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and the Picard group of the moduli of
$SL_r/\mu_s$-bundles on a curve

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Linearization of group stack actions and the Picard group of the moduli of $\text{SL}_r/\mu_s$-bundles on a curve

Yves LASZLO (†)

Introduction

Let $G$ be a complex semi-simple group and $\tilde{G} \to G$ the universal covering. Let $\mathcal{M}_G$ (resp. $\mathcal{M}_{\tilde{G}}$) be the moduli stack of $G$-bundles over a curve $X$ of degree $1 \in \pi_1(G)$ (resp. of $\tilde{G}$-bundles. In [B-L-S], we have studied the link between the groups Pic($\mathcal{M}_G$) and Pic($\mathcal{M}_{\tilde{G}}$), the later being well understood thanks to [L-S]. In particular, it has been possible to give a complete description in the case where $G = \text{PSL}_r$ but not in the case $\text{SL}_r/\mu_s$, $s \mid r$, although we were able to give partial results. The reason was that we did not have at our disposal the technical background to study the morphism $\mathcal{M}_{\tilde{G}} \to \mathcal{M}_G$. It turns out that it is a torsor under some group stack, not far from a Galois étale cover in the usual schematic picture. Now, the descent theory of Grothendieck has been adapted to the set-up of fpqc morphisms of stacks in [L-M] and gives the theorem 4.1 in the particular case of a morphism which is torsor under a group stack. We then used this technical result to determine the exact structure of Pic($\mathcal{M}_G$) where $G = \text{SL}_r/\mu_s$ (theorem 5.6).

I would like to thank L. Breen to have taught me both the notion of torsor and linearization of a vector bundle in the set-up of group-stack action and for his comments on a preliminary version of this paper.

Notation

Throughout this paper, all the stacks will be implicitly assumed to be algebraic over a fixed base scheme and the morphisms locally of finite type. We fix once for all a projective, smooth, connected genus $g$ curve $X$ over an algebraically closed field $k$ and a closed point $x$ of $X$. The Picard stack parameterizing families of line bundles of degree $0$ on $X$ will be denoted by $\mathcal{J}(X)$ and the jacobian variety of $X$ by $J_X$. If $G$ is an algebraic group over $k$, the quotient stack $\text{Spec}(k)/G$ (where $G$ acts trivially on $\text{Spec}(k)$) whose category over a $k$-scheme $S$ is the category of $G$-torsors (or $G$-bundles) over $S$ will be denoted by $BG$. If $n$ is an integer and $A = \mathcal{J}(X), J_X$ or $B_{\text{G}_m}$ we denote by $n_A$ the $n$th-power morphism $a \mapsto a^n$. We denote by $\mathcal{J}_n$ (resp. $J_n$) the $0$-fiber $A \times_A \text{Spec}(k)$ of $n_A$ when $A = \mathcal{J}(X)$ (resp. $A = J_X$).

1. Generalities.— Following [Br], for any diagram

$$
\begin{array}{c}
A \xrightarrow{h} B \\
\downarrow^f \\
C \xrightarrow{l} D
\end{array}
$$

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of 2-categories, we'll denote by \( l \star \lambda : l \circ f \Rightarrow l \circ g \) (resp. \( \lambda \star h : f \circ h \Rightarrow g \circ h \)) the 2-morphism deduced from \( \lambda \).

1.1. For the convenience of the reader, let us prove a simple formal lemma which will be useful in the section 4. Let \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) be three 2-categories, a 2-commutative diagram

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{f} & \mathcal{B} \\
\delta_0 \downarrow & & \downarrow \delta_1 \\
\mathcal{C} & \xrightarrow{g} & \mathcal{D}
\end{array}
\]

(1.1.1)

and a 2-morphism \( \mu : \delta_0 \Rightarrow \delta_1 \).

**Lemma 1.2.** Assume that \( f \) is an equivalence. There exists a unique 2-morphism

\[
\mu \star f^{-1} : d_0 \Rightarrow d_1
\]

such that \( (\mu \star f^{-1}) \star f = \mu \).

**Proof:** let \( \epsilon_k, k = 0, 1 \) the 2-morphism \( d_k \circ f \Rightarrow \delta_k \). Let \( b \) be an object of \( \mathcal{B} \). Pick an object \( a \) of \( \mathcal{A} \) and an isomorphism \( \alpha : f(a) \xrightarrow{\sim} b \). Let \( \varphi_\alpha : d_0(b) \xrightarrow{\sim} d_1(b) \) be the unique isomorphism making the diagram

\[
\begin{array}{ccc}
\delta_0(a) & \xrightarrow{\epsilon_0(a)} & d_0 \circ f(a) \xrightarrow{d_0(\alpha)} d_0(b) \\
\mu_\alpha \downarrow & & \downarrow \varphi_\alpha \\
\delta_1(a) & \xrightarrow{\epsilon_1(\alpha)} & d_1 \circ f(a) \xrightarrow{d_1(\alpha)} d_1(b)
\end{array}
\]

commutative. We have to show that \( \varphi_\alpha \) does not depend on \( \alpha \) but only on \( b \). Let \( \alpha' : f(a') \xrightarrow{\sim} b \) be another isomorphism. There exists a unique isomorphism \( i : a' \xrightarrow{\sim} a \) such that \( \alpha \circ f(i) = \alpha' \). The one has the equality \( \varphi_{\alpha'} = d_1(\alpha) \circ \Phi \circ d_0(\alpha)^{-1} \) where

\[
\Phi = [d_1 \circ f(i)] \circ \epsilon_1(a') \circ \mu_{a'} \circ \epsilon_0(a')^{-1} \circ [d_0 \circ f(i)]^{-1}.
\]

