G}^{-\infty}(M)$ as the cohomology of this subcomplex.

If $\phi : N \to M$ is a $G$-equivariant map between two $G$-manifolds, then the pull-back of differential forms induces a cochain map

$$\phi^* : (\mathcal{A}_G^{-\infty}(M), d_{\mathfrak{g}}) \to (\mathcal{A}_G^{-\infty}(N), d_{\mathfrak{g}}),$$

in particular a map in cohomology (again denoted by)

$$\phi^* : H_G^{-\infty}(M) \to H_G^{-\infty}(N).$$

Thus the correspondence $M \to H_G^{-\infty}(M)$ is a contravariant functor from the category of $G$-manifolds and $G$-equivariant maps to the category of $\mathbb{Z}/2$-graded $S(\mathfrak{g})^G$-modules.

Similarly the correspondence $M \to H_{\text{cpt},G}^{-\infty}(M)$ is a contravariant functor from the category of $G$-manifolds and $G$-equivariant proper maps to the category of $\mathbb{Z}/2$-graded $S(\mathfrak{g})^G$-modules.

If $M$ is oriented such that the action of any $g \in G$ is orientation preserving, integration over $M$ defines a map $\int_M$ from $\mathcal{A}_{\text{cpt},G}^{-\infty}(M)$ to $C^{-\infty}(\mathfrak{g})^G$:

$$(\int_M \alpha, \Phi) := \int_M (\alpha, \Phi), \quad \text{for any test density } \Phi \text{ on } \mathfrak{g}.$$

This map induces a map from $H_{\text{cpt},G}^{-\infty}(M)$ to $C^{-\infty}(\mathfrak{g})^G$.

**Definition 4** Let $p : M \to B$ be a $G$-equivariant fibration of $G$-manifolds. Then the map $p$ is said to have $G$-oriented fibers if the fibers of $p$ are oriented and if the $G$-action on $M$ preserves the orientation of all the fibers.

If $p : M \to B$ is a $G$-equivariant fibration of $G$-manifolds with $G$-oriented fibers, then the integration over the fibers gives a cochain map denoted by $\int_{M/B}$ or by $p_*$ from $\mathcal{A}_{\text{cpt},G}^{-\infty}(M)$ to $\mathcal{A}_{\text{cpt},G}^{-\infty}(B)$:

$$(\int_{M/B} \alpha, \Phi) := \int_{M/B} (\alpha, \Phi), \quad \text{for any test density } \Phi \text{ on } \mathfrak{g}.$$
In particular, we get an induced map in cohomology (again denoted by) $p_*$ or $\int_{\mathcal{M}/B}$ (depending on the choice of a $G$-orientation):

$$p_* : H^{\infty}_{\text{cpt},G}(M) \to H^{\infty}_{\text{cpt},G}(B).$$

Similarly if, in addition, $p$ is a proper map, we get the integration map

$$p_* : H^{-\infty}_G(M) \to H^{-\infty}_G(B).$$

Observe that if $\alpha \in H^{\infty}_{\text{cpt},G}(M)$, $\beta \in H^{-\infty}_G(B)$, then $\alpha \wedge p^* \beta \in H^{-\infty}_{\text{cpt},G}(M)$ and we have

$$p_* (\alpha \wedge p^* \beta) = p_* \alpha \wedge \beta. \quad (8)$$

(Our sign convention for $p_*$ is as in [3], chapter 1.)

**Definition 5** If $Q$ is a continuous operator on $\mathcal{A}(\mathcal{M})$, we still denote by $Q$ the operator on $C^{-\infty}(\mathfrak{g},\mathcal{A}(\mathcal{M}))$ defined by

$$(Q \cdot \alpha, \Phi) = Q \cdot (\alpha, \Phi), \quad \text{for all test densities } \Phi \text{ on } \mathfrak{g}.$$

We will often write the above as $(Q\alpha)(X) = Q \cdot (\alpha(X))$ and say that $Q$ is the pointwise extension of $Q$ to $C^{-\infty}(\mathfrak{g}, \mathcal{A}(\mathcal{M}))$.

We start to compute the space $H^{-\infty}_G$ in some elementary situations.

Let $p : \mathcal{V} \to B$ be a $G$-equivariant real vector bundle over $B$. Let $i : B \to \mathcal{V}$ be the inclusion from $B$ to $\mathcal{V}$ as the zero section.

**Proposition 6** The canonical maps

$$i^* : H^{-\infty}_G(\mathcal{V}) \to H^{-\infty}_G(B)$$

and

$$p^* : H^{-\infty}_G(B) \to H^{-\infty}_G(\mathcal{V})$$

are inverses to each other. In particular, both of them are isomorphisms.
Proof: The proof is similar to the Poincaré lemma for the de Rham complex. Let $\mathcal{R}$ be the vertical Euler vector field on $\mathcal{V}$. Extend the operator $\mathcal{L}(\mathcal{R})$ pointwise to $C^{-\infty}(\mathfrak{g}, \mathcal{A}(\mathcal{V}))$. Extend similarly the operator $i(\mathcal{R})$ to $C^{-\infty}(\mathfrak{g}, \mathcal{A}(\mathcal{V}))$. As $i(E^i)i(\mathcal{R}) + i(\mathcal{R})i(E^i) = 0$, for all $i$, Cartan relation implies
\[
\mathcal{L}(\mathcal{R}) = d_g \iota(\mathcal{R}) + \iota(\mathcal{R})d_g
\]
on $C^{-\infty}(\mathfrak{g}, \mathcal{A}(\mathcal{V}))$. As $\mathcal{R}$ commutes with the action of $G$, the operators $\mathcal{L}(\mathcal{R})$, $i(\mathcal{R})$ preserve $A_G^{-\infty}(\mathcal{V})$. Let $h_t(v) = tv$. Let $\beta \in \mathcal{A}(\mathcal{V})$. Then $h_t^* \beta = \beta$ for $t = 1$, while $h_t^* \beta = p^* i^* \beta$ for $t = 0$. We compute
\[
\frac{d}{dt} h_t^* \beta = t^{-1} \mathcal{L}(\mathcal{R}) h_t^* \beta.
\]
(As $\mathcal{R}$ vanishes at 0, the right hand side depends smoothly on $t \in \mathbb{R}$.) Clearly, the same relation persists for differential forms with parameters. Thus for $\beta \in A_G^{-\infty}(\mathcal{V})$, we have
\[
\frac{d}{dt} h_t^* \beta = t^{-1} \mathcal{L}(\mathcal{R}) h_t^* \beta = t^{-1} (\iota(\mathcal{R})d_g + d_g \iota(\mathcal{R})) h_t^* \beta.
\]
Define $H : A_G^{-\infty}(\mathcal{V}) \to A_G^{-\infty}(\mathcal{V})$ by
\[
H \beta = \int_0^1 h_t^*(\iota(\mathcal{R}) \beta) t^{-1} dt.
\]
Then we obtain
\[
\beta - p^* i^* \beta = \int_0^1 \frac{d}{dt} h_t^* \beta dt = (d_g H + H d_g) \beta,
\]
for all $\beta \in A_G^{-\infty}(\mathcal{V})$. Thus we see that $p^* i^* = I$ in cohomology, where $I$ is the identity operator. Of course, $pi = I$, in particular $i^* p^* = I$. This proves the proposition. □

Considering the case of a vector space $V$, we have

**Corollary 7** Let $G$ be a Lie group and $V$ be a finite dimensional real representation space for $G$, then
\[
H_G^{-\infty}(V) = C^{-\infty}(\mathfrak{g})^G.
\]
We now prove the Thom isomorphism for compactly supported cohomology of vector bundles.

A \( G \)-equivariant vector bundle is said to be \( G \)-oriented if its fibres are \( G \)-oriented.

**Definition 8** Let \( G \) be a compact Lie group and let \( p : \mathcal{V} \to B \) be a \( G \)-equivariant \( G \)-oriented vector bundle over a compact base \( B \).

An element \( u \in H_{cpt,G}(\mathcal{V}) \) such that \( p_\ast u = 1 \) in \( H_G(B) \) will be called a Thom class.

Given a \( G \)-orientation \( o \) on \( \mathcal{V} \), recall [16] that there exists a unique Thom class \( u_o \in H_{cpt,G}(\mathcal{V}) \). Multiplication by \( u_o \) induces a map \( m_o : m_o(\alpha) = u_o \wedge p^\ast \alpha \) from \( H_G(B) \) to \( H_{cpt,G}(\mathcal{V}) \) and the map \( m_o \) is an isomorphism. Similarly, as \( u_o \in H_{cpt,G}(\mathcal{V}) \), we can define the map \( m_o(\alpha) = u_o \wedge p^\ast \alpha \) from \( H_{\acute{c}pt}^{-\infty}(B) \) to \( H_{\acute{c}pt,G}^{-\infty}(\mathcal{V}) \).

**Proposition 9.** Let \( G \) be a compact Lie group and let \( \mathcal{V} \to B \) be a \( G \)-equivariant \( G \)-oriented real vector bundle over a compact base \( B \). Then the maps

\[
m_o : H_{G}^{-\infty}(B) \to H_{\acute{c}pt,G}^{-\infty}(\mathcal{V})
\]

and

\[
p_\ast : H_{\acute{c}pt,G}^{-\infty}(\mathcal{V}) \to H_{G}^{-\infty}(B)
\]

are inverses to each other. In particular both of them are isomorphisms.

**Remark 10** In the above proposition, we have assumed the base \( B \) to be compact, just in order to simplify notation. If \( B \) is not necessarily compact, the same proof will lead to isomorphism of \( H_{\acute{c}pt,G}^{-\infty}(B) \) with \( H_{\acute{c}pt,G}^{-\infty}(\mathcal{V}) \). It is also clear that a similar Thom isomorphism will hold between \( H_{\acute{c}pt}^{-\infty}(B) \) and the cohomology \( H_{\acute{c}pt,G}^{-\infty}(\mathcal{V}) \) of the complex of equivariant forms on \( \mathcal{V} \) with compact support along the fibers.

**Proof:** The proof is similar to the proof of the Thom isomorphism in equivariant cohomology given in [17]. For \( \alpha \in \mathcal{A}(\mathcal{V}) \), we denote by \( \hat{\alpha} \) the image of \( \alpha \) under the automorphism \( x \to -x \) of \( \mathcal{V} \). Let us consider the bundle \( \mathcal{V} \oplus \mathcal{V} \) over \( B \) and let \( \sigma(x,y) = (y,-x) \) be the automorphism of \( \mathcal{V} \oplus \mathcal{V} \). Denote by \( \sigma_t, t \in \mathbf{R} \) the transformation

\[
\sigma_t(x,y) = ((\cos t)x + (\sin t)y, -(\sin t)x + (\cos t)y)
\]
of the fibers of $V \oplus V$. Then $\sigma_0$ is the identity, while $\sigma_{\pi/2}$ is equal to $\sigma$. Let

$$S = \left( \frac{d}{dt} \sigma_t \right)$$

be the vector field on $V \oplus V$ induced by the group of transformations $\sigma_t$. We have

$$\mathcal{L}(S) = dt(S) + \iota(S)d$$

on $A_{\text{cpt}}(V \oplus V)$. Extending these transformations pointwise (cf. Definition 5) to $C^{-\infty}(\mathfrak{g}, A_{\text{cpt}}(V \oplus V))$ we obtain the relation:

$$\mathcal{L}(S) = d_g \iota(S) + \iota(S)d_g$$

on $C^{-\infty}(\mathfrak{g}, A_{\text{cpt}}(V \oplus V))$. The transformations $\sigma_t$ commute with the action of $G$. Thus $\iota(S)$ and $\mathcal{L}(S)$ preserve $A_{\text{cpt},G}^{-\infty}(V \oplus V)$.

Define $H : A_{\text{cpt},G}^{-\infty}(V \oplus V) \to A_{\text{cpt},G}^{-\infty}(V \oplus V)$ by

$$H \nu = \int_0^{\pi/2} (\sigma_t^* \iota(S)\nu)dt.$$ 

We obtain, as in the proof of the preceding proposition (6)

$$\sigma^* \nu - \nu = (d_g H + H d_g) \nu$$

for any $\nu \in A_{\text{cpt},G}^{-\infty}(V \oplus V)$.

Let $p_i : V \oplus V \to V, i = 1, 2$ be the natural maps obtained by projections on the first or second component respectively. Consider $\alpha \in A_{\text{cpt},G}^{\infty}(V)$ and $\beta \in A_{\text{cpt},G}^{-\infty}(V)$. Then $p_2^* \alpha \land p_2^* \beta$ is a well defined element of $A_{\text{cpt},G}^{-\infty}(V \oplus V)$. If $\alpha, \beta$ are closed equivariant forms, then $p_2^* \alpha \land p_2^* \beta$ is closed and is in the same cohomology class as $\sigma^*(p_1^* \alpha \land p_2^* \beta) = p_2^* \alpha \land p_1^* \beta$. Let us integrate over the fibers of $p_2$. It is clear that $(p_2)_*(p_1^* \alpha \land p_2^* \beta) = p^*(p_* \alpha) \beta$. Thus the equality in cohomology

$$p_1^* \alpha \land p_2^* \beta \equiv p_2^* \alpha \land p_1^* \beta$$

implies (if $N$ is the rank of $V$, and $|\alpha| \in \{0, 1\}$ is the parity of $\alpha$):

$$p^*(p_* \alpha) \land \beta \equiv (-1)^{|\alpha|} \alpha \land p^*(p_* \beta)$$
in cohomology. In particular let \( \alpha = u_o \) be the equivariant Thom class of \( V \to B \), which is of parity \( N \). We obtain the relation
\[
\beta \cong u_o \wedge p^* p_* \beta
\]
in \( H^{-\infty}_{\mathrm{equiv},G}(V) \). Thus, we see that \( m_o p_* = I \) in cohomology. Of course \( p_* m_o = I \). This proves the proposition. 

We finish this section by giving an example of a non-trivial closed equivariant form with generalised coefficients.

We need a notation.

Let \( V \) be a real vector space of dimension \( n \). Let \( \nu' \) be a non-zero element in \( \Lambda^n V' \). We denote by \( |\nu'|^{-1} \delta_V \) the element of \( C^{-\infty}(V) \) defined by
\[
\int_V |\nu'|^{-1} \delta_V(X) \Phi(X) dX = \Phi(0), \quad \text{for any test function } \Phi \text{ on } V,
\]
where \( dX \) is the Euclidean density on \( V \) determined by \( \nu' \).

We denote by \( \delta_{V,o} \in C^{-\infty}(V) \otimes \Lambda^n V' \) the element:
\[
\delta_{V,o} = |\nu'|^{-1} \delta_V \otimes \nu'.
\]
The element \( \delta_{V,o} \) depends only on the orientation \( o \) of \( V \) determined by \( \nu' \).

Let \( G \) be a Lie group. Consider the action of \( G \) on itself by left translations. Let \( n = \dim G \). Fix an orientation \( o \) on \( g \). Let \( \nu' \in \Lambda^n g' \) be a positive element. Let \( dg \) be the unique left invariant form of maximal degree on \( G \) such that \( (dg)_e = \nu' \), where \( e \) is the identity element of \( G \).

**Lemma 11** The form
\[
\alpha_{G,o}(X) := |\nu'|^{-1} \delta_g(X) \otimes |\det g| dg
\]
is a closed equivariant form on \( G \), which depends only on the choice of \( o \).

**Proof:** It is easy to see that \( \alpha_{G,o} \) is equivariant, as \( g_0 \cdot (|\nu'|^{-1} \delta_g) = |\det g_0| |\nu'|^{-1} \delta_g \) and \( |\det g (g_0^{-1} g)| = |\det g_0|^{-1} |\det g| \). It is immediate to see that \( \alpha_{G,o} \) depends only on \( o \). As \( \alpha_{G,o} \) is of maximal degree, \( d\alpha_{G,o} = 0 \). Also, as the generalised function \( |\nu'|^{-1} \delta_g \) is annihilated by multiplication by all the coordinates functions on \( g \), we see that \( \iota \alpha_{G,o} = 0 \).

We will prove in section 5 that \( \overline{H}_G^\infty(G) = R \alpha_{G,o} \).
3 Koszul complexes

Our main aim in sections 4 and 5 will be the study of the cohomology of “perturbed” Koszul complexes. Thus, in this section, we recall some well-known facts on Koszul complexes.

Let $V$ be a finite dimensional real vector space of dimension $n$ with basis $e^i, 1 \leq i \leq n$ and dual basis $e_i \in V'$. Let $S(V')$ be the ring of polynomial functions on $V$. Let $L$ be a $S(V')$-module. We still denote by $e_i$ the action of $e_i \in V'$ on $L$. We denote by $i(e_i) : \Lambda^* V' \to \Lambda^{*-1} V'$ the contraction by the vector $e_i \in V$.

Consider the space $L \otimes \Lambda^* V'$ which is $\mathbb{Z}_+$-graded by the exterior degree. On this space, the operator

$$j_L = \sum_{i=1}^{n} e_i \otimes i(e_i)$$

is an operator of degree $-1$ and its square is zero. We denote by $H(j_L)$ the homology space of $j_L$. It is a $\mathbb{Z}_+$-graded vector space. If $L = S(V')$ (considered as a $S(V')$-module under multiplication), we denote the operator $j_L$ by $j_V$. We denote:

$$A^* = S(V') \otimes \Lambda V'.$$

The following proposition is basic

**Proposition 12** Consider the operator $j_V$ on $A^*$. We have:

1. If $i > 0$, $H_i(j_V) = 0$.

2. If $i = 0$, the map $\phi \to \phi(0)$ from $A^0 = S(V')$ to $\mathbb{R}$ induces an isomorphism from $H_0(j_V)$ with $\mathbb{R}$.

Even though this proposition is well known, we give a proof as we will use the explicit homotopy given below in the rest of the article.

**Proof:** We identify the space $A^* = S(V') \otimes \Lambda^* V'$ with the space of differential forms with polynomial coefficients on $V$. We write an element
$x \in V$ as $x = \sum x_i e^i$, so that $e_i(x) = x_i$. If $I = (i_1, i_2, \ldots, i_k)$ is a multi-index: $1 \leq i_1 < \cdots < i_k \leq n$, we identify $e_{i_1} \wedge e_{i_2} \cdots \wedge e_{i_k} \in \Lambda^k V$ with the $k$-form $dx_I = dx_{i_1} dx_{i_2} \cdots dx_{i_k}$.

Consider the partial derivative $\partial^i$ in the direction of $e^i \in V$. Let $E_V$ be the Euler vector field $E_V = \sum x_i \partial^i$. We denote the contraction operator $\iota(e^i)$ on $\Lambda^k V$ by $i^i$. Thus

$$j_V = \sum_i x_i i^i = \iota(E_V).$$

We denote by $\epsilon_i$ the multiplication by $dx_i$. Let $d_V = \sum \partial^i \otimes e_i$ be the de Rham differential on $A^*$. Let $L_V$ be the Euler operator on $S(V') \otimes \Lambda^k V$ given by the Lie derivative action of $E_V$.

$$L_V(f dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = (\sum_i x_i \partial^i f + k f)dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

for $f \in S(V')$.

Then, Cartan relation implies $L_V = d_V j_V + j_V d_V$.

Consider the subcomplex $A^*_0$ of $A^*$ such that $A^*_0 = A^k$, if $k > 0$, while $A^*_0 = \{ \phi \in S(V'), \phi(0) = 0 \}$. The operator $L_V$ keeps $A^*_0$ stable and induces an invertible operator of degree 0 on $A^*_0$. We can give an integral formula for the inverse $F_V$ of the Euler operator $L_V$: Define $F_V$ on $A^*_0$ by

1. If $f dx_I \in A^k$, $k > 0$,

$$F_V(f dx_I) = \left( \int_0^1 f(tx) t^{k-1} dt \right) dx_I.$$

2. If $f \in A^*_0$

$$F_V f = \int_0^1 f(tx) t^{-1} dt.$$

It is well defined as $f(tx)$ vanishes for $t = 0$.

It is immediate to see that $F_V L_V \phi = \phi$ for every $\phi \in A^*_0$. The operator $F_V = L_V^{-1}$ commutes with $j_V$ and $d_V$.

Let $h_V := F_V d_V$, then, if $\phi \in A^*_0$,
\[ \phi = (h_V j_V + j_V h_V) \phi. \]

This formula clearly implies the proposition. \[ \qed \]

Let \( L, N \) be two \( S(V') \)-modules. The tensor product space (over \( \mathbb{R} \)) \( L \otimes N \) is given a structure of \( S(V') \)-module by defining the action of an element \( f \in V' \) to be \( f \cdot (m \otimes n) = f m \otimes n - m \otimes f n \).

Consider the operator \( j_{L \otimes N} \) on \( L \otimes N \otimes \Lambda V' \). The homology space \( H_0(j_{L \otimes N}) \) in degree 0 is the quotient of \( L \otimes N \) by the subspace spanned by elements of the form \( f m \otimes n - m \otimes f n \), for \( f \in V' \). This quotient is by definition \( L \otimes_{S(V')} N \):

\[ H_0(j_{L \otimes N}) = L \otimes_{S(V')} N. \]

If \( N \) is a \( S(V') \)-module, denote by \( N^0 \) the space \( N \) with the trivial action of \( V' \).

**Lemma 13** Let \( N \) be a \( S(V') \)-module. The operator \( R := \exp \sum \partial^i \otimes e_i \) gives an isomorphism of the \( S(V') \)-module \( S(V') \otimes N \) with the \( S(V') \)-module \( S(V') \otimes N^0 \).

**Proof:** It is sufficient to check this assertion when \( V \) is a 1-dimensional vector space, where it is checked easily. \[ \qed \]

**Corollary 14** If \( L \) is a free \( S(V') \)-module, then \( L \otimes N \) is also free; hence \( H_i(j_{L \otimes N}) = 0 \) if \( i > 0 \).

**Lemma 15** For any \( S(V') \)-modules \( L \) and \( N \), the homology space of the operator \( j_{L \otimes N} \) on the complex

\[ L \otimes N \otimes \Lambda V' \]

is equal to the torsion group \( \text{Tor}^{S(V')}(L, N) \), i.e.,

\[ H_i(j_{L \otimes N}) = \text{Tor}_i^{S(V')}(L, N). \]

**Proof:** Let us consider the complex

\[ 0 \to S(V') \otimes N \otimes \Lambda^0 V' \xrightarrow{\partial} \ldots \xrightarrow{\partial} S(V') \otimes N \otimes V' \xrightarrow{\partial} S(V') \otimes N \xrightarrow{m} N \to 0 \]
where the last map is the surjective map $m : S(V') \otimes N \to N : m(\phi \otimes n) = \phi n$. By the preceding corollary, this complex is exact. Furthermore if we endow the space $S(V') \otimes N \otimes \Lambda V'$ with the $S(V')$ module structure $S(V') \otimes (N \otimes \Lambda V')^0$, the homomorphisms $j_{S(V') \otimes N}$ and $m$ are $S(V')$-module morphisms. Thus the complex above is a free resolution of $N$ as a $S(V')$-module. We may calculate the torsion group $\text{Tor}^{S(V')}(L, N)$ using this resolution. The space $L \otimes_{S(V')} S(V') \otimes (N \otimes \Lambda^* V')^0$ is isomorphic with $L \otimes N \otimes \Lambda^* V'$. The operator $I \otimes_{S(V')}(j_{S(V') \otimes N})$ under this isomorphism becomes the operator $j_{L \otimes N}$. This proves the lemma.

We now introduce another vector space $P$ considered as a parameter space. Let $W = V \oplus P$. We write an element $w \in W$ as $w = x + y$, with $x \in V$, $y \in P$.

Let us consider the space

$$A^\infty_* = C^\infty(W) \otimes \Lambda^* V'.$$

The multiplication by the coordinate function $x_i$ is an operator on $C^\infty(W)$. Thus the operator $j^\infty_V := \sum_{i=1}^n x_i \otimes \iota(e^i)$ is an operator of degree $-1$ on $A^\infty_*$ and $(j^\infty_V)^2 = 0$. Let $r_P : C^\infty(W) \to C^\infty(P)$ be the restriction map.

**Proposition 16** Consider the operator $j^\infty_V$ on the complex $A^\infty_*$. We have

1. If $i > 0$, $H_i(j^\infty_V) = 0$.

2. If $i = 0$, the map $r_P$ from $A^0 = C^\infty(W)$ to $C^\infty(P)$ induces an isomorphism from $H_0(j^\infty_V)$ with $C^\infty(P)$.

**Proof:** The method of proof is identical to the proof of Proposition 12. For simplicity, we denote $j^\infty_V$ by $j_V$. Let

$$d_V = \sum_i \partial^i \otimes \epsilon_i$$

be the partial de Rham differential in the direction of $V$ on the complex $A^\infty_*$. Let $\mathcal{L}_V$ be the Euler operator on $A^\infty_*$ with respect to the variables $x_i, dx_i$:

$$\mathcal{L}_V(fdx_{i_1} \wedge \cdots \wedge dx_{i_k}) = (\sum_i x_i \partial^i f + kf)dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$
if $f \in C^\infty(W)$.

Then, $\mathcal{L}_V = d_Vj_V + j_Vd_V$.

Let $A_0^{\infty,*}$ be the subcomplex of $A^{\infty,*}$ defined by $A_0^{\infty,k} = A^{\infty,k}$ if $k > 0$ and $A_0^{\infty,0} = \{ \phi \in C^\infty(W); r_P\phi = 0 \}$.

The Euler operator $\mathcal{L}_V$ induces an operator of degree 0 on $A_0^{\infty,*}$, which is invertible. Its inverse $F_V$ is given explicitly by an integral formula as in the proof of Proposition 12:

**Definition 17** Let us consider the operator $F_V$ of degree 0 on $A_0^{\infty,*}$ defined by

1. If $f dx_I \in A_0^{\infty,k}$, $k > 0$,

   $$F_V(f dx_I) = \int_{0}^{1} f(tx + y)t^{k-1}dt dx_I.$$

2. If $f \in A_0^{\infty,0}$,

   $$F_V f = \int_{0}^{1} f(tx + y)t^{-1}dt.$$

   It is well defined as $f(tx + y)$ vanishes for $t = 0$.

The operator $F_V$ commutes with $d_V$, $j_V$. It is easy to prove

$$F_V \mathcal{L}_V \phi = \phi$$

for every $\phi \in A_0^{\infty,*}$. Thus if

(13)

$$h_V = F_Vd_V,$$

$$\phi = (h_Vj_V + j_Vh_V)\phi, \quad \text{for } \phi \in A_0^{\infty,*}.$$

Thus $h_V$ is a homotopy for the complex $A_0^{\infty,*}$. The existence of $h_V$ implies that the subcomplex $A_0^{\infty,*}$ is exact. This in turn implies the proposition. $\blacksquare$

Observe that if $f \in C^\infty_{\text{cpt}}(W) \otimes \Lambda V'$ is compactly supported, then $F_Vf$ is not necessarily compactly supported.
Remark 18 If a Lie group $G$ acts linearly on $V$ and $P$, the operator $j^\infty_V$ commutes with the action of $G$. Thus $(A^{\infty,*})^G$ is a subcomplex of $A^{\infty,*}$. The homotopy $h_V$, that we have constructed above, commutes with the action of $G$. It results that the homology of the subcomplex $(A^{\infty,*})^G$ of $A^{\infty,*}$ exists only in degree 0 and in degree 0 is isomorphic to $C^\infty(P)^G$.

Now, consider the space:

$$A^{-\infty,*} = C^{-\infty}(W) \otimes \Lambda^* V'.$$

The operator $j^{-\infty}_V$ is similarly defined on $A^{-\infty,*}$. The complex $j^{-\infty}_V : A^{-\infty,*} \rightarrow A^{-\infty,*-1}$ is called the Koszul complex with $C^\infty$-coefficients (with the space $P$ as a parameter space).

Choose an orientation $o$ on $V$. By Formula (10) of section 2, this determines an element $\delta_{V,o}$ of $C^{-\infty}(V) \otimes \Lambda^n V$. If $f \in C^{-\infty}(P)$ is a generalised function on $P$, the product $\delta_{V,o}(x)f(y)$ is in $C^{-\infty}(W) \otimes \Lambda^n V'$.

It is easy to identify the homology of $j^{-\infty}_V$ in top degree.

Lemma 19 The kernel of $j^{-\infty}_V$ on $A^{-\infty,n} = C^{-\infty}(W) \otimes \Lambda^n V'$ is equal to the space $\delta_{V,o} \otimes C^{-\infty}(P)$.

