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Mihael PERMAN
Wendelin WERNER

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Wendelin WERNER

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Laboratoire de Mathématiques de l’Ecole Normale Supérieure
45 rue d’Ulm 75230 PARIS Cedex 05
Tel : (33)(1) 44 32 00 00
Adresse électronique : wwerner@dmi.ens.fr

*Institute for Mathematics, Physics and Mechanics
University of Ljubljana, Jadranska 19
61111 Ljubljana, Slovenia
Adresse électronique :mihael.perman@uni-lj.si
Perturbed Brownian motions

Mihael Perman and Wendelin Werner

University of Ljubljana and C.N.R.S.

Abstract

The paper deals with one-dimensional Brownian motion perturbed when it hits its minimum and/or its maximum. It first presents some features of perturbed reflected Brownian motion defined as $|B| - \mu \ell$ where $B$ is standard Brownian motion, $\ell$ its local time at 0 and $\mu$ a positive constant; in particular it is shown that the positive and negative excursions in the sense of Itô of this “one-sided” perturbed Brownian motion are two independent point processes, which in turn is used to construct two-sided perturbed Brownian motion. It is then shown that the constructed process is almost surely the unique solution of the implicit stochastic equation studied by Le Gall [11], Carmona, Petit and Yor [5] and Davis [6], even when no restrictions are imposed on the partial reflections. Finally, the Hausdorff dimension of sets of exceptional points for perturbed Brownian motion such as points of monotonicity are computed.

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1 Introduction

Over the last few years, there has been an increased interest in “perturbed reflecting Brownian motion” which has been the subject of numerous papers. Perturbed reflecting Brownian motion is a process which behaves just as Brownian motion, except when it hits its minimum. More precisely, let $B$
denote standard linear Brownian motion, \( \ell \) its local time process at level 0, and \( \mu > 0 \) a fixed positive constant: then perturbed reflecting Brownian motion \( X \) is defined by

\[
X = |B| - \mu \ell.
\]

Note that Lévy’s identity (see e.g. [18], Theorem VI-(2,3)) shows that \( X \) is in fact linear Brownian motion when \( \mu = 1 \). When the perturbation factor \( \mu \) is greater than 1, the perturbation is “self-repelling” and when \( \mu < 1 \), it is “self-attracting”.

One of the main features of \( X \) is that its local times in the space variable taken at suitable stopping times are squared Bessel processes and one gets versions for perturbed reflecting Brownian motion of the well-known Ray-Knight theorems; this also yields generalizations for \( X \) of Lévy’s Arc-sine law for Brownian motion. Yor [23] provides an overview of the results; see also Abraham-Mazliak [1], Carmona, Petit and Yor [4], Perman [16], Petit [17], Shi-Werner [19] and Werner [22].

In the present paper we first prove (in section 2) an independence result for positive and negative parts of \( X \), which completes a result in [23] where the claim is established for \( \mu < 2 \). This also completes the arguments in [22] to give a simple proof of the two Ray-Knight Theorems for the local times of \( X \) initially derived by Le Gall and Yor [12] and Carmona, Petit and Yor [4].

In the third section of this paper, we use this independence result to construct and derive some results concerning Brownian motion perturbed at both extrema referred to as two-sided perturbed Brownian motion. Such processes are among the simplest continuous generalisations of self-interacting random walks in one dimension (see Tóth [21] and Davis [6]). Considering two independent perturbed reflecting Brownian motions \( X \) and \( X' \) (with two possibly different perturbation factors \( \mu \) and \( \nu \) respectively) and their point process of negative excursions \( e \) and \( e' \) respectively, we construct the two-sided perturbed Brownian motion \( \tilde{X} \) by sliding together \(-e \) and \( e' \), according to the local times at zero (just as Brownian motion is reconstructed from two reflected Brownian motions). We then identify \( \tilde{X} \) with the process that has been constructed in the case where \( |(\mu - 1)(\nu - 1)| \leq 1 \) by Le Gall [11], Carmona, Petir and Yor [5] and Davis [6] as the unique strong solution of the implicit stochastic equation,

\[
Y_t = W_t + (1 - 1/\mu) \sup_{s \in [0,t]} Y_s + (1 - 1/\nu) \inf_{s \in [0,t]} Y_s, \quad Y_0 = 0, \quad (1)
\]
where $W$ is linear Brownian motion started from 0; their construction is based on a fixed-point argument which fails as soon as $|((\mu - 1)(\nu - 1))| > 1$, see [6]. We then generalise features (Arc-sine laws, Ray-Knight theorems) of $X$ that have been derived by Carmona, Petit and Yor [5] in the case $|((\mu - 1)(\nu - 1))| < 1$, to all values $\mu > 0$, $\nu > 0$.

Davis also showed that when $|((\mu - 1)(\nu - 1))| > 1$, the equation (1) always has at least one solution, but that is has strictly more than one solution, for some well-chosen deterministic functions $W$. In the fourth section of this paper, we study this implicit stochastic equation and we show that, in the case where $W$ is a linear Brownian motion, it almost surely has only one solution (for any fixed $\mu > 0$ and $\nu > 0$), and that this solution has the same law than $\sim X$.

Finally, in the last section of the paper we study some fine path properties of perturbed reflecting Brownian motion and two-sided perturbed Brownian motion, such as the existence of points of monotonicity and related questions, using Ray-Knight theorems for perturbed Brownian motions. Even though we are not using their results, these problems have many similarities with Brownian “slow points” (see e.g. Davis-Perkins [7] and the references therein).

