Towards the Maximal Number of Components of a Non-singular Surface of Degree 5 in $RP^3$

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1 Introduction

The problem of determining the maximal number of connected components of a surface of given degree $m$ in $RP^3$ was posed by D. Hilbert yet in 1900 (see the 16-th problem of his famous list). Despite of developments in the last decades on the topology of real algebraic varieties the response is still not known, except in the trivial cases $m \leq 3$ and the case $m = 4$. In this last case the maximal number of components is equal to 10 (surfaces with 10 components were constructed by K. Rohn [Ro] in 1886, a proof of the maximality was given by V. Kharlamov [Kh 1] in 1972).

It is well known, that to determine the maximal number of components it is enough to consider non-singular surfaces: by a small variation, any singular surface can be replaced by a non-singular one having at least the same number of components.

A standard application of the Smith and Comessatti inequalities (see §3) gives an estimate: the number of components of a non-singular surface of degree $m$ in $RP^3$ is less or equal to $(5m^3 - 18m^2 + 25m)/12$. In particular, it can not be more than 25 for $m = 5$.

V. Kharlamov [Kh 2] constructed a surface of degree 5 in $RP^3$ with 21 components. (The surface constructed is a M-surface: the total $Z_2$-homology group has same rank as that of its complexification; see §3)

In the present paper we construct a non-singular surface of degree 5 in $RP^3$ with 22 components. We follow the scheme of [Kh 2] and use, in addition, some elements of Itenberg’s recent construction [It] of counter-examples to the Ragsdale conjecture (see [Ra]).

2 Construction

2.1 An equivariant analogue of Horikawa’s theorem

By a real algebraic (or analytic) manifold we mean a complex one supplied with complex conjugation. $X$ being a real variety, denote its real point set by $RX$ and the complex point one by $CX$.

Let $\Sigma_2$ be the standard non-singular model of the real cone defined in $P^3$ by the equation $x_0^2 + x_1^2 = x_2^2 + x_3^2$. Following Horikawa [Ho], consider an irreducible curve $B$ on $\Sigma_2$ verifying the following conditions:
(i) its intersection number with a linear generator of $\Sigma_2$ is equal to 6;
(ii) its intersection number with the inverse image of the vertex of the cone is equal to 1;
(iii) it has only 2 singular points, these points are ordinary triple points and they lie both on the same linear generator $L$.

We will call it the Horikawa curve.

Denote by $\tilde{W}$ the surface obtained from $\Sigma_2$ by blowing-up the two singular points of a Horikawa curve $B$ and by $\tilde{L}$ and $\tilde{B}$ the proper transforms of $L$ and $B$ under this blowing-up. Then take a double covering $\tilde{S} \to \tilde{W}$ with branch locus $B \cup L$. Such a covering exists because of (i-iii) and it is unique.

The inverse image of $L$ is a non-singular rational curve with self-intersection number $-1$. Contracting it to a point, we get a non-singular surface; denote it by $S$.

If the Horikawa curve is real, the surface $\tilde{S}$ gets, in a usual way, 2 canonical real structures. They are liftings of the complex conjugation of $W$. They differ by the covering transformation and both can be projected to $S$; call also canonical the two resulting real structures on $S$.

**Proposition 1** Let the Horikawa curve $B$ be real and let $S$ be supplied with one of its canonical real structures. Then there exists an equivariant deformation of $S$ to a non-singular surface of degree 5 in $\mathbb{R}P^3$.

**Proof.** Take a versal deformation $p : L \to M$ of $S$. By Horikawa’s theorem (see [Ho, Th. 3]), $M$ consists of 2 smooth irreducible components $M_0$ and $M_1$ intersecting normally; $\dim_{\mathbb{C}}M_0 = \dim_{\mathbb{C}}M_1 = 40$, $\dim_{\mathbb{C}}M_0 \cap M_1 = 39$. Points of $M_0 \setminus M_0 \cap M_1$ correspond to quintic surfaces and points of $M_1 \setminus M_0 \cap M_1$ to coverings of $P^1 \times P^1$. The standard versality arguments show that the deformation may be made equivariant. It rests to remark that the corresponding antiholomorphic involution does not interchange irreducible components of $M$ (they are of different nature) and that $RM_0$ and $RM_1$, as fixed point sets of an antiholomorphic involution on smooth complex manifolds, are smooth connected manifolds; they intersect normally and

$$\dim_{\mathbb{R}}M_0 = \dim_{\mathbb{R}}M_1 = 40, \quad \dim_{\mathbb{R}}M_0 \cap M_1 = 39 \quad \blacksquare$$

### 2.2 A special case of the Viro theorem

Let $P$ be a convex polygon in $\mathbb{R}^2$ with integer vertices and that verifies the following condition: it is contained in the triangle

$$\Delta = \{x \geq 0, \ y \geq 0, \ x + y \leq m\}, \ m \in \mathbb{N}$$

and it contains vertices $x = 0, y = m$ and $x = m, y = 0$. In the sequel, such a polygon will serve as a Newton polygon of curves of degree $m$ with a singularity at the origin prescribed by Newton polygonal line $\Gamma(P)$, which is, by definition, the union of sides of $P$ facing the origin.