Proof: As $x_i\delta_{V,o}(x) = 0$ for all $i$, the subspace $\delta_{V,o}(x) \otimes C^{-\infty}(P)$ of $C^{-\infty}(W) \otimes \Lambda^n V'$ is in $\text{Ker}(j^{-\infty}_V)$. Reciprocally if $f \otimes \nu' \in C^{-\infty}(W) \otimes \Lambda^n V'$ is such that $j^{-\infty}_V(f \otimes \nu') = 0$, we see that $x_if(x + y) = 0$ for all $i$. Thus $f$ is the product of the $\delta$-function on the transverse subspace $V$ with a generalised function on $P$. \hfill \Box

Proposition 20 We have

1. If $i \neq n$, $H_i(j^{-\infty}_V) = 0$.

2. $H_n(j^{-\infty}_V) = \delta_{V,o} \otimes C^{-\infty}(P)$

Remark 21 Let $K$ be a compact group acting linearly on $W$ and preserving the direct sum decomposition $W = V \oplus P$. Thus the group $K$ acts on $A^{-\infty,*}$ and the operator $j_V$ commutes with the action of $K$. Hence $(A^{-\infty,*})^K$ is a subcomplex of $A^{-\infty,*}$. Let $\chi(k) := \det_V(k)$. Then, $K$ being compact,
\( \chi(k) = \pm 1 \) and moreover \( k \cdot \delta_{V,0} = \chi(k)\delta_{V,0} \). By averaging over \( K \) the equation \( \alpha = j_V^{-\infty} \beta \), we see that the homology of the subcomplex \((A^{-\infty,*})^K\) of \( A^{-\infty,*} \) is also equal to 0, except in top degree \( n \), while in top degree \( H_n((A^{-\infty,*})^K) = \delta_{V,0} \otimes C^{-\infty}(P)^\times \), where \( C^{-\infty}(P)^\times = \{ f \in C^{-\infty}(P); k \cdot f = \chi(k)f, \text{for all } k \in K \} \).

We now give a proof of Proposition 20.

**Proof:** Let us consider the subcomplex \( L^* = C_{\text{cpt}}^{-\infty,V}(W) \otimes \Lambda^*V' \) of \( A^{-\infty,*} \), where \( C_{\text{cpt}}^{-\infty,V}(W) \) denotes the space of generalised functions \( \phi \) on \( W \), with support contained in a set of the form \( F \oplus P \) where \( F \) is a compact subset of \( V \) ( \( F \) depending upon the generalised function \( \phi \)).

**Lemma 22** The inclusion \( L^* \to A^{-\infty,*} \) induces an isomorphism in homology.

**Proof:** Let us denote the operator \( j_V^{-\infty} \) by \( j_V \). Choose a scalar product on \( V \) and an orthonormal basis \( e^i \) of \( V \). Let \( e_i \in V' \) be the dual basis and let us denote by \( e_i \) the exterior multiplication by \( e_i \) on \( \Lambda^*V' \). Let \( \epsilon_V \) be the operator of degree +1 on \( A^{-\infty,*} \) defined by \( \epsilon_V = \sum_i x_i e_i \). It is easily verified that \( \epsilon_V j_V + j_V \epsilon_V = |x|^2 I \).

Let \( \chi \) be a smooth function on \( V \) such that \( \chi(x) = 1 \) for \( |x| < 1/2 \) and \( \chi(x) = 0 \) for \( |x| > 1 \). Extend \( \chi \) to a smooth function on \( W \) by setting \( \chi(x+y) = \chi(x) \). The multiplication by the function \( \chi \) on \( A^{-\infty} \) commutes with \( j_V \). It sends \( A^{-\infty} \) to \( L \). The function \( (1 - \chi(x))(|x|^2)^{-1} \) is a \( C^\infty \) function on \( V \), thus on \( W \). We write for \( \phi \in C^{-\infty}(W) \otimes \Lambda^*V' \)

\[ \phi = \phi_0 + \phi_1 \]

with \( \phi_0 = \chi \phi \) and

\[ \phi_1 = \frac{1 - \chi(x)}{|x|^2} |x|^2 \phi = \frac{1 - \chi(x)}{|x|^2} (j_V \epsilon_V + \epsilon_V j_V) \phi. \]

If \( j_V \phi = 0 \), both elements \( \phi_0 \) and \( \phi_1 \) are annihilated by \( j_V \). Furthermore \( \phi_1 \) is in the image of \( j_V \). Thus each element of \( H(A^{-\infty}) \) has a representative in the subcomplex \( L \) and hence the natural map \( H(L) \to H(A^{-\infty}) \) is surjective. Now let \( \phi \in L \) be such that \( \phi = j_V \alpha \) with \( \alpha \in A^{-\infty} \). We can find a \( C^\infty \) function \( \theta \) on \( W \) such that it is equal to 1 on the support of \( \phi \) and
such that $\theta \alpha \in L$. Thus $\phi = \theta \phi = j_V(\theta \alpha)$ and the homology class of $\phi$ in $H(L)$ is zero. Thus the natural map $H(L) \to H(A^{-\infty})$ is injective.

If $f \otimes \nu' \in L^s$, i.e. $f \in C^-_{\text{cpt}, V}(W)$, we can define $I(f \otimes \nu') \in C^-\infty(P)$ by

$$\int_P I(f \otimes \nu')(y)\phi(y)dy = \int_W f(x, y)\phi(y)dx dy$$

where $dx$ is the positive density on $V$ associated to $\nu'$. If $\ell = \delta_{V, o}(x)g(y)$, then $I(\ell) = g$. We denote by $L_0^*$ the subcomplex of $L^*$ such that $L_0^k = L^k$ if $k \neq n$, while $L_0^n = \{ \phi \in L^n; I(\phi) = 0 \}$. We will construct an explicit homotopy for $j_V^{-\infty}$ on the subcomplex $L_0^*$.

Consider the complex $R^* = C^\infty_{\text{cpt}, p}(W) \otimes \Lambda^* V'$, where $C^\infty_{\text{cpt}, p}(W)$ denotes the space of smooth functions on $W$, with support contained in a set of the form $V + F$ where $F$ is a compact subset of $P$. Define the subcomplex $R_0^*$ by $R_0^k := R_k^p$, if $k > 0$, while $R_0^0 := \{ \phi \in R^0; r_p \phi = 0 \}$, where $r_p$ is the restriction map from $C^\infty(W)$ to $C^\infty(P)$. The operator $F_\nu$ given in Definition 17 preserves $R_0^*$. Thus the homotopy $h_\nu = F_\nu d_\nu$ of $A_0^{\infty,*}$ is also a homotopy for $R_0^*$.

Let us choose a Lebesgue measure $dy$ on $P$. The pairing $(\cdot, \cdot)$ between the complexes $L^*$ and $R_*^{\infty}$ defined by

$$(\phi dx_I, f dx_J) = \int_W \phi(x + y)f(x + y)(dx_I \wedge dx_J)dy$$

is a non degenerate pairing. The space $L_0^n$ is the orthogonal of the subspace $1 \otimes C^\infty_{\text{cpt}}(P)$ of functions on $W$ constant in the $x \in V$ variables. Thus $(\cdot, \cdot)$ induces a non degenerate pairing between $L_0^*$ and $R_0^{*-\infty}$.

The operator $j_V$ satisfies

$$(j_V \alpha, \beta) + (-1)^{|\alpha|}(\alpha, j_V \beta) = 0$$

for $\alpha \in L^*$, $\beta \in R_*^{\infty-*}$.

Thus we can transpose the homotopy for $R_0^*$ and obtain a homotopy for $L_0^*$. More explicitly, define the operator $U_\nu$ of degree 0 on $L_0^*$ by

$$(U_\nu \alpha, \beta) = (\alpha, F_\nu \beta)$$

for $\alpha \in L_0^*$, $\beta \in R_0^{*-\infty}$. We extend the partial de Rham differential $d_\nu$ from $A_*^{\infty,*}$ to $A_*^{-\infty,*}$ (again denoted by $d_\nu$) by the same formula (12). The
operator $d_V$ also satisfies
\[(d_V \alpha, \beta) + (-1)^{|d_V|}(\alpha, d_V \beta) = 0.\]

Thus $d_V L^{n-1} \subset L^n_0$ and $d_V$ is an operator of degree 1 on $L^n_0$. The operator $U_V$ commutes with $d_V$. Let $k_V = U_V d_V = d_V U_V$. Then, for $\alpha \in L_0$, $\beta \in R_0$,
\[(k_V \alpha, \beta) + (-1)^{|d_V|}(\alpha, h_V \beta) = 0.\]
Thus, for $\alpha \in L_0$,
\[\alpha = (k_V j_V + j_V k_V) \alpha\]
as follows from the transpose relation $\beta = (h_V j_V + j_V h_V) \beta$.

The complex $L^n_0$ is thus exact and this implies the proposition. \[\blacksquare\]

Let us now consider a Lie group $G$ with Lie algebra $\mathfrak{g}$. Recall the definition of the Koszul differential $c_L$ on the space $T^* = L \otimes \Lambda^* \mathfrak{g}'$ calculating the cohomology of a $\mathfrak{g}$-module $L$: For $\alpha \in L \otimes \Lambda^p \mathfrak{g}'$, $c_L \alpha \in L \otimes \Lambda^{p+1} \mathfrak{g}'$ is defined by
\begin{equation}
(c_L \alpha)(X_1, \ldots, X_{p+1}) = \\
\sum (-1)^{i+1} X_i \cdot \alpha(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}) \\
+ \sum (-1)^{i+j} \alpha([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1}),
\end{equation}
where $X_1, \ldots, X_{p+1}$ are elements of $\mathfrak{g}$.

Consider $\iota(X) : T^* \rightarrow T^{-1}$, the contraction by an element $X \in \mathfrak{g}$ and $\mathcal{L}(X)$ the action of $\mathfrak{g}$ by tensor product on $T$. It is not difficult to verify the relation $\mathcal{L}(X) = c_L \iota(X) + \iota(X) c_L$. The space $\Lambda \mathfrak{g}'$ acts by exterior multiplication on $T$ and $c_L$ satisfies the Leibniz's rule: $c_L(\alpha \xi) = c(\alpha) \xi + (-1)^{|\alpha|} \alpha c_L(\xi)$, where $c$ in this formula denotes the Koszul differential of the complex $\Lambda \mathfrak{g}'$ (corresponding to the trivial one dimensional representation $L$).

Let $K$ be a Lie subgroup of $G$. Assume that $L$ is a $(\mathfrak{g}, K)$-module. Consider the subspace
\[T^*_K = (L \otimes \Lambda^*(\mathfrak{g}/K))^K\]
of $L \otimes \Lambda g'$. From the relation $\mathcal{L}(X) = c_L \iota(X) + \iota(X)c_L$, it is easy to see that $(T_K^*, c_L)$ is a subcomplex of $(L \otimes \Lambda^* g', c_L)$. The cohomology of the subcomplex $(T_K^*, c_L)$ is by definition the relative Lie algebra cohomology $H^*(g, K, L)$ of the $(g, K)$-module $L$.

Consider the algebra $D(g)$ of differential operators on $g$ with polynomial coefficients. Then the adjoint action of $g$ on $g$ determines a Lie algebra homomorphism $\tau$ from $g$ into $D(g)$. If $L$ is a $D(g)$-module, then $L$ is a $g$-module, via the adjoint action. Furthermore, as $D(g)$ contains $S(g')$, the module $L$ is a $S(g')$-module.

Let $e_i$ be a basis of $g$, $e_i \in g'$ the dual basis. We consider the element $\Omega = \sum_i e_i \tau(e^i)$ of $D(g)$.

**Lemma 23** The element $\Omega \in D(g)$ is identically 0.

**Proof:** For $X \in g$, denote by $\partial_X$ the constant coefficient vector field on $g$ equal to $X$. The adjoint vector field $\tau(e^i)$ is given by: $\tau(e^i) = -\sum_j x_j \partial_{[e^i, e^j]}$ and hence

$$\Omega = -\sum_{i,j} x_i x_j \partial_{[e^i, e^j]}$$

which is equal to zero, as the vector field $-\sum_{i,j} x_i x_j \partial_{[e^i, e^j]}$ is the vector field equal at the point $X \in g$ to $[X, X] = 0$.

Let $L$ be a $D(g)$-module. Consider the space

$$A^* := L \otimes \Lambda g'.$$

As $L$ is a $S(g')$-module, we can consider the operator $j = j_L : A^* \to A^{*-1}$ given (as in Formula (11)) by $j = \sum_i e_i \otimes \iota(e_i)$.

On the other hand, the $g$-module structure on $L$ gives rise to the Koszul differential $c : L^* \to L^{*-1}$.

**Lemma 24.** Let $L$ be a $D(g)$-module. The operator $j + c$ satisfies

$$(j + c)^2 = 0$$

**Proof:** As $j^2 = 0$, $c^2 = 0$, we have to verify that $jc + cj = 0$. We have, (setting $e^i = \iota(e^i)$) and using the Leibniz’s rule:

$$jc + cj = \sum_i (e_i e^i c + ce_i e^i) = \sum_i e_i (e^i c + ce_i e^i) = \sum_i e_i (\mathcal{L}(e^i) + c(e_i) e^i).$$
The action $\mathcal{L}(e^i)$ is by the tensor product action $\tau(e^i) \otimes I + I \otimes \mathcal{L}_\Lambda(e^i)$, where $\mathcal{L}_\Lambda(e^i)$ is the action of $\mathfrak{g}$ on $\Lambda \mathfrak{g}'$ induced from the adjoint representation. Thus, as $\sum_i e_i \tau(e^i) = 0$ from the preceding lemma, it remains to see that $\sum_i e_i \mathcal{L}_\Lambda(e^i) \xi + \sum_i c(e_i)c' \xi = 0$ for $\xi \in \Lambda \mathfrak{g}'$. Writing $\xi$ as a product of elements $\alpha \in \mathfrak{g}'$, it is sufficient to prove this relation for $\xi \in \mathfrak{g}'$ where this is checked easily.

The spaces $L = C^b_\infty(\mathfrak{g})$ have natural $D(\mathfrak{g})$-module structures. Thus on $L \otimes \Lambda \mathfrak{g}'$, the operators $j$ and $c$ are defined and satisfy $(j + c)^2 = 0$. We will see in section 5 that we obtain this example of perturbed Koszul complex when computing the $G$-equivariant cohomology of a Lie group $G$ provided with the free action of $G$ on itself given by left translation. We compute the cohomology of slightly more complicated complexes in the next section.

4 Induction of equivariant differential complexes

Let $(A, d)$ be a differential complex, i.e. a $\mathbb{Z}/2\mathbb{Z}$-graded vector space over $\mathbb{R}$, with a differential $d$ of odd degree. We will assume that $A$ is a Frechet space and that $d$ is continuous. (In most of the applications, $A$ will be the space of smooth differential forms on a $G$-manifold $M$.) Let $G$ be a Lie group acting on $A$. We assume that the action of $G$ on $A$ is differentiable.

As in H. Cartan [9], we say that $(A, d)$ is a $G$-differential complex, if

1. The action of $G$ preserves the $\mathbb{Z}/2\mathbb{Z}$-grading and commutes with $d$.

2. There are given continuous contraction operators $\iota(X), X \in \mathfrak{g}$ of odd degree satisfying $\iota(X)\iota(Y) + \iota(Y)\iota(X) = 0$ for all $X, Y \in \mathfrak{g}$ and $g\iota(X)g^{-1} = \iota(gX)$, for all $g \in G, X \in \mathfrak{g}$.

3. The Lie derivative action $\mathcal{L}(X)$ of the action of $G$ on $A$ satisfies $\mathcal{L}(X) = d\iota(X) + \iota(X)d$, for all $X \in \mathfrak{g}$.

If $A$ is a $G$-differential complex, define the space $A_{\text{hor}}$ of $G$-horizontal elements to be

$$A_{\text{hor}} = \{ \alpha \in A; \iota(Y)\alpha = 0, \text{ for all } Y \in \mathfrak{g} \}.$$
The space $A_{basG}$ of $G$-basic elements is by definition the space of elements of $A$ which are $G$-invariant and horizontal:

$$A_{basG} = A^G_{horG}.$$ 

Remark that the differential $d$ leaves the space $A_{basG}$ stable.

Let $(A, d)$ be a $G$-differential complex. We call $(A, d)$ a $G$-differential algebra if, in addition, $A$ has a $\mathbb{Z}/2\mathbb{Z}$-graded algebra structure satisfying the following:

1. The action of any $g \in G$ on $A$ is by algebra automorphisms.

2. The operator $d$ and the operators $\iota(X)$, $X \in g$, are odd derivations of the algebra $A$.

If $G$ acts smoothly on a manifold $M$, then $A^\bullet(M) = A^{even}(M) \oplus A^{odd}(M)$ is a $G$-differential algebra. If $L$ is a differentiable $G$-module, the tensor product $G$-module $L \otimes \Lambda g'$ together with the Koszul differential $c_L$ and the contraction operators $I \otimes \iota(X)$ is a $G$-differential complex. In particular, taking $L$ to be the trivial one dimensional $G$-module, $\Lambda g'$ is a $G$-differential complex, in fact a $G$-differential algebra.

The tensor product (over $\mathbb{R}$) of two $G$-differential complexes is canonically a $G$-differential complex. Thus, for any $G$-differential complex $A$, we can form the $G$-differential complex $A \otimes \Lambda g'$. We write an element $\alpha \in A \otimes \Lambda g'$ as $\alpha = \sum_k \alpha_{[k]}$ with $\alpha_{[k]} \in A \otimes \Lambda^k g'$. We denote by $r : A \otimes \Lambda g' \to A$ the projection of an element $\alpha \in A \otimes \Lambda g'$ on its component $\alpha_{[0]}$ of exterior degree 0.

**Lemma 25** Let $A$ be a $G$-differential complex. The map $r : A \otimes \Lambda g' \to A$ induces an isomorphism from $(A \otimes \Lambda g')_{horG}$ to $A$.

**Proof:** For $X \in g$, we denote by $\iota(X)$ the tensor product contraction on $A \otimes \Lambda g'$. If $\alpha \in (A \otimes \Lambda g')_{horG}$ is such that $\alpha_{[0]} = 0$, it is easy to see by induction on the exterior degree that $\alpha = 0$. Let us prove that $r$ is surjective: Let $E'$ be a basis of $g$ with dual basis $E_i \in g'$. Denote by $e_i$ the exterior multiplication by $E_i$ on $A \otimes \Lambda g'$ from the left. Let $h_i =$
I - \varepsilon_i \iota_i(E^i). It is easy to see that the operators \( h_i \) commute with each other. In particular, the operator \( h = \prod_i (I - \varepsilon_i \iota_i(E^i)) \) acting on \( A \otimes \Lambda g' \) is well defined, i.e. it does not depend upon the order in which the product is taken. We verify: \( \iota_i(E^i)h_i = 0 \), \( \iota_i(E^i)h_i = h_i \iota_i(E^i) \) for \( i \neq j \). Thus \( h \) is a projector from \( A \otimes \Lambda g' \) to the set \((A \otimes \Lambda g')_{\text{hor}}\). If \( a \in A = A \otimes \Lambda^0 g' \), the element \( h(a) = a - (-1)^{|a|-1} \sum_i \iota_i(E^i)a \otimes E_i + \cdots \) is a horizontal element of \( A \otimes \Lambda g' \) with component of exterior degree 0 equal to \( a \). Thus \( r \) is surjective.

**Definition 26** The operator \( h := \prod_i (I - \varepsilon_i \iota_i(E^i)) \) on \( A \otimes \Lambda g' \) is called the horizontal projection.

By the preceding lemma, if \( a \in A \otimes \Lambda g' \), the element \( h(a) \) is the unique horizontal element of \( A \otimes \Lambda g' \) whose component of exterior degree 0 is \( a_{|0|} \).

If \((A, d_A)\) is a \( G \)-differential complex, we can define the spaces \( \mathcal{A}^{\pm \infty} := C^{\pm \infty}(g, A) \). We denote both of them by \( \mathcal{A} \) when there is no need to indicate precisely the smoothness properties of a function \( f : g \to A \) that we assume. The space \( \mathcal{A} \) inherits a \( \mathbb{Z}/2 \)-graded structure from that of \( A \).

For any \( E \in g \), the contraction operator \( \iota(E) \) is extended to \( \mathcal{A} \) pointwise:

\[
(\iota(E)f)(X) = \iota(E)(f(X)).
\]

Similarly the differential \( d_A \) is extended on \( \mathcal{A} \) by

\[
(d_A f)(X) = d_A(f(X)).
\]

Thus we can define on \( \mathcal{A} \) the operator

\[
\iota_g = \sum x_i \iota_i(E^i)
\]

i.e. \( (\iota_g f)(X) = \sum_i x_i(\iota_i(E^i)f(X)) \) and the operator

\[
d_g = d_A - \iota_g.
\]

When \( g \) is understood, we will just write \( \iota \) for \( \iota_g \). If we take \( A = \mathcal{A}(M) \) for a \( G \)-manifold \( M \), then \( \mathcal{A}^{\pm \infty} \) was introduced in section 2 and \( d_g \) here coincides with the operator \( d_g \) of section 2.
Lemma 27 The operator $d_g$ is odd and satisfies $d_g^2 = 0$ on

$$\mathcal{A}_G^{\pm\infty} := C^{\pm\infty}(g, A)^G.$$  

More generally if $\chi$ is a character of $G$ trivial on its connected component, then

$$(\mathcal{A}_G^{\pm\infty}, d_g) := (C^{\pm\infty}(g, A)^\chi, d_g)$$ 

is a complex, where

$$C^{\pm\infty}(g, A)^\chi := \{ f \in C^{\pm\infty}(g, A) : g \cdot f = \chi(g) f, \text{ for all } g \in G \}.$$  

**Proof:** It is easy to see that $i^2 = 0$. So to prove that $i^2 f = 0$ for $f \in \mathcal{A}_G^{\pm\infty}$, it suffices to show that $(d_A + id_{A})f = 0$. Now $((d_A + id_{A})f)(X) = \sum_i x_i L_A(E^i)(f(X))$, where $L_A(E^i)$ is the Lie derivative action on $A$ of the element $E^i \in g$. The invariance condition on an element $f \in C^{\pm\infty}(g, A)^\chi$ implies that $L_A(E^i)(f(X)) = \frac{d}{dt} f(X + \epsilon [E^i, X])$. Thus, for $f \in C^{\pm\infty}(g, A)^\chi$, we obtain in the notation of Lemma 23 (section 3)

$$((d_A + id_{A})f)(X) = \sum_i x_i \frac{d}{dt} f(X + \epsilon [E^i, X]) = \sum_i x_i x_j \partial_{[E^i, E^j]} f(X) = 0.$$ 

\[\blacksquare\]

Let $G$ be a Lie group and let $K$ be a closed subgroup of $G$. Let $(A, d_A)$ be a $K$-differential complex. Let

$$L^{\pm\infty} = C^{\pm\infty}(g, A).$$

As for $\mathcal{A}$, we denote both of these by $L$, when there is no need to indicate the precise smoothness assumption. The $\mathbb{Z}/2$-graded structure on $A$ induces a $\mathbb{Z}/2$-structure on $L$. Consider on $L$ the structure of $G$-module, defined by $(g \cdot f)(X) = f(g^{-1} \cdot X)$, for $f \in C^{\pm\infty}(g, A)$. This structure of $G$-module of course induces a structure of $g$-module on differentiation. Thus on $L \otimes \Lambda g'$, we can define the Koszul differential $c$ (associated to this $g$-module structure on $L$). However, we take in account sign rules in defining $c$: $c$ coincides with the Koszul differential $c_{L, even}$ (see Formula 14 of section 3) on $L^{even} \otimes \Lambda g'$, while we define $c = -c_{L, odd}$ on $L^{odd} \otimes \Lambda g'$.

We extend the operator $d_A$ pointwise on $L$: $(d_A f)(X) = d_A (f(X))$. We still denote by $d_A$ the operator $d_A \otimes I$ on $L \otimes \Lambda g'$.  

Let $E'$ be a basis of $\mathfrak{g}$. We extend the operator $\iota_A(E^i)$ to $L \otimes \Lambda^g'$ following the sign rules (2), (3) given in section 1.

Consider the operator

$$j = \sum x_i \iota_A(E^i)$$

on $L \otimes \Lambda^g'$.

**Lemma 28** The operator $c_g : = d_A + c + j$ satisfies $c_g^2 = 0$.

**Proof:** The $\mathfrak{g}$-module structure on $L$ is induced from its $D(\mathfrak{g})$-module structure via the adjoint representation. It follows from Lemma 24 of section 3 that we have $(c + j)^2 = 0$. Furthermore, as can be easily seen, we have $(c + j)d_A + d_A(c + j) = 0$.

The space $L$ is equipped with the operators still denoted by $\iota_A(E), E \in \mathfrak{g}$, defined pointwise by their action on $A$.

Total contraction operators $\iota_l(Y), Y \in \mathfrak{k}$, are defined by the tensor product contraction on $L \otimes \Lambda^g'$. Let us consider the action of $K$ on $L$ by

$$(k \cdot f)(X) = k \cdot f(k^{-1}X)$$

and the action on $L \otimes \Lambda^g'$ by tensor product. We denote by $\mathcal{L}_l(Y), Y \in \mathfrak{k}$, the corresponding infinitesimal action of $Y \in \mathfrak{k}$.

We define $(L \otimes \Lambda^g')_{\text{hor} K} = \{ \alpha \in L \otimes \Lambda^g' ; \iota_l(Y)\alpha = 0, \text{for all } Y \in \mathfrak{k} \}$. The subspace $(L \otimes \Lambda^g')_{\text{bas} K}$ of $K$-basic elements is the subspace of elements of $L \otimes \Lambda^g'$ which are horizontal and invariant under $K$:

$$(L \otimes \Lambda^g')_{\text{bas} K} = ((L \otimes \Lambda^g')_{\text{hor} K})^K.$$

**Lemma 29** The operators $c_A : = c + d_A$ and $j$ preserve the subspace of $K$-basic elements.

**Proof:** The operator $j$ clearly preserves the space of horizontal elements. It commutes with the action of $K$, thus it preserves the space of $K$ basic elements.

Using the relation $\iota_l(Y)c_A + c_A\iota_l(Y) = \mathcal{L}_l(Y)$, we see that the space of $K$-basic elements is stable under $c_A$. 


Definition 30 Let $A$ be a $K$-differential complex. Define the induced complex $\text{Ind}_{G/K}^{\pm \infty} A$ from $K$ to $G$ of the $K$-differential complex $A$ to be the space

$$\text{Ind}_{G/K}^{\pm \infty} A = (C^{\pm \infty}(g, A) \otimes \Lambda g')_{bas,K}$$

with the differential $c_g = c + d_A + j$.

If $(A, d_A) = (R, 0)$, then $\text{Ind}_{G/K}^{\pm \infty} R = (C^{\pm \infty}(g) \otimes \Lambda (g/k)')^K$ with differential $c_g = c + j$, where $c$ is the Koszul differential of the $(g, K)$-module $C^{\pm \infty}(g)$.

Our aim is to compute the cohomology of the complex $(\text{Ind}_{G/K}^{\pm \infty} A, c_g)$ in terms of the cohomology of the complex $(C^{\pm \infty}(k, A)^K, d_k)$.

The cohomology of the complex $\text{Ind}_{G/K}^{\pm \infty} A$ is determined in [12]. Recall the results:

Theorem 31 The restriction map

$$r_k : C^{\infty}(g, A) \otimes \Lambda g' \to C^{\infty}(k, A)$$

given by

$$(r_k \alpha)(Y) = \alpha(Y) \quad \text{for } \alpha \in C^{\infty}(g, A), Y \in k$$

$$(r_k \alpha) = 0 \quad \text{if } \alpha \in C^{\infty}(g, A) \otimes \Lambda^{\geq 1} g'$$

defines a cochain map from the complex $(\text{Ind}_{G/K}^{\infty} A, c_g)$ to the complex $(C^{\infty}(k, A)^K, d_k)$. Furthermore if the principal bundle $G \to G/K$ possesses a $G$-invariant connection, then the restriction map $r_k$ induces an isomorphism in cohomology.

In particular, if $A = R$, we obtain a map $r_k : \text{Ind}_{G/K}^{\infty} R \to C^{\infty}(k)^K$ and this map is an isomorphism in cohomology when $K$ is compact.

Remark here that the restriction map does not extend to $\text{Ind}_{G/K}^{-\infty} (A)$ as generalised functions cannot usually be restricted to a subspace.

The assumption that $G \to G/K$ possesses a $G$-invariant connection is satisfied for example when $K$ is a reductive Lie group, in particular when $K$ is a compact subgroup.