Notation. The following notation will be valid throughout the paper. For $\mu > 0$, the processes $X$, $B$ and $\ell$ are defined as above. For $x \in R$ and $t \geq 0$, let $(L^x_t, t \geq 0, x \in R)$ denote the semi-martingale local time process of $X$ defined for times $t \geq 0$ and all levels $x$. Further, define for all $a \in R$ the hitting time of $a$ by $X$:

$$T_a = \inf\{t \geq 0, X_t = a\}.$$ 

For any process $(Y_t, t \geq 0)$, $s$ will be called a time of decrease for $Y$ if there exists $\varepsilon > 0$, such that for all $u \in (0, \varepsilon)$, $Y_{s-u} > Y_s > Y_{s+u}$. Similarly, we define times of increase and of monotonicity for $Y$. A point $x$ will be called a point of decrease $Y$ if $Y_t = x$ for some time of decrease. Points of increase or of monotonicity are defined similarly.

$\cong$ will denote identity in law between two processes or two random variables.

When $f : [0, \infty) \to R$ is a continuous function, we define

$$f^\ast(t) = \sup_{s \in [0, t]} f(s) \text{ and } f^\#(t) = \inf_{s \in [0, t]} f(s).$$

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As in [23] and in [5], for $a > 0$ and $b > 0$, $Z_a$ will denote a gamma random variable with parameter $a$ and $Z(a, b)$ a Beta random variable with parameters $a, b$:

$$P(Z_a \in dt) = \frac{t^{a-1}e^{-t}dt}{\Gamma(a)} \text{ for } t > 0$$

and

$$P(Z(a, b) \in dt) = \frac{t^{a-1}(1-t)^{b-1}dt}{\text{B}(a, b)} \text{ for } t \in (0, 1),$$

where $\Gamma$ (resp. $\text{B}$) is the Gamma (resp. Beta) function.

Finally, $\text{dim}(H)$ will denote the Hausdorff dimension of a set $H$.

### 2 An independence result

Define the processes $A^+$ and $A^-$ as follows:

$$A^+_t = \int_0^t 1(X_s > 0) \, ds \quad \text{ and } \quad A^-_t = \int_0^t 1(X_s < 0) \, ds.$$  

They count the time spent positive or negative up to time $t$ by $X$. Define the right-continuous inverses of $A^+$ and $A^-$ as

$$\kappa^+(t) = \inf\{u > 0, \ A^+_u > t\} \quad \text{ and } \quad \kappa^-(t) = \inf\{u > 0, \ A^-_u > t\}.$$  

**Proposition 1** Let $X$ be perturbed reflecting Brownian motion. The two processes $X^+$ and $X^-$ obtained from $X$ by time changes

$$X^+_t = X_{\kappa^+(t)} \quad \text{ and } \quad X^-_t = X_{\kappa^-(t)}$$

are independent. Moreover, $Y^+$ is reflecting Brownian motion.

A proof of this result, in case $\mu < 2$, can be found in Yor [23]; see also [13].

**Proof:** The idea is to approximate the processes $X^+$ and $X^-$ using up- and downcrossings by $X$: For all $\varepsilon > 0$, we define the stopping times (with respect to the filtration generated by $(B, \ell)$), $T^\varepsilon_n$ and $S^\varepsilon_n$ as follows: $T^\varepsilon_1 = 0$ and for all $n \geq 1$,

$$S^\varepsilon_n = \inf\{t > T^\varepsilon_n, \ X_t = \varepsilon\},$$

$$T^\varepsilon_n = \inf\{t > T^\varepsilon_{n-1}, \ X_t = \varepsilon\},$$

and

$$\ell(t) = \inf\{u \in \mathbb{R}, \ X_u = 0, \ X_t = \varepsilon\}.$$
We also define the clocks corresponding to the up- and downcrossings from level 0 to \( \varepsilon \) by \( X \):

\[
A_\varepsilon^+(t) = \sum_{n \geq 1} \{\min(t, T_{n+1}^\varepsilon) - \min(t, S_n^\varepsilon)\},
\]

\[
A_\varepsilon^-(t) = \sum_{n \geq 1} \{\min(t, S_n^\varepsilon) - \min(t, T_n^\varepsilon)\}.
\]

Let \( \kappa_\varepsilon^+ \) and \( \kappa_\varepsilon^- \) denote their respective right-continuous inverses, and put, for all \( t \geq 0 \):

\[
X^{\varepsilon+}(t) = X(\kappa_\varepsilon^+(t)) \quad \text{and} \quad X^{\varepsilon-}(t) = X(\kappa_\varepsilon^-(t)).
\]

One now just needs to notice that

1. For all \( \varepsilon > 0 \), the two processes \( X^{\varepsilon+} \) and \( X^{\varepsilon-} \) are independent, which is readily derived from the strong Markov property for the process \( (X, \ell) \) (notice that \( \ell_{T_{n+1}^\varepsilon} = \ell_{S_n^\varepsilon} \), because \( B_t > 0 \) for all \( t \in [S_n^\varepsilon, T_n^\varepsilon] \)).

2. Almost surely,

\[
\lim_{\varepsilon \to 0^+} A_\varepsilon^+ = A^+
\]

for instance in the sense of uniform convergence on compact intervals: Almost surely, for all \( t \in [0, T] \),

\[
|A_\varepsilon^+(t) - A^+(t)| = |A_\varepsilon^-(t) - A^-(t)| \leq \int_0^T 1_{\{X,t \in [0, \varepsilon]\}} ds \xrightarrow{\varepsilon \to 0} 0.
\]

This yields that for fixed \( t_1, \ldots, t_n \), almost surely,

\[
\lim_{\varepsilon \to 0} (X^{\varepsilon+}(t_1), X^{\varepsilon-}(t_1), \ldots, X^{\varepsilon-}(t_n)) = (X^+(t_1), X^-(t_1), \ldots, X^-(t_n)).
\]

Hence, for the pair of continuous processes \( (X^{\varepsilon+}, X^{\varepsilon-}) = (X \circ \kappa_\varepsilon^+, X \circ \kappa_\varepsilon^-) \) the finite dimensional marginal distributions converge to those of \( (X^+, X^-) = (X \circ \kappa^+, X \circ \kappa^-) \).

