Shock Profiles for the Perthame-Tadmor Kinetic Model

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1. INTRODUCTION.

B. Perthame and E. Tadmor [PT] have considered the following kinetic model, which is a caricature of the BGK model:

$$\partial_t f + v \partial_x f = 1_{0 \leq v \leq \rho(t,x)} - f,$$

with the notation

$$\rho(t, x) = \int_{\mathbb{R}} f(t, x, v) dv.$$ 

This class of models is reminiscent of kinetic schemes used to solve hydrodynamic equations which have been proposed by Kaniel and received a lot of attention in the last recent years ([Br1-2-3], [GM], [Pe], [LPT], [PT]). It is also closely related to the kinetic formulation of conservation laws studied by Lions-Perthame-Tadmor [LPT].

In [LPT], the following result is proven: given any initial data $0 \leq \rho^0 \in L^\infty(\mathbb{R})$, let $f_\epsilon$ be the family indexed by $\epsilon \in [0, 1]$ of solutions of (1) on $\mathbb{R}^+ \times \mathbb{R}_x \times \mathbb{R}_v$ corresponding to the initial number densities

$$f_\epsilon(0, x, v) = 1_{0 \leq v \leq \rho^0(x/\epsilon)}.$$ 

This family has the following "hydrodynamic limit":

$$f_\epsilon(\epsilon t, \epsilon x, v) \rightharpoonup 1_{0 \leq v \leq \rho(t,x)}$$ 

where $\rho$ is the entropic solution of the Hopf equation

$$\partial_t \rho + \partial_x \left( \frac{\rho^2}{2} \right) = 0, \quad x \in \mathbb{R}, \ t > 0,$$

with initial data

$$\rho(0, x) = \rho^0(x), \quad x \in \mathbb{R}.$$ 

On the other hand, Nicolaenko-Thurber [NTh] and Caffarelli-Nicolaenko [CN] have proved the existence of "shock profiles" for the Boltzmann equation in the weak shock regime. The existence of shock profiles without assumption on the strength of the profile is still an open problem for the Boltzmann equation as well as for the BGK model or for general discrete velocity models [BIU]. The only instance of a discrete velocity model for which the existence of shock profiles has been demonstrated in full generality is the Broadwell model: see [Ca].

Since the kinetic model (1) described above is the only one for which the inviscid hydrodynamic limit has been demonstrated globally in time (and therefore beyond the apparition of shock waves for the limiting conservation law), it is a logical step to investigate the shock profile problem on (1). However, we contend that this model (1) possesses too many particular features which more realistic kinetic models do not have. While studying the shock profile problem for (1) might give some ideas on how to deal with some difficulties proper to shock profiles, it seems that crucial ideas are still missing to treat the corresponding problem for the Boltzmann or BGK equations.
Finally, there are striking analogies between the shock profile problem for the kinetic model (1) and the transport-collapse method proposed by Brenier [Br1-2].

2. THE SHOCK PROFILE PROBLEM.

A shock wave for the Hopf equation (4) propagating at speed $s$ is a weak solution of (4) of the form

$$\rho(t, x) = \rho_\pm 1_{x-s t \leq x_0} + \rho_0 1_{x-s t > x_0}. \quad (5)$$

A necessary and sufficient condition for (5) to define a weak solution of (4) is that the parameters $\rho_\pm$, $\rho_0$ and $s$ be related by the Rankine-Hugoniot relation [Sm]

$$s = \frac{1}{2}(\rho_- + \rho_+). \quad (6)$$

Hence, formula (5) with the value of $s$ given by (6) defines a solution of the Hopf equation (4) in the sense of distributions. It is well-known that (4) can have more than one solution in the sense of distributions. However, there exists only one “entropic” solution of (4). An “entropic solution” of (4) is a locally bounded function $u$ which is a solution of (4) in the sense of distributions and such that, for any convex function $\eta : \mathbb{R} \to \mathbb{R}$

$$\partial_t \eta(u) + \partial_u q(u) \leq 0 \quad (7)$$

where $q$ is a primitive of $u \mapsto u \eta'(u)$: see [Sm] for more details on these notions.

It is classical (and yet instructive) to see what follows from applying the entropy condition (7) to a shock wave solution defined by (5)-(6). It is easily seen that (7) implies

$$q(\rho_+) - q(\rho_-) - \frac{\rho_+ + \rho_-}{2}(\eta(\rho_+) - \eta(\rho_-)) \leq 0. \quad (8)$$

Since (7) is to be verified for any convex function $\eta$, we can take the example $\eta(u) = \frac{1}{2} u^2$, $q(u) = \frac{3}{2} u^3$; (8) becomes then

$$\frac{1}{2}(\rho_+^2 - \rho_-^2) - \frac{1}{2}(\rho_+ + \rho_-) \frac{\rho_+^2 + \rho_-^2}{2} = (\rho_+ - \rho_-)[\frac{1}{2}(\rho_+^2 + \rho_+ \rho_- + \rho_-^2) - \frac{1}{2}(\rho_+ + \rho_-)^2] = (\rho_+ - \rho_-) \cdot \frac{1}{2}(\rho_+ - \rho_-)^2. \quad (8')$$