We now consider $\text{Ind}_{G/K}^{-\infty} A$. Let $r = g/k$. We identify $r'$ with $k^\perp = \{ f \in g', f|k = 0 \}$. Let $n = \dim r$. Consider the character $\chi_{G/K}(k) = \text{sign}(\det r k)$
of $K$. As $G$ and $K$ are fixed, we denote $\chi_{G/K}$ simply by $\chi$. Let $\nu' \in \Lambda^n r'$ be a nonzero element. The element $\nu'$ determines an Euclidean measure $|d\nu'|$ and an orientation $o$ on $r$. If $dX$ is an Euclidean measure on $\mathfrak{g}$, we denote by $dY$ the Euclidean measure on $k$ such that $dX = |d\nu'|dY$.

Remark that, as $n = \dim \mathfrak{g}/k$, the space $\Lambda k' \otimes \Lambda^n r'$ is naturally embedded as a subspace of $\Lambda^2$. Consider the horizontal projection operator (see Definition 26) $h_K : A \to (A \otimes \Lambda k')_{\text{hor}K}$. Thus, for $a \in A$, the element $h_K(a) \wedge \nu'$ belongs to $(A \otimes \Lambda g')_{\text{hor}K}$.

**Definition 32** Choose $\nu'$ a nonzero element of $\Lambda^n r'$. Let $f \in A_{K,N}^{-\infty} := C^{-\infty}(k, A)^\wedge$. We define $\text{Ind}_{G/K, o} f \in \text{Ind}_{G/K} A = (C^{-\infty}(\mathfrak{g}, A) \otimes \Lambda g')_{\text{bas}K}$ by

$$(\text{Ind}_{G/K, o} f, \Phi dX) = h_K\left( \int_{k} f(Y)\Phi(Y)dY \right) \wedge \nu'$, for any test function $\Phi$ on $\mathfrak{g}$$

(with $dX = dY|d\nu'|$).

It is easy to see that the map $\text{Ind}_{G/K, o}$ depends only on the orientation $o$ of $\mathfrak{g}/r$ determined from $\nu'$. Thus we write $\text{Ind}_{G/K} A$ for $\text{Ind}_{G/K, o}$. Remark that the map $\text{Ind}_{G/K, o}$ is injective.

**Proposition 33** The map $\text{Ind}_{G/K}$ is a cochain map of parity degree equal to $\dim G/K$ from the $\mathbb{Z}/2$-cochain complex $(A_{K,N}^{-\infty}, d_A)$ to the $\mathbb{Z}/2$-cochain complex $(\text{Ind}_{G/K} A, c_A)$.

**Proof:** Let $L = C^{-\infty}(\mathfrak{g}, A)$ and let $L_{\text{det}} \subset L$ be the subset of elements $F \in L$ satifying $k \cdot F = (\det_+ k)F$. If $F \in L$, define $h_K(F) \in L \otimes \Lambda k'$ by $h_K(F)(X) = h_K(F(X))$. The map $F \to h_K(F) \wedge \nu'$ sends $L$ to $(L \otimes \Lambda k')_{\text{hor}K} \wedge \nu'$. It sends $L_{\text{det}}$ to $(L \otimes \Lambda g')_{\text{bas}K} = \text{Ind}_{G/K} A$.

**Lemma 34** For every $F \in L_{\text{det}}$, we have $c_A(h_K F \wedge \nu') = h_K(d_A F) \wedge \nu'$, where $c_A$ is the operator on $\text{Ind}_{G/K} A$ defined in Lemma 29.

**Proof:** We compute $c_A(h_K(F) \wedge \nu')$ for $F \in L_{\text{det}}$. Let $K^i$ be a basis of $k$ with dual basis $K_i$. It is easy to see that $c(\nu') = -\sum (\text{Tr}_A K_i) K_i \wedge \nu'$. Thus, by Leibniz’s rule,

$$c_A(h_K(F) \wedge \nu') \in L \otimes (\Lambda g' \wedge \nu') = L \otimes (\Lambda k' \wedge \nu').$$
As $c_A$ preserves $K$-basic elements (cf. Lemma 29), $c_A(h_K(F) \wedge \nu')$ is $K$-basic, in particular is horizontal. The space $(L \otimes (\Lambda g' \wedge \nu'))_{h_K}$ is isomorphic with $L$ by the map $\alpha \wedge \nu' \mapsto \alpha_{[0]} \in L$ for $\alpha \in L \otimes \Lambda g'$, where $\alpha_{[0]}$ is the component of $\alpha$ in the zeroth exterior degree. Thus, if $c_A(h_K(F) \wedge \nu') = G \wedge \nu'$, then $c_A(h_K(F) \wedge \nu') = h_K(G_{[0]} \wedge \nu')$.

We have $G = d_A(h_K(F)) + c(h_K F) + (-1)^{|F|+1} h_K F \wedge \sum_t(\text{Tr} \pi_t K^t) K_t$. Looking at the term of zeroth-exterior degree, it follows that $G_{[0]} = d_A F$. This proves the lemma.

For $f \in C^{-\infty}(k, A)^n$, the element $F \in C^{-\infty}(g, A)$ defined by $(F, \Phi dX) = \int_k f(Y) \Phi(Y) dY$ belongs to $L^{\text{det}}$ and $\text{Ind}_{G/K, o} f = h_K(F) \wedge \nu'$. The preceding lemma implies $c_A \text{Ind}_{G/K, o} f = \text{Ind}_{G/K, o} d_A f$. Proposition 33 is now a consequence of the following

**Lemma 35**

$$
j(\text{Ind}_{G/K, o} f) = -\text{Ind}_{G/K, o} (i_k f).
$$

**Proof:** Let $E^i$ be a basis of $g$ such that the first elements form a basis of $k$. Let $E_i$ be the dual basis. Then the last $n$ coordinates $x_i$ vanish on $k$. We denote by $y_i$, $1 \leq i \leq \text{dim } k$ the coordinates on $k$ corresponding to the basis of $k'$ dual to the basis $E^i$ of $k$. We have then

$$(j(\text{Ind}_{G/K, o} f), \Phi dX) = \sum_{i=1}^{\text{dim } k} (\iota_A(E^i) h_K(\int_k f(Y) y_i \Phi(Y) dY)) \wedge \nu'.
$$

As $h_K$ is a projector on the $K$-horizontal elements for the tensor product contraction, it satisfies for $E^i \in k$ and $a \in A$, $\iota_A(E^i)(h_K a) + \iota_A(E^i)(h_K a) = 0$. But $\iota_A(E^i) h_K = h_K \iota_A(E^i)$, and we obtain the lemma. 

The proof of this lemma completes the proof of Proposition 33.

If $A$ is a $K$-differential algebra, then $C^{-\infty}(k, A)^n$ is a module over $C^\infty(k, A)^K$. Similarly $\text{Ind}_{G/K, o} A$ is a module over $\text{Ind}_{G/K} A$. Remark the following relation between the maps $r_k$ and $\text{Ind}_{G/K, o}$

**Lemma 36** If $\alpha \in \text{Ind}_{G/K} A$ and $s \in C^{-\infty}(k, A)^n$, then

$$
\alpha \text{Ind}_{G/K, o} s = \text{Ind}_{G/K, o} ((r_k \alpha) s).
$$

The main result of this section is the following
Theorem 37 Assume that the group $K$ is compact. Then the map $\text{Ind}_{G/K,o} : (A^{\infty}_{K,G}, d_K) \to (\text{Ind}_{G/K} A, c_g)$ induces an isomorphism in cohomology.

Proof: We can choose a $K$-invariant decomposition:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{r}.$$ 

To this direct sum decomposition is associated the tensor product decomposition $\Lambda^2 \mathfrak{g} = \Lambda^1 \mathfrak{k} \otimes \Lambda^1 \mathfrak{r}$.

Let $h_K : L \to L \otimes \Lambda_k^1$ be the projection on $K$-horizontal vectors for the tensor product contraction (see definition 26). The map $W(v \otimes \xi) = h_K(v) \wedge \xi$

for $v \in L$ and $\xi \in \Lambda^1 \mathfrak{r}$ is an isomorphism from the space $L \otimes \Lambda^1 \mathfrak{r}$ to the space $(L \otimes \Lambda^1 \mathfrak{g})_{\text{hor}K}$. The map $W$ commutes with the action of $K$ and allows us to identify the space

$$T_K := (L \otimes \Lambda^1 \mathfrak{r})^K$$

with the space

$$\text{Ind}_{G/K} A = (C^{-\infty}(\mathfrak{g}, A) \otimes \Lambda^1 \mathfrak{g})_{\text{bas}K}.$$ 

On the space $T_K$, we will use the $\mathbb{Z}_2$-gradation given by the exterior degree $T_K = \bigoplus_{p=0}^n T^p_K = \bigoplus_{p=0}^n (L \otimes \Lambda^p \mathfrak{r})^K$.

Let $R^i$ be a basis of $\mathfrak{r}$ with dual basis $R_i$ and let $K^j$ be a basis of $\mathfrak{k}$ with dual basis $K_j$. We write $X \in \mathfrak{g}$ as $X = Y + R$, with $R = \sum_i x^i R^i$, $Y = \sum_j y^j K^j$.

The operator $\iota_k = \sum_j y^j \iota_A(K^j)$ is defined on $L = C^{-\infty}(\mathfrak{g}, A)$. Let $j_r : L \otimes \Lambda^1 \mathfrak{r} \to L \otimes \Lambda^{-1} \mathfrak{r}$ be given by $j_r = \sum_i x^i \iota_A(R^i)$. It is easy to see that $\iota_k$ and $j_r$ commute with the action of $K$ on $L \otimes \Lambda^1 \mathfrak{r}$.

Lemma 38 For all $\alpha \in L \otimes \Lambda^1 \mathfrak{r}$,

$$jW(\alpha) = W(j_r \alpha - \iota_k \alpha).$$

Proof: We have, for $v \in L$ and $\xi \in \Lambda^1 \mathfrak{r}$,

$$j(h_K v \wedge \xi) = \sum_i x^i \iota_A(R^i)(h_K v \wedge \xi) + \sum_j y^j (\iota_A(K^j) h_K(v)) \wedge \xi.$$ 

Further, for $K^j \in \mathfrak{k}$, $\iota_A(K^j) h_K(v) + h_K(\iota_A(K^j)v) = 0$, and we obtain the lemma. ■
Lemma 39  If $\alpha \in T_K^n$, then $W^{-1}c_AW(\alpha) \in T_K^n \oplus T_K^{n+1} \oplus T_K^{n+2}$.

Proof:  This is obvious since, $k$ being a subalgebra, $\alpha(\Lambda^qk' \otimes \Lambda^p r') \subset (\Lambda^{q+1}k' \otimes \Lambda^p r') \oplus (\Lambda^qk' \otimes \Lambda^{p+1} r') \oplus (\Lambda^{q+1}k' \otimes \Lambda^{p+2} r')$.  

Let us now prove Theorem 37.  By the map $W$, we identify $Ind^{-\infty}_GA$ with $T_K$ and write still $c_g$ for the operator $W^{-1}c_gW$ on $T_K$.  As we have seen in Proposition 20 and Remark 21 (of section 3), the homology groups of the operator $\mu_r : T^{2n}_K \to T^{2n-2}_K$ are equal to zero except in maximal degree $n = \dim r$.  Furthermore, if $\alpha \in T^{2n}_K$ is such that $\mu_r \alpha = 0$, then $\alpha = \delta_r(x) \otimes f(y)$ for some $f \in C^{-\infty}(k, A)$ by Lemma 19 and Remark 21 (of section 3).  But then, by definition of $Ind_{G/K, o}$ we obtain $\alpha = Ind_{G/K, o} f$.

Now, if $\alpha \in T^{2n}_K$ is such that $c_g \alpha = 0$, writing this equation component-wise, we see that $\alpha$ satisfies the relation $\mu_r \alpha = 0$.  Thus $\alpha$ is of the form $Ind_{G/K, o} f$.  As $c_g Ind_{G/K, o} f = Ind_{G/K, o} d_k f$ (by Proposition 33), we see that $f \in \text{Ker} d_k$, since $Ind_{G/K, o}$ is an injective map.  Consider now an element $\alpha = \sum_{k \geq k_0} \alpha_{[k]}$ in the kernel of $c_g$.  From the degree consideration, we see that its component of minimal exterior degree $k_0$ is annihilated by $J_r$.  If $k_0$ is less than $n$, there exists an element $\beta \in T^{2n+1}_K$ such that $\alpha_{[k_0]} = j_r \beta$.  The element $\alpha - c_g \beta$ is in the same cohomology class as $\alpha$ and all its non-zero exterior degree components are of degree strictly greater than $k_0$.  By induction, $\alpha$ has a representative in $T^n_K$ and we see that the map $Ind_{G/K, o}$ is surjective in cohomology.

Suppose now that $Ind_{G/K, o} f = c_g \beta$ with $\beta = \sum_{k \geq k_0} \beta_{[k]}$.  Writing this equation component-wise, we see that $j_r \beta_{[k_0]} = 0$.  If $k_0 < n$, changing $\beta$ to $\beta' = \beta - c_g \gamma$ with $\gamma \in T^{k_0+1}_K$ and $j_r \gamma = \beta_{[k_0]}$, we still have $Ind_{G/K, o} f = c_g \beta'$.  By choice, $\beta' = \sum_{k > k_0} \beta_{[k]}$.  Hence, by induction, we obtain an element $\tilde{\beta} \in T^n_K$ such that $Ind_{G/K, o} f = c_g \tilde{\beta}$.  From degree consideration, $j_r \tilde{\beta} = 0$.  But then $\tilde{\beta} = Ind_{G/K, o} g$ for some $g \in C^{-\infty}(k, A)$.  The equation $c_g Ind_{G/K, o} g = Ind_{G/K, o} f$ reads as $Ind_{G/K, o} d_k g = Ind_{G/K, o} f$.  But $Ind_{G/K, o}$ being an injective map, we get $d_k g = f$.  This proves that the map $Ind_{G/K, o}$ is injective in cohomology, thereby completing the proof of Theorem 37.

As a particular case, if $K$ is compact and if $(A, d_A) = (R, 0)$, we obtain that the map $Ind_{G/K, o}$ induces an isomorphism from $C^{-\infty}(k)$ to the cohomology of the complex $((C^{-\infty}(g) \otimes \Lambda(g/k)\hat{j})_K, j + c)$.

We apply these calculations in the next section to the calculation of the equivariant cohomology of fibre bundles over homogeneous spaces.
5 Equivariant cohomology of Homogeneous spaces

Let $G$ be a Lie group and let $K$ be a closed subgroup of $G$. Let $D = G/K$. Let $e \in D$ be the base point of $D$. We identify the tangent space of $G$ at a point $g \in G$ with $\mathfrak{g}$ by sending $X \in \mathfrak{g}$ to the tangent vector to the curve $g \exp tX$ in $G$. Let $n = \dim D$. Let $r = \mathfrak{g}/k$. The tangent space $T_e D$ at $e \in D$ is identified with $r$. If $g \in G$, we again denote by $L_g$ the map $T_e D \to T_{g e} D$ induced from the left multiplication map $L_g : D \to D$. For $\alpha \in C^{\pm \infty}(\mathfrak{g}, \mathcal{A}(D)) := \sum_{p=0}^{\dim D} C^{\pm \infty}(\mathfrak{g}, \mathcal{A}^p(D))$, $\alpha_p$ is the $p$-th component of $\alpha$. If $x \in D$ and $\xi_1, \cdots, \xi_p \in T_x D$, then $(\alpha_p)_x(\xi_1, \cdots, \xi_p)$ is a function (maybe generalised) on $\mathfrak{g}$ given by $X \to (\alpha_p)(X)_x(\xi_1, \cdots, \xi_p)$.

Let $\alpha \in \mathcal{A}^{\pm \infty}_G(D) = C^{\pm \infty}(\mathfrak{g}, \mathcal{A}(D))^G$. Let $R^1, \ldots, R^p \in r$, $X \in \mathfrak{g}$ and $g \in G$. As $\alpha$ is an equivariant form

$$\alpha_p(Ad(g)(X)) = (L_g R^1, \ldots, L_g R^p) = \alpha_p(X)(R^1, \ldots, R^p).$$

Let $\tilde{\alpha}(X) = \alpha(X)_e$. Thus $\tilde{\alpha}$ is a function on $\mathfrak{g}$ with values in $\Lambda r'$ and the map $\alpha \to \tilde{\alpha}$ is an isomorphism from the space $\mathcal{A}^{\pm \infty}_G(D)$ to the space $T_K = (C^{\pm \infty}(\mathfrak{g}) \otimes \Lambda r')^K$, where the action of $K$ on both $C^{\pm \infty}(\mathfrak{g})$ and $\Lambda r'$ is induced from the adjoint representation.

Let $R^i$ be a basis of $r = \mathfrak{g}/k$, with dual basis $R^i \in r'$. Let $x_i = R^i(X)$. Let $j_r$ be the operator on $T$ given by

$$j_r = \sum_i x_i t_A(R^i).$$

Let $c$ be the Koszul differential on $T_K$ (see Formula 14 of section 3). From [12], we have

**Lemma 40** For $\alpha \in \mathcal{A}^{\pm \infty}_G(D)$, we have $(d^g_\alpha) = (c + j_r)\tilde{\alpha}$.

For example, the complex $\mathcal{A}^{-\infty}_G(G)$ becomes isomorphic, under evaluation at $e$, to $C^{-\infty}(\mathfrak{g}) \otimes \Lambda \mathfrak{g}'$ and the differential $d^\mathfrak{g}$ becomes the perturbed differential $e + j$.

In the notation of the preceding section, if $(A, d_A) = (\mathbb{R}, 0)$, we have $\mathcal{A}^{\pm \infty}_G(D) \cong Ind^{\pm \infty}_G/_{/K}(\mathbb{R})$, as cochain complexes.
Equivariant cohomology with generalised coefficients

Let $\chi_G/K(k) := \text{sign}(\det_r k)$. As $G$ and $K$ are fixed in this section, we denote $\chi_G/K$ by $\chi$. Recall the definition (see Theorem 31 and Definition 32 of section 4) of the maps

$$r_k : \text{Ind}_{G/K}^\infty(\mathbb{R}) \to C^\infty(\mathfrak{k})^K$$

and

$$\text{Ind}_{G/K,o} : C^{-\infty}(\mathfrak{k})^\chi \to \text{Ind}_{G/K}^{-\infty}(\mathbb{R}).$$

Using the identification $\alpha \to \hat{\alpha}$, we get maps again denoted by $r_k$ and $\text{Ind}_{G/K,o}$:

$$r_k : \mathcal{A}_G^\infty(\mathfrak{d}) \to C^\infty(\mathfrak{k})^K$$

and

$$\text{Ind}_{G/K,o} : C^{-\infty}(\mathfrak{k})^\chi \to \mathcal{A}_K^{-\infty}(\mathfrak{d}).$$

Let us describe explicitly these maps.

The point $e$ is a $K$-stable submanifold of $\mathfrak{d}$. The successive restriction maps $\mathcal{A}_G^\infty(\mathfrak{d}) \to \mathcal{A}_G^\infty(\mathfrak{d}) \to \mathcal{A}_K^\infty(e)$ are well defined. The composed map $\mathcal{A}_G^\infty(\mathfrak{d}) \to \mathcal{A}_K^\infty(e) = C^\infty(\mathfrak{k})^K$ coincides obviously with the map $r_k$:

$$(r_k \alpha)(Y) = (\alpha \hat{0})_e(Y)$$

for $Y \in \mathfrak{k}$.

We now describe the map $\text{Ind}_{G/K,o}$ from $C^{-\infty}(\mathfrak{k})^\chi$ to $\mathcal{A}_G^{-\infty}(G/K)$. Let $\nu' \in \Lambda^n \mathfrak{r}'$ be a non-zero element. The element $\nu'$ determines an orientation $o$ and an Euclidean measure $|d\nu'|$ on $\mathfrak{g}/\mathfrak{k}$. If $dX$ is an Euclidean measure on $\mathfrak{g}$, we denote by $dY$ the Euclidean measure on $\mathfrak{k}$ such that $dX = |d\nu'|dY$.

We identify the space $\mathcal{A}(G/K)$ with $C^\infty(G, \Lambda \mathfrak{r}')^K$. An element in $C^\infty(G, \Lambda \mathfrak{r}')^K$ is a smooth function $\phi : G \to \Lambda \mathfrak{r}'$ such that $\phi(gk) = k^{-1}\phi(g)$, for all $k \in K$.

**Proposition 41** The map $\text{Ind}_{G/K,o} : C^{-\infty}(\mathfrak{k})^\chi \to \mathcal{A}_G^{-\infty}(G/K)$ is given, for $f \in C^{-\infty}(\mathfrak{k})^\chi$, by

$$\left(\int_{\mathfrak{g}} (\text{Ind}_{G/K,o} f)(X) \phi(X) dX\right)(g) = |\det_{\mathfrak{g}}(g)|\left(\int_{\mathfrak{k}} f(Y) \phi(gY) dY\right) \nu',$$

where $\phi$ is any test function on $\mathfrak{g}$ and $g \in G$.

Moreover, for any $f \in C^{-\infty}(\mathfrak{k})^\chi$, the equivariant form $\text{Ind}_{G/K,o}(f)$ is $d_{\mathfrak{g}}$-closed.
Proof: It is easy to verify that $\text{Ind}_{G/K,0}f$ defined by the Formula 15 is indeed an element of $A_{G}^{-\infty}(D)$. It obviously coincides with the map given in Definition 32 of section 4 (denoted also by $\text{Ind}_{G/K,0}$) at $g = e$. The fact that $\text{Ind}_{G/K,0}f$ is $dg$ closed follows from Proposition 33 of section 4. It is also easy to check it directly. 

Assume that $G/K$ is compact and $G$-oriented. Thus $\chi = 1$. We can integrate over $G/K$ an equivariant cohomology class and we obtain then a $G$-invariant generalised function on $g$. The next formula is just the integration over $G/K$ of the formula given in Proposition 41 for $\text{Ind}_{G/K,0}f$. However, it indicates the analogy between $\int_{G/K} \text{Ind}_{G/K,0}f$ and characters of induced representations.

Proposition 42 Assume that $G/K$ is compact and $G$-oriented. Let $f \in C^{-\infty}(k)^K$. Then

$$\int_{G/K,0} (\text{Ind}_{G/K,0}f, \Phi dX) = \int_{G/K,0} |\det g(g)| \left( \int_{k} f(Y) \Phi(gY) dY \right) dg/dk,$$

for any test function $\Phi$ on $g$ and compatible choices of the left-invariant Haar measure $dg$ on $G$, of the $G$-invariant measure $dg/dk$ on $G/K$ and of the Euclidean measure $dY$ on $k$.

We still denote by $r_k$ the map $H_{G}^{-\infty}(G/K) \rightarrow C^{\infty}(k)^K$ induced from the map $r_k$ at the cohomology level and by $\text{Ind}_{G/K,0}$ the map $C^{-\infty}(k)^\chi \rightarrow H_{G}^{-\infty}(G/K)$ induced from the map $\text{Ind}_{G/K,0}$ at the cohomology level.

As a particular case of [12] (see Theorem 31 of section 4), we have the following

Proposition 43 Assume that $K$ is compact. Then the map $r_k$ gives an isomorphism from $H_{G}^{-\infty}(G/K)$ with $C^{\infty}(k)^K$.

Further Theorem 37 of section 4 gives as an immediate corollary the following

Theorem 44 Assume that $K$ is compact. Then the map $\text{Ind}_{G/K,0}$ gives an isomorphism from $C^{-\infty}(k)^\chi$ with $H_{G}^{-\infty}(G/K)$. The map $\text{Ind}_{G/K,0}$ is of even (resp. odd) degree if $\dim G/K$ is even (resp. odd).
When $K$ is compact, let us give a formula for the map $\text{Ind}_{G/K,o}$ in terms of generalised functions.

Choose a $K$-invariant decomposition

$$g = k \oplus r$$

and let $pr_k$ (resp. $pr_r$) be the projection of $g$ on $k$ (resp. on $r$) determined by this decomposition.

With the notation of Formula 9 of section 2, we have (by Proposition 41), for $f \in C^{-\infty}(k)^\times$,

$$(\text{Ind}_{G/K,o} f)(X) = |\nu'|^{-1} \delta_r(pr_r X) f(pr_k X) \nu'.$$

where we have identified the space $\mathcal{A}(G/K)$ with $C^\infty(G, \mathcal{A}r')^K$.

Consider the case where $K$ is the trivial subgroup. Recall that we have defined the element

$$\alpha_{G,o}(X) = |\nu'|^{-1} \delta_k(X) \otimes |\det g|dg$$

in Lemma 11 of section 2. From Formula 16, we see that it is also equal to the element $\text{Ind}_{G,o} 1$. Thus we obtain from Theorem 44:

**Lemma 45** Let $G$ be a Lie group acting on itself by left translations, then

$$H_G^{-\infty}(G) \cong \mathbb{R} \alpha_{G,o}.$$

A more general result is proved in Theorem 86 in section 9.

Now we are going to generalize Proposition 43 and Theorem 44 as follows. Let $M$ be a $K$-manifold. Consider the product manifold $G \times M$. The group $K$ acts freely on the right on $G \times M$ by $(g, m)k = (gk, k^{-1}m)$. Consider the fiber space $\mathcal{M} = G \times_K M$ of orbits of the $K$-action. The group $G$ acts on the left on $\mathcal{M}$. When $M$ is a point, the space $\mathcal{M}$ is $D = G/K$. The projection $(g, m) \mapsto g$ induces a map $\mathcal{M} \to D$. Thus the space $\mathcal{M}$ is a fiber space over $D$ with fiber $M$.

If $\alpha \in \mathcal{A}(\mathcal{M}) \subset \mathcal{A}(G \times M)$, and $g \in G$, then $\alpha(g)$ is an element of $(A g' \otimes \mathcal{A}(M))_{\text{hor}K}$, where $g'$ is identified with left invariant 1-forms on $G$. 


Thus
\[ A(\mathcal{M}) = C^\infty(G, (\Lambda g' \otimes \mathcal{A}(M))_{\text{hor}K})^K \]
where $K$-invariants are taken with respect to the action of $K$ by right multiplication on $G$, left action on $M$ and adjoint action on $\Lambda g'$.

If $\alpha(X) \in A(\mathcal{M})$, then $\tilde{\alpha}(X) := \alpha(X)_e \in (\Lambda g' \otimes \mathcal{A}(M))_{\text{hor}K}$.

By $G$-invariance, the space $A_G^\pm(\mathcal{M})$ is thus identified with
\[ \text{Ind}_{G/K}^\pm(\mathcal{A}(M)) := (C^\pm(\mathfrak{k}, \mathcal{A}(M)) \otimes \Lambda g')_{\text{bas}K}. \]

Let $A = \mathcal{A}(M)$ be our $K$-differential complex, then $\text{Ind}_{G/K}^\pm(\mathcal{A}(M))$ is provided with a differential $c_g$. As before, the map $\alpha \rightarrow \tilde{\alpha}$ is an isomorphism from $A_G^{\pm\infty}(G \times K M)$ to $\text{Ind}_{G/K}^{\pm\infty}(A)$ and the following lemma ([12]) is a generalisation of Lemma 40.

**Lemma 46** For any $\alpha \in A_G^{-\infty}(\mathcal{M})$,
\[ d_{\tilde{g}} \tilde{\alpha} = c_g \tilde{\alpha}. \]

Thus the complex $(A_G^{\pm\infty}(\mathcal{M}), d_g)$ is identified with the complex $(\text{Ind}_{G/K}^{\pm\infty}(\mathcal{A}(M)), c_g)$ of section 4. The complex $(C^{\infty}(k, A)^K, d_k)$ defined in section 4 is the complex $(A_K^{\infty}(M), d_k)$ of the $K$-equivariant cohomology of $M$. As $\chi$ is trivial on the connected component of $K$, the operator $d_k$ on $C^{\infty}(k, \mathcal{A}(M))^\chi$ still satisfies $d_k^2 = 0$, as follows from Lemma 27 of section 4.

**Definition 47** Let us denote by $A_K^{-\infty}(\mathcal{M})$ the space $C^{-\infty}(k, \mathcal{A}(M))^\chi$. We define $H^{-\infty}_{K,\chi}(\mathcal{M})$ to be the cohomology of the complex $(C^{-\infty}(k, \mathcal{A}(M))^\chi, d_k)$.