Combining 1 and 2 completes the proof of independence. The proof that \( Y^+ \) is reflecting Brownian motion is given in Yor [23].
3 Two–sided perturbed Brownian motion

As mentioned before, one can say informally that $X$ behaves like Brownian motion while away from its past minimum. Some of the results in this section will give a more precise meaning to this statement. A natural extension would be a process that behaves like standard Brownian motion while away from either its past minimum or its past maximum whereas at its past extrema the process mimics the behaviour of one-sided perturbed reflecting Brownian motion. Several authors have considered such processes; see Le Gall [11], Carmona, Petit and Yor [5] and Davis [6].

To construct such a process, let $X$ be perturbed reflecting Brownian motion and associate a point process $e$ with $X$ the same way the excursion process is associated to a diffusion with a recurrent point. This point process is defined on the space $(0, \infty) \times U$ with

$$U = \{ f \in C[0, \infty), \exists R > 0, f(s) \neq 0 \text{ if and only if } s \in (0, R) \}.$$ 

with the Borel $\sigma$–field inherited from the compact–open topology of $C[0, \infty)$. To be more precise, define $\{\tau_s^0, s > 0\}$ as the inverse local time at level 0 of $X$: $\tau_s^0 = \inf\{u > 0, L_u^0 > s\}$. It can be shown by a routine excursion calculation that almost surely $\lim_{t \to \infty} L_t = \infty$. The countably many points of $e$ are hence of the form $(s, e_s)$ where

$$e_s(u) = \begin{cases} X_{s-} + u & \text{for } 0 \leq u \leq \tau_s^0 - \tau_s - \\ 0 & \text{else} \end{cases}$$

for all $s$ for which $\tau_s - \tau_s - > 0$. It is easily seen that this point process is discrete but it is not a Poisson point process (except when $\mu = 1$) because $(X_t, t \geq 0)$ is not Markov. First, a few simple properties of this point process are proved.

**Proposition 2** Let $e$ be the point process defined above, and let $e^+$ and $e^-$ be the processes of positive and negative excursions respectively defined in the obvious way.

(i) The two processes $e^+$ and $e^-$ are independent point processes.

(ii) $e^+$ has the same distribution as the point process of positive excursions of standard Brownian motion.
(iii) Let \( I(e_s) = \inf\{e_s(u) : u \geq 0\} \) be the infimum of an excursion. Conditionally on \( \{\inf_{s \leq t} I(e_s) = m\} \), the mean measure of the point process \( e^- \) restricted to \((t, \infty) \times U_m\) where \( U_m = \{f \in U, m \leq \inf_x f(u) < 0\} \) is just \((1/2)\lambda \times n_m\) where \( \lambda \) is Lebesgue measure and \( n_m \) is Itô’s excursion law restricted to \( U_m \). Furthermore, the conditional mean measure of the point process on \((t, \infty)\) with points \( \{u > t, (u, e_u) \in e, I(e_u) < m\} \) is \( du/(2|m|) \).

Proof: (i) The independence of \( e^+ \) and \( e^- \) is a simple consequence of the independence of \( X^+ \) and \( X^- \) because \( e^+ \) and \( e^- \) can be recovered from \( X^+ \) and \( X^- \) respectively.

(ii) \( e^+ \) is the excursion process of \( X^+ \) which is reflecting Brownian motion.

(iii) Let \( m < 0 \) and \( T_m = \inf\{t > 0, X_t = m\} \). The random variable \( T_m \) is a stopping time for the reflecting Brownian motion in the definition of \( X = |B| - \mu \ell \) (it is the hitting time of \( m/\mu \) by the local time of \( B \) at level 0). The strong Markov property shows that the process \( Y_u = X_{T_{m+u}} - m \) is also perturbed reflecting Brownian motion, and hence its “positive part” is reflecting Brownian motion. It can now easily be argued that the excursions from level \( |m| \) of that reflected Brownian motion that do not touch 0 are just like Brownian excursions. This also proves the second statement in (iii).

Remark: This proposition, combined with the arguments developed in Werner [22], provides a simple proof of the generalised Ray-Knight Theorems for the process \( X \) (see [4], [23], and also [1]), which will be used later in this paper.

The idea of constructing the two-sided perturbed Brownian motion is to replace the point process \( e^+ \) by an independent copy of \(-e^-\) with a possibly different \( \mu \), and then reconstruct a continuous process from this new point process. As \( e \) is not a Poisson point process, it needs to be argued that the reconstruction does give a continuous process (this is not true in general). Fix \( \mu, \nu > 0 \) and take two independent point processes \( e^{(1)} \) and \( e^{(2)} \) with the law of \( e^- \) for \( \mu \) and \( \nu \) respectively. Let \( e \) be the point process with points \( (s, -e_s^{(1)}) \) and \( (s, e_s^{(2)}) \). Note that (iii) of the above proposition yields that there are no “ties”; the combined point process \( e \) does not have two different points with the same “local time” \( s \). Call the measurable transformation that reconstructs a path from the point process of excursions by \( \Xi \). Such a reconstruction is possible in a measurable manner and is unique but we have
to check that the resulting path is continuous. We also have to check whether \( \hat{X} \) corresponds to the two-sided perturbed Brownian motion constructed by Le Gall [11], Carmona, Petit and Yor [5] and Davis [6] via the functional equation mentionned in the introduction. From now on, we put

\[ \alpha = 1 - 1/\mu \text{ and } \beta = 1 - 1/\nu. \]

**Proposition 3** Let \( (\hat{X}_t, t \geq 0) = \Xi \varepsilon. \)

(i) \( \hat{X} \) is a continuous process with the Brownian scaling property.

(ii) The process \( (\hat{X}_t, \hat{X}^*(t), \hat{X}^\#(t)) \) is a Markov process.

(iii) The process

\[ W_t = \hat{X}_t - \alpha \hat{X}^*(t) - \beta \hat{X}^\#(t), \]

is a standard Brownian motion with respect to the filtration generated by the Markov process in (ii).