The functoriality of \( \epsilon_i \) and \( \mu \) ensures that one has the equalities

\[
d_k \circ f(i) \circ \epsilon_k(a') = \epsilon_k(a) \circ \delta_k(i)
\]

and

\[
\mu_\alpha = \delta_1(i) \circ \mu_{a'} \circ \delta_0(i)^{-1}.
\]

This shows that

\[
\Phi = \epsilon_1(a) \circ \mu_\alpha \circ \epsilon_0(a)^{-1}
\]

which proves the equality \( \varphi_\alpha = \varphi_{\alpha'} \). We can therefore define \( \mu_b \) to be the isomorphism \( \varphi_\alpha \) for one isomorphism \( \alpha : f(a) \xrightarrow{\sim} b \). One checks that the construction is functorial in \( b \) and the lemma follows. \( \blacksquare \)
2. Linearizations of line bundles on stacks.—— Let us first recall following [Br] the notion of torsor in the stack context.

2.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a faithfully flat morphism of stacks. Let us assume that an algebraic $gr$-stack $\mathcal{G}$ acts on $f$ (the product of $\mathcal{G}$ is denoted by $m_\mathcal{G}$ and the unit object by 1). Following [Br], this means that there exists a 1-morphism of $\mathcal{Y}$-stacks $m : \mathcal{G} \times \mathcal{X} \to \mathcal{X}$ and a 2-morphism $\mu : m \circ (m_\mathcal{G} \times \text{Id}_\mathcal{X}) \Rightarrow m \circ (\text{Id}_\mathcal{G} \times m)$ such that the obvious associativity condition (see the diagram (6.1.3) of [Br]) is satisfied and such that there exists a 2-morphism $\epsilon : m \circ (1 \times \text{Id}_\mathcal{X}) \Rightarrow \text{Id}_\mathcal{X}$ which is compatible to $\mu$ in the obvious sense (see (6.1.4) of [Br]).

Remark 2.2.— To say that $m$ is a morphism of $\mathcal{Y}$-stacks means that the diagram

$$
\begin{array}{ccc}
\mathcal{G} \times \mathcal{X} & \xrightarrow{m} & \mathcal{X} \\
\downarrow & & \downarrow \\
\mathcal{Y} & & \\
\end{array}
$$

is 2-commutative. In other words, if we denote for simplicity the image of a pair of objects $m(g, x)$ by $g \cdot x$, this means that there exists a functorial isomorphism $\iota_{g, x} : f(g \cdot x) \to f(x)$.

2.3. Suppose that $\mathcal{G}$ acts on another such $f' : \mathcal{X} \to \mathcal{Y}$. A morphism $p : \mathcal{X} \to \mathcal{X}$ will be said equivariant if there exists a 2-morphism

$$q : m \circ (\text{Id} \times p) \Rightarrow p \circ m'
$$

which is compatible to $\mu$ (as in [Br] (6.1.6)) and $\epsilon$ (which is implicit in [Br]) in the obvious sense.

Definition 2.4.— With the above notations, we say that $f$ (or $\mathcal{X}$) is a $\mathcal{G}$-torsor over $\mathcal{Y}$ if the morphism $\text{pr}_2 \times m : \mathcal{G} \times \mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$ is an isomorphism (of stacks) and the geometrical fibers of $f$ are not empty.

Remark 2.5.— In down to earth terms, this means that if $\iota : f(x) \to f(x')$ is an isomorphism in $\mathcal{Y}$ ($x, x'$ being objects of $\mathcal{X}$), there exist an object $\xi$ of $\mathcal{G}$ and a unique isomorphism $(x, g \cdot x) \sim (\iota, x')$ which induces $\iota$ thanks to $\iota_{g, x}$ (cf. 2.2).

Example 2.6.— If $\mathcal{M}_X(G_m)$ is the Picard stack of $X$, the morphism $\mathcal{M}_X(G_m) \to \mathcal{M}_X(G_m)$ of multiplication by $n \in \mathbb{Z}$ is a torsor under $B\mu_n \times J_n(X)$ (cf. (3.1)).

2.7. Let a $\mathcal{L}$ be a line bundle on $\mathcal{X}$. By definition, the data $\mathcal{L}$ is equivalent to the data of a morphism $l : \mathcal{X} \to B\mu_m$ (see [L-M].prop. 6.15). If $\mathcal{L}, \mathcal{L}'$ are 2 line bundles on $\mathcal{X}$ defined by $l, l'$, we will view an isomorphism $\mathcal{L} \approx \mathcal{L}'$ as a 2-morphism $l \Rightarrow l'$.