Recall the definition of the maps
\[ r_k : \text{Ind}_{G/K}^{\infty}(\mathcal{A}(M)) \rightarrow A_K^{\infty}(\mathcal{M}) \]
and
\[ \text{Ind}_{G/K,0} : A_K^{-\infty}(\mathcal{M}) \rightarrow \text{Ind}_{G/K}^{-\infty}(\mathcal{A}(M)) \]
from section 4. Using the identification $\alpha \rightarrow \tilde{\alpha}$, we get maps again denoted by $r_k$ and $\text{Ind}_{G/K,0}$:
\[ r_k : A_G^{\infty}(\mathcal{M}) \rightarrow A_K^{\infty}(\mathcal{M}) \]
and

$$\text{Ind}_{G/K, o} : \mathcal{A}_{K, \infty}^-(M) \to \mathcal{A}_{G}^-(\mathcal{M}).$$

Let us describe explicitly these maps.

The fiber of $\mathcal{M} \to \mathcal{D}$ over the point $e \in \mathcal{D}$ is canonically identified with $\mathcal{M}$ and is a $K$-stable submanifold of $\mathcal{M}$. The successive restriction maps $\mathcal{A}_{G}^-(\mathcal{M}) \to \mathcal{A}_{K}^-(\mathcal{M}) \to \mathcal{A}_{K}^-(\mathcal{M})$ are well defined. The composed map $\mathcal{A}_{G}^-(\mathcal{M}) \to \mathcal{A}_{K}^-(\mathcal{M})$ coincides obviously with the map $r_k$.

The following proposition provides an explicit description of the map $\text{Ind}_{G/K, o}$ generalising Proposition 41, whose proof is identical (and hence is omitted).

**Proposition 48** For any $f \in \mathcal{A}_{K, \infty}^-(\mathcal{M})$, $\text{Ind}_{G/K, o}f \in \mathcal{A}_{G}^-(\mathcal{M})$ is given by

$$(\text{Ind}_{G/K, o}f, \Phi dX)(g) = |\det g| h_K(\int f(Y)\Phi(gY) dY) \wedge \nu'$$

where $\Phi$ is any test function on $g$ and $dX = dY |d\nu'|$.

We still denote by $r_k$ the map from $H_{G}^-(\mathcal{M})$ to $H_{K}^-(\mathcal{M})$ induced from the map $r_k$ at the cohomology level. As a generalization of Proposition 43, we have the following Theorem ([12]), (got from Theorem 31 of section 4):

**Theorem 49** Let $K$ be a compact subgroup of a Lie group $G$ and let $M$ be a $K$-manifold. Then the map $r_k$ gives an isomorphism from $H_{G}^-(\mathcal{M})$ with $H_{K}^-(\mathcal{M})$.

We still denote by $\text{Ind}_{G/K, o}$ the map from $H_{K, \infty}^-(\mathcal{M})$ to $H_{G}^-(G \times_K M)$ induced from the map $\text{Ind}_{G/K, o}$ at the cohomology level. As a corollary of Theorem 37 of section 4, we get the main result of this section which generalizes Theorem 44.

**Theorem 50** Let $K$ be a compact subgroup of a Lie group $G$ and let $M$ be a $K$-manifold. Then the map $\text{Ind}_{G/K, o}$ gives an isomorphism in cohomology from $H_{K, \infty}^-(\mathcal{M})$ to $H_{G}^-(G \times_K M)$. This map is of even (resp. odd) degree, if $\dim (G/K)$ is even (resp. odd).

When $K$ is compact, let us write the map $\text{Ind}_{G/K, o}$ in terms of generalised functions. We identify $\mathcal{A}(\mathcal{M})$ with the subspace of $K$-basic elements of $\mathcal{A}(G \times M)$. 


Choose a $K$-invariant decomposition
\[ g = k \oplus r \]
and let $pr_k$ (resp. $pr_r$) be the projection of $g$ on $k$ (resp. on $r$) determined by this decomposition. Let $\nu' \in \Lambda^n r'$ be a positive element. With the help of the decomposition $g = k \oplus r$, we consider $\nu'$ as an element of $\Lambda^n g'$. With the notation of Formula 9 of section 2, we have by Proposition 48, for any $m \in M$ and $f \in \mathcal{A}_{K, \xi}^{-\infty}(M)$,
\[ ((\text{Ind}_{G/K, o} f)(X))_{(e, m)} = |\nu'|^{-1} \delta_r(pr_r X)(h_K(f(pr_k X)))_m \wedge \nu'. \]

Assume that $K$ is compact. Choose an orientation $o$ on $r$. Then there are canonical isomorphisms
\[ r_k : H_G^\infty(G/K) \cong C^\infty(k)^K \]
and
\[ \text{Ind}_{G/K, o} : C^{-\infty}(k)^\wedge \cong H_G^{-\infty}(G/K) \]
guaranteed by Proposition 43 and Theorem 44 respectively.

The natural map $H_G^\infty(G/K) \to H_G^{-\infty}(G/K)$ thus gives rise (using the above two identifications) to a map
\[ M_o : C^\infty(k)^K \to C^{-\infty}(k)^\wedge. \]
Comparing the $Z/2Z$ degree of the maps, we see that this map is identically zero if $G/K$ is not of even dimension.

**Definition 51** Define the Chern-Weil map
\[ w_{G/K} : C^\infty(k)^K \to H_G^\infty(G/K) \]
as the inverse of $r_k$.

We denote $w_{G/K}$ simply by $w$ if $G$ and $K$ are understood. An explicit formula for $w_{G/K}$ can be given in terms of the curvature of the principal bundle $G \to G/K$, see [19].
To compute $M_o$, we need also an explicit inverse of the map $\text{Ind}_{G/K, o}$.

We will be able to do it only under some restrictive assumptions.

By definition, for $p \in C^\infty(k)^K$ the element $q = M_o p$ is the unique $q \in C^\infty(k)^K$ such that $w(p) \cong \text{Ind}_{G/K, o} q$ in the space $H_G^\infty(G/K)$. More generally, let $p \in C^\infty(k)^K$ and $f \in C^\infty(g)^G$. As $w(p) \in H_G^\infty(G/K)$, we can multiply $w(p)$ by $f$ and we obtain an element $\alpha = fw(p) \in H_G^\infty(G/K)$.

Thus there exists a unique element $S_o(\alpha) \in C^\infty(k)^K$ such that in the space $H_G^\infty(G/K)$, we have the equality $\alpha \cong \text{Ind}_{G/K, o} S_o(\alpha)$. For $f \in C^\infty(g)^G$, we denote by $q^*f$ the element $fw(1) \in H_G^{-\infty}(G/K)$ (the map $q^* : C^\infty(g)^G \to H_G^{-\infty}(G/K)$ is induced from the map $q : G/K \to \text{point}$).

Thus we get a map (still denoted by $S_o$)

$$S_o : C^\infty(g)^G \to C^\infty(k)^K$$

taking $f$ to $S_o(q^*f)$. Lemma 36 of section 4 implies that, if $f \in C^\infty(g)^G$ and $p \in C^\infty(k)^K$,

$$S_o(fw(p)) = S_o(f)p$$

(18)

If $f \in C^\infty(g)^G$ is a smooth function, then $q^*f \cong w(\overline{\tau}_K f)$ in $H_G^\infty(G/K)$, where $\overline{\tau}_K : C^\infty(g)^G \to C^\infty(k)^K$ is the restriction map. Thus

$$S_o(f) = M_o(\overline{\tau}_K f)$$

In particular, $S_o(f)$ depends only on the restriction of $f$ to $k$.

The map $S_o$ exists, under the assumption that $K$ is compact. However, we are able to compute this map explicitly only when $G$ itself is compact.

Let $G$ be a compact connected Lie group. We also assume that $\chi = 1$ so that $G/K$ is orientable (this is only for convenience). We choose the orientation on $G/K$ given by $o$.

**Definition 52** If $\Phi \in C^\infty(k)^K$, define $C_o(\Phi) \in C^\infty(g)^G$ by

$$C_o(\Phi)(X) = \int_{G/K, o} (w\Phi)(X)$$

where $w$ is the Chern-Weil homomorphism.

Define

$$F_o : C^\infty(g)^G \to C^\infty(k)^K$$
as the transpose of the map $C_o$:

$$\text{vol}(G/K, dg/dk) \int_k F_o(f)(Y)\Phi(Y)dY = \int_g f(X)C_o(\Phi)(X)dX$$

for any $\Phi \in C^\infty_{cp}(k)^K$, and where the measures $dX$ on $g$, $dY$ on $k$ and $dg/dk$ on $G/K$ are chosen in a compatible way.

It is easy to see that $C_o$ sends invariant compactly supported functions on $k$ to invariant compactly supported functions on $g$, hence the map $F_o$ is well defined.

The orientation $o$ determines a square root $\det_{g/k,o}^{1/2}(Y)$. The normalisation of this square root is as in ([11], Formula 12).

Lemma 53 If $f \in C^\infty(g)^G$, then $F_o(f)(Y) = (-2\pi)^{\dim(G/K)/2} \det_{g/k,o}^{1/2}(Y) f(Y)$, for $Y \in k$.

(If $\dim G/K$ is odd, then the function $\det_{g/k}^{1/2}$ is identically 0.)

Proof: The integral formula ([11], page 43) for equivariant cohomology classes gives the lemma.  

Proposition 54 Assume that $G$ is a compact connected Lie group and $K$ a closed subgroup of $G$ such that $G/K$ is oriented. Then for every $f \in C^{-\infty}(g)^G$ and $p \in C^\infty(k)^K$,

$$fw(p) \sim \text{Ind}_{G/K,o}(F_o(f)p)$$

in $H^{-\infty}_G(G/K)$.

Proof: Using Formula 18, it is sufficient to prove the formula of this proposition when $p = 1$.

Let $\Phi$ be a $G$-invariant test function on $g$ and let $p' \in C^\infty(k)^K$. Let us compute

$$\int_g \int_{G/K,o} \Phi(X)f(X)w(p')(X)dX = \int_g \Phi(X)f(X)C_o(p')(X)dX.$$
If \( q^* f \cong \text{Ind}_{G/K, o} u \), for \( u \in C^{-\infty}(k)^K \), then \( f w(p') \cong \text{Ind}_{G/K, o}(p' u) \) by Lemma 36 of section 4. Thus, using Proposition 42, we have:

\[
\int_G \int_{G/K,o} \Phi(X)f(X)w(p')(X)dX = \int_G \int_{G/K,o} \Phi(X)(\text{Ind}_{G/K, o}(p' u))(X)dX
\]

\[
= \int_{G/K,o} \left( \int_G \Phi(X) \text{Ind}_{G/K, o}(p' u)(X)dX \right)
\]

\[
= \int_{G/K,o} \left( \int_k \Phi(gY)p'(Y)u(Y)dY \right)dg/dk
\]

Let \( p' \) be compactly supported. Taking \( \Phi \) with sufficiently large support, we obtain

\[
\int_G f(X)C_o(p')(X)dX = \text{vol}(G/K, dg/dk) \int_k p'(Y)u(Y)dY
\]

which is what we needed to prove.

The preceding proposition determines the inverse \( S \) of the map \( \text{Ind}_{G/K, o} \) on the subspace of \( H^{-\infty}_G(G/K) \) spanned by elements of the form \( f \alpha \) where \( f \in C^{-\infty}(g)^G \) and \( \alpha \in H^\infty_G(G/K) \). We will see in section 6 that this space is equal to \( H^{-\infty}_G(G/K) \) provided that \( G \) and \( K \) have equal rank.

**Corollary 55** Assume that \( G \) is a compact connected Lie group and let \( K \) be a closed subgroup of \( G \), such that \( G/K \) is oriented. Then, for every \( p \in C^\infty(k)^K \),

\[
M_o(p)(Y) = (-2\pi)^{\dim(G/K)/2} \det_{g/k,o}^{1/2}(Y) p(Y)
\]

for \( Y \in k \).

In particular if \( f \in C^\infty(g)^G \),

\[
S_o(f)(Y) = M_o(f|k)(Y) = (-2\pi)^{\dim(G/K)/2} \det_{g/k,o}^{1/2}(Y) f(Y)
\]

for \( Y \in k \).

**Remark 56** The map \( S_o : C^{-\infty}(g)^G \rightarrow C^{-\infty}(k)^K \) is defined when \( K \) is compact (but \( G \) arbitrary, with \( G/K \) oriented). On the open set where \( \det_{g/k} Y \neq 0 \), the \( G \)-orbits are transverse to \( k \). Thus the restriction to \( k \) of an invariant generalised function on \( g \) has a meaning. In view of Corollary 55 it is natural to conjecture that \( S_o(f) \) coincide with

\[
(-2\pi)^{\dim(G/K)/2} \det_{g/k,o}^{1/2}(Y) f(Y)
\]

on this open set.
We compute even more explicitly the map $F_0$ when $G$ is a compact connected Lie group and $K = T$ is a maximal torus of $G$.

Let $W$ be the Weyl group of the pair $(\mathfrak{g}, \mathfrak{t})$. Let $C^{\pm \infty}(\mathfrak{t})^c$ be the space of $W$-anti-invariant smooth functions (resp. anti-invariant generalised functions) on $\mathfrak{t}$. Recall the definitions of the maps $C_0$ and $F_0$ from Definition 52.

**Lemma 57** The restriction of the map $C_0$ to $C^{\infty}(\mathfrak{t})^c$ gives an isomorphism between $C^{\infty}(\mathfrak{t})^c$ and $C^{\infty}(\mathfrak{g})^G$.

Furthermore the image of $F_0$ is contained in $C^{-\infty}(\mathfrak{t})^c$ and the map $F_0$ gives an isomorphism between $C^{-\infty}(\mathfrak{g})^G$ and $C^{-\infty}(\mathfrak{t})^c$.

**Proof:**

If $\phi \in C^{\infty}(\mathfrak{t})^c$, then $\phi$ is divisible by $\det_{\mathfrak{g}/\mathfrak{t},0}(Y)^{1/2}$ and the restriction of $C_0\phi$ to $\mathfrak{t}$ is equal to $|W|(-2\pi)^{n/2} \det_{\mathfrak{g}/\mathfrak{t},0}(Y)^{-1/2} \phi(Y)$, where $n := \dim(G/T)$. Thus the first assertion follows from Chevalley’s theorem for $C^{\infty}$-functions, see for example [11]. The second is a consequence of the first, as $C_0$ preserves the subspace of compactly supported functions.

Let us describe $F_0 f$, when $f$ is the $\delta$-function: Let $\delta_{\mathfrak{g}}(X)$ (resp. $\delta_{\mathfrak{t}}(Y)$) be the $\delta$ function on $\mathfrak{g}$ (resp. on $\mathfrak{t}$), given with respect to the Euclidean measure on $\mathfrak{g}$ (resp. $\mathfrak{t}$) associated to the Killing form. Let $\alpha \in \mathfrak{i}t'$ be a root. Using the identification of $\mathfrak{t}$ with $\mathfrak{i}t'$ determined by the Killing form, we can consider the differential operator $\prod_{\alpha > 0} \partial_\alpha$ on $\mathfrak{t}$, where the product is taken over all the positive roots of $\mathfrak{g}$ for an order compatible with the orientation $o$ as defined in [11]. Then $F_0 \delta_{\mathfrak{g}} = \prod_{\alpha > 0} \partial_\alpha \delta_{\mathfrak{t}}$.

### 6 Kunneth formula and applications

Let $K$ be a Lie group. Let $D$ and $M$ be $K$-manifolds. Consider the complex $\mathcal{A}_K(D)$ of $K$-equivariant forms on $D$ with polynomial coefficients and its cohomology $H_K(D) = Z_K(D)/B_K(D)$ defined in section 2, Definition 1. Recall that these spaces are $\mathbb{Z}_4$-graded. The evaluation map $E$ at zero taking $\alpha \rightarrow \alpha(0)$ gives a map from $H_K(D)$ to the usual De Rham cohomology $H(D)$ of $D$. Consider the map $m$ from $\mathcal{A}_K(D) \otimes \mathcal{A}_K(M)$ to $\mathcal{A}_K(D \times M)$ given by $m(\alpha \otimes \beta)(X) = \alpha(X) \wedge \beta(X)$. It induces a map (still denoted by $m$) from $H_K(D) \otimes H_K(M)$ to $H_K(D \times M)$.
Similarly, we can also consider the map $m^{-\infty}$ from $A_K(D) \otimes \mathcal{A}_K^{-\infty}(M)$ to $\mathcal{A}_K^{-\infty}(D \times M)$ given by $m^{-\infty}(\alpha \otimes \beta)(X) = \alpha(X) \wedge \beta(X)$: this is well defined as we can multiply a generalised function by a polynomial function. It induces a map (still denoted by $m^{-\infty}$) from $H_K(D) \otimes H_K^{-\infty}(M)$ to $H_K^{-\infty}(D \times M)$.

**Proposition 58** Let $K$ be a compact connected Lie group. Let $D$ be a compact $K$-manifold. Assume that the evaluation map $E : H_K(D) \to H(D)$ is surjective. Then, for any $K$-manifold $M$, the multiplication map

$$m^{-\infty} : H_K(D) \otimes H_K^{-\infty}(M) \to H_K^{-\infty}(D \times M)$$

is surjective.

We remark that the surjectivity of the multiplication map

$$m : H_K(D) \otimes H_K(M) \to H_K(D \times M)$$

is known (under the hypothesis of the proposition); and follows from the same argument as that for $m^{-\infty}$ given below.

**Proof:**

Let $\mathcal{A}^{(p)}(D \times M) = \Gamma(D \times M, \Lambda^p T^* D \otimes \Lambda T' M)$, where $\Gamma$ denotes the space of smooth sections and $T'M$ denotes the cotangent bundle of $M$.

We write $A(D \times M) = \bigoplus_{p=0}^{\dim D} \mathcal{A}^{(p)}(D \times M)$. The total exterior differential $\partial_{D \times M}$ on $\mathcal{A}(D \times M)$ breaks up into the sum $\partial_D + \partial_M$ of partial exterior differential $\partial_D$ along $D$

$$\partial_D : \mathcal{A}^{(p)}(D \times M) \to \mathcal{A}^{(p+1)}(D \times M)$$

and partial exterior differential $\partial_M$ along $M$

$$\partial_M : \mathcal{A}^{(p)}(D \times M) \to \mathcal{A}^{(p)}(D \times M).$$

Let us consider the complex $\mathcal{A}_K^{-\infty}(D \times M) = C^{-\infty}(k, \mathcal{A}(D \times M))^K$. We write

$$B^p = C^{-\infty}(k, \mathcal{A}^{(p)}(D \times M))^K.$$

The operator $d_k$ can be written as a sum of homogeneous operators $d_k = d_1 + r_0 + r_{-1}$, with

$$d_1 : B^p \to B^{p+1}, r_0 : B^p \to B^p, r_{-1} : B^p \to B^{p-1}.$$
We have $d_1 = d_D$, $r_0 = d_M - \sum_i x_i(E^i_M)$, and $r_{-1} = -\sum_i x_i(E^i_D)$, where $E^i$ is a basis of $k$. We write $d = d_D$.

Let us choose a $K$-invariant metric on $D$ and consider $D$ as a Riemannian manifold. This endows the space $A(D)$ with an inner product. Let $d^* : A^p(D) \to A^{p-1}(D)$ be the adjoint operator to $d = d_D$. Let $\mathcal{H}(D) = \text{Ker}(d) \cap \text{Ker}(d^*)$ be the space of harmonic forms on $D$. It is a $K$-invariant finite dimensional space of $d$-closed forms on $D$. The map $\mathcal{H}(D) \to H(D)$ is an isomorphism. As $K$ is connected, $K$ acts trivially on $H(D)$. Thus every element of $\mathcal{H}(D)$ is $K$-invariant.

**Lemma 59** For any $\alpha_0 \in \mathcal{H}(D)$, there exists $\alpha \in Z^K(D)$ such that

$$\alpha_0 - \alpha(X) \in \sum_{j \leq p} A^j(D), \quad \text{for every } X \in k.$$

**Proof:** Our hypothesis implies that if $\alpha_0 \in A^p(D)$ is $d$-closed, we can find $\gamma \in A^{p-1}(D)$ and $\lambda \in Z^K(D)$ such that $\alpha_0 - d\gamma = \lambda(0)$. If $\alpha_0$ is $K$-invariant, we may assume (eventually after averaging this equation by the action of $K$) that $\gamma$ is $K$-invariant. Take $\alpha = \lambda + d_k\gamma$, then $\alpha \in Z^K(D)$ and $\alpha(0) = \alpha_0$. The complex $A_K(D)$ is $\mathbb{Z}_4$-graded by its total equivariant degree. We may thus assume that $\alpha$ is of total degree $p$. Thus $\alpha(X) - \alpha(0) \in A^{p-2}(D) \oplus A^{p-1}(D) \oplus \cdots$. This proves the lemma. 

We continue with the proof of Proposition 58.

Let $P$ be the orthogonal projection of $A(D)$ onto $\mathcal{H}(D)$. We have $Pd = dP = 0$. Let $G : A^p(D) \to A^{p-1}(D)$ be the Green kernel. It satisfies $Gd + dG = I - P$. We can extend the operator $P$, by the formula $P(\alpha)(X) = P(\alpha(X))$, to an operator still denoted by $P$,

$$P : B^p \to \mathcal{H}(D) \otimes A^{-\infty}_k(M).$$

Similarly we can extend pointwise the operator $G$

$$G : B^p \to B^{p-1}.$$

Let $r = r_0 + r_{-1}$ and let $N = Gr + rG$. The operator $N$ decreases strictly the exterior degree in $D$. Let $\nu \in B = \sum B^j$. The equation $Gd + dG = I - P$ gives the perturbed equation

$$Gd_k\nu + d_kG\nu = \nu - (P - N)\nu.$$
Assume $d_k \nu = 0$. Let us write $\nu = \sum_{j \leq p} \nu_j$, with $\nu_j \in B^j$. We will prove by induction on $p$ that $\nu$ has a representative in $Z_K(D) \otimes Z_K^\infty(M)$. The equation above implies that $\nu \cong \nu' := (P - N)\nu$. We have $\nu'_p = P\nu_p \in \mathcal{H}^p(D) \otimes A_K^\infty(M)$. Let us write the equation $d_k \nu' = 0$ component by component. We obtain, in particular, the equation $r_0 \nu'_p + dv'_{p-1} = 0$. Applying $P$, we get $Pr_0 \nu'_p + Pdv'_{p-1} = 0$. As $Pd = 0$, this implies that $Pr_0 \nu'_p = r_0 P\nu'_p = r_0 \nu'_p = 0$. Thus $\nu'_p \in \mathcal{H}^p(D) \otimes Z_K^\infty(M)$. The preceding lemma allows us to find an element $\xi \in Z_K(D) \otimes Z_K^\infty(M)$ such that $\nu' - \xi \in \sum_{j < p} B^j$. We conclude Proposition 58 by the induction hypothesis.

**Theorem 60** Let $K$ be a compact connected Lie group. Let $D$ be a compact $K$-manifold. Assume that $H_K(D)$ is a free module over $H_K(\text{point})$. Then, for any $K$-manifold $M$, the multiplication map $m^{-\infty}$ induces an isomorphism

\[ \hat{m}^{-\infty} : H_K(D) \otimes H_K(\text{point}) H_K^{-\infty}(M) \cong H_K^{-\infty}(D \times M). \]

(As is well known, from the Kunneth spectral sequence (see [14]; Proposition 6.1, page 50), the map $m$ induces an isomorphism

\[ \hat{m} : H_K(D) \otimes H_K(\text{point}) H_K(M) \cong H_K(D \times M) \]

under the hypothesis of the theorem.)

**Proof:**

Since $H_K(D)$ is a free module over $H_K(\text{point})$, the Eilenberg-Moore spectral sequence (see [14], chapter 3, section 1) degenerates at the $E^2$-term, giving rise to the canonical isomorphism $H(\text{point}) \otimes H_K(\text{point}) H_K(D) \rightarrow H(D)$, where $H(\text{point}) = R$ is considered as an $H_K(\text{point}) = S(K)^K$ module, via the evaluation at 0. In particular the evaluation map $E : H_K(D) \rightarrow H(D)$ is surjective. So, in view of Proposition 58, to prove the theorem, it suffices to show that $\hat{m}^{-\infty}$ is injective.

Let $P^p(X)$ be a homogeneous basis for the $H_K(\text{point})$-module $H_K(D)$. Then $P^p(0)$ is a basis of $H(D)$. In particular the set $P^p$ is finite. For $n \in \mathbb{Z}_+$, let $P^n$ be the elements of the basis $P^p$ such that $\deg(P^n) = n$. The elements $P^n(0)$ form a basis of $H^n(D)$. We can write an element
By Poincaré duality, there exist the elements \( P^b \in H^{\dim D-n}(D) \) such that \( \int_D P^a(0) \wedge Q_b = \delta^a_b \). Let us choose elements \( P_b \) of \( H_K(D) \) of total degree \((\dim D - n)\) such that \( P_b(0) = Q_b \). Then we still have \( \int_D P^a(X)P_b(X) = \delta^a_b \) as only the top exterior degree term can contribute to the integral, and this top term is precisely \( P^a(0) \wedge P_b(0) \). Furthermore if the total degree of \( P^k \) is strictly less than \( n \), then \( \int_D P^k(X)P_b(X) = 0 \).

Let \( \nu \in H_K^{-\infty}(D \times M) \). For any fixed \( a \), consider the map \( \nu \to \int_D P_a(X)\nu(X) \). This is a well defined map from \( H_K^{-\infty}(D \times M) \to H_K^{-\infty}(M) \). We have

\[
\int_D P_a(X)(\hat{m}^{-\infty}a)(X) = \nu_a(X)
\]

for all \( a \) such that \( \deg P^a = n \). Thus if \( \hat{m}^{-\infty}(a) = 0 \), \( \nu_a(X) = 0 \), for all \( a \) such that \( \deg P^a = n \), which is a contradiction to the choice of \( n = n(\alpha) \). Hence \( \alpha = 0 \). □

Applying Theorem 60 to the case where \( M \) is a point, we obtain the following

**Corollary 61** Let \( K \) be a compact connected Lie group. Let \( D \) be a compact \( K \)-manifold. Assume that \( H_K(D) \) is free over \( H_K(\text{point}) \). Then

\[
H_K^{-\infty}(D) \cong C^{-\infty}(k)^K \otimes_{S(k)^K} H_K(D).
\]

The following well-known lemma gives some examples of compact \( K \)-manifolds \( D \) such that \( H_K(D) \) is free over \( H_K(\text{point}) = S(k')^K \).

**Lemma 62** Let \( K \) be a compact connected Lie group and let \( L \) be a closed subgroup of \( K \). Then \( H_K(K/L) \) is free over \( H_K(\text{point}) \) if and only if \( K \) and \( L \) have the same rank.

**Proof:** Let \( D = K/L \). Recall that the equivariant cohomology \( H_K(D) \) is isomorphic to \( H_L(\text{point}) = S(V)^L \) by the map \( r_1 \), induced from the inclusion of the base point \( e \in K/L \). The restriction of polynomial functions on \( k \) to polynomial functions on \( l \) gives a homomorphism from \( S(k')^K \) to \( S(V)^L \). If \( H_K(K/L) \) is free over \( H_K(\text{point}) \), then in particular the homomorphism \( S(k')^K \to S(V)^L \) is injective. This implies that \( K \) and \( L \) have the same
rank. Conversely, suppose now that $L$ is a closed subgroup of $K$ with equal rank. Let $T$ be a maximal torus of $L$ (and hence of $K$). Let $W = N_K(T)/T, W_L = N_L(T)/T$ be the respective Weyl groups of $(K, T)$ and $(L, T)$. By Chevalley's theorem, $S(Y)^L$ is isomorphic to $S(t')^{W_L}$, while the ground ring $S(k')^K$ is isomorphic to $S(t')^W$. Further $S(t')^{W_L}$ is a free module over $H_K(\text{point}) \cong S(t')^W$.

The case where $D = K/T$ is a particularly important example. Considering this case we will see that we obtain as a consequence of Theorem 60 the following:

**Proposition 63** Let $K$ be a compact connected Lie group with maximal torus $T$ and Weyl group $W$. Let $C^{-\infty}(t)^c$ be the space of $W$-anti-invariant generalised functions on $t$. Then the map from

$$S(t') \otimes_{S(V)^W} C^{-\infty}(t)^c \to C^{-\infty}(t)$$

given by $P \otimes f \to P f$, for $P \in S(t')$, $f \in C^{-\infty}(t)^c$, is an isomorphism.