**Proof:** (i) Continuity follows from the continuity of the two processes obtained from \( \hat{X} \) by sliding together the positive and negative excursions. The two processes obtained this way are just the negative parts of one-sided perturbed Brownian motion. To prove the Brownian scaling property note that \( X \) has the same property and it follows that the property is inherited by \( \hat{X} \).

(ii) This property is a direct consequence of the fact that the process

\[ (X^-_t, \inf_{s \leq t} X^-_s) \]

is a Markov process, which follows from the fact that \( (|B|, \ell) \) is a Markov process.

(iii) The idea of the proof is simple. Introduce the stopping times (for the process \( (\hat{X}, \hat{X}^#, \hat{X}^*) \)) \( \hat{T}^\varepsilon_1 = \inf\{t \geq 0, \hat{X}_t = \varepsilon\} \), and for all \( n > 0 \),

\[ \hat{S}^\varepsilon_n = \inf\{t > \hat{T}^\varepsilon_n, \hat{X}_t = 0\}, \]

\[ \hat{T}^\varepsilon_{n+1} = \inf\{t > \hat{S}^\varepsilon_n, \hat{X}_t = \varepsilon\}. \]

On the time interval \([\hat{S}^\varepsilon_n, \hat{T}^\varepsilon_{n+1}]\) (resp. \([\hat{T}^\varepsilon_{n}, \hat{S}^\varepsilon_{n+1}]\), the process \( \hat{X}^*_s \) (resp. \( \hat{X}^\#_s \)) is constant, and \( \hat{X} \) behaves exactly like perturbed reflecting Brownian motion.
(resp. the inverse of perturbed reflecting Brownian motion). Consequently, on each of these intervals,

\[ \tilde{X}_t = \alpha \tilde{X}_s^* - \beta \tilde{X}_s^# \]

is a Brownian motion. Hence, the process \( W_{T^*} - W_{T^1} \) is a Brownian motion, which is independent from \((\tilde{X}_t, t \in [0, T^2_1])\). Letting \( \varepsilon \to 0 \) completes the proof.

We now state some properties of \( \tilde{X} \). From now on, \( T_\alpha \) will denote the hitting time

\[ T_\alpha = \inf\{t > 0, \tilde{X}_t = a\} \]

of \( a \in \mathbb{R} \) by \( \tilde{X} \), and \( \tilde{L} \) will denote the local times of the process \( \tilde{X} \).

**Proposition 4** (i) *(The first Ray-Knight theorem for \( \tilde{X} \))* Suppose that \( a > 0 \). The process \((\tilde{L}^{-a+}_x, x \in [0, a])\) is a squared Bessel process of index \( 2/\nu \) started from 0, and reflected at 0. The process \((\tilde{L}^0_{T^*}, x \geq 0)\) is a squared Bessel process of index \( 2 - 2/\mu \) started from \( \tilde{L}^0_{T^*} \) and absorbed at 0. In particular, the law of \( \tilde{L}^0_{T^*} \) is that of \( 2aZ_{1/\nu} \).

(ii) The random variable \( \tilde{A}_t^*: = t^{-1} \int_0^t 1(\tilde{X}_s > 0)ds \) has the distribution \( Z(1/(2\nu), 1/(2\mu)) \).

(iii) If \( a > 0, b > 0 \), then

\[ P(T_{-a} < T_b) = P \left( Z(1/\mu, 1/\nu) > \frac{b}{a+b} \right). \]

(iv) The explicit formulas for the density of the random variables \( T_\alpha, \tilde{X}_t, \tilde{X}_t^* \) and \( \tilde{X}_t^# \) stated in paragraph 3.3 in [5] hold for any \( \alpha < 1 \) and \( \beta < 1 \). So does also Proposition 3.7 in [5].

**Remarks**-

- We do not state the second Ray-Knight theorem for \( \tilde{X} \) as it is an even more immediate consequence of the second Ray-Knight Theorem for \( X \) (see e.g. [5] for a statement).
• (i) and (ii) are stated when \(|(1 - \mu)(1 - \nu)| < 1\) in Carmona-Petit-Yor [5]. In particular, (ii) shows that any Beta-random variable can be constructed as the time spent above zero, by a perturbed Brownian motion.

• (iii) for \(\mu = \nu\) is the continuous time version of some results of Nester [15]. When \(\mu = 1\), (iii) is exactly the hitting time property for the process \(X\) (see [22]).

Proof- (i) is a direct consequence of the two Ray-Knight theorems for the process \(X\); see e.g. Yor [23]. The last statement in (i) is another way to express (a) of Corollary 9.1.1 in Yor [23].

(ii) Let \((\sigma_s : s \geq 0)\) be the inverse local time of two–sided Brownian motion. By the scaling property it follows that

\[
\hat{A}_1^+ \overset{\mathcal{D}}{=} \frac{\hat{A}_{\sigma_1}^+}{\hat{A}_{\sigma_1}^+ + \hat{A}_{\sigma_1}^-},
\]

just like in the Brownian case. See e.g. [23], p. 104. The two random variables \(\hat{A}_{\sigma_1}\) and \(\hat{A}_{\sigma_2}\) are independent, and it is known from [23] that \(\hat{A}_{\sigma_1}^+ \overset{\mathcal{D}}{=} (8Z_{1/(2\mu)})^{-1}\) and \(\hat{A}_{\sigma_1}^- \overset{\mathcal{D}}{=} (8Z_{1/(2\nu)})^{-1}\).

(iii) The last statement of (i) and the independence between the positive and negative parts of \(\hat{X}\) imply that

\[
P(T_{-a} < T_b) = P(\hat{L}_{T_{-a}}^0 < \hat{L}_{T_b}^0) = P(aZ_{1/\nu} < bZ'_{1/\mu})
\]

(where \(Z'\) is another Gamma random variable independent of \(Z\)), and (iii) follows easily.

(iv) follows from (i) and the second Ray-Knight theorem exactly as in [5].

