Diffusion Approximation and
Hyperbolic Automorphisms of the Torus

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Abstract.

In this article a diffusion equation is obtained as a limit of a reversible kinetic equation
scaled appropriately. This limiting diffusion is produced by the collisions of the particles
with the boundary. Indeed, these particles follow a reversible reflection law having conve-
nient mixing properties. This model, based on “Arnold's cat map” can be handled with
Fourier series instead of the symbolic dynamics associated to a Markov partition. As a con-
sequence, optimal convergence results can be obtained by elementary means and illustrate
the apparition of irreversibility in macroscopic limits.

0. — Introduction.

“Irreversibility” is a fundamental notion in Statistical Mechanics. It is also a mislead-
ing one, and we feel useful to recall some basic facts. In doing so, we follow suggestions
from the referee, whom we thank for his help in clarifying the presentation of this paper.

Consider a mechanical system made of a (large) number $N$ of identical point par-
ticles; denote by $x_j(t)$ and $v_j(t)$ respectively the position of the $j$-th particle at time $t$
and its velocity. Starting at time $t = 0$ with an initial state of the system denoted by
$(x_j(0), v_j(0))_{1 \leq j \leq N}$, one lets the system evolve until time $t > 0$ into $(x_j(t), v_j(t))_{1 \leq j \leq N}$. 

One then changes instantaneously the velocity of each particle into its opposite, which leads to the new state $(x_j(t), -v_j(t))_{1 \leq j \leq N}$, and lets the system evolve further until time $2t$. If the final state thus reached is identical to $(x_j(0), v_j(0))_{1 \leq j \leq N}$, the dynamics considered will be called “reversible”; otherwise it will be called irreversible.

One of the central issues in non-equilibrium statistical mechanics is the derivation of macroscopic dynamics such as hydrodynamics from first principle equations. For example, the Euler or Navier-Stokes systems for a perfect ideal gas can be derived from the Boltzmann equation for, say, a gas of hard spheres, which in turn can be derived from the Liouville equation (or equivalently the system of Hamilton’s equations) for the N-body classical hamiltonian with hard-sphere interactions. These derivations are mostly “formal”, except in particular regimes where mathematical proofs have been given: see for example [Ni], [Ca], [BGL1-2], [BU], [DEL], [La], [IP], [CIP]. There are structural obstacles to a global understanding of these questions. One difficulty is that the partial differential equations (PDEs) involved are mostly nonlinear. Another conceptual difficulty is related to the notion of irreversibility. Indeed, whereas the Liouville equation is reversible in the sense described above, neither the Boltzmann equation nor the Euler and Navier-Stokes systems are. Indeed, these models contain in some sense Carnot’s “Second Principle of Thermodynamics”, and this can be seen most easily by noticing that these models have a natural notion of entropy which increases until the system reaches an equilibrium.

It appears therefore desirable to isolate the difficulties described above, and for one thing, to discuss the apparition of irreversibility on macroscopic dynamics on examples which can be analyzed as explicitly as possible (which, by the way, almost rules out dynamics described by nonlinear PDEs). Such an example has been studied in a remarkable series of papers by Bunimovich-Sinai and more recently Bunimovich-Sinai-Chernov: [BS1-2], [BSC1-2]. They have considered the periodic Lorentz gas, i.e. the dynamics of a population of point particles interacting with a periodic array of infinitely heavy circular obstacles through elastic collisions. They showed that, under the assumption that the
billiard has the property of “finite horizon” (see [BS2] for more details) the long time, large scale limiting dynamics of these particles is described by a linear diffusion equation with diffusion constant given by the Green-Kubo formula. However, their analysis relies on sophisticated tools from ergodic theory, namely the construction of a “Markov sieve” for the billiard dynamics thus generated.

The goal of the present paper is to give another example of a “reversible” dynamics, which, after some appropriate scaling, leads to an “irreversible” limiting dynamics. In particular, we believe that it might be of some interest to provide such an example that could be completely analyzed by elementary and explicit techniques and without appealing to general and abstract constructions from ergodic theory, as in the work of Bunimovich-Sinai-Chernov.

Before describing the particular (class of) models described in this paper, we want to make as clear as possible some very simple remarks concerning irreversibility. We refer in particular to [Le] and [Ga] for a precise and yet elementary presentation of these notions.