Definition 2.8.— A $\mathcal{G}$-linearization of $\mathcal{L}$ is a 2-morphism $\lambda : l \circ m \Rightarrow l \circ \text{pr}_2$ such that the two diagrams of 2-morphisms

\begin{equation}
\begin{array}{ccc}
l \circ m \circ (m_\mathcal{G} \times \text{Id}_\mathcal{X}) & \xrightarrow{l \circ \mu} & l \circ m \circ (\text{Id}_\mathcal{G} \times m) \\
\downarrow \lambda \circ (m_\mathcal{G} \times \text{Id}_\mathcal{X}) & & \downarrow \lambda \circ (\text{Id}_\mathcal{G} \times m) \\
l \circ \text{pr}_2 \circ (m_\mathcal{G} \times \text{Id}_\mathcal{X}) & \xleftarrow{l \circ \text{pr}_2 \circ \text{pr}_2} & l \circ \text{pr}_2 \circ (\text{Id}_\mathcal{G} \times m) = l \circ m \circ \text{pr}_2
\end{array}
\end{equation}
Remark 2.9. – In $g_1, g_2$ are objects of $\mathcal{G}$ and $d$ of $\mathcal{X}$, the commutativity of the diagram (2.8.1) means that the diagram

$$
\begin{array}{ccc}
\mathcal{L}_{g_1,g_2}\times_{X} & \xrightarrow{\sim} & \mathcal{L}_{g_2}\
\downarrow & & \downarrow \\
\mathcal{L}_{g_1} & & \mathcal{L}_{g_2}
\end{array}
$$

is commutative and the commutativity of (2.8.2) that the two isomorphisms $\mathcal{L}_{1,x} \simeq \mathcal{L}_x$ defined by the linearization $\lambda$ and $\epsilon$ respectively are the same.

3. An example. – Let me recall that a closed point $x$ of $X$ has been fixed. Let $S$ be a $k$-scheme. The $S$-points of the jacobian variety of $X$ are by definition isomorphism classes of line bundles on $X_S$ together with a trivialization along $\{x\} \times S$ (such a pair will be called a rigidified line bundle). For the convenience of the reader, let me state this well known lemma which can be found in SGA4, exp. XVIII, (1.5.4)

**Lemma 3.1.** – The Picard stack $\mathcal{J}(X)$ is canonically isomorphic (as a $k$-group stack) to $X \times \mathbb{B}G_m$.

*Proof:* let $f : \mathcal{J}(X) \to X \times \mathbb{B}G_m$ be the morphism which associates

- to the line bundle $L$ on $X_S$ the pair $L \otimes L^{-1}_{\mid \{x\} \times S}, L_{\mid \{x\} \times S}$ (thought as an object of $X \times \mathbb{B}G_m$ over $S$);

- to an isomorphism $L \sim L'$ on $X_S$ its restriction to $\{x\} \times S$.

Let $f' : X \times \mathbb{B}G_m \to \mathcal{J}(X)$ be the morphism which associates

- to the pair $(L, V)$ where $L$ is a rigidified bundle on $X_S$ and $V$ a line bundle on $S$ (thought as an object of $X \times \mathbb{B}G_m$ over $S$), the line bundle $L \otimes_{X_S} V$;

- to an isomorphism $(l, v) : (L, V) \sim (L', V')$ the tensor product $l \otimes_{X_S} v$.

The morphisms $f$ and $f'$ are (quasi)-inverse each other and are morphisms of $k$-stacks. $\blacksquare$

We will identify from now $\mathcal{J}(X)$ and $X \times \mathbb{B}G_m$. Let $\mathcal{L}$ (resp. $\mathcal{P}$ and $\mathcal{T}$) be the universal bundle on $X \times \mathcal{J}(X)$ (resp. on $X \times X$ and $\mathbb{B}G_m$) and let $\Theta = (\det R \mathcal{P})^{-1}$ be the theta line bundle on $X$. The isomorphism $\mathcal{L} \sim \mathcal{P} \otimes \mathcal{T}$ yields an isomorphism

$$
\det R \mathcal{L}^n(m,x) \sim \Theta^{-n^2} \otimes \mathcal{T}^{m+1-g}.
$$

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4. Descent of $\mathcal{G}$-line bundles.--- The object of this section is to prove the following statement.

**Theorem 4.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ a $\mathcal{G}$-torsor as above. Let $\text{Pic}^\mathcal{G}(\mathcal{X})$ be the group of isomorphism classes of $\mathcal{G}$-linearized line bundles on $\mathcal{X}$. Then, the pull-back morphism $f^* : \text{Pic}(\mathcal{Y}) \to \text{Pic}^\mathcal{G}(\mathcal{X})$ is an isomorphism.