4 The implicit stochastic equation

Suppose that \(f\) is a continuous function with \(f(0) = 0\) and consider the equation:

\[
g(t) = f(t) + \alpha g^s(t) + \beta g^\#(t), \quad g(0) = 0. \tag{2}
\]

Then, it is known that:
• This equation has at least one solution \( g \), for all \( \alpha < 1, \beta < 1 \) and \( f \) (see Davis [6]).

• If \( |\rho| \leq 1 \), then this solution is unique (see Carmona-Petit-Yor [5], Davis [6]).

• If \( |\rho| > 1 \), then one can find functions \( f \) such that (2) has at least two distinct solutions (see Davis [6]).

The following result shows however, that when \( f \) is a linear Brownian motion, then the solutions are unique even if \( |\rho| > 1 \).

**Proposition 5** When \( f \) is a linear Brownian motion started from \( 0 \), then any two solutions of (2) are almost surely identical (for any fixed \( \alpha < 1 \) and \( \beta < 1 \)). Moreover, their law is that of \( \tilde{X} \).

We will first derive the following weaker statement:

**Proposition 6** When \( f \) is a linear Brownian motion started from \( 0 \), then any solution of (2) is identical in law to \( \tilde{X} \) (for any fixed \( \mu > 0 \) and \( \nu > 0 \), i.e. for any fixed \( \alpha < 1 \) and \( \beta < 1 \)).

There are various ways of deriving Proposition 6. One possibility would be to show that the positive and negative parts of a solution are independent, and to check that the positive (resp. negative) excursion process is identical to that of \( \tilde{X} \). We opt here for a direct proof, which does not use our construction of \( \tilde{X} \): Before proving the proposition, we first state two useful lemmas:

**Lemma 1** Suppose that \( f \) is a continuous function, such that \( f(0) = 1 - \alpha \), and that \( \alpha < 1, \beta < 1 \). Then the equation

\[
g(t) = f(t) + \alpha g^+(t) + \beta \min(g^+(t), 0), \quad g(0) = 1 \tag{3}
\]

has a unique solution.

**Proof of Lemma 1** The idea behind this lemma is similar to the ideas used by Davis [6]. Recall from [5], that the functional equation (2) has a unique solution, when \( \alpha = 0 \) or when \( \beta = 0 \). Suppose that \( g \) solves (3) and define \( t_1 = \inf\{t > 0, \ g(t) = 0\} \). Let \( g_1 \) denote the solution of the equation

\[
g_1(t) = f(t) + \alpha g^+_1(t), \quad g_1(0) = 1, \tag{4}
\]
and let \( t'_1 = \inf\{t > 0, g_1(t) = 0\} \). As the solution of the equation (4) on \([0, \min(t_1, t'_1))\) is unique, \( g = g_1 \) on the interval \([0, \min(t_1, t'_1))\) and consequently, \( t_1 = t'_1 \). Then, let \( g_2 \) denote the unique solution of the equation

\[
g_2(t) = f(t + t_1) + \alpha g_1^*(t_1) + \beta g_2^*(t), \quad g_2(0) = g_1(t_1) = 0,
\]

and \( t_2 = \inf\{t > 0, g_2^*(t) = g_1^*(t_1)\} \). Again, one necessarily has

\[
g(t_1 + t) = g_2(t) \text{ for all } t \in [0, t_2].
\]

Note also that for all \( M > 0 \), \( f \) is uniformly continuous on \([0, M]\), and therefore, it is easy to see that there exists \( \varepsilon > 0 \), such that either \( t_1 + t_2 > M \) or \( \min(t_1, t_2) > \varepsilon \). An easy induction completes the proof. We safely leave this to the reader.

**Lemma 2** Suppose \( W \) denotes a linear Brownian motion started from 0 and define for all \( t > 0 \), \( z(t) = \sup\{s < t, W(s) = 0\} \). There exists a sequence \((u_n, n \geq 1)\) of stopping times (for \( W \)) such that almost surely,

\[
\lim_{n \to \infty} u_n = 0+,
\]

\[
W(u_n) = W^*(u_n) \geq n|W^#(u_n)|
\]

and

\[
2W^*(z(u_n)) \leq W^*(u_n)
\]

for all \( n \geq 1 \).

**Proof of Lemma 2** Using the 0-1 law and the scaling property of Brownian motion, it is very easy to check that for all \( n \geq 1 \), the set

\[
A_n = \{t > 0; n|W^#(t)| < W^*(t) \text{ and } 2W^*(z(t)) < W^*(t)\}
\]

is such that \( 0 \in \overline{A}_n \) almost surely. Let \( \tau \) denote the inverse local time of \( W \) at level 0. For all \( a > 0 \), \( A_n \cap [0, \tau(a)] \neq \emptyset \) a.s. Define now a deterministic sequence of positive numbers \( a_n \) in such a way that \( a_0 = 1 \), and for all \( n \geq 1 \),

\[
P(A_n \cap [\tau(a_n), \tau(a_{n-1})] = \emptyset) < e^{-n} \text{ and } P(\tau(a_n) \geq (n + 1)^{-1}) < e^{-n}.
\]

For all \( n > 1 \), we then define the stopping time

\[
u_n = \inf\{t > \tau(a_n), W_t \in A_n\}.
\]

Borel-Cantelli’s lemma implies that there exists a.s. \( n_0 > 1 \) such that for all \( n > n_0 \), \( u_n < \tau(a_{n-1}) < e^{-n} \), and the definition of \( u_n \) ensures that (5) and (6) are satisfied.
Proof of Proposition 6. We will focus only on the cases \( \{ \alpha > 0 \} \) and \( \{ \beta > 0 \} \), since the cases \( \{ \alpha \leq 0 \text{ and } \beta \leq 0 \} \) are already dealt with in Carmona, Petit and Yor [5] (in this case, \( |\rho| < 1 \)). With no loss of generality, we can assume that \( \alpha > 0 \) (change \( X \) into \(-X\) if \( \alpha \leq 0 \) and \( \beta > 0 \)).