**Proof:** Let us consider $D = K/T$ and $M = \text{point}$ in Theorem 60. We thus obtain an isomorphism

$$\hat{m}^{-\infty} : H_K(K/T) \otimes_{H_K(\text{point})} C^{-\infty}(k)^K \to H_K^{-\infty}(K/T).$$

We have $H_K(\text{point}) \cong S(t')^W$. Recall the isomorphisms

$$w_{K/T} : S(t') \to H_K(K/T)$$

where $w_{K/T}$ is the Chern-Weil homomorphism (see Definition 51 of section 5),

the isomorphism $F_o$ proved in Lemma 57

$$F_o : C^{-\infty}(k)^K \to C^{-\infty}(t)^c,$$

and the isomorphism (cf. Theorem 44)

$$\text{Ind}_{K/T,o} : C^{-\infty}(t) \to H_K^{-\infty}(K/T).$$

Using the above identifications, the isomorphism (19) gives an isomorphism, again denoted by

$$\hat{m}^{-\infty} : S(t') \otimes_{S(V)^W} C^{-\infty}(t)^c \to C^{-\infty}(t).$$
By Proposition 54 of section 5, we have for $a \in S(t')$, $f \in C^{-\infty}(k)^K$

$$w_{K/T}(a)f = \text{Ind}_{G/K_0}(aF_0 f).$$

Hence we obtain our proposition. ■

7 Equivariant cohomology and subgroups

Let $K$ be a compact connected Lie group and let $M$ be a $K$-manifold. If $L$ is a compact subgroup of $K$ of equal rank, then $H_K(K/L)$ is free over $H_K(\text{point})$. In this section, we will use Theorem 60 to compare $H_K^{-\infty}(M)$ and $H_L^{-\infty}(M)$ (and also $H_K(M)$ and $H_L(M)$).

Let $D = K/L$. The space $H_K(D)$ is isomorphic to $H_L(\text{point}) = S(V)^L$ by the Chern-Weil isomorphism:

$$(24) \quad w_D : H_L(\text{point}) = S(V)^L \rightarrow H_K(D).$$

Consider the natural restriction map $r_L : H_K(M) \rightarrow H_L(M)$ of a $K$-equivariant form on $M$ to a $L$-equivariant form on $M$.

Let us recall the following proposition (see [14]; page 38). We include a proof for completeness.

**Proposition 64** Let $K$ be a compact connected Lie group and let $M$ be a $K$-manifold. Let $L$ be a compact subgroup of $K$ of equal rank. Then the map

$$I_M : H_L(\text{point}) \otimes_{H_K(\text{point})} H_K(M) \rightarrow H_L(M)$$

given by

$$P \otimes \omega \rightarrow P(r_L \omega)$$

for $P \in H_L(\text{point})$ and $\omega \in H_K(M)$ is an isomorphism.

In particular, $H_L(M)$ is generated by $r_L H_K(M)$ over the ground ring $H_L(\text{point}) = S(V)^L$.

**Proof:** Consider the manifold $\mathcal{M} = K \times_L M$. By Theorem (49), the restriction map induced from the inclusion $i_{L,M}$ of the fiber $\mathcal{M}$ at $e \in D$ in the fibered space $\pi : \mathcal{M} \rightarrow D$ induces an isomorphism

$$(25) \quad i_{L,M}^* : H_K(\mathcal{M}) \rightarrow H_L(M).$$
Consider the map $\mu : [k, m] \to km$ from $\mathbb{K} \times \mathbb{M}$ to $\mathbb{M}$. The map $t = (\pi \times \mu)$ given by $t([k, m]) = (kL, k \cdot m)$ is a $\mathbb{K}$-equivariant isomorphism from $\mathcal{M}$ to $D \times M$, where $\mathbb{K}$ acts on $D \times M$ as the diagonal action. Thus we have an isomorphism

$$t^* : H_K(D \times M) \to H_K(\mathcal{M}).$$

As follows from Theorem 60 and Lemma 62, the multiplication map $\hat{m}$ is an isomorphism:

$$\hat{m} : H_K(D) \otimes_{H_K(\text{point})} H_K(M) \to H_K(D \times M).$$

Using the isomorphisms (24) - (27), we obtain an isomorphism

$$I : H_L(\text{point}) \otimes_{H_K(\text{point})} H_K(M) \to H_L(M).$$

It remains to show that $I$ is equal to $I_M$. For this, we have to compute for $P \in S(\mathbb{V})^L$ and $\omega \in H_K(M)$

$$i_{L,M}^*(\pi^*(w_DP) \wedge \mu^*\omega) = i_{L,M}^*\pi^*(w_D\omega) \wedge i_{L,M}^*\mu^*\omega = P \wedge r_L\omega.$$  

This proves the proposition.

In particular, consider $M = \mathbb{K}/U$ for a closed subgroup $U$ of $\mathbb{K}$. Then the equivariant cohomology space $H_K(M)$ is equal to $H_L(\text{point})$ via the Chern-Weil homomorphism $w_M$ and we obtain:

**Proposition 65** Let $\mathbb{K}$ be a compact connected Lie group. Let $L$ be a closed subgroup of $\mathbb{K}$ of same rank. Let $U$ be a closed subgroup of $\mathbb{K}$. Then the map $P \otimes A \to P(r_L(w_M A))$ for $P \in S(\mathbb{V})^L$, $A \in S(\mathbb{u}')^U$ determines an isomorphism:

$$S(\mathbb{V})^L \otimes_{S(k')^K} S(\mathbb{u}')^U \cong H_L(\mathbb{K}/U)$$

Remark that by our hypothesis, $H_K(\mathbb{K}/L) = S(\mathbb{V})^L$ is free over $S(k')^K$ with rank equal to $\dim H(\mathbb{K}/L)$. Thus as a vector space

$$H_L(\mathbb{K}/U) \cong H(\mathbb{K}/L) \otimes_{\mathbb{R}} S(\mathbb{u}')^U.$$
Corollary 66 If $U$ also has the same rank as $K$, then $H_L(K/U)$ is a free (finitely generated) $H_L(\text{point})$-module.

Proof: As $S(u')^U$ is free over $S(k')^K$, the $S(Y)^L$-module $S(Y)^L \otimes_{S(k')} S(u')^U$ is free with rank equal to $\dim H(K/U)$. 

We turn now to the determination of the equivariant cohomology $H_L^-(M)$ in the case where $M$ is a $K$-manifold. We denote by $\chi_{K/L}$ the character of $L$ with values in $\pm 1$ given by $\gamma \mapsto \det_{k/L} \gamma$. Recall the definition of $H_{L,\chi_{K/L}}^-(M)$ from Definition 47 of section 5.

Theorem 67 Let $K$ be a compact connected Lie group and let $M$ be a $K$-manifold. Let $L$ be a compact subgroup of $K$ of equal rank. Choose an orientation $o$ on $k/l$. Then, there is a natural isomorphism of $H_L(\text{point})$-modules:

$$I_{M,o}^*: H_L(\text{point}) \otimes_{H_K(\text{point})} H_K^-(M) \rightarrow H_{L,\chi_{K/L}}^-(M).$$

In particular, for $M = K/U$, we obtain

$$H_{L,\chi_{K/L}}^-(M) \cong S(Y)^L \otimes_{S(k')} C^{-\infty}(u)^{\chi_{K/L}}.$$

Proof: The proof is almost the same as the proof of Proposition 64. Using the same notation, we use the chain of isomorphisms:

$$Ind_{K/L,o} : H_{L,\chi_{K/L}}^-(M) \rightarrow H_K^-(M).$$

The isomorphism $t$ of $M$ with $D \times M$ induces an isomorphism

$$t^* : H_K^-(D \times M) \rightarrow H_K^-(M).$$

As follows from Theorem 60 and Lemma 62, the multiplication map $\hat{m}^{-\infty}$ is an isomorphism:

$$\hat{m}^{-\infty} : H_K(D) \otimes_{H_K(\text{point})} H_K^-(M) \rightarrow H_K^-(D \times M).$$

Using the isomorphisms (24), (30)-(32), we obtain an isomorphism

$$I_{M,o}^* : H_L(\text{point}) \otimes_{H_K(\text{point})} H_K^-(M) \rightarrow H_{L,\chi_{K/L}}^-(M).$$
The description of $I_{M,0}^\infty$ is as follows: for $P \in H_L(point) = S(V)^L$ and $\omega \in H_K^\infty(M)$, the element $\alpha = I_{M,0}^\infty(P \otimes \omega)$ is the unique element in $H_L^\infty(M)$ such that $\text{Ind}_{K/L,0}\alpha = \pi^*(w_D P) \mu^* \omega$. It follows from Lemma 36 of section 4 that $\pi^*(w_D Q) \text{Ind}_{K/L,0}\alpha = \text{Ind}_{K/L,0} Q \alpha$, for any $Q \in H_L(point)$. Hence $I_{M,0}^\infty$ is an isomorphism of $H_L(point)$-modules. The last statement in Theorem 67 follows easily from Theorem 44.

The isomorphism $I_{M}^\infty$ is not so easy to determine explicitly as the isomorphism $I_M$. We determine it as much as we can.

Consider $H_K^\infty(M)$ as a $H_K^\infty(M)$-module and $H_L^\infty(M)$ as a $H_L^\infty(M)$-module.

**Lemma 68** If $\alpha \in H_K(M)$ and $\beta \in H_K^\infty(M)$, then

$$I_{M,0}^\infty(1 \otimes \alpha \beta) = I_M(1 \otimes \alpha) I_{M,0}^\infty(1 \otimes \beta).$$

**Proof:** If $I_{M}^\infty(1 \otimes \beta) = \omega$, we have $\mu^* \beta = \text{Ind}_{K/L,0}\omega$. Then $\mu^* \alpha \otimes \mu^* \beta = \text{Ind}_{K/L,0}(r_L \alpha) \omega$ by Lemma 36 of section 4. This proves the lemma.

We assume that $K/L$ is orientable so that $\chi_{K/L} = 1$. Recall the map $F_{\alpha} : C^\infty(k)^K \to C^\infty(l)^L$ given in Definition 52 of section 5. Then

**Lemma 69** Let $f \in C^\infty(k)^K$, $\alpha \in H_K(M)$, then $I_{M,0}^\infty(1 \otimes f \alpha) = (F_{\alpha} f) r_L \alpha$.

**Proof:** This follows immediately from Proposition 54 of section 5.

Thus we know $I_{M}^\infty$ on the subspace $S(V)^L \otimes_{H_K(point)} C^\infty(k)^K H_K(M)$ of $S(V)^L \otimes_{H_K(point)} H_K^\infty(M)$.

If $M$ is compact and $H_K(M)$ is a free module over $H_K(point)$, then $H_K^\infty(M)$ is equal to $C^\infty(k)^K \otimes_{H_K(point)} H_K(M)$ (cf. Corollary 61 of section 6). Thus in this case, the isomorphism $I_{M,0}^\infty$ is entirely determined by the knowledge of $F$.

### 8 Reduction to the maximal torus

Let $K$ be a compact connected Lie group and let $T$ be its maximal torus. Let $W = N(T)/T$ be the Weyl group.

Let $M$ be a $K$-manifold. It is well known (cf. [14]: chapter 3, section 1, Proposition 1) that the natural restriction map $\mathcal{A}_K(M) \to \mathcal{A}_T(M)$ induces
an isomorphism between $H_K(M)$ and $H_T(M)^W$. In this section, we prove a similar statement for the generalised $K$-cohomology $H^{-\infty}_K(M)$.

We first need to define a map from $H^{-\infty}_T(M)$ to $H^{-\infty}_K(M)$: Choose a non-
zero $K$-invariant form $\nu'$ on $K/T$ of maximal exterior degree. In particular, $\nu'$ determines an Euclidean measure $|d\nu'|$ on $k/t$ and an orientation $o$. For $f \in C^{-\infty}(t, \mathcal{A}(M))$, define $A_o(f) \in C^{-\infty}(k, \mathcal{A}(K \times M))$ by: If $\Phi$ is a test function on $k$

\[(A_o(f), \Phi dX)_{(k,m)} := \nu' \wedge (k \cdot (\int_t f(Y) \Phi(k \cdot Y) dY))_m\]

where $dY$ is the Euclidean measure on $t$ which is quotient of $dX$ by $|d\nu'|$ on $k/t$. In particular $A_o$ depends only on the orientation $o$ on $K/T$ associated to $\nu'$.

**Lemma 70** Let $D = K/T$. Consider $D \times M$ as a $K$-manifold under the diagonal action. Then the map $A_o$ defines a cochain map from $(\mathcal{A}^{-\infty}_T(M), d_t)$ to $(\mathcal{A}^{-\infty}_K(D \times M), d_k)$.

**Proof:** It is not difficult to check that if $f$ is $T$-invariant, then $A_o(f)$ is in $\mathcal{A}^{-\infty}_K(D \times M)$. Now, as $\nu'$ is of maximal degree, we see that $d_{D \times M}(A_o(f)) = A_o(d_Mf)$. It is also easy to prove that $i_k A_o(f) = A_o(i_t f)$.

Consider the projection $\pi : D \times M \to M$ with fiber $D$. We denote the map $\pi_* : \mathcal{A}^{-\infty}_K(D \times M) \to \mathcal{A}^{-\infty}_K(M)$ by $\int_D$ (cf. Formula 7 of section 2).

Define $B(f) \in \mathcal{A}^{-\infty}_K(M)$ by

\[B(f) = \int_{D,o} A_o(f),\]

where the orientation on $D$ is the orientation $o$. In particular, we see that $B$ does not depend on $o$. If we denote by $dk/dt$ the positive density on $K/T$ associated to $\nu'$, we have for $f \in \mathcal{A}^{-\infty}_T(M)$ and $\Phi$ a test function on $k$,

\[\int_k B(f)(X) \Phi(X) dX = \int_{K/T} k \cdot (\int_t f(Y) \Phi(k \cdot Y) dY) dk/dt.\]

The map $B$ is a cochain map from $(\mathcal{A}^{-\infty}_T(M), d_t)$ to $(\mathcal{A}^{-\infty}_K(M), d_k)$. The Weyl group $W$ canonically acts on $\mathcal{A}^{-\infty}_T(M) = C^{-\infty}(t, \mathcal{A}(M)^T)$. 
Theorem 71 Let $K$ be a compact connected Lie group and let $T$ be its maximal torus. Let $W$ be the Weyl group of $K$.

The restriction of the cochain map $B$ to $A^{-\infty}_T(M)^W$ induces an isomorphism in cohomology

$$b : H^{-\infty}_T(M)^W \to H^{-\infty}_K(M).$$

Proof:

Again, this theorem is an easy consequence of Theorem 60 of section 6. As in the proof of Theorem 67, we consider $M = K \times_T M$ and we use the isomorphism (cf. Theorem 50 of section 5):

$$\text{Ind}_{K/T,\circ} : H^{-\infty}_T(M) \to H^{-\infty}_K(M).$$

Composing this with the isomorphism (see Formula 31 of section 7):

$$t^* : H^{-\infty}_K(D \times M) \to H^{-\infty}_K(M),$$

we obtain an isomorphism

$$\text{Ind}_{K/T,\circ} \circ \text{Ind}_{K/T,\circ} : H^{-\infty}_T(M) \to H^{-\infty}_K(D \times M).$$

It is not difficult to see that $(t^*)^{-1} \text{Ind}_{K/T,\circ} f = A_\circ f$, for $f \in H^{-\infty}_T(M)$.

Let $\epsilon$ be the character of $W$ given by $\epsilon(w) = \det_t(w)$. Let $r$ be the action of $W$ on $K/T \times M$ given by $w \cdot (kT, m) = (kw^{-1}T, m)$. This action commutes with the diagonal action of $K$ and hence induces an action still denoted by $r$ on $H^{-\infty}_K(D \times M)$. Under the isomorphism (36), the natural action of $W$ on $H^{-\infty}_K(M)$ becomes the action $r \otimes \epsilon$.

The $K$-equivariant cohomology $H_K(D) \cong S(t')$ of $D$ is free over $H_K(\text{point}) \cong S(t')^W$. Hence, as follows from Theorem 60, the multiplication map $\hat{m}^{-\infty}$ is an isomorphism:

$$\hat{m}^{-\infty} : H_K(D) \otimes H_K(\text{point}) H^{-\infty}_K(M) \to H^{-\infty}_K(D \times M).$$

For the action of the group $W$ on $H_K(D)$, induced by the action of $W$ by right translation on $D = K/T$, the subspace $H_K(D)^W$ of $H_K(D)$ is a free $H_K(\text{point})$-module of rank one, in fact $H_K(D)^W = H_K(\text{point})w_D(\chi)$,
where \( w_D(\chi) \in H_K(D) \) is the image under the Chern-Weil homomorphism \( w_D \) of the \( W \)-anti-invariant polynomial function

\[
\chi(Y) = (2\pi)^{-\dim(D)/2} |W|^{-1} \det_{k/\mathfrak{t},\alpha}^{1/2}(Y), \quad Y \in \mathfrak{t}.
\]

We have \( \int_D w_D(\chi)(X) = 1 \), for all \( X \in k \). The space \( H_K(D) \) is isomorphic to \( H_K(\text{point}) \) under \( \alpha \to \int_D \alpha(X) \). Thus, by (37), the map \( \alpha \to \int_D \alpha \) induces an isomorphism (depending on the choice of an orientation on \( D \))

\[
H_K^{-\infty}(D \times M)^* \cong H_K^{-\infty}(M).
\]

and hence, by (36),

\[
H_T^{-\infty}(M)^W \cong H_K^{-\infty}(M).
\]

The above isomorphism is given by the restriction to \( H_T^{-\infty}(M)^W \) of the map \( B = \int_{D,\alpha} A_* \). Thus we obtain the formula of the theorem. \( \blacksquare \)

In particular, when \( M = \text{point} \), the isomorphism given by Theorem 71 is the well known isomorphism \( b : C^{-\infty}(\mathfrak{t})^W \to C^{-\infty}(k)^K \) given by:

\[
(b(f), \Phi dX) = \text{vol}(K/T, dk/dt)(\int f(Y)\Phi(Y)dY)
\]

if \( \Phi \) is a \( K \)-invariant test function on \( k \).

## 9 The case of a free action

Let \( G \) be a Lie group. Let \( P \) be a right \( G \)-manifold (i.e. \( G \) acts on \( P \) from the right). (Of course any left \( G \)-manifold can be thought of as a right \( G \)-manifold under \( x \cdot g := g^{-1} \cdot x \), for \( x \in P \) and \( g \in G \).)

**Definition 72** Let \( P \) be a right \( G \)-manifold. We will say that the action of \( G \) on \( P \) is principal (or that \( G \) acts principally on \( P \)) if the orbit space \( P/G \) is a smooth manifold such that \( P \to P/G \) is a smooth principal \( G \)-bundle.

If \( G \) is compact, then a right action is principal if and only if the action is free.

Let \( P \) be a right \( G \)-manifold. If \( G \) acts principally on \( P \), it is known [10] that the \( G \)-equivariant de Rham cohomology of \( P \) is isomorphic to the de Rham cohomology of the quotient space \( P/G \). In this section, we
prove similarly that the space $H_G^{-\infty}(P)$ is isomorphic to $H(P/G)$. We also consider the following more general situation:

(S): Let $G$ be a Lie group and let $N$ be a closed normal subgroup of $G$. Let $P$ be a right $G$-manifold. Assume that the subgroup $N$ acts principally on $P$.

We ask the question: Under what hypothesis $H_G^{-\infty}(P)$ is isomorphic to $H_{G/N}(P/N)$. In this section, we prove this affirmatively when $G$ is compact connected.

An important example of this situation (S) is the following:

Example 73 Let $U$ and $K$ be two Lie groups and let $G := U \times K$ be the direct product. Let $L$ be a $U \times K$-manifold. For convenience, we assume that $U$ acts on the left and $K$ on the right. Assume the right action of $K$ on $L$ is principal. Let $M$ be a $K$-manifold and let $P := L \times M$. Define the action of an element $g = (u, k) \in G = U \times K$ on $P$ by $(x, m) \cdot (u, k) = (u^{-1}xk, k^{-1}m)$, for $x \in L$, $m \in M$. Then the action of the (normal) subgroup $K$ of $G$ is principal on $P$. The quotient manifold $(L \times M)/K$ is the left $U$-space $\mathcal{M} = L \times_K M$ fibered over $L/K$ with fiber $M$.

Consider the quotient map $q : P \to P/N$ under the situation (S). Recall Definition 4 of a $G$-equivariant fibration with $G$-oriented fibers. Later in the section, we will need to impose the following conditions (74) and (75)

Condition 74 There exists a $G$-orientation $\omega$ for the fibers of $q$.

This condition (74) is satisfied, for example, when $\det_n g > 0$, for all $g \in G$. In particular this is satisfied if $G$ is connected.

Condition 75 There exists a $G$-invariant connection $\omega$ for the principal $N$-bundle $q : P \to P/N$.

This condition (75) is always satisfied when $G$ is compact.

It is proved in [12] that the canonical map $q^* : H_{G/N}^\infty(P/N) \to H_G^\infty(P)$ is an isomorphism, when the condition (75) is satisfied. Furthermore, an explicit formula for the inverse of $q^*$ is given in terms of the equivariant curvature of $\omega$. The reader should however be warned that the natural map
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$q^*$ is sometimes equal to zero when applied to the equivariant cohomology with generalised coefficients.

Whenever the conditions (74) and (75) are satisfied, we will construct a natural map (cf. Proposition 79)

$$m_0 : H^\infty_{G/N}(P/N) \to H^\infty_G(P)$$

and will show that $m_0$ is an isomorphism, if either $N = G$ or $G$ is compact.

We begin by constructing a natural element $\gamma_0 \in H^\infty_G(P)$ (assuming the validity of conditions (74) and (75)):

Let $B := P/N$ be the space of $N$-orbits. Consider the projection $q : P \to B$. The vertical tangent bundle $V$ is a $G$-equivariant real vector bundle over $P$. By assumption, the bundle $V$ is a $G$-orientable vector bundle. As the group $N$ acts principally, the bundle $V$ is a trivial bundle over $P$ canonically isomorphic to $P \times \mathfrak{n}$. The isomorphism is obtained by sending $(x, X) \in P \times \mathfrak{n}$ to the vertical tangent vector $(X)_x$. The action of an element $g \in G$ on $V = P \times \mathfrak{n}$ is given by $(x, Y) \cdot g = (xg, g^{-1} \cdot Y)$ for $x \in P, Y \in \mathfrak{n}$. (Observe that if $\det g \cdot \mathfrak{n}(g) > 0$, for all $g \in G$, then any choice of orientation of $\mathfrak{n}$ gives rise to a $G$-orientation of the vector bundle $V$, i.e., in this case the condition (74) is satisfied.)

Let us choose a $G$-invariant connection form $\omega \in (\mathcal{A}^1(P) \otimes \mathfrak{n})^G$. Using $\omega$, we obtain a $G$-invariant decomposition

$$TP = V \oplus H$$

of the tangent bundle as sum of vertical and horizontal subbundles.

Similarly, using $\omega$, we have an isomorphism

$$(38) \quad U : P \times \mathfrak{g} \to P \times \mathfrak{n} \times \mathfrak{g}/\mathfrak{n}$$

given by $U(x, X) = (x, Y, Q)$, where $Q \in \mathfrak{g}/\mathfrak{n}$ is the projection of $X \in \mathfrak{g}$ and where $Y = \omega_x(X_p) \in \mathfrak{n}$.

Consider the dual bundle $V' = P \times \mathfrak{n'}$ to the vertical tangent bundle $V$. The projection $TP \to V'$, given by the connection $\omega$, determines a $G$-invariant injection $s_\omega$ of $V'$ in the cotangent bundle $T'P$. Consider the
canonical 1-form $\alpha$ on the manifold $T'P$. Let $\alpha_{\omega} := s^*_\omega \alpha \in \mathcal{A}^1(V')$. It is a $G$-invariant differential form on $V'$. The form

$$\beta_{\omega} := e^{i\text{id}_\omega} \in A^\infty_G(V') \otimes_{\mathbb{R}} \mathbb{C}$$

is a closed $G$-equivariant differential form. Consider the projection $p : V' \to P$. We will prove below that if $\Phi$ is a test function on $\mathfrak{g}$, then $\int_{\mathfrak{g}} \beta_{\omega}(X) \Phi(X) dX$ is a differential form on $V' = P \times \mathfrak{n}'$ rapidly decreasing in the direction $\mathfrak{n}'$. Thus, as the vector bundle $V'$ is $G$-oriented, we may define $p_* \beta_{\omega}$ as an element of $C^{-\infty}(\mathfrak{g}, \mathcal{A}(P)) \otimes_{\mathbb{R}} \mathbb{C}$ by setting:

$$\int_{\mathfrak{g}} (p_* \beta_{\omega})(X) \Phi(X) dX = p_* (\int_{\mathfrak{g}} \beta_{\omega}(X) \Phi(X) dX)$$

(the map $p_*$ depends on the choice of $\omega$).

A representative of the element $\gamma_0$ will be defined as the integral of $\beta_{\omega}$ over the fibers of $p$, normalised in order that $\gamma_0(X)$ is a differential form on $P$ with real coefficients.

**Proposition 76** Let $p : V' \to P$ be the projection as above. Let us choose a $G$-orientation $o$ on the vector bundle $V'$. Let $n = \dim \mathfrak{n}$. Let $c_n = 1$, if $n$ is even, and $c_n = -i$ if $n$ is odd. Define

$$\gamma_{\omega, o} := c_n (2\pi)^{-n} p_*(e^{i\text{id}_\omega}).$$

Then $\gamma_{\omega, o}$ is an element of $\mathcal{A}^{-\infty}_G(P)$ and is $d_{\mathfrak{g}}$-closed.

The cohomology class of $\gamma_{\omega, o}$ in $H^\infty_G(P)$ is independent of the choice of the $G$-invariant connection $\omega$. It depends only on the $G$-orientation $o$. We denote it by $\gamma_0$.

**Proof:**

Writing $\beta_{\omega} = e^{i\text{id}_\omega}$, we compute $p_* \beta_{\omega}$.

Let $E^j$ be a basis of $\mathfrak{n}$ with dual basis $E_j$. We write an element of $\mathfrak{n}'$ as $y = \sum_j y^j E_j$. Let

$$\omega = \sum_j \omega_j E^j \in (\mathcal{A}^1(P) \otimes \mathfrak{n})^G$$

be the connection form. By definition $(\omega_k, E^j_P) = \delta_k^j$ and $\omega$ of course by definition vanishes on the horizontal vectors. Under the identification $V' \cong P \times \mathfrak{n}'$, the 1-form

$$\alpha_{\omega} = \sum_j y^j \omega_j.$$
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Define, as in ([3], chapter 7), the moment \( \mu \in g' \otimes C^\infty(P) \otimes n \) of the connection \( \omega \) by setting, for any \( X \in g \)

\[
\mu(X) = -\omega(X_P).
\]

Thus \( \mu(X)x \) is an element of \( n \) and \( \mu(X + Y)x = \mu(X)x - Y \) for all \( Y \in n \) and \( x \in P \). Let us compute

\[
d_X\alpha_\omega := (d_g\alpha_\omega)(X).
\]

We obtain

\[
d_X\alpha_\omega = (y, \mu(X)) + \sum_j (dy^j\omega_j + y^j d\omega_j).
\]

We have \( e^{idy\omega} = e^{i(y, \mu(X))}A(y, dy, \omega, d\omega) \), where \( A \) is a polynomial expression in \( y^j, dy^j, \omega_j, d\omega_j \). If \( \Phi \) is a test function on \( g \), the integral

\[
\int_g e^{i(y, \mu(X))}\Phi(X)dX
\]

is a function on \( V' \) rapidly decreasing over the fiber \( n' \). This can be seen as follows: Consider the isomorphism (38) of \( P \times g \) with \( P \times n \times g/n \).

For \( x \in P \), let

\[
q_x := \{ X \in g; \omega_x(X_P) = 0 \}.
\]

Thus \( q_x \) is isomorphic to \( g/n \) under the natural projection \( g \to g/n \) and

\[
(39) \quad g = n \oplus q_x.
\]

We fix \( x \in P \), and write \( q_x = q \). Let \( X \in g \). Using the decomposition \( g = n \oplus q \), we write \( X = Y + Q \), with \( Y \in n, Q \in q \). We have \( (y, \mu(X)) = -(y, Y) \). Writing \( \Phi(X)dX = \Phi(Y, Q)dQdY \), we get

\[
\int_g e^{i(y, \mu(X))}\Phi(X)dX = \int_n e^{-i(y, Y)}\Psi(Y)dY,
\]

where \( \Psi(Y) = \int_q \Phi(Y, Q)dQ \). Clearly \( \Psi(Y) \) is a \( C^\infty \)-function with compact support on \( n \). As Fourier transform of test functions are rapidly decreasing, it follows that

\[
\int_g e^{iX\alpha_\omega}\Phi(X)dX = A(y, dy, \omega, d\omega) \int_n e^{-i(y, Y)}\Psi(Y)dY
\]
is a form on $V'$ rapidly decreasing over the fiber $n'$ of the projection $V' \to P$.