The descent theory of Grothendieck has been adapted in the case of algebraic 1-stacks in [L-M], essentially in the proposition (6.23). Let $\mathcal{X}_{\bullet} \to \mathcal{Y}$ be the (augmented) simplicial complex of stacks coskeleton of $f$ (as defined in [De] (5.1.4) for instance). By proposition (6.23) of [L-M], one just has to construct a cartesian $\mathcal{O}_{\mathcal{X}_{\bullet}}$-module $\mathcal{L}_{\bullet}$ such that $L_0$ is the $\mathcal{O}_{\mathcal{X}_{\bullet}}$-module $\mathcal{L}$ to prove the theorem. The $n$-th piece $\mathcal{X}_n$ is inductively defined by $\mathcal{X}_0 = \mathcal{X}$, $\mathcal{X}_n = \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}_{n-1}$ for $n > 0$. Let $p_n : \mathcal{X}_n \to \mathcal{X}$ be the projection on the first factor. It is the simplicial morphism associated to the map

$$\tilde{p}_n : \Delta_0 \to \Delta_n$$

We define $L_n$ by the morphism

$$l_n : \mathcal{X}_n \xrightarrow{p_n} \mathcal{X} \xrightarrow{l} \text{BG}_m$$

4.2. Let $\delta_i$ (resp. $s_j$) be the face (resp. degeneracy) operators (see [De] (5.1.1) for instance) (by abuse of notation, we use the same notation for $\delta_j, s_j$ and their image by $\mathcal{X}_{\bullet}$). The category $(\Delta)$ is generated by the face and degeneracy operators with the following relations (see for instance the proposition VII.5.2 page 174 of [McL])

$$\delta_i \circ \delta_j = \delta_{j+1} \circ \delta_i \quad i \leq j$$

(4.2.1)

$$s_j \circ s_i = s_i \circ s_{j+1} \quad i \leq j$$

(4.2.2)

$$\begin{cases} s_j \circ \delta_i = \delta_i \circ s_{j-1} & i < j \\ = 1 & i = j, i = j + 1 \\ = \delta_{i-1} \circ s_j & i > j + 1. \end{cases}$$

(4.2.3)

Therefore, the data of a cartesian $\mathcal{O}_{\mathcal{X}_{\bullet}}$-module $\mathcal{L}_{\bullet}$ is equivalent to the data of isomorphisms $\alpha_j : \delta_j^* \mathcal{L}_n \cong \mathcal{L}_{n+1}$, $j = 0, \ldots, n + 1$ and $\beta_j : s_j^* \mathcal{L}_{n+1} \cong \mathcal{L}_n$, $j = 0, \ldots, n$ (where $n$ is a non negative integer) which are compatible with the relations 4.2.1, 4.2.2 and 4.2.3. Let $n$ be a non negative integer.
4.3. We have first to define for \( j = 0, \ldots, n+1 \) an isomorphism \( \alpha_j : \delta_j^* \mathcal{L}_n \xrightarrow{\sim} \mathcal{L}_{n+1} \). The line bundle \( \delta_j^* \mathcal{L}_n \) is defined by the morphism \( l \circ p_n \circ \delta_j : \mathcal{X}_{n+1} \to \text{BG}_m \) and \( \delta_j^* \mathcal{L}_n \) is associated to the map

\[
\begin{cases}
\Delta_0 & \to \Delta_{n+1} \\
0 & \to \delta_j(0)
\end{cases}
\]

If \( j \neq 0 \), one has therefore \( \tilde{p}_n \circ \delta_j = \tilde{p}_{n+1} \) and \( \delta_j^* \mathcal{L}_n = \mathcal{L}_{n+1} \). We define \( \alpha_j \) by the identity in this case.

Suppose now that \( j = 0 \). Let \( \pi_n : \mathcal{X}_n \to \mathcal{X}_1 \) be the projection on the 2 first factors (associated to the canonical inclusion \( \Delta_1 \to \Delta_n \)). The commutativity of the 2 diagrams

\[
\begin{array}{ccc}
\mathcal{X}_{n+1} & \xrightarrow{\delta_0} & \mathcal{X}_n \\
\downarrow \pi_{n+1} & & \downarrow p_n \\
\mathcal{X}_1 & \xrightarrow{\delta_0} & \mathcal{X}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\mathcal{X}_{n+1} & \xrightarrow{p_{n+1}} & \mathcal{X} \\
\downarrow \pi_{n+1} & & \downarrow \delta_1 \\
\mathcal{X}_1 & \xrightarrow{\delta_1} & \mathcal{X}
\end{array}
\]

allows to reduce the problem to the construction of an isomorphism

\[
\delta_0^* \mathcal{L} \xrightarrow{\sim} \delta_1^* \mathcal{L}
\]

are the face morphisms or, what is amounts to the same, to the construction of a 2-morphism \( \nu : l \circ \delta_0 \Rightarrow l \circ \delta_1 \) (the morphism \( \alpha_j \) will be \( \alpha_j = \nu \circ \pi_{n+1} \)). Now the diagram

\[
\begin{array}{ccc}
\text{BG}_m & & \\
\downarrow l \circ m & & \downarrow l \circ \delta_0 \\
\mathcal{G} \times \mathcal{X} & \xrightarrow{pr_2 \times m} & \mathcal{X} \times_{Y} \mathcal{X} \\
\downarrow l \circ pr_2 & & \downarrow l \circ \delta_1 \\
\text{BG}_m & & 
\end{array}
\]

(4.3.1)

is strict commutative and \( pr_2 \times m \) is an equivalence by the definition of a torsor. By the lemma 1.2, the 2-morphism \( \lambda \) induces a canonical 2-morphism \( \lambda \ast (pr_2 \times m)^{-1} : l \circ \delta_0 \Rightarrow l \circ \delta_1 \) which is the required 2-morphism \( \nu \).