Suppose first that

\[
V(t) = W(t) + \alpha V^*(t) + \beta V^#(t), \quad V(0) = 0
\]  

(7)

for all \( t \geq 0 \), where \( W \) is a linear Brownian motion started from 0. Define the sequence \( (u_n, n \geq 1) \) as in lemma 2. (7) and the fact that \( \alpha > 0 \) clearly implies that

\[
V^#(u_n) \geq W^#(u_n) + \max(\beta, 0) V^#(u_n)
\]

and hence, if we put \( \beta^+ = \max(0, \beta) \), (using (5)),

\[
|V^#(u_n)| \leq \frac{W^*(u_n)}{n(1 - \beta^+)}.
\]

Combined with (7), this yields

\[
W(u_n) + \alpha V^*(u_n) \geq V(u_n) \geq W(u_n) \left(1 - \frac{1}{n(1 - \beta^+)}\right) + \alpha V^*(u_n).
\]

(6) then implies easily that

\[
V^*(z(u_n)) \leq \frac{W^*(z(u_n))}{1 - \alpha} \leq \frac{W(u_n)}{2(1 - \alpha)} \leq \frac{V(u_n)}{2(1 - (n(1 - \beta^+))^{-1})};
\]

hence, for \( n \) large enough, \( V^* \) increases on the time interval \( [z(u_n), u_n] \). It is then easy (as \( V \) can not hit its minimum on this interval), to conclude that for \( n > n_0 \) (\( n_0 \) is a deterministic integer)

\[
V(u_n) = V^*(u_n),
\]

and therefore that

\[
W(u_n) \geq (1 - \alpha) V(u_n) \geq W(u_n) \left(1 - \frac{1}{n(1 - \beta^+)}\right).
\]

Finally, this shows that almost surely,

\[
\lim_{n \to \infty} \frac{V^*(u_n) - V^#(u_n)}{W(u_n)} = (1 - \alpha)^{-1},
\]  

(8)
and
\[
\lim_{n \to \infty} \frac{V^\#(u_n)}{W(u_n)} = 0. \tag{9}
\]

Note also that
\[
W(u_n) + (\alpha - 1)V^\#(u_n) + \beta V^\#(u_n) = 0. \tag{10}
\]

We now define the process (for fixed large \(n\)),
\[
\tilde{V}(u) = \frac{V(u_n + (V(u_n) - V^\#(u_n))^2 u) - V^\#(u_n)}{V(u_n) - V^\#(u_n)}.
\]

Define
\[
t = t(u) = u_n + (V(u_n) - V^\#(u_n))^2 u.
\]

Note that
\[
\tilde{V}^\#(u) = \frac{V^\#(t) - V^\#(u_n)}{V(u_n) - V^\#(u_n)}. \tag{11}
\]

and that
\[
\min(\tilde{V}^\#(u), 0) = \frac{V^\#(t) - V^\#(u_n)}{V(u_n) - V^\#(u_n)}. \tag{12}
\]

Define also the process
\[
\tilde{W}(u) = (1 - \alpha) + \frac{W(t) - W(u_n)}{V(u_n) - V^\#(u_n)}.
\]

As \(u_n\) is a stopping time for \(W\), the strong Markov property and the scaling property imply that \(\tilde{W}\) is a Brownian motion started from \((1 - \alpha)\), which is independent of \((W_n, u \in [0, u_n])\). Using (11), (12) and (10), one easily gets
\[
(V(u_n) - V^\#(u_n)) \left( \tilde{W}(u) + \alpha \tilde{V}^\#(u) + \beta \min(\tilde{V}^\#(u), 0) \right)
\]
\[= W(t) - W(u_n) + (1 - \alpha)(V(u_n) - V^\#(u_n)) + \alpha(V^\#(t) - V^\#(u_n)) \]
\[+ \beta(V^\#(t) - V^\#(u_n)) \]
\[= W(t) + \alpha V^\#(t) + \beta V^\#(t) - W(u_n) + (1 - \alpha)V(u_n) + (-1 - \beta)V^\#(u_n) \]
\[= V(t) - V^\#(u_n). \]
Hence,
\[ \bar{V}(u) = \tilde{W}(u) + \alpha \tilde{V}^*(u) + \beta \min(\bar{V}^*(u), 0), \]
and the lemma shows that the law of \( \bar{V} \) is defined in a unique way. In other words, if \((U(u), u \geq 0)\) denotes another solution to the functional equation (7), then (with obvious notation), the two processes \( \bar{U} \) and \( \bar{V} \) are identical in law. Hence, for all large enough \( n \),
\[ \frac{V(u_n + \cdot) - V^*(u_n)}{V(u_n) - V^*(u_n)} \geq \frac{U(u_n + \cdot) - U^*(u_n)}{U(u_n) - U^*(u_n)}. \]
Letting \( n \to \infty \) and using the estimates (8) and (9) then readily implies that \( U \) and \( V \) are identical in law.

**Proof of Proposition 5** - We are now ready to derive Proposition 5: Suppose that \( f \) is a linear Brownian motion started from 0, and that \( g_1 \) and \( g_2 \) are two solutions of (2). Suppose furthermore that \( g_1 \neq g_2 \) with strictly positive probability. It is then easy to see that this implies that for some fixed deterministic time \( S > 0 \),
\[ P(g_1(S) \neq g_2(S)) > 0. \]
Suppose for instance that
\[ P(g_1(S) < g_2(S)) > 0. \]
We set \( E = 1 \) if \( g_1(S) \geq g_2(S) \), and \( E = 2 \) if \( g_2(S) > g_1(S) \) (note that a priori, \( g_E \) and \( \max(g_1, g_2) \) are not necessarily always equal, since \( g_1 \) and \( g_2 \) could cross). Clearly, \( g_E \) is a solution of (2). The law of \( g_E \) is different than that of \( g_1 \) because \( g_E(S) \geq g_1(S) \) and \( P(g_E(S) > g_1(S)) > 0 \). On the other hand, as the solutions of (2) are unique in law, the laws of \( g_E(S) \) and \( g_1(S) \) are identical, which contradicts the previous statement. Hence, \( g_1 = g_2 \) almost surely.