This proves that $p_*\beta_\omega$ exists, as an element of $C^{-\infty}(g,\mathcal{A}(P)) \otimes_R \mathbb{C}$. It is clearly $G$-invariant, so that $p_*\beta_\omega \in \mathcal{A}^{-\infty}_G(P) \otimes_R \mathbb{C}$. Furthermore $p_*\beta_\omega$ is $d_g$-closed:

$$(d_g p_* \beta_\omega, \Phi dX) = p_*(d_g \beta_\omega, \Phi dX) = 0.$$ 

If $\omega_t$ is a one-parameter smooth family of $G$-invariant connections, we denote $\alpha_{\omega_t}, \beta_{\omega_t}$ by $\alpha_t, \beta_t$ respectively. We have

$$\frac{d}{dt} \beta_t = \frac{d}{dt}(e^{id\alpha_t}) = idg((\frac{d}{dt}\alpha_t) \wedge \beta_t).$$

The integral of $(\frac{d}{dt}\alpha_t \wedge \beta_t)$ over the fiber $n'$ exists in the sense of generalised functions and

$$(40) \quad \frac{d}{dt} p_* \beta_t = idg p_*(\frac{d}{dt}\alpha_t \wedge \beta_t).$$

Thus the cohomology class of $p_* \beta_t$ is independent of the choice of the connection $\omega_t$.

We now compute explicitly the element $\gamma_{\omega,0}$ defined in Proposition 76 and show, in particular, that $\gamma_{\omega,0}(X)$ is a differential form with real coefficients.

Let us fix $x \in P$. The orientation $o$ on the vector bundle $V$ gives rise to an orientation $o_x$ on $n$ (which may depend on the connected component of $x \in P$). Let $E^j$ be an ordered basis of $n$. We will say that this order is $o$-compatible, if this basis is of orientation $o_x$. The exterior product $\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n$ of the components $\omega_i$ of the connection $\omega$ is a vertical form on $P$ of maximum dimension.

Let $\nu' \in \Lambda^n n'$ be such that $(\nu', E^1 \wedge \cdots \wedge E^n) = 1$. The element $\nu'$ also determines an Euclidean measure $dY$ on $n$ and a $\delta$-function $|\nu'|^{-1}\delta_n(Y) \in C^{-\infty}(n)$ (cf Section 2, Formula 9).

We can also write

$$|\nu'|^{-1}\delta_n(Y) = (2\pi)^{-\dim n} \int_{n'} e^{i(y,Y)}dy,$$

where $dy$ is the measure on $n'$ dual to the Euclidean measure $dY$ on $n$. Let $\Omega = d\omega + \frac{1}{2}[\omega,\omega]$ be the curvature of the connection $\omega$. Thus $\Omega \in$
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$(\mathcal{A}^j(P) \otimes \mathfrak{n})^G$. We write

$$\Omega = \sum_j \Omega_j E^j.$$ 

Define the equivariant curvature of $\omega$ (as in [3], chap 7) by

$$\Omega(X) = \mu(X) + \Omega.$$ 

To simplify notation, we will use $\nu'$ to identify generalised functions and distributions and write $\delta_n$ instead of $|\nu'|^{-1}\delta_n$.

Let us show that the generalised function $\delta_n(\Omega(X)) \in C^{-\infty}(g, \mathcal{A}(P))$ is well defined. We describe, at each point $x \in P$, $\delta_n(\Omega(X))_x$ as a generalised function on $g$ with values in the vector space $\Lambda T_x^*P$. In the decomposition $g = \mathfrak{n} \oplus \mathfrak{q}_x$ given by formula 39, we write $X = Y + Q$. Then $\Omega(Y + Q)_x = -Y + \Omega_x$ and we define $\delta_n(\Omega(X))_x$ by its Taylor expansion:

$$\delta_n(\Omega(X))_x = \delta_n(-Y) + \sum_j \Omega_j(\partial_{E^j}\delta_n)(-Y) + \cdots$$

**Proposition 77** Let $x \in P$ and let $E^j$ be an ordered basis of $\mathfrak{n}$, with an order compatible with the orientation $\omega_x$. Then

$$\gamma_{\omega,\circ}(X)_x = |\nu'|^{-1}\delta_n(\Omega(X))_x \wedge (\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n)_x.$$ 

**Proof:** The highest degree component of $e^{i\omega \cdot \omega} = e^{i(\omega_1 \cdot y + \cdots + \omega_n \cdot \omega)}$ in $dy_j$’s is equal to

$$c_n^{-1} dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n \wedge \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n e^{i(y \cdot \omega)}.$$ 

The curvature $\Omega$ is equal to $d\omega$ modulo terms in $\omega_j$. Thus

$$\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n e^{i(y \cdot \omega)} = \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n e^{i(y \cdot \Omega)}$$

and

$$p_{y,\circ,\omega}(X)_x = c_n^{-1} \left( \int_{\mathfrak{n}} e^{i(y \cdot (\mu(x) + \Omega))} dy \right) \wedge (\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n)_x$$

$$= c_n^{-1}(2\pi)^n |\nu'|^{-1}\delta_n(\Omega(X))_x(\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n)_x.$$ 

This proves the proposition. ■

In particular, we see that $\gamma_{\omega,\circ}$ indeed belongs to $\mathcal{A}_G^{-\infty}(P)$ and we obtain Proposition 76. ■
Remark 78. It is easy to check directly from the formula of the above proposition that $\gamma_{\omega_0}$ is $d_g$ closed.

Fix $x \in P$ and consider the decomposition $g = n \oplus q$ given by (39). The space $q$ is isomorphic to $g/n$. The generalised function $\gamma_{\omega_0}(X)_x$ is constant in the direction $q$: $\gamma_{\omega_0}(Y + Q)_x = \gamma_{\omega_0}(Y)_x$. Thus, if $f(Q)$ is a generalised function on $g/n$, we can multiply $\gamma_{\omega_0}(Y)_x$ by $f(Q)$ and we obtain a generalised function on $g$ with values in $\Lambda T'_x P$.

Define the map

$$m_{\omega_0} : A_{G/N}^{-\infty}(P/N) \to A_{G}^{-\infty}(P)$$

by setting

$$m_{\omega_0}(\alpha) = q^*(\alpha) \wedge \gamma_{\omega_0},$$

for $\alpha \in A_{G/N}^{-\infty}(P/N)$.

The above discussion shows that $m_{\omega_0}$ is well defined. It is a cochain map of differential complexes

$$m_{\omega_0} : (A_{G/N}^{-\infty}(P/N), d_{g/n}) \to (A_{G}^{-\infty}(P), d_g).$$

Proposition 79. The induced map in cohomology

$$H_{G/N}^{-\infty}(P/N) \to H_{G}^{-\infty}(P),$$

from the cochain map

$$m_{\omega_0}(\alpha) = q^*(\alpha) \wedge \gamma_{\omega_0},$$

does not depend upon the choice of the $G$-invariant connection $\omega$ on the principal $N$-bundle $q : P \to P/N$. We will denote it by $m_\omega$.

Proof: This follows easily from Formula 40 on the variation of $p_\omega \beta_\omega$. □

We describe now the properties of $m_{\omega_0}$ in relation to the $A_{G}^{\infty}(P)$-module structure on $A_{G}^{-\infty}(P)$.

If $\alpha \in A_{G}^{\infty}(P)$, we can define $\beta \in A_{G}^{\infty}(P)$, by setting

$$\beta(X) := \alpha(X + \Omega(X)).$$
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Explicitly, for \( x \in P, Y \in n, Q \in q \), \( X = Y + Q \), then \( X + \Omega(X) = Y + Q + \Omega_x - Y = Q + \Omega_x \), thus \( \beta(X)_x = \alpha(Q + \Omega)_x \) is defined by its Taylor expansion

\[
\alpha(Q + \Omega)_x = \alpha(Q)_x + \sum_j \Omega_j(\partial_{E_j} \alpha)(Q)_x + \cdots
\]

Thus \( \beta(X + Y)_x = \beta(X)_x \), for \( X \in g, Y \in n \). Hence \( \beta \in C^\infty(g/n, A(P)) \).

Let \( \Gamma \subset A(P) \) be the subspace of horizontal forms. The group \( G \) acts on \( \Gamma \). The connection \( \omega \) defines a horizontal projector \( h : A(P) \to \Gamma \) which commutes with the action of \( G \). Define, for \( \alpha \in A^\infty_G(P) \),

\[
W_\omega(\alpha)(X) := h(\alpha(\Omega(X))).
\]

As \( W_\omega \alpha = h(\beta) \), \( W_\omega(\alpha) \in C^\infty(g/n, \Gamma) \). The \( G \)-invariance implies that \( W_\omega(\alpha) \in C^\infty(g/n, \Gamma)^G \). As \( N \) is a normal subgroup of \( G \), \( N \) acts trivially on \( g/n \), in particular \( W_\omega(\alpha) \in C^\infty(g/n, \Gamma^N) \). The space \( \Gamma^N \) is the space of forms on \( B := P/N \), and we think of \( W_\omega(\alpha) \) as an element of \( C^\infty(g/n, A(B)) \). The \( G \)-invariance implies that \( W_\omega(\alpha) \in A^\infty_{G/N}(P/N) \). Thus we have obtained a map

\[
W_\omega : A^\infty_G(P) \to A^\infty_{G/N}(P/N).
\]

**Remark 80** The map \( W_\omega \) is a generalisation of the Chern-Weil map: If \( N = G \), let \( \phi \in C^\infty(g)^G \) and consider \( \phi(X)1 \in A^\infty_G(P) \). Then \( W_\omega(\phi(X)1) = \phi(\Omega) \) is the characteristic form on \( P/G \) associated to \( \phi \) by the classical Chern-Weil homomorphism.

**Proposition 81** If \( \alpha \in A^\infty_G(P) \), then

\[
\alpha \wedge \gamma_{\omega, \alpha} = q^*(W_\omega \alpha) \wedge \gamma_{\omega, \alpha} = m_{\omega, \alpha}(W_\omega(\alpha)).
\]

In particular, for \( \beta \in A^\infty_{G/N}(P/N) \),

\[
\alpha \wedge m_{\omega, \alpha}(\beta) = m_{\omega, \alpha}(W_\omega \alpha \wedge \beta).
\]

**Proof:** The proof follows easily from our formula given in Proposition 77 for \( \gamma_{\omega, \alpha} \). We have, for \( x \in P, X = Y + Q, Y \in n, Q \in q \).
\[
\begin{align*}
\alpha(X)_x \wedge \gamma_{\omega,0}(X)_x &= \alpha(Y + Q)_x \delta_n(\Omega - Y)_x(\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n)_x \\
&= \alpha(\Omega + Q)_x \delta_n(\Omega - Y)_x(\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n)_x \\
&= \alpha(X + \Omega(X))_x \wedge \gamma_{\omega,0}(X)_x.
\end{align*}
\]

As \(\gamma_{\omega,0}\) is already of top degree in the vertical directions, we see that

\[
\alpha(X + \Omega(X))_x \wedge \gamma_{\omega,0}(X)_x = h(\alpha(X + \Omega(X)))_x \wedge \gamma_{\omega,0}(X)_x
\]

which is the formula we want. \(\blacksquare\)

The following proposition is proved in [12]. We include a proof for completeness.

**Proposition 82** Let \(G\) be a Lie group and let \(N \subset G\) be a closed normal subgroup of \(G\). Let \(P\) be a right \(G\)-manifold such that the action of \(N\) is principal. Assume that the principal \(N\)-bundle \(q : P \to P/N\) admits a \(G\)-invariant connection \(\omega\). Then the map

\[
W_\omega : \mathcal{A}_G^{\infty}(P) \to \mathcal{A}_G^{\infty}(P/N)
\]

defined above is a cochain map:

\[
W_\omega d_\omega = d_{\omega/n}W_\omega.
\]

Furthermore, if \(\beta \in \mathcal{A}_G^{\infty}(P/N)\) then

\[
W_\omega q^* \beta = \beta.
\]

**Proof:** The last equation follows from the definition of \(W_\omega\), as \(\alpha(Y + Q)_x = (q^* \beta)(Q)\) is independent of the variable \(Y \in n\), thus \(\alpha(\Omega + Q)_x = \alpha_x(Q)\) and is horizontal.

The fact that \(W_\omega\) commutes with differentials is proved in [12]. We give here a proof following easily from Proposition 81: Let \(\alpha \in \mathcal{A}_G^{\infty}(P)\) and write \(\gamma\) instead of \(\gamma_{\omega,0}\) and \(W\) instead of \(W_\omega\). Let us compute \(d_\omega(\alpha \wedge \gamma)\):

As \(\gamma\) is \(d_\omega\)-closed, we obtain from Proposition 81,

\[
d_\omega(\alpha \wedge \gamma) = d_\omega \alpha \wedge \gamma = q^*(W(d_\omega \alpha)) \wedge \gamma.
\]
We also have
\[
\begin{align*}
   d_g(\alpha \wedge \gamma) &= d_g(q^*W(\alpha) \wedge \gamma) \\
   &= d_g(q^*W(\alpha)) \wedge \gamma \\
   &= q^*(d_{g/n}W(\alpha)) \wedge \gamma.
\end{align*}
\]

Thus
\[
m_{\omega,\gamma}(W(d_g\alpha)) = m_{\omega,\gamma}(d_{g/n}W(\alpha)).
\]

But the map \(m_{\omega,\gamma}\) is easily seen to be injective and hence we obtain
\[
W(d_g\alpha) = d_{g/n}W(\alpha).
\]

We also have ([12])

**Theorem 83** Let the notation and assumptions be as in the above proposition (82). Then the cochain map
\[
q^* : A^\infty_{G/N}(P/N) \to A^\infty_G(P)
\]
induces an isomorphism in cohomology.

As \(W_\omega q^* = I\), where \(I\) is the identity operator, the map \(W_\omega\) provides an explicit inverse for \(q^*\) in cohomology.

Let us consider example 73 for the following special case: The manifold \(L\) is equal to the Lie group \(U\) and \(K\) is a closed subgroup of \(U\). The manifold \(L = U\) is a \(U \times K\) manifold, where the action of \((u, k) \in U \times K\) on \(x \in L\) is given by \(x \cdot (u, k) = u^{-1}xk\). Let \(G = U \times K\) and \(P = U \times M\), for a \(K\)-manifold \(M\). The action of both of the (normal) subgroups \(U\) and \(K\) of \(G\) on \(P\) are principal.

Consider first the action of \(U\). The space \(P/U\) is our \(K\)-manifold \(M\) we started with. Consider the canonical Maurer-Cartan connection \(\omega_U \in (A^1(U) \otimes \mathfrak{u})^U\) defined by \(\omega_U(X_U) = X\) for every \(X \in \mathfrak{u}\). Here \(X_U\) is the vector field associated to the action of \(U\) on \(L\) by left translation, i.e \((X_U)_x\) is the tangent vector to the curve \(\exp(-\varepsilon X)x\). We have \(\omega_U(Y_L)_x = -x \cdot Y\), for \(x \in L, Y \in \mathfrak{k}\), as \(K\) acts by right translations on \(L\). The connection \(\omega_U\)
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extends trivially to a \((U \times K)\)-invariant connection \(\omega_U \otimes 1\) for the principal \(U\)-bundle \(q_U : U \times M \to M\).

As before, trivialize the vertical bundle \(V_U \cong P \times \mathfrak{u}\) (for \(q_U\)) by the map defined by sending the vertical tangent vector \((X_{U \times M})_{x,m}\) to \((x, m, X)\), for \(x \in U, m \in M, X \in \mathfrak{u}\). Choose an orientation \(o_u\) on \(\mathfrak{u}\). We obtain a \(U\)-invariant orientation \(o_U\) on \(V_U\), by setting \((o_U)_{x,m} = \text{sign}(\det u_x) o_u\). This orientation is \(U \times K\) invariant, if and only if \(\det u_k > 0\), for every \(k \in K\). Thus, under this condition, we get a map

\[ m_{o_U} : H^{-\infty}_K(M) \to H^{-\infty}_{U \times K}(U \times M). \]

Consider now the action of \(K\). The quotient space \(U \times M\) by the action of \(K\) is the induced space

\[ \mathcal{M} = U \times_K M \]

with left action of \(U\), that we considered in section 5. Assume there exists a \(U\)-invariant connection \(\omega \in \mathcal{A}^1(U) \otimes \mathfrak{k}\) for the principal \(K\)-bundle \(U \to U/K\). Then the form \(\omega \otimes 1\) on \(U \times M\) is a \(U\)-invariant connection form for the principal \(K\)-bundle \(q_K : U \times M \to U \times_K M\). Assume that \(\det k \neq 0\) for every \(k \in K\). Choose an orientation \(o_k\) on \(\mathfrak{k}\), then the fibration \(q_K\) has a unique \(U \times K\)-invariant orientation \(o_K\) given by \(o_{e,m} = o_k\) for each point \(m \in M\), where \(e\) is the identity of \(U\). Thus, under these conditions, there exists a map

\[ m_{o_K} : H^{-\infty}_U(U \times K M) \to H^{-\infty}_{U \times K}(U \times M). \]

Given orientations \(o_u, o_k\) of \(\mathfrak{u}, \mathfrak{k}\) respectively, they determine an orientation \(o\) on \(\mathfrak{u}/\mathfrak{k}\). We have \(\det u_k > 0\) for all \(k \in K\), as both numbers \(\det u_k\) and \(\det k\) are > 0, by assumption. Recall the map

\[ \text{Ind}_{U/K,o} : H^{-\infty}_K(M) \to H^{-\infty}_U(U \times K M) \]

from Section 5, Proposition 48

**Lemma 84** Let \(U\) be a Lie group and let \(K\) be a closed subgroup of \(U\), such that the principal \(K\)-bundle \(U \to U/K\) admits a \(U\)-invariant connection. Assume \(\det k > 0, \det u_k > 0\) for all \(k \in K\). Let \(o_u, o_k, o\) be compatible orientations on \(\mathfrak{u}, \mathfrak{k}, \mathfrak{u}/\mathfrak{k}\), then

\[ m_{o_U} = m_{o_K} \text{Ind}_{U/K,o}. \]
Proof: First, we explicitly compute the map \( m_{\alpha,\nu} \). Consider the canonical connection \( \omega_U \). Its curvature is 0. The equivariant curvature of \( \omega_U \) is given by 

\[
\Omega_U(X, Y)(e, m) = -X + Y.
\]

Let \( \ell = \dim U \). Let \( \nu'_U \in \Lambda^\ell u' \) be a positive element (with respect to the orientation \( \alpha_U \)). Let \( dx \) be the unique left \( U \)-invariant form on \( U \), such that \( (dx)_e = \nu' \). Then \( \nu'_U \) determines a \( \delta \)-function \( |\nu'_U|^{-1}\delta_u \) on \( u \), and for \( X \in u, Y \in k \),

\[
\gamma_{\omega_U,\alpha_U}(X, Y)(e, m) = |\nu'_U|^{-1}\delta_u(Y - X)(dx)_e.
\]

Thus, for \( \alpha \in \mathcal{A}_K^\infty(M) \),

\[
(m_{\omega_U,\alpha_U})(X, Y)(e, m) = |\nu'_U|^{-1}\delta_u(Y - X)(\alpha(Y) \wedge dx)(e, m).
\]

More explicitly, for \( \Phi_1 \) a test function on \( u \), \( \Phi_2 \) a test function on \( k \),

\[
\left( \int_{u \times k} (m_{\omega_U,\alpha_U})(X, Y)\Phi_1(X)\Phi_2(Y)dXdY \right)(e, m) = \\
\left( \int_k \alpha(Y)\Phi_1(Y)\Phi_2(Y)dY \right)(e, m)
\]

where \( dX \) is the Euclidean density on \( u \) determined by \( \nu' \).

Now let us analyse the action of \( K \) on \( U \times M \). Let \( k = \dim K \), \( n = \dim(U/K) \). Let \( E^i \) be an oriented basis of \( k \) with dual basis \( E_i \in k' \). Let \( \nu'_K = E_1 \wedge \cdots \wedge E_k \). The connection \( \omega \) determines a \( K \)-invariant decomposition \( u = k \oplus r \). Thus \( E_i \) can be thought of as an element of \( u' \) vanishing on \( r \). If \( \omega = \sum_i \omega_i E^i \), the form \( \omega_i \) is the unique left \( U \)-invariant 1-form on \( U \) such that \( (\omega_i)_e = E_i \). Let \( pr_k \) (resp. \( pr_r \)) be the projection from \( u \) to \( k \) (resp. \( r \)) determined by \( \omega \).

Consider the connection \( \tilde{\omega} := \omega \otimes 1 \) for the principal \( K \)-bundle \( q_K : U \times M \to U \times_K M \). Let \( \Omega \in \mathcal{A}^2(U) \otimes k \) be the curvature of \( \omega \). The equivariant curvature of \( \tilde{\omega} \) at the point \((e, m) \in U \times M \) is given, for \( X \in u, Y \in k \) by

\[
\Omega(X, Y)(e, m) = (pr_KX - Y) + \Omega_e.
\]

Thus the element \( \gamma_{\omega,\alpha} \) is given by

\[
(\gamma_{\omega,\alpha})(X, Y)(e, m) = |\nu'_K|^{-1}\delta_k((pr_KX - Y) + \Omega_e)(\omega_1 \wedge \cdots \wedge \omega_k)_e.
\]
Let $\mu' \in \Lambda^r r'$ be such that $\mu' \wedge \nu'_K = \nu'_E$. Let $dr$ be the unique left $U$-invariant $n$-form on $U$ such that $(dr)_e = \mu'$. The element $\mu'$ determines also a $\delta$-function $|\mu'|^{-1}\delta_r$ on the vector space $r$. Let $\alpha \in \mathcal{A}_K^\infty(M)$. By definition, for $X \in u$, $(\text{Ind}_{U/K, o} \alpha)(X)_{e, m}$ is the projection (on the horizontal elements for the diagonal $K$-action) of $|\mu'|^{-1}\delta_r(pr_{-}X)(\alpha(pr_{K}X) \wedge dr)_{e, m}$. As $\gamma_{\omega, o_K}$ is already of top vertical dimension in the direction $K$, we have
\[
((m_{\omega, o_K} \text{Ind}_{U/K, o} \alpha)(X, Y))_{e, m} = |\mu'|^{-1}\delta_r(pr_{-}X)(\alpha(pr_{K}X) \wedge dr \wedge |\mu'|^{-1}\delta_r((pr_{K}X - Y) + \Omega) \wedge \omega_1 \wedge \omega_k)_{e, m}.
\]

Now $dr \wedge \omega_1 \ldots \wedge \omega_k$ is the form $dx$ of top degree on $U$ and $\Omega$ is a form on $U$, so that $dx \wedge \delta_r((pr_{K}X - Y) + \Omega) = dx \wedge \delta_r(pr_{K}X - Y)$. Thus, if $\Phi_1$ is a test function on $u$ and $\Phi_2$ a test function on $k$, we obtain
\[
(\int_{u \times k} (m_{\omega_{U}, o_K} \alpha)(X, Y)\Phi_1(X)\Phi_2(Y)dXdY)_{e, m} = \\
(\int_{k} \alpha(Y)\Phi_1(Y)\Phi_2(Y)dY) \wedge dx)_{e, m}.
\]

Comparing with the preceding calculation, we obtain the equality, for $X \in u, Y \in k, m \in M$,
\[
(m_{\omega_{U}, o_K} \alpha)(X, Y)_{e, m} = (m_{\omega, o_K} \text{Ind}_{U/K, o} \alpha)(X, Y)_{e, m}.
\]
By $U$-invariance we obtain the equality at each point $(x, m) \in U \times M$ and the lemma is proved.

**Proposition 85** Let $U$ be a Lie group. Let $K$ be a compact subgroup of $U$ such that $\det_{uk} = 1$, $\det_k k = 1$ for all $k \in K$. Then for any $K$-manifold $M$, the maps
\[
m_{o_U} : H^{-\infty}_K(M) \rightarrow H^{-\infty}_{U \times K}(U \times M)
\]
and
\[
m_{o_K} : H^{-\infty}_U(U \times K M) \rightarrow H^{-\infty}_{U \times K}(U \times M),
\]
(defined earlier) are both isomorphisms.

**Proof:** Let $G = U \times K$. The space $P = U \times M$ is also the induced space $G \times_{\Delta(K)} M$, where $K$ is embedded in $G = U \times K$ by the diagonal map $\Delta$. It is easy to see from the explicit calculation above that the map $m_{o_U}$
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coincides with the map $\text{Ind}_{G/K, o}$. As $K$ is compact, Theorem 50 of section 5 implies that $m_{ov}$ is an isomorphism. As $\text{Ind}_{U/K, o}$ is also an isomorphism, Lemma 84 gives us the proposition.

Let us return to the general situation $(S)$ of a right $G$-manifold $P$, with principal action of a normal subgroup $N$ of $G$, satisfying Conditions (74) and (75). Then the map $m_o : H^\infty_{G/N}(P/N) \to H^\infty_G(P)$ is defined. Although it would be desirable to know that $m_o$ is always an isomorphism, we are able to prove it only under additional hypotheses.

First consider the case where $G = N$. Thus $H^\infty_{G/N}(P/N)$ is simply equal to $H(P/G) = H(B)$. As $P \to B = P/G$ is a principal bundle with group $G$, we can find a ($G$-invariant) connection $\omega$ for the bundle $P \to B$. We assume that this fibration has $G$-oriented-fibers. Thus we can construct a canonical element (up to the $G$-orientation $o$) $\gamma_o \in H^\infty_G(P)$ and the map $m_o$.

Let $\Omega$ be the curvature of the connection $\omega$ for $P \to P/G$. If $\phi(X) \in C^\infty(g)^G$, then $\phi(\Omega) \in H(B)$ and is independent of the choice of $\omega$. We thus define a structure of $C^\infty(g)^G$-module on $H(B)$ via the Chern-Weil homomorphism: $\phi \cdot \beta = \phi(\Omega) \wedge \beta$, for $\phi \in C^\infty(g)^G$ and $\beta \in H(B)$.

**Theorem 86** Let $G$ be a Lie group acting principally (from the right) on a manifold $P$. Assume further that the quotient map $q : P \to P/G$ has $G$-oriented fibers under an orientation $o$. Then the map

$$m_o : H(P/G) \to H^\infty_G(P)$$

given by $m_o(\alpha) = q^* \alpha \wedge \gamma_o$ is an isomorphism of $C^\infty(g)^G$-modules.

**Proof:** The fact that $m_o$ is a morphism of $C^\infty(g)^G$-modules, follows readily from Proposition 81.

It remains to see that $m_o$ is an isomorphism of vector spaces:

If $P = G \times B$ is the direct product of $G$ and $B$, with the action of an element $g_0 \in G$ given by $(g, m) \cdot g_0 = (g^{-1}g, m)$, for $g \in G, m \in B$, then by Proposition 85 (for $U = G, K = e, M = B$), the equivariant cohomology $H^\infty_G(P)$ is isomorphic with $H(B)$ under the map $m_o$. Thus our theorem is true when the fibration $P \to B$ is trivial. (Remark: a trivial principal
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A $G$-bundle is usually trivialised as $G \times B$, where the action of $G$ is on the right $(g, m) \cdot g_0 = (gg_0, m)$. We can use the isomorphism $(g, m) \to (g^{-1}, m)$ to change this usual trivialisation to the trivialisation used above.

Let us now return to the general situation. Choose a $(G$-invariant) connection form $\omega$ for the principal $G$-bundle $q$. Consider the element $\gamma_o \in H^\infty_G(P)$ given in Proposition 76 (with respect to the given $G$-orientation $o$). Let $U$ be an open subset of $B$. Denote by $\gamma_U$ the restriction of $\gamma_o$ to $q^{-1}(U)$. We denote by $m_U : H(U) \to H^\infty_G(q^{-1}(U))$ the map $m_o$ restricted to $q^{-1}(U)$: $m_U(\alpha) = q^*\alpha \wedge \gamma_U$.

**Lemma 87** Let $U$ and $V$ be two open subsets of $B$. Assume that the maps $m_U, m_V, m_{U \cap V}$ are isomorphisms, then $m_{U \cup V}$ is an isomorphism.

**Proof:** This lemma is proved by a standard Mayer-Vietoris argument: Both the sequences

\[ 0 \to A(U \cup V) \to A(U) \oplus A(V) \to A(U \cap V) \to 0, \]

\[ 0 \to A^\infty_G(q^{-1}(U \cup V)) \to A^\infty_G(q^{-1}(U)) \oplus A^\infty_G(q^{-1}(V)) \]

\[ \to A^\infty_G(q^{-1}(U \cap V)) \to 0 \]

are exact. The surjectivity of the last map can be seen as follows: Choose a partition of unity $f_U, f_V$ for $U \cup V$ subordinate to $U, V$. Then the functions $q^* f_U, q^* f_V$ are $G$-invariant functions. If $\beta \in A^\infty_G(q^{-1}(U \cap V))$, then $\beta$ is the image of $(-(q^* f_U) \beta, (q^* f_V) \beta) \in A^\infty_G(q^{-1}(U)) \oplus A^\infty_G(q^{-1}(V))$.