In other words, (8) with this particular choice of $\eta$ is equivalent to

$$\rho_+ \leq \rho_-. \quad (9)$$

On the other hand, if (9) holds, we claim that (8) holds for any convex $C^1$ entropy $\eta$ and associated entropy flux $q$ (i.e. a primitive of $u \mapsto u \eta'(u)$). Indeed, if $\eta$ and $f$ are $C^1$ convex functions defined on $\mathbb{R}$ and if $q$ is a $C^1$ function such that $q' = f' \eta'$, then for all $x, y \in \mathbb{R}$, one has

$$(y - x)(q(y) - q(x)) \geq (f(y) - f(x))(\eta(y) - \eta(x)).$$

We apply this inequality with $f(u) = \frac{1}{2} u^2$, $y = \rho_-$ and $x = \rho_+$:

$$0 \leq (\rho_+ - \rho_-)(q(\rho_-) - q(\rho_+)) - \frac{1}{2}(\rho_+^2 - \rho_-^2)(\eta(\rho_-) - \eta(\rho_-)) = (\rho_+ - \rho_-) \left[ q(\rho_+) - q(\rho_-) - \frac{\rho_+ + \rho_-}{2}(\eta(\rho_+) - \eta(\rho_-)) \right]$$

Finally, there are striking analogies between the shock profile problem for the kinetic model (1) and the transport-collapse method proposed by Brenier [Br1-2].
and (8) follows immediately.

A shock profile for (1) is a solution of (1) of the form
\[ F(t, x, v) = f(x - st, v) \]  \hspace{1cm} (10)

satisfying the boundary conditions
\[ \lim_{z \to -\infty} f(z, v) = 1_{0 \leq z \leq \rho_-}, \quad \lim_{z \to +\infty} f(z, v) = 1_{0 \leq z \leq \rho_+}. \]  \hspace{1cm} (11)

Hence the function \( f \) satisfies
\[ (v - s)\partial_z f = 1_{0 \leq v \leq \rho_f(z)} - f, \quad \rho_f(z) = \int_{\mathbb{R}} f(z, v)dv. \]  \hspace{1cm} (12)

We shall stay deliberately loose for the moment on the sense in which the limits (11) are to be understood.

If such a shock profile exists, it satisfies
\[ \partial_z \int_{\mathbb{R}} (v - s)f(z, v)dv = 0, \]
and, provided that the limits (11) hold in a strong enough topology
\[ \int_{\mathbb{R}} (v - s)f(z, v)dv = \int_{\mathbb{R}} (v - s)1_{0 \leq v \leq \rho_-} dv = \int_{\mathbb{R}} (v - s)1_{0 \leq v \leq \rho_+} dv \]
\[ = \frac{1}{2} \rho_-^2 - s \rho_- = \frac{1}{2} \rho_+^2 - s \rho_+. \]
This shows that \( \rho_- \), \( \rho_+ \) and \( s \) are related by the Rankine-Hugoniot relation (6), giving the value of \( s \). When this value is substituted in the equality above, it is seen that, if a shock profile \( f \) satisfying (12) and (11) (where the limits are taken in a strong enough topology) exists, it should satisfy
\[ \int_{\mathbb{R}} (v - s)f(z, v)dv = -\frac{1}{2} \rho_+^2. \]  \hspace{1cm} (13)

The main result of this paper (existence of shock profile for (1), that is, of a solution of (11)-(12) when \( s \) is given by (6)) is stated as Theorem 8 in section 5. More precisely, Theorem 8 applies to an equivalent formulation of the shock profile problem described in (19)-(20)-(21).

3. THE H THEOREM.

The kinetic model (1) possesses an analogue of Boltzmann’s H Theorem (for the Boltzmann equation); see [PT] [LPT]. Before dealing with this important feature of (1), we observe that (1) satisfies \( L^\infty \) a priori bounds. We refer to these bounds improperly as “the Maximum Principle”; although these bounds cannot be considered as a maximum principle for (1), they result from the maximum principle applied to the streaming operator in (1).

**Maximum Principle.** Let \( f \) be a solution of (11)-(12). Then, for all \( v \neq s \) and \( z \in \mathbb{R} \), one has \( 0 \leq f(z, v) \leq 1 \). For all \( v < 0 \) and \( z \in \mathbb{R} \), one has \( f(z, v) = 0 \).

**Proof.** If \( f \) satisfies (12), one has, for all \( v \neq s \)
\[ (v - s)\partial_z f + f = (v - s)e^{z/(v-s)}\partial_z \left( e^{z/(v-s)}f \right) \geq 0 \]
which shows that

\[ i) \forall v > s, \text{ the map } z \mapsto e^{z/(\nu-s)} f(z, v) \text{ is nondecreasing}; \]

\[ ii) \forall v < s, \text{ the map } z \mapsto e^{z/(\nu-s)} f(z, v) \text{ is nonincreasing.} \]

Let us assume that, for any given \( v > s \), there exists a sequence \( z_n(v) \to -\infty \) such that

\[ f(z_n(v), v) \to 1_{0 \leq v \leq s}. \]

Therefore, for all \( z \geq z_n(v) \)

\[ e^{z/(\nu-s)} f(z, v) \geq e^{z_n(v)/(\nu-s)} f(z_n(v), v). \]

Hence, letting \( n \to +\infty \), one sees that

\[ e^{z/(\nu-s)} f(z, v) \geq 0 \]

for all \( z \in \mathbb{R} \). The same method shows that \( f(z, v) \geq 0 \) for all \( z \in \mathbb{R} \) and \( v < s \). Changing \( f \) into \( 1 - f \) shows that \( f(z, v) \leq 1 \) for all \( z \in \mathbb{R} \) and \( v \neq s \).