A consequence of this result is the following:

**Proposition 7** Let \( \hat{X} \) denote the perturbed Brownian motion constructed in the previous section, and let
\[ W = \hat{X} - \alpha \hat{X}^* - \beta \hat{X}^# \]
denote the linear Brownian motion defined in Proposition 3-(iii). Then, the filtrations generated by \( \hat{X} \) and \( W \) are almost surely identical.
Proof. Define $W$ as in (13). Davis’ construction of a solution of the implicit stochastic equation shows that there exists a solution of

$$g = W + \alpha g^* + \beta g^\#,$$  

that is measurable with respect to the filtration generated by $W$. As the solution to this equation is a.s. unique, the statement follows.

Remark. Davis [6] has pointed out that the discrete version of two-sided perturbed Brownian motion converges towards $\bar{X}$, as soon as $|\rho| = |\alpha \beta/((1 - \alpha)(1 - \beta))| = |(\mu - 1)(\nu - 1)| < 1$. It is reasonable to expect that this is the case even if $|\rho| \geq 1$, but we do not tackle this problem here.

5 Fine properties of $X$ and $\bar{X}$

Let $\mu > 0$, $\nu > 0$ and $\alpha = 1 - 1/\mu$, $\beta = 1 - 1/\nu$, $X$ and $\bar{X}$ defined as above. We will in fact only prove results for $X$. We safely leave the analogous proofs for $\bar{X}$ to the reader. It is well known that Brownian motion does not have any times of monotonicity (see e.g. Burdzy [3] and the references therein). As perturbed reflecting Brownian motion behaves just like Brownian motion when away from its past minimum any time of monotonicity is necessarily a time where $X$ equals its past minimum. Since $X = |B| - \mu \ell$, times of monotonicity for $X$ must be zeros of $B$ and consequently $-\mu \ell_t$ must be a point of decrease (as it can not be a point of increase) of $X$ for all such $t$.

When $\mu = 1$, Lévy’s identity shows that $X$ is in fact itself a Brownian motion and $\bar{X}$ has therefore almost surely no points of monotonicity. Hence, almost surely, for all $t \geq 0$ such that $B_t = 0$,

$$\limsup_{u \to 0^+} \frac{|B_{t+u}|}{\ell_{t+u} - \ell_t} \geq 1. \quad (14)$$

Consequently, if $\mu < 1$, a.s. for all $t \geq 0$, such that $B_t = 0$,

$$\limsup_{u \to 0^+} \frac{|B_{t+u}|}{\ell_{t+u} - \ell_t} > \mu$$

and $X$ has no points of decrease either.

We are going to see that when $\mu > 1$, then $X$ almost surely has points of decrease. More precisely, let $H$ denote the set of points of decrease for $X$ and $D$ the set of times of decrease. Then:
**Proposition 8** If \( \mu > 1 \), then \( D \) and \( H \) are almost surely non-empty. Moreover, the Hausdorff dimension of \( H \) is \( \alpha \) and that of \( D \) is \( \alpha/2 \). Almost surely, \( D \) and \( H \) are empty, for all \( \mu \leq 1 \).

Let us first make a few of remarks:

1. This implies the following weaker result: Almost surely,
   \[
   \inf_{t \in [0,1]} \limsup_{u \to 0^+} \frac{|B_{t+u}|}{\ell_{t+u} - \ell_t} = 1,
   \]
   which complements \((14)\). This strongly recalls the local ‘self-normalisation’ problems (see e.g. Knight \([10]\), Khoshnevisan \([9]\) and the references therein). Of course, for fixed \( t = 0 \),
   \[
   \limsup_{u \to 0^+} \frac{|B_u|}{\ell_u} = \infty \quad \text{a.s.}
   \]
   (using for instance Lévy’s identity).

2. When \( \mu \to 1^+ \), the Hausdorff dimensions of the sets \( D \) and \( H \) tend to zero, as one would expect. Also, when \( \mu \to \infty \), the Hausdorff dimension of \( D \) converges towards \( 1/2 \), which is the Hausdorff dimension of the set \( \{t > 0, B_t = 0\} \).

3. Let \( \tau \) denote the right-continuous inverse of \( \ell \). \( \tau \) is a stable subordinator of index \( 1/2 \). Necessarily, for \( t \in D \), one has \( X_t = -\mu \ell_t \), i.e. \( \tau(-X_t/\mu) = t \); hence
   \[
   \tau(-\Pi/\mu) = D. \tag{15}
   \]
   Hence, using the results of \([8]\), it is actually sufficient to prove that
   \[
   \dim(H) = 1 - \frac{1}{\mu},
   \]
   since \((15)\) then implies that \( \dim(D) = \dim(H)/2 \).

   The following similar result for \( \tilde{X} \) is an immediate consequence of Proposition 8:
**Proposition 9** If \( \max(\alpha, \beta) > 0 \), then the Hausdorff dimension of the set of points (resp. times) of monotonicity of \( \tilde{X} \) is \( \max(\alpha, \beta) \) (resp. \( \max(\alpha, \beta)/2 \)). If \( \alpha \leq 0 \) (resp. \( \beta \leq 0 \)), then there are no times of increase (resp. decrease) for \( \tilde{X} \). If \( \alpha > 0 \) (resp. \( \beta > 0 \)) then the Hausdorff dimension of the set of point(s) of increase (resp. decrease) for \( \tilde{X} \) is \( \alpha \) (resp. \( \beta \)), and that of times of increase (resp. decrease) is \( \alpha/2 \) (resp. \( \beta/2 \)).