4.4. We have then to define for \( j = 0, \ldots, n \) an isomorphism \( \beta_j : s_j^* \mathcal{L}_{n+1} \xrightarrow{\sim} \mathcal{L}_n \). The line bundle \( s_j^* \mathcal{L} \) is defined by the morphism \( l \circ p_{n+1} \circ s_j \) and \( p_{n+1} \circ s_j \) is associated to the canonical inclusion \( \Delta_0 \hookrightarrow \mathcal{X}_n \) which means \( p_{n+1} \circ s_j = p_n \). Therefore, \( s_j^* \mathcal{L}_{n+1} = \mathcal{L}_n \) and we define \( \beta_j \) to be the identity.

4.5. We have to show that the data \( \mathcal{L}_*, \alpha_j, \beta_j, j \geq 0 \) defines a line bundle on the simplicial stack \( \mathcal{X}_* \) as explained in (4.2). Notice that the fact that the definition of the \( \beta_j \) ’s is compatible with the relations 4.2.2 is tautological (\( \beta_j \) is the identity on the relevant \( \mathcal{L}_n \)).

4.6. Relation 4.2.1: in terms of \( l \), this relation means the following. We have the 2 strictly commutative diagrams

\[
\alpha_i \circ (\delta_i \ast \alpha_j) : l \circ p_n \circ \delta_j \circ \delta_i \xrightarrow{\sim} l \circ p_{n+1} \circ \delta_i \xrightarrow{\alpha_i} l \circ p_{n+2}
\]
The relation $\alpha_i \circ (\alpha_j \circ \delta_i) = \alpha_{i+1} \circ (\alpha_i \circ \delta_{i+1})$, $i \leq j$.

If $j = 0$, the relation 4.2.1' is just by definition of $\alpha_j$ the condition 2.8.1 (see remark 2.9).

If $j > 0$, both the 2 isomorphisms $\alpha_j$ and $\alpha_{j+1}$ are the relevant identity and the relation 4.2.1' is tautological.

4.7. Relation 4.2.3: the only non tautological relation in (4.2.3) is the associated to the equality $s_0 \circ \delta_0 = 1$ in (\Delta) which means as before that $\alpha_0 \circ \delta_0$ is the identity functor of $l \circ p_n = l \circ p_n \circ \delta_0 = s_0$. But, this is exactly the meaning of the relation 2.8.2 (see remark 2.9).

5. Application to the Picard groups of some moduli spaces. — Let us chose 3 integers $r, s, d$ such that

$r \geq 2$ and $s \mid r \mid ds$.

If $G$ is the group $\text{SL}_r / \mu_s$ we denote as in [B-L-S] by $\mathcal{M}_G(d)$ the moduli stack of $G$-bundles on $X$ and by $\mathcal{M}_{\text{SL}_r}(d)$ the moduli stack of rank $r$ vector bundles and determinant $\mathcal{O}(d,x)$. If $r = s$ (i.e. $G = \text{PSL}_r$), the natural morphism of algebraic stacks

$$\pi : \mathcal{M}_{\text{SL}_r}(d) \to \mathcal{M}_G(d)$$

is a $\mathcal{J}_r$-torsor (see the corollary of proposition 2 of [Gr] for instance). Let me explain how to deal with the general case.

5.1. Let $E$ be a rank $r$ vector bundle on $X_S$ endowed with an isomorphism $\tau; D^r/s \xrightarrow{\sim} \text{det}(E)$ where $D$ is some line bundle. Let me define the $\text{SL}_r / \mu_s$-bundle $\pi(E)$ associated to $E$ (more precisely to the pair $(E, \tau)$).
Definition 5.2.— An $s$-trivialization of $E$ on the étale neighborhood $T \to X_S$ is a triple $(M, \alpha, \sigma)$ where $\alpha : D \to M^s$ is an isomorphism ($M$ is a line bundle on $T$); $\sigma : M^r \to E_T$ is an isomorphism; $\det(\sigma) \cdot \alpha^r : D^r \to \det(E)$ is equal to $\tau$.

Two $s$-trivializations $(M, \alpha, \sigma)$ and $(M', \alpha', \sigma')$ of $E$ will be said equivalent if there exists an isomorphism $\iota : M \to M'$ such that $\iota^* \cdot \alpha = \alpha'$.

The principal homogeneous space

\[ T \mapsto \{ \text{equivalence classes of } s\text{-trivializations of } E_T \} \]

defines the $\text{SL}_r / \mu_s$-bundle $\pi(E)$ \(^\dagger\). Now, the construction is obviously functorial and therefore defines the morphism $\pi : \mathcal{M}_{\text{SL}_r}(d) \to \mathcal{M}_G(d)$ (observe that an object $E$ of $\mathcal{M}_{\text{SL}_r}(d)$ has determinant $\mathcal{O}(\frac{d}{2}x^r \cdot s)$). Let $L$ be a line bundle and $(M, \alpha, \tau)$ an $s$-trivialization of $E_T$. Then, $(M \otimes L, \alpha \otimes \text{Id}_L, \sigma \otimes \text{Id}_L)$ is an $s$-trivialization of $E \otimes L$ (which has determinant $(D \otimes L^s)^{r/s}$). This shows that there exists a canonical functorial isomorphism

\begin{equation}
\pi(E) \sim \pi(E \otimes L)
\end{equation}

In particular, $\pi$ is $J_s$-equivariant.