To prove the second statement above, i.e. that \( f(z, v) = 0 \) for all \( z \in \mathbb{R} \) and \( v < 0 \), it suffices to apply the same line of reasoning to the function \( g = 1_{s < v} f \) which satisfies

\[ (v - s) \partial_z g + g = 0, \]

\[ g(z, v) \to 0, \quad \text{as } |z| \to +\infty. \]

This completes the proof of the Maximum Principle. //

**N.B.** In fact the \( L^\infty \) bounds above hold when one replaces the condition (11) by

\[ \forall v \in \mathbb{R}, \exists x_n(v) \to +\infty \text{ and } y_n(v) \to -\infty \quad \text{s.t.} \]

\[ f(x_n(v), v) \to 1_{0 \leq v \leq \rho}, \quad \text{and } f(y_n(v), v) \to 1_{0 \leq v \leq \rho}. \quad (11') \]

This will allow to use the Maximum Principle without making more precise the meaning of the convergence in (11) and relying only on the weakened limiting behavior (11').

The analogue of Boltzmann’s H Theorem for model (1) is recalled in the following Lemma.

**Lemma 1.** Let \( f \in L^1(\mathbb{R}^+) \) such that \( 0 \leq f \leq 1 \) a.e., and let \( \rho = \int_{\mathbb{R}^+} f(v) dv \). For any nondecreasing function \( h : \mathbb{R}^+ \to \mathbb{R} \)

\[ \int_{\mathbb{R}^+} 1_{0 \leq v \leq \rho} h(v) dv \leq \int_{\mathbb{R}^+} f(v) h(v) dv. \]

Moreover, if \( h(v) > h(\rho) \) for all \( v > \rho \) and \( h(v) < h(\rho) \) for all \( v < \rho \), the inequality above is an equality if and only if

\[ f(v) = 1_{0 \leq v \leq \rho} \text{ a.e..} \]

**Proof.** One can think of \( 1_{0 \leq v \leq \rho} \) as the nonincreasing rearrangement of \( f \) under the constraint \( 0 \leq f \leq 1 \). Here is a simple proof for the above inequality:

\[ \int_{\mathbb{R}^+} (f(v) h(v) - 1_{0 \leq v \leq \rho} h(v)) dv = \]

\[ \int_{\rho}^{+\infty} f(v) h(v) dv - \int_{0}^{\rho} (1 - f(v)) h(v) dv = \]

\[ \int_{\rho}^{+\infty} f(v) (h(v) - h(\rho)) dv + \int_{0}^{\rho} (1 - f(v)) (h(\rho) - h(v)) dv \geq 0. \]

The case where equality holds follows immediately from the equalities above. //
Therefore, if \( f \) is a solution of (12) and if \( h \) is a nondecreasing function on \( \mathbb{R}^+ \), it follows from the Maximum Principle and Lemma 1 that
\[
\partial_z \int_{\mathbb{R}^+} (v - s)f(z, v)h(v)dv = -\int_{\mathbb{R}^+} (f(z, v) - 1_{0 \leq v \leq \rho_f(z)})h(v)dv \leq 0.
\]
(14)

Hence the function
\[
z \mapsto \int_{\mathbb{R}^+} (v - s)f(z, v)h(v)dv
\]
is nonincreasing
(15)
and one has
\[
0 \leq \int_X \int_{\mathbb{R}^+} (f(z, v) - 1_{0 \leq v \leq \rho_f(z)})h(v)dv = -\int_{\mathbb{R}^+} (v - s)f(Y, v)h(v)dv + \int_{\mathbb{R}^+} (v - s)f(X, v)h(v)dv.
\]
(16)

**Corollary 2.** Let \( 0 < \rho_+ < \rho_- \) and \( s = \frac{1}{2}(\rho_+ + \rho_-) \). Let \( f \in L^1(\mathbb{R}^+) \) be such that \( f \geq 0 \) a.e. and
\[
\int_{\mathbb{R}^+} (v - s)f(v)dv = -\frac{1}{2}\rho_+\rho_-.
\]
Then
\[
\rho_+ \leq \rho_f = \int_{\mathbb{R}^+} f(v)dv \leq \rho_-
\]
(17)

**Proof.** Apply Lemma 1 with \( h(v) = v - s \). Then
\[
\int_{\mathbb{R}^+} (v - s)1_{0 \leq v \leq \rho_f} = \frac{1}{2}\rho_0^2 - s\rho_f \leq \int_{\mathbb{R}^+} (v - s)f(v)dv = -\frac{1}{2}\rho_+\rho_-,
\]
which is equivalent to
\[
\frac{1}{4}(\rho_f - \rho_-)(\rho_f - \rho_+) \leq 0.
\]
The announced inequality follows directly from here. //

**Corollary 3.** Let \( f \) satisfy (11')-(12)-(13). Then for almost every \( v > \rho_- \) or \( v < 0 \) and \( z \in \mathbb{R} \) one has \( f(z, v) = 0 \) and for almost every \( 0 \leq v < \rho_+ \) and \( z \in \mathbb{R} \) one has \( f(z, v) = 1 \).