Suppose that $P$ is triangulated, that the vertices of the triangles are integer and that some distribution of signs, $a_{i,j} = \pm$ at the vertices of the triangulation, is given. Then there arises a naturally associated piecewise-linear curve $L$ in $\mathbb{R}P^3$. 

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The construction of $L$ is the following.

Take copies

$$P_x = s_x(P), \quad P_y = s_y(P), \quad P_{xy} = s(P)$$

and

$$\Delta_x = s_x(\Delta), \quad \Delta_y = s_y(\Delta), \quad \Delta_{xy} = s(\Delta)$$

of $P$ and $\Delta$, where $s = s_x \circ s_y$ and $s_x, s_y$ are reflections with respect to the coordinate axes. Extend the triangulation of $P$ to a symmetric triangulation of $P_\times = P \cup P_x \cup P_y \cup P_{xy}$ and extend the distribution of signs to a distribution on the vertices of the extended triangulation which verifies the modular property: $g^*(a_{i,j}x^iy^j) = a_{g(i,j)}x^iy^j$

for $g = s_x, s_y$ and $s$ (other words, the sign at a vertex is the sign of the corresponding monomial in the quadrant containing the vertex).

If a triangle of the triangulation has vertices of different signs, select a midline separating them. If a midline comes to $\Gamma(P)$ at a point $b$, select also the segment joining $b$ to the origin. Denote by $L_\times$ the union of the selected midlines and segments. It is contained in $T_\times = \Delta \cup \Delta_x \cup \Delta_y \cup \Delta_{xy}$. Glue by $s$ the sides of $T_\times$. The resulting space $T$ is homeomorphic to $RP^2$. Take the curve $L$ to be the image of $L_\times$ in $T$.

A pair $(T, L)$ is called a chart of a real plane algebraic curve $A$, if there exists a homeomorphism $(T, L) \to (RP^2, RA)$ which maps each segment of $L$ joining the origin and an edge of $\Gamma(P)$ to a branch of $A$ corresponding to this edge.

Call a curve $A$ regular, if it does not have singular points outside of the origin.

Let us introduce 2 additional assumptions: the considered triangulation of $P$ is primitive and convex. The first condition means that all triangles are of area $1/2$ (or, equivalently, that all integer points of $P$ are vertices of the triangulation). The second one means that there exists a convex piecewise-linear function $P \to R$ which is linear on each triangle of the triangulation and not linear on the union of 2 triangles.

The following statement is the special case of Viro's theorem [Vi 1, Th. 1.4]:

Proposition 2 Under the assumptions made above on the polygon $P$ and its triangulation, there exists a real regular curve $A$ in $RP^2$ with chart $(T, L)$.

2.3 Lemma

We say that a real curve in $P^2$ is of class $H$, if

(i) its Newton diagram is the pentagon $\Pi$ presented in Figure 1,

(ii) there are 3 branches corresponding to the edge $BC$ of the diagram, they are smooth and each of them is tangent with simple inflexion to another.

We say that a real curve in $P^2$ is of class $\tilde{H}$, if

(i) its Newton diagram is the quadrangle $\Pi$ presented in Figure 2,

(ii) the truncation $f_{BD}$ of a polynomial $f$ defining the curve is equal to

$$\gamma x(y - ax^3)(y - bx^3)(y - cx^3)$$

where $\gamma \neq 0, a, b, c$ are real numbers and $a, b, c$ are pairwise different.
Curves of class $\overline{H}$ have 4 branches at the origin, 3 of them, similarly for class $H$, are smooth and pairwise tangent with simple inflexion. (The principle difference between these two classes is that a curve of class $H$ is degenerated at the origin with respect to the Newton diagram and a curve of class $\overline{H}$ is non-degenerated.)

**Lemma 3** Up to homeomorphism of the plane, each real regular curve $\tilde{C}$ of class $\overline{H}$ is equivalent to a real regular curve $C$ of class $H$. Moreover, a homeomorphism may be chosen to transform 3 tangent branches of $\tilde{C}$ to 3 tangent branches of $C$.