**Lemma 5.3.**— The natural morphism of algebraic stacks $\pi : \mathcal{M}_{\text{SL}_r}(d) \to \mathcal{M}_G(d)$ is a $J_s$-torsor.

**Proof:** let $E, E'$ be two rank $r$ vector bundles on $X_S$ (with determinant equal to $\mathcal{O}(d \cdot x)$) and let $\iota : \pi(E) \sim \pi(E')'$ an isomorphism. As in the proof of the lemma 13.4 of [B-L-S], we have the exact sequence of sets

\[ 1 \to \mu_s \to \text{Isom}(E, E') \to \text{Isom}(\pi(E), \pi(E'))' \to \pi_{E,E'} \to H^1_{et}(X_S, \mu_s). \]

Let $L$ be a $\mu_s$-torsor such that $\pi_{E,E'}(\iota) = [L]$. Then, $\pi(E \otimes L)$ is canonically equal to $\pi(E)$ and $\pi_{E \otimes L, E'} = 0$ and $\iota$ is induced by an isomorphism $E \otimes L \sim E'$ well defined up to multiplication by $\mu_s$. The lemma follows. \(\blacksquare\)

5.4. Let $U$ be the universal bundle on $X \times \mathcal{M}_{\text{SL}_r}(d)$. We would like to know which power of the determinant bundle $D = (\det R\Gamma U)^{-1}$ on $\mathcal{M}_{\text{SL}_r}(d)$ descends to $\mathcal{M}_G(d)$. As in I.3 of [B-L-S], the rank $r$ bundle $\mathcal{F} = L^{\otimes (r-1)} \oplus L^{1-r}(d \cdot x)$ on $X \times J(X)$ has determinant $\mathcal{O}(d \cdot x)$ and therefore defines a morphism

\[ f : J(X) = JX \times BG_m \to \mathcal{M}_{\text{SL}_r}(d) \]

which is $J_s$-equivariant.

\(^\dagger\) We see here a $G$-bundle as a formal homogeneous space under $G$.\[ 8 \]

\(\blacksquare\)
The vector bundle $\mathcal{F}' = \mathcal{O}^{\oplus(r-1)} \oplus \mathcal{L}^{-r/s}(d_x)$ on $X \times \mathcal{J}(X)$ has determinant $[\mathcal{L}^{-1}(d_x/x)]^{r/s}$. The $G$-bundle $\pi(\mathcal{F}')$ on $X \times \mathcal{J}(X)$ defines a morphism $f': \mathcal{J} \to \mathcal{M}_G(d)$. The relation $\mathcal{L} \otimes (\text{Id}_X \times s_{\mathcal{J}})^* (\mathcal{F}') = \mathcal{F}$ and (5.2.1) gives an isomorphism $\pi(\mathcal{F}) = (\text{Id}_X \times s_{\mathcal{J}})^* \pi(\mathcal{F}')$ which means that the diagram

\[
\begin{array}{ccc}
\mathcal{J}(X) & \xrightarrow{f} & \mathcal{M}_{\text{SL}_r}(d) \\
\downarrow s_{\mathcal{J}} & & \downarrow \pi \\
\mathcal{J}(X) & \xrightarrow{f'} & \mathcal{M}_G(d)
\end{array}
\]

(5.4.1)

is 2-commutative. Exactly as in I.3 of [B-L-S], let me prove the

**Lemma 5.5.** The line bundle $f^* \mathcal{D}^k$ on $\mathcal{J}(X)$ descends through $s_{\mathcal{J}}$ if and only if $k$ multiple of $s/(s,r/s)$.

**Proof:** let $\chi = r(g - 1) - d$ be the opposite of the Euler characteristic of $(k)$-points of $\mathcal{M}_{\text{SL}_r}(d)$. By (3.1), one has an isomorphism $f^* \mathcal{D}^k \cong \Theta^{kr(r-1)} \otimes \mathcal{T}^{k\chi}$. The theory of Mumford groups says that $\Theta^{kr(r-1)}$ descends through $s_{\mathcal{J}}$ if and only if $k$ is a multiple of $s/(s,r/s)$. The line bundle $\mathcal{T}^{k\chi}$ on $\text{BG}_m$ descends through $s_{\text{BG}_m}$ if and only if $k\chi$ is a multiple of $s$. The lemma follows from the above isomorphism and from the observation that the condition $s \mid r \mid ds$ forces $s\chi$ to be a multiple of $s$.

Let me recall that $\mathcal{D}$ is the determinant bundle on $\mathcal{M}_{\text{SL}_r}(d)$ and $G = \text{SL}_r/\mu_s$.

**Theorem 5.6.** Assume that the characteristic of $k$ is 0. The integers $k$ such that $\mathcal{D}^k$ descends to $\mathcal{M}_G(d)$ are the multiple of $s/(s,r/s)$.