**Proof.** One has
\[
(v - s)\partial_z(f1_{v > \rho_-}) + (f1_{v > \rho_-}) = 1_{0 \leq v \leq \rho_f}1_{v > \rho_-} = 0
\]
according to Corollary 2 above. In the same way
\[
(v - s)\partial_z[(1 - f)1_{0 \leq v < \rho_+}] + [(1 - f)1_{0 \leq v < \rho_+}] = (1 - 1_{0 \leq v \leq \rho_f})1_{0 \leq v < \rho_+} = 0.
\]
One concludes then as in the proof of the Maximum Principle. //

Corollaries 2 and 3 suggest to modify the shock profile problem above by a galilean tranformation which will set to 0 the speed of the associated shock wave. We define
\[
\alpha = \frac{1}{2}(\rho_- - \rho_+), \quad \mu = v - s,
\]
and we introduce a new unknown function \( g \) defined by
\[
g(z, \mu) = f(z, \mu + s) - 1_{0 \leq \mu + s < \rho_+}.
\]
It follows from the Maximum Principle, Corollaries 2 and 3 that
\[
0 \leq g(z, \mu) \leq 1, \text{ a.e., } g(z, \mu) = 0, \text{ for } |\mu| > \alpha.
\]
Moreover
\[ \rho_g(z) = \int_{\mathbb{R}} g(z, \mu) d\mu = \int_{-\alpha}^{\alpha} g(z, \mu) d\mu = \rho_f(z) - \rho_+. \]

Finally
\[ f(z, \mu + s) - \mathbf{1}_{0 \leq \mu + s \leq \rho_+} = (g(z, \mu) + \mathbf{1}_{0 \leq \mu + s < \rho_+} - (\mathbf{1}_{\rho_+ \leq \mu + s \leq \rho_f(z)} + \mathbf{1}_{0 \leq \mu + s < \rho_+}) = g(z, \mu) - \mathbf{1}_{0 \leq \mu + s \leq \rho_+} \]

since
\[ s - \rho_+ = \frac{1}{2}(\rho_+ + \rho) - \rho_+ = \frac{1}{2}(\rho_+ - \rho) = \alpha. \]

All in all, (11)-(12) can be recast as
\[ \mu \partial_z g + g = \mathbf{1}_{0 \leq \mu + \alpha \leq \rho_0}, \quad z \in \mathbb{R}, \quad |\mu| < \alpha, \quad (19) \]
\[ \lim_{z \to -\infty} g(z, \mu) = 1, \quad |\mu| < \alpha, \quad (20) \]

Finally, the flux condition (13) reads
\[ \int_{-\alpha}^{\alpha} \mu g(z, \mu) d\mu = 0. \quad (21) \]

4. THE SLAB PROBLEM.

Let \( Z > 0 \); consider the problem
\[ \mu \partial_z g + g = \mathbf{1}_{0 \leq \mu + \alpha \leq \rho_0}, \quad |z| < Z, \quad |\mu| < \alpha, \quad (22) \]
\[ g(-Z, \mu) = 1, \quad 0 < \mu < \alpha, \quad (23) \]
\[ g(Z, \mu) = 0, \quad -\alpha < \mu < 0. \quad (24) \]

The Maximum Principle applies here to show that
\[ 0 \leq g(z, \mu) \leq 1, \text{ for a.e. } z \in ]-Z, Z[ \times ]-\alpha, \alpha[, \quad (25) \]

(same proof as in section 3). Therefore
\[ |\mu \partial_z g(z, \mu)| \leq 2, \text{ for a.e. } z \in ]-Z, Z[ \times ]-\alpha, \alpha[, \quad (26) \]

Applying the Velocity Averaging Theorem in \( L^\infty \) and with space dimension 1 (see [GLPS]) shows that
\[ |\rho_g(x) - \rho_g(y)| \leq 4(|\log \alpha| + 2)|x - y| \log |x - y|, \]
\[ \forall x, y \in [-Z, Z], \quad |x - y| \leq \frac{1}{\varepsilon}. \quad (27) \]

Consider now the set
\[ C = \{ \rho \in C([-Z, Z]) \text{ s.t. } 0 \leq \rho \leq 2\alpha \}; \]
\[ C \text{ is obviously a closed convex subset of } C([-Z, Z]). \]

For \( \theta \in C \), we define \( g_\theta \) as the unique solution of
\[ \mu \partial_z g_\theta + g_\theta = \mathbf{1}_{0 \leq \mu + \alpha \leq \theta}, \quad |z| < Z, \quad |\mu| < \alpha, \quad (29) \]
The Maximum Principle holds again for the problem above and shows that
\[ 0 \leq g_\theta \leq 1, \text{ for a.e. } z \in [-Z, Z], \alpha, \alpha'. \]
Therefore
\[ 0 \leq \int_{-\alpha}^{\alpha} g_\theta(z, \mu) d\mu \leq 2\alpha, \quad z \in [-Z, Z]. \]
Using the same argument as in (26)-(27), one can see that
\[ T : \theta \mapsto \int_{-\alpha}^{\alpha} g_\theta d\mu \]
maps \( C \) into \( C \). As in (27), one has
\[ |T[\theta](x) - T[\theta](y)| \leq C|x - y| \log |x - y|, \quad |x - y| \leq \frac{1}{2}, \]
whence \( T \) is compact. Schauder’s fixed point theorem (see for example [Sm]) applies to show the existence of \( \rho \in C \), a fixed point of \( T \). Let \( g = g_\rho \); it is clear that \( g \) is a solution of (22)-(24). To summarize, we have proved