First, we insist that, although the model treated by Bunimovich-Sinai-Chernov or ours give some insight on the apparition of irreversibility when macroscopic limits are taken, it is likely that the apparition of irreversibility when the Boltzmann equation is derived from Classical Mechanics is governed by a different mechanism. Indeed, in this latter case, there are two different notions of entropy involved:

a] the Gibbs entropy

\[ S_G(t) = - \int f_N(x_1, v_1, ..., v_N) \log f(x_1, v_1, ..., v_N) dx_1 dv_1 ... dx_N dv_N \]

(where \( f_N \) is the solution of the \( N \)-body Liouville equation);

b] the Boltzmann \( H \) function

\[ H(t) = - \int F_t(x_1, v_1) \log F(x_1, v_1) dx_1 dv_1 \]

(where \( F \) is the number density which is the solution of the Boltzmann equation).
It is believed (and in some cases proved) that, under the assumption that the number of particles tends to infinity as the radius of each particle tends to zero according to the very particular Boltzmann-Grad limit (see [CIP]), and if the initial state is factorized, i.e.

\[ f_0^N(x_1, v_1, ..., x_N, v_N) = F_0(x_1, v_1) ... F_0(x_N, v_N) \]

then at further times the solution of the Liouville equation will remain approximately so with \( F \) being a solution of the Boltzmann equation. In other words, the first marginal of the solution of the \( N \)-body Liouville equation follows approximately the dynamics prescribed by the Boltzmann equation. However

— like any integral of any function of \( f^N \), \( S_G(t) = S_G(0) \) (a consequence of Liouville’s theorem for Hamiltonian dynamics);

— by Boltzmann’s H theorem, \( H(t) \) increases until \( F_t \) reaches an equilibrium state.

Observe however that for a factorized state

\[
S_G(t) = - \int F_t(x_1, v_1) ... F_t(x_N, v_N) \log (F_t(x_1, v_1) ... F_t(x_N, v_N)) \, dx_1 \, dv_1 ... dx_n \, dv_N
\]

\[ = N H(t) \left( \int F_t(x, v) \, dx \, dv \right)^{N-1} \]

so that, in the limit as \( N \to +\infty \), one cannot infer the behavior of the Boltzmann entropy from that of the Gibbs entropy. And, as explained in [Le], irreversibility is described by the Boltzmann entropy and not by the Gibbs entropy. Notice however, that one can hardly speak of apparition of irreversibility in this case, since irreversibility is introduced from the beginning by considering the marginals of the solution of the Liouville equation.

After this lengthy introduction on irreversibility, let us explain in which sense the model that we analyze below (as well as the one studied by Bunimovich-Sinai-Chernov) might provide some understanding of the apparition of irreversibility as macroscopic limits are taken.
Our model will be reversible in the following sense: it is described by a continuous unitary group of $L^2(X,\mu)$ where $X$ is a phase space of the form $\mathbb{R}^d \times Y$ and $\mu$ a measure on $X$ whose integral along the fibers $Y$ gives the Lebesgue measure on $\mathbb{R}^d$. When appropriately scaled, its long time, large scale limiting evolution is governed by the heat equation on $\mathbb{R}^d$ which is the archetype of irreversible systems and generates a continuous contraction semigroup on $L^2(\mathbb{R}^d)$. Moreover, as is well known, the $L^2$ norm of the solution of the heat equation decreases unless the solution is in equilibrium (the only equilibrium solution in $L^2(\mathbb{R}^d)$ being 0). What happens can be explained very easily in terms of weak topologies: denote by $G_t$ the unitary group corresponding to the scaled reversible evolution and by $\exp(tD\Delta)$ the diffusion semigroup with diffusion constant $D$ (here $\epsilon > 0$ denotes a small scaling parameter, and the macroscopic limit corresponds to letting $\epsilon \to 0$). For all $t > 0$ and all smooth initial data $\phi \in C^\infty_0(\mathbb{R}^d)$, the scaled family $G_t' \cdot \phi$ converges weakly to $\exp(tD\Delta) \cdot \phi$, but not in the norm topology. As a consequence, nonlinear functionals of $G_t' \cdot \phi$ do not converge to the corresponding expressions with $\exp(tD\Delta) \cdot \phi$ in place of $G_t' \cdot \phi$. This applies in particular to the $L^2$ norm, and this explains how the $L^2$ norm of the limiting solution decreases while that of the scaled microscopic dynamics (the one corresponding to the unitary group $G_t$) remains constant under the evolution.