**Proof:** by lemma 5.5 and diagram (5.4.1), we just have to proving that $\mathcal{D}^k$ effectively descends where $k = s/(s,r/s)$. By theorem 4.1 and lemma 5.3, this means exactly that $\mathcal{D}^k$ has a $\mathcal{J}_s$-linearization. We know by lemma 5.5 that the pull-back $f^* \mathcal{D}^k$ has such a linearization.

**Lemma 5.7.** The pull-back morphism $\text{Pic}(\mathcal{J}_s \times \mathcal{M}_{\text{SL}_r}(d)) \to \text{Pic}(\mathcal{J}_s \times \mathcal{J}(X))$ is injective.

**Proof:** by lemma 3.1, one is reduced to prove that the natural morphism 

$$\text{Pic}(B\mu_s \times \mathcal{M}_{\text{SL}_r}(d)) \to \text{Pic}(B\mu_s \times \mathcal{J}(X))$$

is injective. Let $\mathcal{X}$ be any stack. The canonical morphism $\mathcal{X} \to \mathcal{X} \times B\mu_s$ is a $\mu_s$-torsor (with the trivial action of $\mu_s$ on $\mathcal{X}$). By theorem 4.1, one has the equality 

$$\text{Pic}(\mathcal{X} \times B\mu_s) = \text{Pic}^{\mu_s}(\mathcal{X}).$$

Assume further that $H^0(\mathcal{X}, \mathcal{O}) = k$. The later group is then canonically isomorphic to 

$$\text{Pic}(\mathcal{X}) \times \text{Hom}(\mu_s, G_m) = \text{Pic}(\mathcal{X}) \times \text{Pic}(B\mu_s).$$
All in all, we get a functorial isomorphism

\[(5.7.1) \quad \iota_X : \text{Pic}(\mathcal{X} \times B\mu_s) \xrightarrow{\sim} \text{Pic}(\mathcal{X}) \times \text{Pic}(B\mu_s).\]

By [L-S], the Picard group of \(\mathcal{M}_{SL_r}(d)\) is the free abelian group \(\mathbb{Z}\mathcal{D}\) and the formula (3.1) proves that

\[f^* : \text{Pic}(\mathcal{M}_{SL_r}(d)) \rightarrow \text{Pic}(\mathcal{J}(X))\]

is an injection. The lemma follows from the commutative diagram

\[
\begin{array}{ccc}
\text{Pic}(\mathcal{M}_{SL_r}(d)) \times \text{Pic}(B\mu_s) & \xrightarrow{\iota_M} & \text{Pic}(\mathcal{J}(X)) \times \text{Pic}(B\mu_s) \\
\iota_M \downarrow & & \iota_J \downarrow \\
\text{Pic}(\mathcal{M}_{SL_r}(d) \times B\mu_s) & \rightarrow & \text{Pic}(\mathcal{J}(X) \times B\mu_s)
\end{array}
\]

Let \(\mathcal{H}\) (resp. \(\mathcal{H}_J\)) be the line bundle on \(\mathcal{J}_s \times \mathcal{M}_{SL_r}(d)\) (resp. \(\mathcal{J}_s \times \mathcal{J}(X)\))

\[\mathcal{H} = \mathcal{H}om(m^*_{\mathcal{M}}\mathcal{D}^k, \text{pr}^*_{\mathcal{J}}\mathcal{D}^k)\text{ resp. } \mathcal{H}_J = \mathcal{H}om(m^*_{\mathcal{M}}f^*\mathcal{D}^k, \text{pr}^*_{\mathcal{J}}f^*\mathcal{D}^k).\]

Let us choose a \(\mathcal{J}_s\)-linearization \(\lambda_J\) of \(f^*\mathcal{D}^k\). It defines a trivialization of the line bundle \(\mathcal{H}_J\). The equivariance of \(f\) implies (cf. 2.3) that there exists a (compatible) 2-morphism

\[q : m^*_{\mathcal{M}} \circ (\text{Id} \times f) \Rightarrow f \circ m_J\]

making the diagram

\[
\begin{array}{ccc}
\mathcal{J}_s \times \mathcal{J}(X) & \xrightarrow{m_J} & \mathcal{J}(X) \\
\text{Id} \times f \downarrow & & f \downarrow \\
\mathcal{J}_s \times \mathcal{M}_{SL_r}(d) & \xrightarrow{m_M} & \mathcal{M}_{SL_r}(d)
\end{array}
\]

2-commutative. The 2-morphism \(q\) defines an isomorphism from the pull-back \(m^*_{\mathcal{M}}\mathcal{D}^k\) on \(\mathcal{J}_s \times \mathcal{J}(X)\) to \(m^*_{\mathcal{J}}(f^*\mathcal{D}^k)\). The pull-back of \(\text{pr}^*_{\mathcal{J}}\mathcal{D}^k\) on \(\mathcal{J}_s \times \mathcal{J}(X)\) is tautologically isomorphic to \(\text{pr}^*_{\mathcal{J}}(f^*\mathcal{D}^k)\). The preceding isomorphisms induce an isomorphism

\[(\text{Id} \times f)^*\mathcal{H} \xrightarrow{\sim} \mathcal{H}_J.\]