**Theorem 4.** The problem (22)-(24) has a solution satisfying the properties (25)-(27).

**N.B.** In full generality, the analogous problem is still open in the case of the Boltzmann equation. Some very interesting partial results have been obtained: [U] (on the BGK model), [AN] and [ACI] for the Boltzmann equation (at the expense of unphysical truncations on the collision cross-sections).

5. **The Shock Profile Problem.**

Let \( g^Z \) be a family of solutions of (22)-(24) indexed by \( Z \) and satisfying (25)-(27). First, the constant flux
\[ \int_{-\alpha}^{\alpha} \mu g^Z(z, \mu) d\mu = A_Z \in [0, \frac{1}{2} \alpha^2] \quad (32) \]
since
\[ A_Z = \int_{-\alpha}^{\alpha} \mu g^Z(Z, \mu) d\mu = \int_{0}^{\alpha} \mu g^Z(Z, \mu) d\mu \geq 0 \]
\[ = \int_{-\alpha}^{\alpha} \mu g^Z(-Z, \mu) d\mu \leq \int_{0}^{\alpha} \mu d\mu = \frac{1}{2} \alpha^2. \quad (33) \]
Let \( g^* \) be a limit point of \( g^Z \) for the weak-* topology of \( L^\infty(\mathbb{R} \times [-\alpha, \alpha]) \) as \( Z \to +\infty \). (The existence of such limit points is obtained by writing \( \mathbb{R} = \bigcup_{n \geq 1} [-n, n] \), extracting successively weakly-* converging subsequences on \([-n, n] \times [-\alpha, \alpha] \) and finally applying the diagonal extraction procedure). In particular, it follows that \( A^* = \int_{-\alpha}^{\alpha} \mu g^*(z, \mu) d\mu \) is a limit point of \( A_Z \) as \( Z \to +\infty \); therefore, one has \( 0 \leq A^* \leq \frac{1}{2} \alpha^2 \).

On the other hand, the bound (27) and Ascoli-Arzelà’s theorem show that, modulo extraction of a subsequence
\[ \rho^Z = \int_{-\alpha}^{\alpha} g^Z d\mu \to \int_{-\alpha}^{\alpha} g^* d\mu =: \rho^* \quad (34) \]
uniformly on compact subsets of $\mathbb{R}$. Therefore

$$1_{0 \leq \mu + \alpha \leq \rho} \to 1_{0 \leq \mu + \alpha \leq \rho},$$

in $L^\infty(\mathbb{R}; L^p([-\alpha, \alpha]))$ (equipped with the strong topology) for all $1 \leq p < +\infty$. Then one has

$$\mu \partial_z g^* + g^* = 1_{0 \leq \mu + \alpha \leq \rho}, \quad z \in \mathbb{R}, \; |\mu| < \alpha. \quad (35)$$

Multiplying this equality by $\mu$ and integrating over $[-\alpha, \alpha]$ gives

$$\partial_z \int_{-\alpha}^{\alpha} \mu^2 g^*(z, \mu)d\mu + A^* = \int_{-\alpha}^{\rho - \alpha} \mu d\mu \leq 0. \quad (36)$$

Integrating further with respect to $z \in [Y, X]$:

$$A^*(X - Y) \leq \int_{-\alpha}^{\alpha} \mu^2 g^*(Y, \mu)d\mu - \int_{-\alpha}^{\alpha} \mu^2 g^*(X, \mu)d\mu \leq \alpha^2, \quad \forall Y < X \in \mathbb{R}. \quad (36)$$

Therefore, letting $X \to +\infty$ and $Y \to -\infty$ leads to

**Lemma 5.** Let $g^*$ be a limit point of $g^\rho$ in $L^\infty(\mathbb{R} \times [-\alpha, \alpha])$ weak-$*$ as $Z \to +\infty$. Then

$$A^* = \int_{-\alpha}^{\alpha} \mu g^*(z, \mu)d\mu = 0. \quad (36)$$

The formulation of the shock profile problem is clearly translation invariant. It is therefore highly desirable to somehow fix a distinguished point in the profile. We shall make the arbitrary choice of fixing the point where the density takes the mean of the densities at both tails of the shock. To prepare for this, we begin with