Before describing in detail the mechanical system that we shall analyze in this paper, we conclude this impressionistic survey of the notion of irreversibility as related to our model by recalling for readers who might have more background in Physics than in Mathematics the notion of weak convergence which is central to our discussion, as can be understood from the explanation above.

We believe that the most intuitive way of understanding weak topologies in Hilbert spaces consists in looking at the particular model $H = L^2(S^1)$, the space of square-integrable functions on the circle (or equivalently defined on the real line $\mathbb{R}$ and periodic of period, say, one). Then, functions can be thought as superpositions of elementary waves via the theory of Fourier series. For a function $u \in H$, we denote by $\hat{u}(k)$ its Fourier
coefficient of order \( k \in \mathbb{Z} \). Then a family of functions \((u_\varepsilon)\) indexed by \( \varepsilon \in ]0, 1[ \) is said

i] to converge to \( u \) in the weak topology of \( H \) if and only if, for all \( k \in \mathbb{Z} \), the family of

Fourier coefficients of order \( k \), \( \hat{u}_\varepsilon(k) \) converges to \( \hat{u}(k) \);

ii] to converge to \( u \) in the strong (or norm) topology of \( H \) if and only if

\[
\int_{S^1} |u_\varepsilon(x) - u(x)|^p dx = \sum_{k \in \mathbb{Z}} |\hat{u}_\varepsilon(k) - \hat{u}(k)|^p \to 0.
\]

Another formulation of ii] is as follows:

i'] the family \((u_\varepsilon)\) converges to \( u \) in the weak topology of \( H \) if and only if, for any interval \( I \) of \( S^1 \) one has

\[
\int_I u_\varepsilon(x)dx \to \int_I u(x)dx.
\]

This makes more intuitive the relation between weak convergence and coarse graining.

It is clear from this presentation that strong convergence implies weak convergence, whereas the converse is wrong, unless additional assumptions are made on the decay of the Fourier coefficients at high wave numbers (uniform – in \( \varepsilon \) – smallness as the wave number tends to infinity). Since nonlinear functionals of \( u_\varepsilon \) would involve a nonlocal interaction of the Fourier coefficients of \( u_\varepsilon \), it is also clear that such nonlinear functionals of \( u_\varepsilon \) as, for example, the quadratic mean, are not continuous with respect to the weak topology. We refer to [DS] for a comprehensive introduction to the notion of weak convergence in \( L^p \) spaces.

To conclude this list of remarks, we hope to have made clear what is meant here by reversibility and irreversibility, and how the way these notions are used in the present work differs from the more classical situation considered in statistical mechanics, i.e. the derivation of kinetic theory from hamiltonian dynamics.
1.— Presentation of the model.

Between two horizontal plates, a family of particles evolve as a Knudsen gas i.e. a gas with no interparticle collisions. The vertical components of the velocities of the particles are all assumed to have modulus $c > 0$. The horizontal components of their velocities $ca(\omega)$ are parametrized by $T^2 = R^2/(2\pi Z)^2$. Whenever the particles hit the top or bottom plate, their vertical velocities are changed into their opposite while their horizontal velocities are modified by the right action of an hyperbolic automorphism of $T^2$ (cf. figure 1).

We shall use the following notations: the position of the particles is denoted by $(x, z) \in R^d \times (0, h)$ and the vertical component of the velocity of the particles by $\pm c$. The horizontal component of this velocity is given by $ca(\omega), \omega \in T^2$, where $a : T^2 \to R^d$ denotes a smooth zero mean vector field. The nonnegative functions $f_+(t, x, z, \omega)$ (resp. $f_-(t, x, z, \omega)$) represent the number density of particles which at time $t$ occupy the position $(x, z)$ and move with velocity $(ca(\omega), +c)$ (resp. $(ca(\omega), -c)$).
The following hyperbolic automorphism $T$ of the torus (Arnold’s cat map) defined by

$$T \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \pmod{2\pi}. \quad (1)$$

will be the only case treated here for computational simplicity. However, it will be clear throughout the paper that our method would apply to any hyperbolic automorphism of $T^n$. The map $T : T^2 \rightarrow T^2$ is one-to-one and $C^\infty$; it preserves the measure $d\omega_1d\omega_2/4\pi^2$ and its inverse (which also is a $C^\infty$ map) is given by

$$T^{-1} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \pmod{2\pi}. \quad (2)$$