Thus the above Mayer-Vietoris sequences induce the long exact sequences in cohomology. The lemma follows from the five lemma. $lacksquare$

Now Theorem 86 follows by recalling that there is a finite open cover $U_i$ of the base $B$ such that the bundle $q^{-1}(U_i) \to U_i$ is trivial. $lacksquare$

We now return to the general situation $(S)$. Assume now that $G$ is compact. Then the condition (75) on the existence of a $G$-invariant connection for the map $q : P \to P/N$ is always satisfied. We prove the following theorem
**Theorem 88** Let $G$ be a compact Lie group and let $P$ be a right $G$-manifold $G$. Let $N$ be a closed normal subgroup of $G$ acting freely on $P$. We assume furthermore that the fibers of $q : P \to P/N$ admit a $G$-orientation $o$. Then the map

$$m_o : H_{G/N}^{-\infty}(P/N) \to H_{G}^{-\infty}(P)$$

is an isomorphism.

**Proof:** We choose a $G$-invariant connection form $\omega \in (\mathcal{A}^1(P) \otimes \mathfrak{n})^G$ and use notation of the proof of Proposition 76. Let $\Gamma \subset \mathcal{A}(P)$ be the space of horizontal forms. The action of $G$ on $\mathcal{A}(P)$ preserves the subspace $\Gamma$. Let us consider the algebra homomorphism $C : \Lambda \mathfrak{n}' \to \mathcal{A}(P)$, determined by sending $E_j \in \mathfrak{n}'$ to $\omega_2$. We still denote by

$$C : \Gamma \otimes \Lambda \mathfrak{n}' \to \mathcal{A}(P)$$

the map given by $C(\alpha \otimes \xi) = \alpha \wedge C(\xi)$, for $\alpha \in \Gamma, \xi \in \Lambda \mathfrak{n}'$. The map $C$ is an isomorphism. Furthermore $C$ commutes with the action of $G$.

Recall the isomorphism (9) $U : P \times \mathfrak{g} \to P \times \mathfrak{n} \times \mathfrak{g}/\mathfrak{n}$. Let us explicitly write $U$, using coordinates. Let $n = \dim \mathfrak{n}$. Choose a basis $Q^a, n < a \leq \dim \mathfrak{g}$, of $\mathfrak{g}/\mathfrak{n}$. We choose a basis $G_j$ of $\mathfrak{g}$ such that the first $n$-vectors are the vectors $E_j$ and the last ones are representatives of $Q^a$. Let $Q = \sum_{i=n} x_i Q^i$ be an element of $\mathfrak{g}/\mathfrak{n}$. Define, for $p \in P$,

$$k_j(p, Q) = (\omega_2(\sum_{i=n} x_i G^i)_p, E_j).$$

Then $Q \to k_j(p, Q)$ is a linear function in $Q$ varying smoothly in $p$. Let $X = \sum_i x_i G^i$. In these coordinates $x = (x_i)$, we have

$$U(p, X) = (p, Y(x), Q(x))$$

with

$$Q(x) = \sum_{i=n} x_i Q^i, Y(x) = \sum_{j=1}^n (x_j + k_j(p, Q(x)) E_j).$$

We denote by $U^*$ the isomorphism

$$U^* : C^{-\infty}(\mathfrak{n} \oplus \mathfrak{g}/\mathfrak{n}, C^\infty(P)) \to C^{-\infty}(\mathfrak{g}, C^\infty(P))$$
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given by \((U^*s)(X,p) = s(U(p,X)), \text{ for } s \in C^{-\infty}(\mathfrak{n} \oplus \mathfrak{g}/\mathfrak{n}, C^{\infty}(P))\).

Formula above shows that \(U^*s\) is indeed smooth in \(p\) and generalised in \(X\).

Let

\[ A = C^{-\infty}(\mathfrak{n} \oplus \mathfrak{g}/\mathfrak{n}, \Gamma) \otimes \Lambda^*\mathfrak{n}'. \]

With the help of \(U\) and \(C\), we can define an isomorphism

\[ T : A \to C^{-\infty}(\mathfrak{g}, A(P)) \]

by the following formula: For \(s \in C^{-\infty}(\mathfrak{n} \oplus \mathfrak{g}/\mathfrak{n}, C^{\infty}(P)), \alpha \in \Gamma \otimes \Lambda^*\mathfrak{n}', \)

\[ T(s\alpha) = (U^*s)C(\alpha). \]

The group \(G\) acts on \(A\) by the action induced by the adjoint representation of \(G\) on \(\mathfrak{n}, \mathfrak{g}/\mathfrak{n}\), and its natural action on \(\Gamma\). We denote by \(A_G\) the space of \(G\)-invariants in \(A\). Then \(T\) commutes with the action of \(G\) and induces an isomorphism still denoted by \(T\) between \(A_G\) and \(A_G^{-\infty}(P)\).

Consider the \(\mathbb{Z}_+\)-gradation on \(A\)

\[ A^* = C^{-\infty}(\mathfrak{n} \oplus \mathfrak{g}/\mathfrak{n}, \Gamma) \otimes \Lambda^*\mathfrak{n}'. \]

We still write \(d_g\) for the operator on \(A\) obtained from the operator \(d_g\) on \(C^{-\infty}(\mathfrak{g}, A(P))\) under the isomorphism \(T\).

We write \(Y \in \mathfrak{n}\) as \(Y = \sum y_j E^j\). Let \(i(E^j)\) be the contraction on \(\Lambda^*\mathfrak{n}'\) by the vector \(E^j\). Let \(j_n\) be the operator of degree \(-1\) on \(A\) given by

\[ j_n = \sum_j y_j i(E^j). \]

The components \(\Omega_j\) of the curvature \(\Omega\) are horizontal forms. Thus exterior multiplication by \(\Omega_j\) is an operator on \(\Gamma\). We can consider the operator \(f\) acting on \(C^{-\infty}(\mathfrak{n} \oplus \mathfrak{g}/\mathfrak{n}, \Gamma) \otimes \Lambda^*\mathfrak{n}'\) given by

\[ f = \sum_j \Omega_j \otimes i(E^j). \]

The operator \(f\) is homogeneous of degree \(-1\).

Let us write, using the \(\mathbb{Z}_+\)-grading of \(A\), the operator \(d_g\) on \(A\) as a sum of homogeneous operators \(d_i\) of degree \(i\).
Lemma 89 \ We have \[ d_g = d_{-1} \oplus d_0 \oplus d_1 \] with \( d_{-1} = -J_n + f \).

Proof: \ Let \( Q^a \) be a basis of \( g/\mathfrak{n} \). Let \( p \in P \). At the point \( p \in P \), consider the decomposition

\[ g = n \oplus q_p \]

given in (39). We write \( Q^a_p \in q_p \) for the unique element of \( q_p \) above \( Q^a \in g/\mathfrak{n} \). We have \( \omega_p(Q^a_p) = 0 \). The contraction by \( Q^a_p \) produces an operator \( \iota^a \) on the space of horizontal forms \( \Gamma \). Let \( Q = \sum_a q_a Q^a \in g/\mathfrak{n} \). The coordinate function \( q_a \) acts on \( C^{-\infty}(g/\mathfrak{n}) \) by multiplication. The operator \( \iota_{g/n} = \sum q_a \iota^a \) is an operator of degree 0 on \( C^{-\infty}(\mathfrak{n} \oplus g/\mathfrak{n}, \Gamma) \otimes \Lambda^n' \).

It is easy to see that the operator \( \iota = \iota_{g/n} \) on \( C^{-\infty}(g, A(P)) \) becomes the operator \( \iota_{g/n} \oplus j_n \) on \( A \) under the isomorphism \( T \).

Let us now analyse the differential \( d_P \) under the isomorphism \( T \). If \( I = \{ 1 \leq i_1 < i_2 \ldots < i_k \leq n \} \) is an ordered multiindex, we write \( \omega_I = \omega_{i_1} \wedge \ldots \wedge \omega_{i_k} \). Let \( s \in C^{-\infty}(\mathfrak{n} \oplus g/\mathfrak{n}, C^\infty(P)) \), \( \alpha \in \Gamma \). We compute \( d_P((U^* s) \alpha \wedge \omega_I) \).

As \( d_P \) is a derivation, we analyse the exterior differential of each term of this product.

As \( d\omega + \frac{1}{2}[\omega, \omega] = \Omega \), where \( \Omega \) is horizontal, we see that the differential \( d\omega_j \) of the component \( \omega_j \) of the connection \( \omega \) is the sum of an element of \( C(\Lambda^2 \mathfrak{n}') \) and of \( \Omega_j \in \Gamma \).

The differential \( d_P \) does not necessarily keep the space \( \Gamma \) of horizontal forms stable, but \( d_P(\alpha) \in \Gamma \oplus \Gamma \otimes \mathfrak{n}' \), for \( \alpha \in \Gamma \).

Finally, for \( s(Y, Q, p) \in C^{-\infty}(\mathfrak{n} \oplus g/\mathfrak{n}, C^\infty(P)) \), we have, with \( Y = \sum_j y_j E^j \),

\[ T^{-1}d_P T s = d_P s + \sum_j \partial_{y_j} s(Y, Q, p) d_P k_j(p, Q). \]

Combining all these observations, we see that \( d_P \) becomes a sum of the homogeneous operators \( f_{-1} + f_0 + f_1 \) under the isomorphism \( T \). Furthermore the term \( f_{-1} \) is the operator \( f \). Hence we obtain the lemma. \( \blacksquare \)

We now prove Theorem 88 by an induction argument similar to the argument of the proof of Theorem 37. Actually we will make use of the bigrading

\[ A^{k, q} = C^{-\infty}(\mathfrak{n} \oplus g/\mathfrak{n}, \Gamma^q) \otimes \Lambda^k \mathfrak{n}', \]
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where \( \Gamma^q \) refers to the \( \mathbb{Z}^+ \)-grading on \( \Gamma \) given by the exterior degree.

Each of the spaces \( A^\infty_{k,q} \) is stable by \( G \), since \( N \) is a normal subgroup of \( G \). As the group \( G \) is compact, the proof of Proposition 20 and Remark 21 (of section 3) implies that the homology groups of the operator \( j_n : A^\infty_{G,q} \to A^\infty_{G,q-1} \) are equal to zero, except in maximal degree \( n = \dim n \).

Let \( \alpha \in A^\infty_{G} \) be such that \( d_g \alpha = 0 \). We first show that \( \alpha \) is homologous to an element \( \beta \) divisible by \( \gamma_{\omega,\alpha} \). We work with \( A_G \) and write again \( \alpha \) for the element \( T^{-1}(\alpha) \in A_G \). Let \( \alpha = \sum_{k \geq k_0} \alpha_k \) with \( \alpha_k \in A^k_G \).

From the degree consideration (cf. Lemma 89), we see that \( (j_n - f)\alpha_{k_0} = 0 \).

Write now \( \alpha_{k_0} = \sum_{q \geq q_0} \alpha_{(k_0,q)} \) with \( \alpha_{(k_0,q)} \in A^{k_0,q} \). The operator \( f \) sends \( A^{k_0,q} \) to \( A^{k_0-1,q+2} \). Thus, we see again from degree considerations in \( q \) that \( j_n(\alpha_{k_0,q_0}) = 0 \). So if \( k_0 < n \), there exists \( \beta \in A^{k_0+1,q_0} \) such that \( \alpha_{k_0,q_0} = j_n\beta \). The element \( \alpha + d_g \beta \) is homologous to \( \alpha \) and its term of degree \( k_0 \) is in \( \sum_{q > q_0} A^{k_0,q} \). (Of course \( \alpha + d_g \beta \) has no term of degree strictly less than \( k_0 \).) By successive approximations, we thus see that we can construct a representative of \( \alpha \) in \( A^\infty_G \). Now, let \( \alpha \in A^\infty_G \) be such that \( d_g \alpha = 0 \). In particular, \( (j_n - f)(\alpha) = 0 \). We can write at the point \( p \in P \) \( \alpha(Y,Q)_p = \lambda(Y,Q,p)(\omega_1 \wedge \cdots \wedge \omega_n)_p \) where \( \lambda(Y,Q,p) \in C^{-\infty}(n \oplus g/n) \otimes \Lambda H^*_p \), where \( H^*_p \) is the space of horizontal vectors. Let us write \( Y = \sum_j y_j \hat{E}_j \). For every \( j, 1 \leq j \leq n \), the equation \( (j_n - f)(\alpha) = 0 \) implies

\[
(y_j - \Omega_j)\lambda(Y,Q,p) = 0.
\]

It is not difficult to see (using for example the translation \( \lambda(Y,Q,p) \to \lambda(Y + \Omega, Q,p) \)) that \( \lambda(Y,Q,p) = \delta_n(-Y + \Omega)\beta(Q,p) \) where \( \beta(Q,p) \) is a generalised function on \( g/n \) with values in \( \Lambda H^*_p \). This way, we construct an element \( \beta \in C^{-\infty}(g/n, \Gamma) \). As \( \alpha \in A_G \) is \( G \)-invariant, \( \beta \in (C^{-\infty}(g/n, \Gamma))^G \).

As \( N \) is normal, the group \( N \) acts trivially on \( g/n \). Thus, we see that \( \beta \in (C^{-\infty}(g/n, \Gamma^N))^{G/N} = A^{-\infty}_{G/N}(P/N) \) and \( \alpha = q^* \beta \wedge \gamma_{\omega,\alpha} \).

The equation \( d_g \alpha = 0 \) and the injectivity of the map \( m_{\omega,0} \) at the cochain level (cf. Proof of Proposition 82) implies that \( d_g m_{\beta} = 0 \). Thus \( \beta \) is closed and the map \( m_{\omega} \) is surjective. The injectivity is proved by a similar argument.

Let \( K \) and \( U \) be compact subgroups of a Lie group \( L \). Then \( L \) can be thought of as a \( U \times K \)-manifold under \( x \cdot (u,k) = u^{-1}xk \), for \( x \in L \),
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$u \in U$ and $k \in K$. If $M$ is a $K$-manifold, we consider the $U \times K$ manifold $P = L \times M$, with twisted action as in Example 73.

Thus, specializing Theorem 88 to this example, we obtain

**Proposition 90** Let $K, U$ be compact subgroups of a Lie group $L$. Assume that there exists a $U \times K$-invariant orientation $o_K$ for the principal $K$-bundle $q_K : L \times M \to L \times_K M$. Then the map

$$m_{o_K} : H^{-\infty}_U(L \times_K M) \to H^{-\infty}_{U \times_K}(L \times M)$$

is an isomorphism. In particular, when $M = \text{point}$,

$$H^{-\infty}_U(L/K) \cong H^{-\infty}_{U \times_K}(L).$$

In the case of a free action, we have seen that $H^{-\infty}_G(M)$ is isomorphic to $H_G(M)$ under the multiplication by $\gamma_o$. However it may happen that the natural inclusion $H_G(M) \to H^{-\infty}_G(M)$ is identically 0. This is for example the case for the action of $G$ on itself, at least when $G$ is compact: The element $1 \in H_G(G) = \mathbb{R}$ has integral zero over $G$, while the integral of $\gamma_o \in H^{-\infty}_G(G)$ is equal to $\text{vol}(G, dg)\delta_g(X)$, as follows from the explicit formula for $\gamma_o$ given above.

Assume that $M$ is compact and oriented. Thus $f_M$ defines a map from $H^{-\infty}_G(M)$ to $C^{-\infty}(g)^G$. It is clear from the formula, given in Proposition 77 for the generator $\gamma_o$ that if $\alpha = \beta \wedge \gamma_o$ with $\beta \in H_G(M)$, then $f_M \alpha$ is a derivative $P(\partial)\delta_g$ of the $\delta_g$-function on $g$. Moreover, the order of the derivative is less or equal that of $\text{dim}(B)/2$. We will determine explicitly this map in a special case.

Let $K$ be a compact connected semi-simple Lie group and let $T$ be its maximal torus. Let $W$ be the Weyl group of $(K, T)$. Let $S(t)^W$ be the subalgebra of $W$-invariants in $S(t)$. Let $I$ be the ideal in $S(t)$ generated by all the invariants of positive degree. Similarly, let $S(t')^W$ be the subalgebra of $W$-invariants in $S(t')^W$ and let $J$ be the ideal in $S(t')$ generated by all the invariants of positive degree. Let $\delta_t$ be the $\delta$ function on $t$ determined by the Euclidean measure on $t$ associated to the Killing form.
If $f \in C^{-\infty}(t)$ and $Q \in S(t)$, then the derivative $Q(\partial)f$ of $f$ by the constant coefficient differential operator $Q(\partial)$ is well defined.

Similarly, if $P \in S(t)$ is a polynomial function on $t'$ and $Q \in S(t')$, we can define $Q(\delta)P$. An element $P \in S(t)$ is called harmonic if $Q(\delta)P = 0$, for all $Q \in J$. We denote by $\mathcal{H}$ the set of harmonic elements of $S(t)$.

**Lemma 91** Let

$$J := \{ f \in C^{-\infty}(t), Pf = 0, \text{for all } P \in J \}$$

be the set of generalised functions on $t$ annihilated by all the $W$-invariant functions $P$ without constant terms under multiplications. Then $J$ is equal to

$$J = \{ Q(\partial)\delta_t; Q \in \mathcal{H} \}.$$

**Proof:** Choose a $W$-invariant norm $|x|$ on $t$. If $f \in J$, then $f$ is annihilated by the invariant polynomial function $|x|^2$. Hence $f$ is supported at the origin and there exists a $Q \in S(t)$ such that $f = Q(\partial) \cdot \delta_t$. The equation $J \cdot Q(\partial) \cdot \delta_t = 0$ implies, by Fourier transform, that $Q$ is harmonic.

Let $k = t \oplus r$ be the $T$-invariant decomposition of $k$, and let $n = \dim r$. Choose compatible orientations $\alpha_k, \alpha_r, \alpha_t$ on $k, r, t$. Let $\kappa', \mu', \nu'$ be the forms of maximal degree on $k, r, t$ respectively, associated to the Killing form $(,)$ and our choice of orientations. We denote also by $\kappa'$ the left $K$-invariant form on $K$ coinciding with $\kappa'$ at the identity $e$ of $K$. Similarly, we extend $\nu'$ (resp. $\mu'$) as the left $K$-invariant $\dim t$-form (resp. $\dim r$-form) on $K$ coinciding with $\nu'$ (resp. $\mu'$) at $e$.

Let $\lambda \in t'$. The bilinear form on $r$ given by $B_\lambda(X, Y) = (\lambda, [X, Y])$ is an element of $\Lambda^2 r'$. Let $\Delta = \{ \alpha \in \imath t' \}$ be the set of roots of $(k, t)$. Choose an order on $\Delta$ compatible with the orientation $\alpha_r$, as in ([11], page 40). Let $U$ be the polynomial function $U(\lambda) = \prod_{\alpha > 0}(\lambda, i\alpha)$. Then $U \in \mathcal{H}$ and the map $P \in S(t') \rightarrow P(\partial)U$ induces an isomorphism from $S(t')/J$ to $\mathcal{H}$. Furthermore it is easy to see that $B_{\lambda}^{n/2} = ((n/2)!)U(\lambda)\mu'$.

Consider the free action of $T$ on $M = K$ by $k \cdot t = kt$. The space $H_T(K)$ is isomorphic to $H(K/T)$. The Chern-Weil map $W : S(t') \rightarrow H(K/T)$ is surjective, with kernel $J$. Thus we identify $H_T(K)$ with the $S(t')$-module $S(t')/J$. 

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Proposition 92 The map

\[ \int_K : H_T^{-\infty}(K) \to C^{-\infty}(t) \]

is an isomorphism from \( H_T^{-\infty}(K) \) to \( J \). Furthermore, we have

\[ \int_K \gamma_\circ = (-1)^{n/2} \text{vol}(K) U(\partial) \cdot \delta_t. \]

Proof:
Consider the curvature \( \Omega \) of \( K \to K/T \), determined by the \( T \)-invariant decomposition \( k = t \oplus r \). It is an element of \( \Lambda^2 r' \otimes S(t) \). Let us compute \( \exp \Omega \) in the algebra \( \Lambda r' \otimes S(t) \). The component of \( \exp \Omega \) of exterior degree \( n \) is given by the formula \( (\exp \Omega)[n] = \mu' \otimes U \). The term of exterior degree \( n \) of \( \delta_t(\Omega - X) \) is thus equal to \( (-1)^{n/2} \mu' \otimes U(\partial) \delta_t \). Formula for \( \gamma_\circ \in A_T^{-\infty}(K) \) given in Proposition 77 shows that the term of maximal exterior degree of \( \gamma_\circ \) is

\[ (\gamma_\circ)[\dim_K](X) = (-1)^{n/2} (U(\partial) \cdot \delta_t)(X) \kappa'. \]

Integrating over \( K \), we obtain the formula for \( \int_K \gamma_\circ \) given in the proposition.

As \( H_T(K) \) is generated by 1 over \( S(t') \), we obtain that \( H_T^{-\infty}(K) = (S(t')/J) \gamma_\circ \). Furthermore, as seen by Fourier transform, the map \( P \to P(U(\partial) \cdot \delta_t) \) induces an isomorphism from \( S(t')/J \) to \( J \) and we obtain our proposition.

Corollary 93 The natural map \( H_T(K) \to H_T^{-\infty}(K) \) is identically 0.

Proof: Elements of \( H_T(K) = H(K/T) \) come from the base, thus have integral zero on \( K \).

Remark 94 If a torus acts on a compact oriented manifold \( M \) without fixed points, every (equivariant) cohomology class in \( H_T(M) \) is of integral equal to zero, as follows from the localisation formula (see [3], chap 7). The preceding example (i.e \( T \) acting on \( K \) by right translations) gives a striking case of an action of \( T \) without fixed points, where any non-zero equivariant cohomology class with generalised coefficients has a non-zero integral.
10 A spectral sequence for $T$-equivariant cohomology

Let $K$ be a compact connected Lie group and $M$ a $K$-manifold. Let $T$ be a maximal torus of $K$. In section 8, we have seen that the $K$-equivariant cohomology $H^\infty_K(M)$ of $M$ can be computed in terms of the $T$-equivariant cohomology of $M$. In this section, we will establish a spectral sequence relating the $S(t')$-modules $H_T(M)$ and $H_T^{-\infty}(M)$.

Let $T$ be an abelian Lie group (not necessarily compact) and let $M$ be a $T$-manifold. Let $\mathfrak{t}$ be the Lie algebra of $T$. Then, as $T$ is abelian, $A_T(M) = S(t') \otimes A(M)^T$.

Similarly $A_T^{-\infty}(M) = C^{-\infty}(\mathfrak{t}, A(M)^T)$.

We can then consider $A_T^{-\infty}(M)$ as obtained from the space $A_T(M)$ by "extension" of coefficients.

Let us consider the space:

$$\Omega = C^{-\infty}(\mathfrak{t}, A(M)^T) \otimes S(t') \otimes \Lambda t'$$

$\mathbb{Z}$-graded by its exterior degree with respect to $\Lambda t'$, i.e.

$$\Omega^p = C^{-\infty}(\mathfrak{t}, A(M)^T) \otimes S(t') \otimes \Lambda^p t'.$$

Let $E^i$ be a basis of $\mathfrak{t}$ with dual basis $E_i$ of $t'$. An element $X \in \mathfrak{t}$ is written as $X = \sum_i x_i E^i$. We can consider an element of

$$V = C^{-\infty}(\mathfrak{t}, A(M)^T) \otimes S(t')$$

as a form $\alpha(X, Y) \in A(M)^T$ depending in a generalised way on the first variable $X \in \mathfrak{t}$ and in a polynomial way on the second variable $Y \in \mathfrak{t}$.

We consider on $V$ the $\mathbb{Z}/2$-grading given by the parity of an element in $A(M)^T$. Consider the $S(t')$-module structure on the space $V$ defined by $(E_i \cdot \alpha)(X, Y) = (x_i - y_i)\alpha(X, Y)$, i.e. $E_i(\theta \otimes P) = x_i \theta \otimes P - \theta \otimes y_i P$, for $\theta \in C^{-\infty}(\mathfrak{t}, A(M)^T)$ and $P \in S(t')$. 
Let $j$ be the Koszul differential of degree $-1$ on $\Omega = \mathcal{V} \otimes \Lambda t'$, got from the $S(t')$-module $V$ (cf. Formula 11 of section 3), i.e.,

$$j = \sum_i (x_i - y_i) \otimes \iota_A(E^i).$$

(as usual, in extending $\iota_A(E^i)$ to the tensor product of the two super-spaces $V$ and $\Lambda t'$, we respect the sign rules (2) and (3) of section 1.)

If $\alpha(X, Y) = \theta(X) \otimes P(Y) \in V$, the restriction $\alpha(X, X) = \theta(X)P(X)$ of $\alpha$ to the diagonal is well defined. Thus, for any $\beta \in \Omega$, the restriction $\beta(X, X)$ of $\beta$ to the diagonal is an element of $C^{-\infty}(t, \mathcal{A}(M)^T) \otimes \Lambda t'$. Let us denote by $r(\beta)$ the component of exterior degree zero of $\beta(X, X)$. Thus the map $r$ is a map from $\Omega$ to $\mathcal{A}_T^{-\infty}(M)$.

We can also write

$$\Omega = C^{-\infty}(t, \mathcal{A}_T(M)) \otimes \Lambda t',$$

where by definition

$$C^{-\infty}(t, \mathcal{A}_T(M)) = \sum_{p \geq 0} C^{-\infty}(t, \mathcal{A}_T^p(M))$$

and $\mathcal{A}_T^p(M)$ refers to the $\mathbb{Z}_+$-grading of $\mathcal{A}_T(M)$ defined in section 2.

Consider the differential $d_0$ on $\mathcal{A}_T(M)$ and extend it pointwise to $C^{-\infty}(t, \mathcal{A}_T(M))$ by defining $(\overline{d_0} f)(X) = d_0(f(X))$. Consider the operator

$$d_0 = \overline{d_0} \otimes I$$

of degree 0 (with respect to the $\mathbb{Z}$-grading of $\Lambda t'$) on $\Omega$.

The operators $j$ and $d_0$ satisfy $j^2 = 0$, $d_0^2 = 0$, $jd_0 + d_0j = 0$.

Consider the $\mathbb{Z}/2$ grading on $\Omega$ given by the parity of forms on $\mathcal{A}(M)^T$ together with the $\mathbb{Z}$-grading of $\Lambda t'$. Then $d_0$ and $j$ are odd operators. Define the operator

$$D = j + d_0.$$

The operator $D$ is an odd operator on $\Omega$ of square equal to 0.

Let $H(\Omega, D)$ be the cohomology space of $D$. It is a $\mathbb{Z}/2$-graded space.

**Proposition 95** The map $r : \Omega \to \mathcal{A}_T^{-\infty}(M)$ satisfies $rD = d_0r$. Moreover $r$ induces an isomorphism in cohomology. Thus the cohomology of the complex $(\Omega, D)$ is isomorphic to $H_T^{-\infty}(M)$. 
Proof:

The first assertion is immediate, as \( r_j = 0 \), and 
\[
 r(\theta(X) \otimes y, \iota(E_M)P(Y)) = 
\] 
\[
 x, \iota(E_M)(\theta(X)P(X)) \text{ if } \theta(X) \in C^{-\infty}(t) \text{ and } P(Y) \in \mathcal{A}_T(M). 
\]

Let \( n = \dim t \). As \( V \) is a tensor product of the free module \( S(t') \) by 
\[
 C^{-\infty}(t, \mathcal{A}(M)^T), 
\]
the space \( V \) is a free \( S(t') \)-module (cf. Corollary 14 of section 3). Thus by Proposition 12 the Koszul complex

\[
 0 \to \Omega^n \xrightarrow{\cdot j} \cdots \xrightarrow{\cdot j} \Omega^1 \xrightarrow{\cdot j} \Omega^0 \xrightarrow{r} \mathcal{A}^{-\infty}_T(M) \to 0
\]
is exact at all the levels \( \Omega^i \), for all \( i > 0 \). Exactness at \( \Omega^0 \) is easy to check.