**Lemma 6.** For all $Z > 0$, there exists $X_Z \in ]-Z, Z[$ such that

$$\rho^Z(X_Z) = \alpha. \quad (36)$$

**Proof.** Apply (16) with $X = -Z$, $Y = Z$ and $h(\mu) = 1_{\mu \geq 0}$. One has

$$0 \leq \int_{-Z}^{Z} \int_{-\alpha}^{\alpha} (g^Z(z, \mu) - 1_{0 \leq \mu + \alpha \leq \rho_z(z)})1_{\mu \geq 0} d\mu dz$$

$$\leq \int_{0}^{\alpha} \mu g^Z(-Z, \mu)d\mu - \int_{0}^{\alpha} \mu g^Z(Z, \mu)d\mu$$

$$= \frac{1}{2} \alpha^2 - \int_{0}^{\alpha} \mu g^Z(Z, \mu)d\mu. \quad (37)$$

If $\rho^Z(Z) = \alpha$, it follows from (24) that $g^Z(Z, \mu) = 1$ for $0 < \mu < \alpha$. In which case (37) shows that

$$0 = \int_{-\alpha}^{\alpha} (g^Z(z, \mu) - 1_{0 \leq \mu + \alpha \leq \rho_z(z)})1_{\mu \geq 0} d\mu \quad \text{for a.e.} \quad z \in \mathbb{R}$$

$$= \int_{-\alpha}^{\alpha} g^Z(z, \mu)d\mu - \sup(\rho^Z(z) - \alpha, 0). \quad (38)$$

With the same arguments as in (26)-(27), one can see that

$$\int_{-\alpha}^{\alpha} g^Z(z, \mu)d\mu - \int_{-\alpha}^{\alpha} g^Z(Z, \mu)d\mu = \alpha, \quad z \to Z^- \quad (39)$$
which, together with (38), shows that
\[ \rho^Z(z) = 2\alpha, \quad z \to Z^- . \tag{40} \]
This contradicts the assumed condition \( \rho^Z(Z) = \alpha \) because of the continuity of \( \rho^Z \) (see (26)-(27)). Therefore \( \rho^Z(Z) < \alpha \). The same argument, but with \( h(\mu) = -1_{\mu \leq 0} \) shows that \( \rho^Z(-Z) > \alpha \). Since \( \rho^Z \) is continuous on \([-Z, Z]\) because of (26)-(27), we deduce the existence of \( X_Z \) from the intermediate values theorem. //

The next step in the construction of the shock profile is to let \( Z \to +\infty \); however, the main difficulty is to avoid that \( g^Z \) converge to a trivial profile (i.e., to some constant “maxwellian” \( 1_{\mu \leq \mu_0} \)). This difficulty is classical whenever one is dealing with travelling waves, and was pointed out in [BIU] as the main obstruction to conclude to the existence of shock profiles of arbitrary strength in the case of discrete velocity models. Before embarking on technicalities, let us outline briefly the argument used to construct the shock profile.

If one uses the result of Lemma 6 and if one manages to center the profile at a point where the density \( \rho \) takes the mean value of the densities at both tails of the shock, one can see that \( g^Z \) cannot converge to a trivial profile without violating the flux condition of Lemma 5. Then one has to rule out the possibility that the point \( X_Z \) stays within a finite distance from either \(-Z\) or \( Z \) as \( Z \to +\infty \); otherwise, the solution so constructed would not be defined for all \( z \in \mathbb{R} \). If such was the case, the limiting profile so constructed would be a solution of a half-space problem; one obtains the desired contradiction from a uniqueness property special to half-space problems.

Let \( Z_n \to +\infty \) be such that \( g^{Z_n} \to g^* \) in \( L^\infty(\mathbb{R} \times [-\alpha, \alpha]) \) weak-* and let \( X_n = X_{Z_n} \). Consider the sequence
\[ g_n(z, \mu) = g^{Z_n}(z + X_n, \mu) . \tag{41} \]
This centers the profile as explained above. Now assume that \( X_n + Z_n \) stays bounded as \( n \to +\infty \). Modulo extracting subsequences, one can see that
\[ X_n + Z_n \to L \tag{42} \]
and
\[ g_n \to g \quad \text{in } L^\infty([-L, +\infty] \times [-\alpha, \alpha]) \text{ weak-*} \tag{43} \]
with
\[ \rho_{g_n} \to \rho_g \quad \text{uniformly on compact subsets of } [-L, +\infty[ \]
because of (26)-(27) and Ascoli-Arzela’s theorem. Hence \( g \) satisfies the following relations:
\[ \mu \partial_z g + g - 1_{\mu \leq \mu_0} = 0 , \quad z > -L , \quad |\mu| < \alpha , \tag{44} \]
\[ g(-L, \mu) = 1 , \quad 0 < \mu < \alpha , \tag{45} \]
and finally, arguing as in the proof of Lemma 5,
\[ \int_{-\alpha}^{0} \mu g(z, \mu) d\mu = 0 , \quad \rho_g(0) = \alpha . \tag{46} \]
Condition (46) shows that
\[ g(-L, \mu) = 1 , \quad -\alpha < \mu < 0 . \tag{47} \]
On the other hand, we can solve (44) for \( g \) in terms of \( \rho_g \) for \( \mu < 0 \): (44) is equivalent to
\[ \mu e^{-i\mu z} \partial_z (e^{i\mu z} g) = 1_{\mu \leq \mu_0 + \alpha \leq \rho_0} \]
and therefore
\[ e^{i\mu z} g(z, \mu) - e^{i\mu z} g(z', \mu) = \int_{z'}^{\infty} \frac{1}{\mu} e^{i\mu z} 1_{\mu \leq \mu + \alpha \leq \rho_0} d\xi ; \tag{48} \]
then we take the limit \( z' \to +\infty \) when \( \mu < 0 \). Since one has \( 0 \leq g \leq 1 \) for all \( 0 < |\mu| < \alpha \), we find
\[ g(z, \mu) = \int_{z}^{+\infty} \frac{1}{|\mu|} e^{-\frac{\xi}{|\mu|}} 1_{\mu \leq \mu + \alpha \leq \rho_0} d\xi . \tag{49} \]
Write (49) for $z = -L$:

$$1 = \int_{-L}^{+(\infty)} \frac{1}{|\mu|} e^{-\frac{4\pi}{\lambda_{\mu} + \alpha}} d\xi = \int_{-L}^{+(\infty)} \frac{1}{|\mu|} e^{-\frac{4\pi}{\lambda_{\mu} + \alpha}} 1_{[0,\lambda_{\mu} + \alpha]} d\xi .$$  \hspace{1cm} (50)