**Proof.** Let

$$Q(x, y) = \sum_{(i,j) \in \overline{H}} a_{i,j} x^i y^j$$

be a polynomial which defines a real regular curve of class $\overline{H}$ and

$$\Gamma(x, y) = \gamma x (y - a x^3)(y - b x^3)(y - c x^3)$$

be the truncation of $Q$ on $BD$.

Take a linear function $\nu$ in 2 variables with integer coefficients vanishing in each point of $BD$ and positive in other points of $\overline{H}$ and put

$$Q'_{t}(x, y) = \sum_{(i,j) \in \overline{H}} a_{i,j} x^i y^j \nu^{(i,j)} + \gamma x (y - x^2 - a x^3)(y - x^2 - b x^3)(y - x^2 - c x^3) - \Gamma(x, y).$$

For any real $t$ curve $Q'_t = 0$ is of class $H$. Following the same lines as in [Vi 2] in the proof of the theorem on smoothing of quasi-homogeneous singularities, one verifies that for any sufficiently small positive value of $t$ there exist $2$ radii, $r_2 > r_1 > 0$ such that the curve $Q'_t = 0$ is approximated

(a) inside of the disc $D_1$ of radius $r_1$ centered at the origin by the curve

$$\gamma x (y - x^2 - a x^3)(y - x^2 - b x^3)(y - x^2 - c x^3) = 0;$$

(b) outside of the disc $D_2$ of radius $r_2$ centered at the origin by the curve

$Q = 0$;

(c) and in the annulus $D_2 \setminus D_1$ by the curve $\Gamma = 0$.

Thus for a sufficiently small positive $t$ the curve $Q'_t = 0$ is regular and topologically equivalent to the initial curve $Q = 0$ and a homeomorphism of the plane, mapping one into another, may be chosen to transform tangent branches into tangent branches. ●

2.4 The curve

**Proposition 4** There exists a real regular curve of class $H$ of the isotopy type represented in Figure 3 (letters $a, b, c$ mark 3 branches with the common tangent).
Proof. By Lemma 3, it is enough to realise the given isotopy type by a real regular curve of class $\tilde{H}$.

Any convex primitive triangulation of a convex part of a convex polygon is extendable to a convex primitive triangulation of the polygon. Inside of part $BKMN$ of the quadrangle $\tilde{I}$, take the convex primitive triangulation shown in Figure 4 and extend it to $\tilde{I}$.

To apply Proposition 2 we need to choose signs on the vertices in $\tilde{I}$, inside of $BKMN$ put signs according to Figure 4, outside, use the following rule: vertex $(i,j)$ gets sign $"-"$, if $i,j$ are even, and sign $"+"$ otherwise.

The corresponding piecewise-linear curve $L$ is of the required isotopy type (see Figure 5) and Proposition 2 gives the desired result. 

2.5 The surface

Theorem 5 There exists a non-singular surface of degree 5 in $\mathbb{RP}^3$ having 21 connected components homeomorphic to the sphere and one component homeomorphic to the sphere with 7 Möbius bands.

Proof. To each real regular curve of class $H$ corresponds a Horikawa curve: make 2 consecutive blowing-ups at the origin, the second one corresponding to the direction of the tangent line $l$ to the parabolic branches, and then contract to a point the proper transform of $l$; thus we get $\Sigma_2$ and a Horikawa curve on it.

Start with the curve $A$ constructed in 2.4. Then, applying Proposition 1, we obtain a non-singular surface of degree 5 in $\mathbb{RP}^3$ homeomorphic to $RS$ (see 2.1). The surface $RS$ is the real part of a non-singular real model of the 2-sheeted covering $Y$ of $P^2$, ramified along $A$. Choosing between 2 canonical ones, an appropriate real structure on $Y$, (namely, take the one giving $RS$ to be situated over the dark regions in Figure 3), we get, according to Proposition 4, exactly 21 connected components homeomorphic to the sphere and one additional component. It is enough to remark that this component is not orientable and to calculate its Euler characteristic retracing blowing-ups:

$$\chi = 2 - 1 + 2(1 - 2) - 3 + 0 + (1 - 2) = -5$$

(on the non-singular model $RS$ of $RY$ the singular point is replaced by a wedge of 2 circles). 