Let \( \Omega' \) be the exact complex for \( j \) defined by \( \Omega^0 = \Omega^i \) if \( i > 0 \) and 
\[
 \Omega^0 = K_{err}. 
\]
Choose any homotopy \( h \) of \( \Omega' \) of degree 1 i.e. \( hj \circ jh = I_{\Omega'} \). Consider \( N = hd_0 + d_0h. \) Then \( N \) is an operator of degree 1 on \( \Omega' \). We have \( hD + Dh = I + N \) on \( \Omega' \), and \( N \) is a nilpotent operator commuting with \( D \). Let us prove that \( r \) is surjective in cohomology: Let \( \theta \in \mathcal{A}^{-\infty}_T(M) \) be such that \( d_4 \theta = 0. \) We lift \( \theta \) as a form in two variables \( \Theta(X, Y) = \theta(X) \) constant in \( Y \), i.e \( \Theta = \theta \otimes 1. \) Then \( r(D\Theta) = 0 \) i.e \( D\Theta = d_0\Theta \in \Omega' \). Thus 
\[
 (I + N)D\Theta = (hD + Dh)D\Theta = DhD\Theta \text{ i.e. } D(\Theta - (I + N)^{-1}hD\Theta) = 0. 
\]
The element \( w(\Theta) := \Theta - (I + N)^{-1}hD\Theta \) still satisfies \( r(w(\Theta)) = \theta \), and is 
a cocycle for \( D \).

Similarly, we prove that \( r \) is injective: Let \( \alpha \in \Omega \) be such that \( D\alpha = 0 \) and 
\[
 r(\alpha) = d_4 \theta. 
\]
Then \( \alpha' := \alpha - D\Theta \) satisfies \( D\alpha' = 0 \) and \( r\alpha' = 0. \) Then 
\[
 \alpha' = D(I + N)^{-1}h\alpha' \text{ is a boundary. This proves the proposition.} 
\]

We give below a more explicit way to construct a representative in 
\( H(\Omega, D) \) of an element in \( H^{-\infty}_T(M) \).

If \( Y \in t \), we define as in section 4 the tensor product contraction \( \iota_Y(\alpha) = \iota_Y(\iota_M + \iota_Y) \) on \( \Omega \). The horizontal space \( \Omega_{hor} \) is then defined as

\[
 \Omega_{hor} = \{ \alpha \in \Omega, \iota_Y(\alpha) = 0 \text{ for all } Y \in t \}. 
\]
The space \( \Omega_{hor} \) is stable by \( D \). There is a canonical projection map (see Definition 26 of section 4) from \( \Omega \) to \( \Omega_{hor} \) given by

\[
 h = \prod_{i}(I - \epsilon_i \iota(E^i)) 
\]
where \( \epsilon_i \) denotes the multiplication by \( E_i \) on \( \Lambda t' \).
We denote by \( w : \mathcal{A}^\infty_T(M) \to \Omega \) the map \( w(\theta) = h(\Theta) \), where \( \Theta \) is the lift of \( \theta \) constant in \( Y \).

We have
\[
w(\theta) = \Theta + (-1)^{|\theta|} \sum_i \iota(E_M^i)\Theta \otimes E_i - \sum_{i < j} \iota(E_M^i)\iota(E_M^j)\Theta \otimes E_i \land E_j + \cdots
\]

**Lemma 96** The map \( w \) satisfies: \( wd\theta = Dw\theta \). Further \( w \) induces an isomorphism in cohomology, inverse to the map in cohomology induced by \( r \).

**Proof:**
Both terms of this equation belong to \( \Omega_{h,or} \). Thus, to prove that they are equal, we need only to compute their terms of zeroth exterior degree. The element \( wd\theta \) has zero-exterior degree term equal to the lift of \( (d\theta)(X) = d_M\theta(X) - \sum_i x_i(t(E_M^i)\theta(X)) \), constant in \( Y \).

The element \( Dw\theta \) has zero-exterior degree term
\[
d_M\theta(X) - \sum_i y_i(t(E_M^i)\theta(X)) + \sum_{i,j} (y_i - x_i)\iota(A_{ij}(E^i_j)\iota(E_M^j)\theta(X))
\]
which is equal to
\[
d_M\theta(X) - \sum_i x_i(t(E_M^i)\theta(X)).
\]

It is clear that \( rw = 1 \). But since the map \( r \) induces an isomorphism in cohomology, we get that \( w \) also induces isomorphism in cohomology inverse to that of \( r \).

The complex \( (\Omega, D) \) admits an increasing filtration \( \mathcal{F} = \{ \mathcal{F}_p \}_{0 \leq p \leq \dim t} \) by the exterior degree in \( \Lambda t' \) i.e. \( \mathcal{F}_p = \bigotimes_{k \leq p} \Omega_k \). This canonically gives rise to a convergent homology spectral sequence \( E^r \) converging to \( H(\Omega, D) \).

**Lemma 97** Assume that \( T \) is compact and \( M \) is a compact \( T \)-manifold. Then
\[
E^1_p = C^{-\infty}(t) \otimes H_T(M) \otimes \Lambda^p t'.
\]

**Proof:** By definition
\[
E^1_p = H(\mathcal{F}_p/\mathcal{F}_{p-1}, D) = H(\mathcal{F}_p/\mathcal{F}_{p-1}, d_0) \quad \text{since } j(\mathcal{F}_p) \subset \mathcal{F}_{p-1},
\]
\[
\cong H(C^{-\infty}(t, \mathcal{A}_T(M)) \otimes \Lambda^p t', d_0) \cong H(C^{-\infty}(t, \mathcal{A}_T(M)), d_t) \otimes \Lambda^p t'.
\]
It is easy to see that $\text{Ker}(d_t) = C^{-\infty}(t, Z_T(M))$ and moreover $\text{Im}(d_t) \subseteq C^{-\infty}(t, B_T(M))$. Further, using Green's operator, we get a continuous splitting of the map

$$\mathcal{A}^{n-1}_T(M) \xrightarrow{\overline{d}_t} B^n_T(M),$$

and hence $\text{Im}(d_t) = C^{-\infty}(t, B_T(M))$. Also $H^*_T(M)$ being finite dimensional (cf. Remark 98), the projection $Z^*_T(M) \to H^*_T(M)$ admits a continuous splitting. From this we easily conclude that

$$H(C^{-\infty}(t, \mathcal{A}^*_T(M)), d_t) \cong C^{-\infty}(t, H^*_T(M)).$$

This proves the lemma.

**Remark 98** If $T$ is compact and $H^*(M)$ is finite dimensional in each degree then so is $H^*_T(M)$. This follows from the Serre spectral sequence for the fibration $M \to E(T) \times_T M \to B(T)$. In particular, for compact $M$, $H^*_T(M)$ is finite dimensional in each degree.

The differential $d^1 : E^1_p \to E^0_p$ of degree $-1$ induced by $D = d_0 + j$ on $E^1 = C^{-\infty}(t) \otimes H_T(M) \otimes \Lambda t'$ is the Koszul differential $j$ associated to the canonical $S(t')$-module structures on $C^{-\infty}(t)$ and $H_T(M)$. Hence, combining Proposition 95, Lemma 97 and Lemma 15 we obtain the main result of this section.

**Theorem 99** Let $T$ be a compact abelian Lie group and let $M$ be a compact $T$-manifold. Then the cohomology group $H^{-\infty}_T(M)$ has an increasing $\mathbb{Z}_+$-filtration $H_p$, and a convergent homology spectral sequence with

$$E^2_p = \text{Tor}^{S(t')}(C^{\infty}(t), H_T(M))$$

and

$$E^\infty_p = H_p / H_{p-1},$$

where $C^{\infty}(t)$ and $H_T(M)$ have their canonical $S(t')$-module structures.

This spectral sequence is functorial with respect to the $T$-equivariant smooth maps. Further the total $\mathbb{Z}/2$-grading given by the standard $\mathbb{Z}_+$ degree on $H^*_T(M)$ together with the $p$ index in Tor is compatible with the $\mathbb{Z}/2$-grading of $H^{-\infty}_T(M)$.

We obtain a number of corollaries:
Corollary 100 Let $M, N$ be compact $T$-manifolds with a $T$-equivariant smooth map $f : M \to N$. Assume that the induced map $f^* : H^\ast_T(N) \to H^\ast_T(M)$ is an isomorphism in $T$-equivariant cohomology. Then the induced map

$$f^* : H^{-\infty}_T(N) \to H^{-\infty}_T(M)$$

is also an isomorphism.

Proof: It follows immediately from the above spectral sequence. 

The following corollary was obtained in section 6 as a consequence of Theorem 60 (cf. Corollary 61).

Corollary 101 For any compact $T$-manifold $M$ such that $H^\ast_T(M)$ is a projective $S(t')$-module, the canonical map

$$\beta_{T,M} : C^{-\infty}(t) \otimes_{S(t')} H_T(M) \to H^{-\infty}_T(M)$$

is an isomorphism.

Proof: Since $H_T(M)$ is $S(t')$-projective, the spectral sequence of Theorem 99 has $E^2_p = 0$, unless $p = 0$. In particular the spectral sequence degenerates at the $E^2$-term itself. Also

$$E^2_0 = \text{Tor}_0^{S(t')}(C^{-\infty}(t), H_T(M)) \approx C^{-\infty}(t) \otimes_{S(t')} H_T(M).$$

This proves the corollary. 

Let $T$ be an abelian Lie group and let $M$ be a $T$-manifold. The $T$-equivariant de Rham complex with generalised coefficients admits a graded subcomplex obtained by forming the algebraic tensor product

$$\mathcal{A}^{-\infty}_T(M) := C^{-\infty}(t) \otimes \mathcal{A}(M)^T.$$

This subcomplex is stable by the action of $S(t')$. We denote the cohomology of this subcomplex by $\tilde{H}^{-\infty}_T(M)$. We have the following comparison:

Proposition 102 Let $T$ be a compact abelian Lie group and let $M$ be a compact $T$-manifold. Then the canonical map $\tilde{H}^{-\infty}_T(M) \to H^{-\infty}_T(M)$ induced from the inclusion $\mathcal{A}^{-\infty}_T(M) \to \mathcal{A}^{-\infty}_T(M)$, is an isomorphism.
Proof: Recall the definition of the complex $(\Omega, D)$ and define a subcomplex
\[ \tilde{\Omega} := C^{-\infty}(t) \otimes \mathcal{A}(M)^{T} \otimes S(t') \otimes \Lambda t'. \]

The cochain map $r : \Omega \to C^{-\infty}(t, \mathcal{A}(M)^{T})$ restricts to a cochain map (denoted by) $\tilde{r} : \tilde{\Omega} \to C^{-\infty}(t) \otimes \mathcal{A}(M)^{T}$. By the same proof as that of Proposition 95, we can easily see that $\tilde{r}$ induces isomorphism in cohomology.

Thus the augmented complex
\[ \tilde{\Omega} \xrightarrow{\tilde{r}} \tilde{\mathcal{A}}_T^{-\infty}(M) \]
maps by the natural inclusion $i$ into the augmented complex
\[ \Omega \xrightarrow{i} \mathcal{A}_T^{-\infty}(M). \]

The filtration $\{ \mathcal{F}_p \}$ of $\Omega$ gives rise to the filtration $\{ \mathcal{F}_p :\Omega \cap \tilde{\Omega} \}$ of $\tilde{\Omega}$. In particular, we get the induced map $\tilde{E}_p^{t} \to E_p^{t}$, where $\tilde{E}_p^{t}$ is the spectral sequence corresponding to the filtration $\{ \mathcal{F}_p \}$. We have $\tilde{E}_1^{t} = C^{-\infty}(t) \otimes H_T(M) \otimes \Lambda^t t'$. In particular, $E_1^{t} \to E_1^{t}$ is an isomorphism, and hence the inclusion $\tilde{\Omega} \to \Omega$ induces isomorphism in cohomology. But then the map $i : \tilde{\mathcal{A}}_T^{-\infty}(M) \to \mathcal{A}_T^{-\infty}(M)$ also induces an isomorphism in cohomology. $\blacksquare$

The spectral sequence obtained in Theorem 99 may sometimes be used to determine the torsion groups $\text{Tor}^{S(t)}_{i}(C^{-\infty}(t), H_T(M))$. For example, if $K$ is a compact connected Lie group with maximal torus $T$ and if $M$ is a $K$-manifold such that $\text{Tor}^{S(t)}_{i}(C^{-\infty}(t), H_T(M))$ is equal to zero except for $i = i_0$ (for some $i_0$), then from the degenerate spectral sequence of Theorem 99 and Theorem 67,
\[ \text{Tor}^{S(t)}_{i_0}(C^{-\infty}(t), H_T(M)) \cong S(t') \otimes S(k)^{\otimes i} H_{K}^{-\infty}(M). \]

As we next show, this hypothesis is valid when $M$ is homogeneous under $K$. Let $U$ be a closed subgroup of $K$. Let us choose a maximal torus $T_U$ of $U$ and let $T$ be a maximal torus of $K$ containing $T_U$.

**Proposition 103** Let $M = K/U$, where $K$ is a compact connected Lie group and $U$ a closed subgroup. Let $\chi : U \to \pm 1$ be the character $\chi(u) := \det_{K/U}u$. Then the group
\[ \text{Tor}^{S(t')}_{i}(C^{-\infty}(t), H_T(M)) = 0 \quad \text{for} \ i \neq d := \dim(T/T_U) \]
Equivariant cohomology with generalised coefficients

and

\[ \text{Tor}^{S(t')}^d(C^{-\infty}(t), H_T(M)) \cong S(t') \otimes_{S(t')^w} C^{-\infty}(u)^\Lambda. \]

**Proof:** Let \( t_U \) be the Lie algebra of \( T_U \). Let \( W_U \subset GL(t_U) \) be the Weyl group of the pair \((U, T_U)\) (i.e., \( W_U = N_U(T_U)/T_U \)). If \( P \in S(t')^W \) is a \( W \)-invariant function, its restriction to \( t_U \) is \( W_U \)-invariant (to see this, use Chevalley’s theorem to conclude that \( P \) is the restriction to \( t \) of a \( K \)-invariant polynomial on \( k \)). From Proposition 65 of section 7, we have

\[ H_T(M) \cong S(t') \otimes_{S(t')^w} S(t'_U)^W. \]

Thus, to prove the vanishing part, it is sufficient to see that \( \text{Tor}^{S(t')}^d(C^{-\infty}(t), H_T(M)) \cong \text{Tor}^{S(t')}^d(C^{-\infty}(t), S(t') \otimes_{S(t')^w} S(t'_U)^W). \)

Consider \( N := S(t') \otimes_{S(t')^w} S(t'_U) \) as a \( (S(t'), W_U) \)-module by the action of \( S(t') \) on the left and the action of \( W_U \) on the right factor. Then \( N^W = S(t') \otimes_{S(t')^w} (S(t')^W) \). The space \( \text{Tor}^{S(t')}^d(C^{-\infty}(t), N) \) thus carries a canonical structure of \( (S(t'), W_U) \)-module and moreover by the standard averaging process

\[ \text{Tor}^{S(t')}^d(C^{-\infty}(t), H_T(M)) \cong \text{Tor}^{S(t')}^d(C^{-\infty}(t), N)^W. \]

Thus, to prove the vanishing part, it is sufficient to see that \( \text{Tor}^{S(t')}^d(C^{-\infty}(t), N) = 0 \) except for \( i = d \).

Let \( t_1 = t/t_U \), so that \( t'_1 \subset t' \). Consider the partial Koszul complex \( j_1 = j_{t_1} : S(t') \otimes \Lambda t'_1 \rightarrow S(t'_U) \). This gives a \( S(t') \)-free resolution of \( S(t'_U) \).

As \( S(t') \) is free over \( S(t')^W \), the complex \( S(t') \otimes_{S(t')^w} (S(t') \otimes \Lambda t'_1) \) with differential \( I \otimes j_1 \) gives a \( S(t') \)-free resolution of \( S(t') \otimes_{S(t')^w} S(t'_U) \). Thus \( \text{Tor}^{S(t')}^d(C^{-\infty}(t), N) \) is the homology of the complex

\[ C^{-\infty}(t) \otimes_{S(t')} (S(t') \otimes \Lambda t'_1)) \cong C^{-\infty}(t) \otimes_{S(t')^w} (S(t') \otimes \Lambda t'_1) \]

i.e., of the complex

\[ (41) \]

\[ 0 \rightarrow C^{-\infty}(t) \otimes_{S(t')^w} (S(t') \otimes \Lambda^d t'_1) \rightarrow C^{-\infty}(t) \otimes_{S(t')^w} (S(t') \otimes \Lambda^{d-1} t'_1) \]

\[ \cdots \rightarrow C^{-\infty}(t) \otimes_{S(t')^w} (S(t') \otimes t'_1) \rightarrow C^{-\infty}(t) \otimes_{S(t')^w} S(t') \rightarrow 0. \]

Consider the isomorphism obtained in Proposition 63 of section 6

\[ C^{-\infty}(t)^r \otimes_{S(t')^w} S(t') \cong C^{-\infty}(t) \]
induced from the multiplication map. Hence, the map $P_1 \otimes F \otimes P_2 \to P_1 \otimes FP_2$ gives an isomorphism of

$$S(t') \otimes_{S(t')^w} C^{-\infty}(\mathfrak{t})^c \otimes_{S(t')^w} S(t')$$

with

$$S(t') \otimes_{S(t')^w} C^{-\infty}(\mathfrak{t}).$$

Thus we obtain an isomorphism of the complex (41) with the complex

$$S(t') \otimes_{S(t')^w} (C^{-\infty}(\mathfrak{t}) \otimes \mathcal{A}_{\mathfrak{t}}')$$

under the differential $I \otimes j^{-\infty}_1$. By Proposition 20 of section 3, the homology of the complex $(C^{-\infty}(\mathfrak{t}) \otimes \mathcal{A}_{\mathfrak{t}}', j^{-\infty}_1)$ is non-zero only in degree $d$. As $S(t')$ is free over $S(t')^w$, we obtain the vanishing part of the proposition. Furthermore, by the remark just before this proposition and Theorem 44 of section 5, we obtain the assertion regarding $\text{Tor}_d$. 

11 Localisation formula

Let $T$ be a torus, i.e. a compact connected abelian Lie group, acting on a compact oriented manifold $M$. Let $\alpha \in H_T^{-\infty}(M)$. The integral $\Theta(X) := \int_M \alpha(X)$ of $\alpha$ is a generalised function on $\mathfrak{t}$. When $\alpha \in H_T^{-\infty}(M)$, the localisation formula (see [3], chapter 7) gives $\Theta(X)$ in terms of the restriction of $\alpha$ to the fixed submanifold $M^T$ of $M$. As shown by Proposition 92 of section 9 (where $M^T$ is empty but the map $\int_M$ is not zero), it is not possible to determine $\int_M \alpha$ in terms of $\alpha|M^T$ in the generalised case. The main reason for the difference between the $C^{\pm\infty}$-cases is that the space $H_T^{-\infty}(\text{point}) = C^{-\infty}(\mathfrak{t})$ is not torsion free over $S(t')$. Indeed, for $\alpha \in H_T^{-\infty}(M)$, we will find a non-zero polynomial $P \in S(t')$ and determine $P(X) \int_M \alpha(X)$ in terms of $\alpha|M^T$, as in the $C^{-\infty}$-case.

The localisation formula, we are going to give in the generalised case, involves choosing a $T$-equivariant embedding of $M$ in a real representation space $V$ of $T$. This is always possible, see ([8], Chap 6, Theorem 4.1). Let $V_0$ be the subspace of $T$-fixed vectors and let

$$V = V_0 \oplus V_1$$
be the $T$-invariant decomposition. Thus $\det_{V_1}(X)$ is a non zero polynomial on $t$. The fixed submanifold $M_0 = M^T$ of $M$ is given by $M \cap V_0$. The space $V_1$ is even dimensional. Let us choose an orientation on $V_1$. This orientation determines a polynomial square root of $\det_{V_1}(X)$. Using a $T$-invariant metric on $V$, we view the normal bundle $\mathcal{N}$ of $M_0$ in $M$ as a $T$-equivariant subbundle of the trivial bundle $M_0 \times V_1$. The bundle $\mathcal{N}$ is $T$-orientable and is of even rank. Let us denote by $\mathcal{Q}$ the supplementary bundle:

$$M_0 \times V_1 = \mathcal{N} \oplus \mathcal{Q}.$$  

The bundle $\mathcal{Q}$ is a $T$-equivariant bundle over $M_0$. We choose orientations of $V_1, \mathcal{N}, \mathcal{Q}$ in a compatible way. Let $u_{\mathcal{N}} \in H_{\text{cpt}, T}(\mathcal{N})$, $u_{\mathcal{Q}} \in H_{\text{cpt}, T}(\mathcal{Q})$ be the $T$-equivariant Thom classes (see Definition 8 section 2) of $\mathcal{N}$, $\mathcal{Q}$ respectively. Let $\chi(\mathcal{N}) \in H_T(M_0)$ (resp. $\chi(\mathcal{Q})$) be the equivariant Euler class of the bundle $\mathcal{N} \to M_0$ (resp. $\mathcal{Q}$). By definition (we differ here from the definition of [3], chapter 7), the restriction of $u_{\mathcal{N}}$ (resp. $u_{\mathcal{Q}}$) to $M_0$ via the zero section is equal to $\chi(\mathcal{N})$ (resp. $\chi(\mathcal{Q})$). We have the following equality in $H_T(M_0)$:

$$(42) \quad (-2\pi)^{-\dim V_1/2} \det_{V_1}^{1/2}(X) \cong \chi(\mathcal{N})(X) \chi(\mathcal{Q})(X).$$

Let us fix an orientation of $M$ and consider the compatible orientation of $M_0$. Following is the localisation formula in generalised cohomology.

**Theorem 104** Let $T$ be a torus acting on a compact oriented manifold $M$. For $\alpha \in H_{T}^{-\infty}(M)$, we have the equality

$$(-2\pi)^{-\dim V_1/2} \det_{V_1}^{1/2}(X) \int_{M_0} \alpha(X) = \int_{M^T} \alpha(X) \chi(\mathcal{Q})(X)$$

as elements of $C^{-\infty}(t)$.

**Proof:** The proof is obtained by imitating the proof in the $C^\infty$-case given in [2]. Consider the Thom class $u_1(X) \in H_{\text{cpt}, T}(V_1)$ of the $T$-vector space $V_1$, thought of as a $T$-equivariant vector bundle $q : V_1 \to \text{point}$. We have

$$(-2\pi)^{-\dim V_1/2} q^* (\det_{V_1}^{1/2}(X)) \sim u_1(X)$$

as elements of $H_T(V_1)$. Consider the map $p : M \to V_1$ induced by the projection of $V = V_0 \oplus V_1$ to $V_1$. Thus

$$(-2\pi)^{-\dim V_1/2} \det_{V_1}^{1/2}(X) \int_{M_0} \alpha(X) = \int_{M} \alpha(X) p^* u_1(X).$$
We can take a representative of \( u_1 \) (as a cohomology class in \( H^*_{cpt,T}(V_1) \)) supported in a sufficiently small neighborhood of 0 in \( V_1 \). Thus we may assume that \( p^*u_1 \) is compactly supported in a \( T \)-stable tubular neighborhood \( U \) of \( M_0 = p^{-1}(0) \). Let \( \pi \) be the \( T \)-equivariant projection of \( U \to M_0 \). Let \( i \) be the inclusion of \( M_0 \) in \( M \). We have seen in section 2, Proposition 6, that the restriction \( \alpha|U \) of \( \alpha \) to \( U \) is equivalent to \( \pi^*i^*\alpha \) in \( H_T^{-\infty}(U) \). As \( p^*u_1 \) is compactly supported in \( U \), we obtain

\[
\int_M \alpha(X)p^*u_1(X) = \int_U \alpha(X)p^*u_1(X) = \int_U \pi^*i^*\alpha(X)p^*u_1(X) = \int_{M_0} \alpha(X)p_1^*u_1(X).
\]

Let \( \beta = p^*u_1 \in H^*_{cpt,T}(U) \). It remains to show that \( p_*\beta(X) = \chi(Q)(X) \) in \( H_T(M_0) \).

The restriction of \( \beta(X) \) to \( M_0 \) is equal to \( (-2\pi)^{-\dim V_1/2} \det_{V_1}^{1/2}(X) \). The tubular neighborhood \( U \) of \( M_0 \) in \( M \) is \( T \)-equivariantly diffeomorphic to the normal bundle \( N \to M_0 \). Let \( u_N \) be the equivariant Thom class of \( N \). Then, it is well known (see the proof of Proposition 9 of section 2) that \( \beta = (\pi^*\pi_*\beta)u_N \) in \( H^*_{cpt,T}(U) \). By restricting this equality to \( M_0 \), we obtain

\[ (-2\pi)^{-\dim V_1/2} \det_{V_1}^{1/2}(X) \cong p_*\beta(X)\chi(N)(X) \]

in \( H_T(M_0) \). As \( \det_{V_1}(X) \) is a non zero polynomial, \( \chi(N)(X) \) is invertible on the open set \( \det_{V_1}(X) \neq 0 \). By Formula (42), we obtain the equality \( p_*\beta(X) = \chi(Q)(X) \). This proves the theorem. \( \blacksquare \)

Let us illustrate the localisation formula in the simple example of \( M = P_1(C) \).

Let \( p_1 \) be the point at infinity of \( M \). Then \( U = M - \{p_1\} \) is isomorphic to \( C \). We consider the action of \( T = \{e^{i\theta}\} \) on \( P_1(C) \) given by \( z \to e^{i\theta}z \). This action has two fixed points \( p_0 = 0 \) and \( p_1 = \infty \). We write still \( p_0, p_1 \) for the injections of \( p_0 \) and \( p_1 \) in \( M \).

We write an element of \( \mathfrak{t} \) as \( X = \theta J \), with \( \exp 2\pi J = 1 \). Let us first describe the \( T \)-equivariant cohomology of \( M \). It is a free \( S(\mathfrak{t}') \)-module with two generators \( \alpha, \beta \). We can normalise these two generators, by requiring

\[ p_0^*\alpha = 1, \quad \int_M \alpha = 0 \]
while
\[ p_0^*(\beta) = 0, \quad \int_M \beta = 1. \]
Identifying \( X = \theta J \) with \( \theta \), some specific representatives of \( \alpha \) and \( \beta \) are
\[ \alpha = 1 \]
\[ \beta(\theta) = (2\pi)^{-1}(\theta|z|^2(1 + |z|^2)^{-1} + i(1 + |z|^2)^{-2}dz \wedge d\overline{z}). \]
The restriction maps \( p_0^*, p_1^*: H_T(M) \to S(t') \) satisfy
\[ 2\pi(p_1^* - p_0^*) = \theta \int_M. \]
Consider now \( H_T^{-\infty}(M) \). Let \( \delta(\theta) \) be the \( \delta \) function at 0. The element
\[ v(\theta) = \delta(\theta)\beta(\theta) \]
is in \( H_T^{-\infty}(M) \).
As \( \theta \delta(\theta) = 0 \), we have
\[ v(\theta) = (-2i\pi)^{-1}\delta(\theta)(1 + |z|^2)^{-2}dz \wedge d\overline{z}. \]
Thus, the element \( v \) does not have component in zero exterior degree, in particular its restriction to \( M^T = \{p_0\} \cup \{p_1\} \) is zero. The integral \( \int_M v(\theta) \) is equal to \( \delta(\theta) \) and is supported at 0. This is compatible with the localisation theorem which asserts that \( \theta \int_M v(\theta) = 0 \).
Let \( P = p_0^* \oplus p_1^* \) be the map:
\[ P: H_T^{-\infty}(M) \to C^{-\infty}(t) \oplus C^{-\infty}(t). \]
Thus \( v \) is in the kernel of \( P \).
In fact, we have the exact sequence
\[ 0 \to C \to H_T^{-\infty}(M) \to C^{-\infty}(t) \oplus C^{-\infty}(t) \to 0. \]
The exactness of this sequence can be seen as follows: As \( H_T(M) \) is free over \( S(t') \), we have
\[ H_T^{-\infty}(M) = C^{-\infty}(t)\alpha + C^{-\infty}(t)\beta. \]
Writing $\nu = f\alpha + g\beta$, we see that if $P\nu = 0$, then $f = 0$ and $\theta g = 0$. Thus $\nu$ is proportional to $v$. Let us see that $P$ is surjective. The restriction maps $p_0^* p_1^* : H^{-\infty}(M) \to C^{-\infty}(t)$ still satisfy

$$(2\pi)(p_1^* - p_0^*) = \theta \int_M.$$

Thus we have $p_0^* \nu = f$ and $(2\pi)p_1^* \nu = \theta g + (2\pi)f$. As it is always possible to divide by $\theta$ in the space $C^{-\infty}(t)$, we see that $P = p_0^* \oplus p_1^*$ is surjective.

References


REFERENCES