Before proceeding to the actual proof of proposition 8, let us first establish and recall some relevant results. We begin with excursion-theory preliminaries:

Let \( e = (e_s, s > 0) \) denote the excursion process of \( B \) just as in Revuz-Yor [18], Ch. 12. For an excursion \( e_s \), let \( M(e_s) = \sup e_s(u) \) denote its height, and \((L^x(e_s), x \geq 0)\) the associated local time process in the space variable.

**Lemma 3** For the point process \( e \) one has:

(i) Almost surely, for all \( s > 0 \) such that \( M(e_s) > 0 \), and for all \( x \in (0, M(e_s)) \),

\[
L^x(e_s) > 0.
\]

(ii) For any fixed \( \mu > 0 \), almost surely, for every \( s < s' \),

\[
M(e_{s'}) \neq M(e_s) + \mu(s' - s).
\]

**Proof of Lemma 3** (i) follows readily from the corresponding result for Brownian motion up to the inverse local time at 0 (using for instance the fact that the probability that only one excursion exceeds the level \( x \) before \( \tau_1 \) is strictly positive.

(ii) is a straightforward consequence of the fact that the number of excursions of \( B \) is countable.

The following consequence of the previous lemma will be used in the sequel.

**Proposition 10** One has

\[
H = \cup_{q \in Q_+} \{ x \in (q, 0), \; L^x_{\tau_q} = 0 \}.
\]
Proof. Suppose that \( x \in \mathbb{H} \) and that for all \( u \in (0, \varepsilon) \), \( X_{u+t} < X_t = x \). Choose \( q \in Q \cap (\inf \{ X_{t+u}, u \in (0, \varepsilon) \}, x) \). As \( \{ s \leq T_q, \ X_s = x \} = \{ t \} \), one has

\[ L_{x, T^+ q} = 0 \]

and hence

\[ \mathbb{H} \subset \cup_{q \in Q_-} \{ x \in (q, 0), L_{x, T^+ q} = 0 \} . \]

Suppose now that \( L_{x, T^+ q} = 0 \) for some \( q < x < 0 \) and that \( x \notin \mathbb{H} \). Let \( t = \inf \{ s, X_s < x \} \). As \( x \notin \mathbb{H} \), \( t \) is not a time of decrease for \( X \), i.e. there exists a sequence \( t_n \to t^+ \) such that \( X_{t_n} \geq x \). The definition of \( t \) implies however that there exists a sequence \( t'_n \to t^+ \) such that \( X_{t'_n} < X_t \). Hence there exists a sequence \( r_n \to t^+ \) such that \( X_{r_n} = x \). We can furthermore assume that \( r_n < T_q \) for all \( n \)'s.

Part (ii) of the previous Lemma implies that \( r_n \)'s are not simultaneously maxima of excursions of \( |B| \). Hence, using Part (i) of the same Lemma and the fact that \( T_q > r_n \),

\[ L_{x, T^+ q} > 0 , \]

which contradicts the hypothesis and concludes the proof of the proposition.

Proof of Proposition 8- We are now ready to show how Proposition 8 follows from the generalized Ray-Knight Theorem for the local times of \( X \) derived by Le Gall-Yor [12] (see also [22]), we now recall:

**Proposition 11** (Le Gall-Yor [12]). For all fixed \( y < 0 \), the process \((L_{y+}^{y-x}, x \in [0,-y])\) is a squared Bessel process of dimension \( 2/\mu \) started from 0.

Bessel processes of dimension \( \delta > 1 \) hit 0 for strictly positive time almost surely. Moreover (see e.g. Barlow-Pitman-Yor [2] and McKean [14]), its zero set has Hausdorff dimension \( 1 - \delta/2 \). Hence, for all \( q \in Q_- \),

\[ \dim ( \{ x \in (q, 0), L_{x, T^+ q} = 0 \} ) = 1 - \frac{1}{\mu} . \]

Proposition 8 follows immediately, using Proposition 10.
Remark. Recall Lévy’s identity
\[(|B|, \ell) = (W^* - W, W^*),\]
where \(W\) is a linear Brownian motion. Consequently, if \(\eta = \mu - 1\),
\[D = \{t > 0, W_t = W^*_t \text{ and } \exists \varepsilon > 0, \forall u \in (0, \varepsilon), W_{u+t} - W_t > -\eta(W^*_{u+t} - W^*_t)\}.

In other words, if we put for all \(t > 0, u > 0\),
\[I^t_u = \inf_{v \in (0, u)} (W_{t+v} - W_t)\]
and
\[S^t_u = \sup_{v \in (0, u)} (W_{t+v} - W_t),\]
then
\[D = \{t > 0, \ W_t = W^*_t \text{ and } \exists \varepsilon > 0, \forall u \in (0, \varepsilon), I^t_u > -\eta S^t_u\}.
\]

\(D\) is therefore a set of times of ‘approximate increase’ for \(W\). One could also have derived the fact that \(\dim (D) = \alpha/2\) using estimates analogous to those of Knight [10] (see also Khoshnevisan [9]) and the ‘classical’ techniques to compute Hausdorff measures (see e.g. the paper on slow points by Davis and Perkins [7] and the references therein). Similarly, one could also focus on the other exceptional sets like for instance
\[D^+ = \{t, \exists \varepsilon > 0, \forall u \in (0, \varepsilon), I^t_u > -\eta S^t_u\},\]
\[D^\pm = \{t, \exists \varepsilon > 0, \forall u \in (-\varepsilon, \varepsilon), I^t_u > -\eta S^t_u\}\]
and compute their Hausdorff dimension.

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M.P.:  
Institute for Mathematics, Physics and Mechanics  
University of Ljubljana  
Jadranska 19, 61111 Ljubljana  
SLOVENIA  
e-mail: mihael.perman@uni-lj.si

W.W.:  
Laboratoire de Mathématiques  
Ecole Normale Supérieure  
45, rue d’Ulm, F-75230 Paris cedex 05  
FRANCE  
e-mail: wwerner@DMI.ENS.FR