The later line bundle being trivial, so is \((\text{Id} \times f)^*\mathcal{H}\). The lemma above proves therefore that \(\mathcal{H}\) itself is trivial. Each \((k-)\)point \(j\) of \(\mathcal{J}_s\) defines a morphism \(\mathcal{M}_{SL_r}(d) \rightarrow \mathcal{J}_s \times \mathcal{M}_{SL_r}(d)\)
let me denote by $\mathcal{H}_j$ (resp. $f^*\mathcal{H}$) the pull-back of $\mathcal{H}$ (resp. $(\text{Id} \times f)^*\mathcal{H}$) by this morphism. The pull-back morphism

$$H^0(\mathcal{J}_s \times \mathcal{M}_{S_{\mathcal{L}_r}}(d), \mathcal{H}) \to H^0(\mathcal{J}_s \times \mathcal{J}(X), (\text{Id} \times f)^*\mathcal{H})$$

can be identified to the direct sum

$$\bigoplus_{j \in \mathcal{J}_j(k)} H^0(\mathcal{M}_{S_{\mathcal{L}_r}}(d), \mathcal{H}_j) \to H^0(\mathcal{J}(X), f^*\mathcal{H}_j).$$

Because

$$(5.7.2), \quad H^0(\mathcal{M}_{S_{\mathcal{L}_r}}(d), \mathcal{O}) = H^0(\mathcal{J}(X), \mathcal{O}) = k$$

this morphism is a direct sum of non-zero morphisms of vector spaces of dimension 1 and therefore an isomorphism. In particular, a linearization $\lambda_\mathcal{J}$ of $f^*\mathcal{D}^k$ defines canonically an isomorphism

$$\lambda_\mathcal{M} : m^*_\mathcal{M}\mathcal{D}^k \overset{\sim}{\to} pr^*_2\mathcal{D}^k$$

such that $(\text{Id} \times f)^*\lambda_\mathcal{M} = \lambda_\mathcal{J}$.

Explicitly, $\lambda_\mathcal{M}$ is characterized as follows: let $x$ be an object of $\mathcal{M}_{S_{\mathcal{L}_r}}(d)$ over a connected scheme $S$ and $g$ an object of $\mathcal{J}_s(S) = \mathcal{J}_s(k)$. The preceding discussion means that the functorial isomorphisms

$$\lambda_\mathcal{M}(g, x) : \mathcal{D}^k_{g, x} \overset{\sim}{\to} \mathcal{D}^k_x$$

are determined when $x$ lies in the essential image of $f$. In this case, let us chose an isomorphism $f(x') \overset{\sim}{\to} x$ (inducing an isomorphism $g \circ f(x') \overset{\sim}{\to} g \circ x$). Then, the diagram of isomorphisms of line bundles on $S$

$$L'_{g, x'} = L_{f(x')} \overset{q_{g, x'}}{\longrightarrow} L_{g \circ f(x')} \overset{\lambda_{\mathcal{M}(g, x)}}{\longrightarrow} L_{g \circ x}$$

is commutative (where $L = \mathcal{D}^k$ and $L' = f^*\mathcal{D}^k$).

Now, the pull-back of $\lambda_\mathcal{M}$ on $\mathcal{J}_s \times \mathcal{J}(X)$ satisfies conditions 2.8.1 and 2.8.2. Using $(5.7.2)$ and the equivariance of of $f$ as above, this shows that $\lambda_\mathcal{M}$ is a linearization. For instance, keeping the notation above, let us check the condition 2.8.2. We have to check that the isomorphism $\iota$ of $L$ induced by $\epsilon$ is the identity. As above, it is enough to check that on $\mathcal{J}(X)$. By definition, with a slight abuse of notations, the diagrams

$$L'_{x'} = L_{f(x')} \overset{\lambda_{\mathcal{J}(1, x')}}{\longrightarrow} L_{x} \quad \text{and} \quad L_x \overset{\iota}{\longrightarrow} L_x$$


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Because $\lambda_J$ is a linearization, condition 2.8.2 gives the commutative diagram

$$\begin{array}{ccc}
L'_{x'} & = & L'_{x'} \\
\lambda_J(1,x') \downarrow & & \uparrow \epsilon'(x') \\
L_{1,x'} & = & L_{1,x'}
\end{array}$$

showing that the equality $\iota = \text{Id}$ remains to prove the commutativity of the diagram

$$\begin{array}{ccc}
L_{f(1,x')} & \xrightarrow{\epsilon'} & L_{f(x')} \\
q_{1,x'} \downarrow & & \uparrow \\
L_{1,f(x')} & \xrightarrow{\epsilon} & L_{f(x')}
\end{array}$$

But this follows from the commutativity of the diagram

$$\begin{array}{ccc}
f(1,x') & \xrightarrow{\epsilon'} & f(x') \\
q_{1,x'} \downarrow & & \uparrow \\
1.f(x') & \xrightarrow{\epsilon} & f(x')
\end{array}$$

which is by definition the fact that $q$ is compatible to $\epsilon$ as required in (2.3). One would check condition 2.8.1 in an analogous way. ■

Remark 5.8.— This linearization can be certainly also deduced from a careful analysis of the first section of [Fa], but the method above seems simpler.

References


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