3 Limits of the method

3.1 Known restrictions on the topological type of a real surface

We mention 3 well known results (see, for example, the survey articles [Wi],[Kh 3]): for a non-singular real projective surface $X$,

$$(a) \chi(RX) \leq h^{13}(CX) - 2(\rho - 1), \quad (\text{Comessatti inequality})$$

where $\chi$ is the Euler characteristic and $\rho$ is the number of linearly independent real algebraic classes in $H_2(CX; \mathbb{R})$;

$$(b) \beta_1(RX) \leq \beta_1(CX) - 2\phi, \quad (\text{Smith inequality})$$


where \( \beta_* \) is the rank of the total \( \mathbb{Z}_2 \)-homology group and \( \rho + \phi \) is the number of linearly independent algebraic classes (not only real ones) in \( H_2(CX; \mathbb{R}) \);

(c) if \( \beta_*(RX) = \beta_*(CX) \), then

\[
\chi(RX) \equiv \sigma(CX) \mod 16 \quad (\text{Rokhlin congruence})
\]

### 3.2 Application to surfaces of degree 5

If \( X \) is a non-singular surface of degree 5, then

\[
h^{1,1}(CX) = 45, \quad \beta_*(CX) = 55 \quad \text{and} \quad \sigma(CX) = -35.
\]

Thus, according to the Smith and Comessatti inequalities, the number of components of a surface of degree 5 in \( RP^3 \) is not greater than 25.

**Proposition 6** The real part \( RS \) of a Horikawa surface \( S \) can not have more than 24 connected components. If the singular points of a Horikawa curve \( B \) are real, then \( RS \) has no more than 23 components.

**Proof.** Consider, first, the case when the singular points are real.

Then the surface \( \tilde{W} \) has at least 4 independent real algebraic cycles: the inverse image of the vertex of the cone, the inverse images of the singular points of \( B \) and a hyperplane section. So this is also the case for \( \tilde{S} \). Thus

\[
\chi(RS) = 1 + \chi(R\tilde{S}) \leq 1 + (h^{1,1}(C\tilde{S}) - 2 \cdot 3) = h^{1,1}(CS) - 4 = 41
\]

and

\[
\beta_0(RS) = (\chi(RS) + \beta_*(RS))/4 \leq 24.
\]

Moreover, if \( \beta_0(RS) = 24 \), then \( \beta_*(RS) = 55 \) and \( \chi(RS) = 41 \). The last combination contradicts the Rokhlin congruence.

If the singular points are imaginary, then \( \rho \geq 3 \) and \( \phi \geq 1 \). Thus

\[
\chi(RS) \leq 1 + (h^{1,1}(C\tilde{S}) - 2 \cdot 2) = 43,
\]

\[
\beta_*(RS) \leq \beta_*(CS) - 2 = 53
\]

and we obtain again the bound \( \beta_0(RS) \leq 24 \). ●

### 3.3 Concluding remarks

**A.** It was conjectured by V. Arnold (see [Vi 3]) that a non-singular surface of degree \( m \) in \( RP^3 \) has at most \( (m^3 - m + 3((-1)^{m+1} + 1))/6 \) components. O. Viro [Vi 3] showed that for any even \( m \geq 6 \) the conjecture is not true. The surface constructed in the present paper provides a counter-example for \( m = 5 \) (for \( m \leq 4 \) the conjecture is true).

**B.** Real double planes \( RY \) ramified along real plane curves constructed by I. Itenberg in [It] have more than \( 4 + \frac{h^{1,1}(CY)}{2} \) components. For a surface \( X \) of degree 5 in \( RP^3 \)

\[
\frac{2 + h^{1,1}(CX)}{2} = 23.5
\]

and one may expect that a clever direct application of Viro’s construction can give examples of surfaces of degree 5 with at least 24 components.
C. The case of $M$-surfaces, $\beta_1(RX) = \beta_1(CX)$, is always of particular interest. By 3.1, a $M$-surface of degree 5 in $RP^3$ may have 5, 9, 13, 17, 21 or 25 connected components. Examples with 5, 9, 13, 17 and 21 components were constructed by V.Kharlamov [Kh 2]. If $M$-surfaces with 25 components really exist, then, again according to 3.1, they may be only of the following topological types

$$24SII P(2), \ 23SII S(1)II P(1), \ 23SII S(2)II P, \ 22SII S(1) II S(1) II P,$$

where $S$ is the sphere, $P$ is the projective plane, $S(q)$ and $P(q)$ are the sphere and the projective plane with $q$ handles. The two last types are not realisable (the forth was excluded by O. Viro, the third by V. Kharlamov; see [Kh 4]). The problem of existence of $M$-surfaces of degree 5 of the two other topological types $24SII P(2), \ 23SII S(1) II P(1)$ is open.

D. Taking on the Horikawa surface constructed in 2.5 the other canonical real structure (see 2.1) one gets a surface with the real part homeomorphic to $SII S(2)II P(20)$. In particular, here $\beta_1 = 45 = h^{1,1}$. The same value is given by $M$-surfaces having 5 components (see C). It would be interesting to construct surfaces of degree 5 with bigger $\beta_1$.

References