It follows from (50) that

$$\rho(\xi) \geq \alpha, \quad \text{for all } \xi \geq -L .$$  \hspace{1cm} (51)

But then, (49) shows that

$$g(z, \mu) = 1, \quad \text{for a.e. } (z, \mu) \in [-L, +\infty \times [-\alpha, 0] .$$  \hspace{1cm} (52)

It follows from the first relation in (46) that

$$g(z, \mu) = 1, \quad \text{for a.e. } (z, \mu) \in [-L, +\infty \times [-\alpha, \alpha] .$$  \hspace{1cm} (53)

But this stands in contradiction with the second relation in (46). Therefore, the assumed convergence (42) cannot hold. In the same way, one can disprove the assumption that $Z_n - X_n$ remains bounded as $n \to +\infty$.

Summarizing the above discussion, we have proved

**Lemma 7.** Let $Z_n$ and $X_n$ be defined as above. Then

$$X_n + X_n \to +\infty, \quad X_n - Z_n \to +\infty .$$

We can now state the main result of the present paper.

**Theorem 8.** There exists a function $g \in L^\infty(\mathbb{R} \times [-\alpha, \alpha])$ such that

$$\mu \partial_z g + g = 1_{[0,\lambda_{\mu} + \alpha]} , \quad z \in \mathbb{R}, \quad |\mu| < \alpha , \quad (19)$$

$$\int_{-\alpha}^{\alpha} \mu g(z, \mu) d\mu = 0 , \quad z \in \mathbb{R} , \quad (21)$$

and

$$\rho_g(0) = \alpha .$$  \hspace{1cm} (54)

Moreover, there exist two sequences $x_n \to +\infty$ and $y_n \to -\infty$ as $n \to +\infty$ such that

$$\int_{-\alpha}^{\alpha} |\mu||g(x_n, \mu)| d\mu \to 0 , \quad (55)$$

$$\int_{-\alpha}^{\alpha} |\mu||g(y_n, \mu) - 1| d\mu \to 0 , \quad (56)$$

**Proof.** Because of Lemma 7, one has clearly

$$g_n \to g , \quad \text{in } L^\infty(\mathbb{R} \times [-\alpha, \alpha]) \text{ weak-*} ,$$

and (21) holds arguing as in the proof of Lemma 5. Then $\rho_{x_n} \to \rho_g$ uniformly on compact subsets of $\mathbb{R}$ because of (26)-(27) and Ascoli-Arzela’s theorem, which shows that (19) holds. For the same reason, (54) holds as a result of the centering procedure (41).

**N.B.** At this point it is clear that $g$ cannot be a trivial profile. Otherwise, (54) would impose $g = 1_{[0,\lambda_{\mu} + \alpha]}$ which is incompatible with the null flux condition (21).

We begin with the following identity on the entropy dissipation.
Lemma 9. For all $f \in L^1([-\alpha, \alpha])$ such that $0 \leq f \leq 1$ a.e. and all $\gamma \in [-\alpha, \alpha]$, one has

$$
\int_{-\alpha}^{\alpha} (f - 1_{\beta_{\mu} + \alpha \leq \beta}) 1_{\mu \geq \gamma} d\mu
\]

$$
= \int_{-\alpha}^{\alpha} |f - 1_{\beta_{\mu} + \alpha \leq \beta}|(1_{\mu \geq \gamma} 1_{\beta_{\mu} - \alpha < \gamma} + 1_{\mu \leq \gamma} 1_{\beta_{\mu} - \alpha \geq \gamma}) d\mu.
(57)
$$

We defer the proof of Lemma 9 until the end of the proof of Theorem 8.

We apply the analogue of the H Theorem with $h(\mu) = 1_{\mu \geq \gamma}$ and proceed as in (16) to obtain

$$
\int_{0}^{+\infty} \int_{-\alpha}^{\alpha} |g - 1_{\beta_{\mu} + \alpha \leq \beta}|(1_{\mu \geq \gamma} 1_{\beta_{\mu} - \alpha < \gamma} + 1_{\mu \leq \gamma} 1_{\beta_{\mu} - \alpha \geq \gamma}) d\mu dz \leq \frac{1}{2}\alpha^2.
(58)
$$

Hence there exists a sequence $x_n \to +\infty$ such that

$$
g(x_n, \mu) - 1_{0 \leq \beta_{\mu} + \alpha \leq \beta}(1_{\mu \geq \gamma} 1_{\beta_{\mu} - \alpha < \gamma} + 1_{\mu \leq \gamma} 1_{\beta_{\mu} - \alpha \geq \gamma}) \to 0
(59)
$$
in $L^1([-\alpha, \alpha])$. On the other hand, $0 \leq g \leq 1$; therefore, modulo extracting a subsequence, one can assume that

$$
g(x_n, \mu) \to \phi(\mu), \quad \text{in } L^\infty([-\alpha, \alpha]) \text{ weak-*}
(60)
$$
and in particular

$$
\rho(x_n) \to \langle \phi \rangle = \int_{-\alpha}^{\alpha} \phi(\mu) d\mu.
(61)
$$

If $\langle \phi \rangle < \alpha$, one has, by letting $\gamma = 0$ in (59), the convergence

$$
g(x_n, \mu) \to 0, \quad \text{in } L^1([0, \alpha])
(62)
$$
and likewise, if $\langle \phi \rangle \geq \alpha$,

$$
g(x_n, \mu) - 1 \to 0, \quad \text{in } L^1([-\alpha, 0]).
(63)
$$

But the relation (63) is impossible: indeed, (63) together with the null flux condition (21) shows that

$$
\int_{-\alpha}^{\alpha} |\rho(x_n, \mu)| g(x_n, \mu) - 1 |d\mu \to 0
$$
and therefore

$$
\int_{0}^{\alpha} \rho g(x_n, \mu) d\mu \to \frac{1}{2}\alpha^2.
$$

But if one uses (16) with $h(\mu) = 1_{\mu \geq \gamma}$, one has

$$
g(z, \mu) = 1, \quad \text{for a.e. } z \in \mathbb{R}, \mu \in [0, \alpha];
$$
and then the null flux condition (21) imposes

$$
g(z, \mu) = 1, \quad \text{for a.e. } z \in \mathbb{R}, \mu \in [-\alpha, \alpha];
$$
which contradicts (54). Hence (62) holds, and we deduce from (62) and the null flux condition (21) that (55) holds. The same type of argument would provide a sequence $y_n$ such that (56) holds. //

Proof of Lemma 9. One has

$$
\int_{-\alpha}^{\alpha} (f - 1_{\beta_{\mu} + \alpha \leq \beta}) 1_{\mu \geq \gamma} d\mu =
$$
would be possible to apply the same strategy as here to prove existence of a solution to the slab problem.

even arguments in step b] could be easier to adapt to this case than those in step a]. In fact, step b]
compactness.

yet, the next fundamental difficulty is to show that, as the width of the slab tends to infinity, the limiting
profile is nontrivial. The Velocity Averaging lemmas do not help at this stage, since they provide only local
compactness.

It might be that, once step a]) (or some analogue) is known in the case of the Boltzmann equation, some
arguments in step b] could be easier to adapt to this case than those in step a]. In fact, step b] could be
summarized as follows: in the case of a half-space problem, if the number density of incoming particles at
the boundary of the half-space is known to be a maximalian, then the only bounded solution of this half-
space problem is precisely the constant maximalian density given by the boundary data; on the contrary,
in the case of a shock profile, the solution switches from one maximalian state to another. This appears
to be the fundamental qualitative difference between shock profiles and half-space problems, and it seems
reasonable to expect that this holds for most kinetic models, and certainly for either the Boltzmann or the
BGK equations. Hence, one could hope to be able to adapt part of the arguments of step b], step a] being
completed (or assumed) in either cases.

To conclude, let us point out that the above methods could be used on other kinetic models of the same
type. There is a model similar to (1)

\[ \partial_t f + a'(v) \partial_x f = 1_{0 \leq v \leq \rho_f} - f \]

whose hydrodynamic limit is the scalar conservation law

\[ \partial_t u + \partial_x a(u) = 0, \]

in the case where \( a : \mathbb{R} \to \mathbb{R} \) is a \( C^1 \) convex function. The arguments given here in the case where \( a(u) = \frac{1}{2} u^2 \)
apply without modification to the general case. Also, there is a similar model whose hydrodynamic limit is
the \( p \)-system with the particular pressure law: \( p(\rho) = \frac{1}{2} \rho^3 \); this system has the particular property that,
when transformed into the system of its Riemann invariants, it reduces to two uncoupled Hopf equations.
It seems likely that the method above would also apply to this case (which is so far the only non scalar case
that can be treated by the method initiated by Lions-Perthame-Tadmor [LPT]).

6. CONCLUSION.

We finish with some concluding remarks on the method used and the results that we have been able to
obtain.

The proof of existence of a shock profile is based on two steps:

a] the existence of stationary solutions in the slab, with uniform bounds as the width of the slab tends
to infinity;

b] a particular uniqueness property for half-space problems.

Step a] is based on the Maximum Principle, in particular the weak continuity of the nonlinear collision
operator is follows from the Velocity Averaging lemma. The Maximum Principle is very special to this
model and is not expected to hold for either the Boltzmann or the BGK models. Even \( L^p \) a priori bounds
(with \( p < +\infty \)) on the number density are not known in either cases. If such bounds were known, it
would be possible to apply the same strategy as here to prove existence of a solution to the slab problem.
Yet, the next fundamental difficulty is to show that, as the width of the slab tends to infinity, the limiting
profile is nontrivial. The Velocity Averaging lemmas do not help at this stage, since they provide only local
compactness.

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REFERENCES.