The densities $f^\pm$ satisfy the Liouville system of equations

$$\partial_t f^\pm + ca(\omega) \cdot \partial_x f^\pm \pm c \partial_z f^\pm = 0, \quad x \in \mathbb{R}^d, \ 0 < z < h, \ \omega \in T^2. \quad (3)$$

with the following boundary conditions on the plates:

$$f^+(t,x,0,\omega) = f^-(t,x,0,T\omega), \quad x \in \mathbb{R}^d, \ \omega \in T^d, \quad (4a)$$

$$f^-(t,x,h,\omega) = f^+(t,x,h,T\omega), \quad x \in \mathbb{R}^d, \ \omega \in T^2. \quad (4b)$$

Their value at $t = 0$ is given by the following initial condition

$$f^\pm(0,x,z,\omega) = \phi(x), \quad x \in \mathbb{R}^d, \ 0 < z < h, \ \omega \in T^2; \quad (5)$$

which is compatible with an approximation, as $h \rightarrow 0$, by an horizontal diffusion, and avoids initial layers in the limiting process.

Since the densities $f^\pm$ satisfy the equation (3), they are constant along the characteristic lines of the system; hence

$$f^\pm(t,x,z,\omega) = \phi \left( x - h \sum_{k=0}^{[ct/h]} a(T^k \omega) \right) + O(h). \quad (6)$$

The asymptotic limit leading to an horizontal diffusion will be obtained by letting $h$ go to zero and observing the system for large positive times. A small parameter $\epsilon$ being
introduced, \( h \) is changed into \( \epsilon h \) (thus letting the collision frequency go to \( \infty \)) and \( t \) into \( t/\epsilon \). The problem of interest \((3)-(4a,b)-(5)\) becomes

\[
e^2 \partial_t f^\pm_{\epsilon} + c e a(\omega) \cdot \partial_x f^\pm_{\epsilon} \pm c \partial_z f^\pm_{\epsilon} = 0, \quad x \in \mathbb{R}^d, \; 0 < z < h, \; \omega \in \mathbb{T}^2. \tag{7}
\]

\[
f^+_{\epsilon}(t,x,0,\omega) = f^-_{\epsilon}(t,x,0,T\omega), \quad x \in \mathbb{R}^d, \; \omega \in \mathbb{T}^2, \tag{8a}
\]

\[
f^-_{\epsilon}(t,x,h,\omega) = f^+_{\epsilon}(t,x,h,T\omega), \quad x \in \mathbb{R}^d, \; \omega \in \mathbb{T}^2, \tag{8b}
\]

\[
f^\pm_{\epsilon}(0,x,z,\omega) = \phi(x), \quad x \in \mathbb{R}^d, \; 0 < z < h, \; \omega \in \mathbb{T}^2. \tag{9}
\]

Its solution is given by (see \((6)\))

\[
f^\pm_{\epsilon}(t,x,z,\omega) = \phi \left( x - \epsilon h \sum_{k=0}^{[\epsilon t/\epsilon^2 h]} a(T^k \omega) \right) + O(\epsilon); \tag{10}
\]

therefore most of the analysis is reduced to studying the limit, as \( \epsilon \to 0 \), of the expression

\[
\psi_{\epsilon}(t,x,\omega) = \phi \left( x - \epsilon h \sum_{k=0}^{[\epsilon t/\epsilon^2 h]} a(T^k \omega) \right). \tag{11}
\]

2.— Notations and Main Results.

The following notation will be systematically used below:

\[
\langle F \rangle = \frac{1}{4\pi^2} \int_{\mathbb{T}^2} F(\omega) d\omega.
\]

The formula \( U_T f = f \circ T \) with the mapping \( T \) given by \((1)\) defines an operator \( U_T \) in the space \( L^2(\mathbb{T}^2) \); this operator is unitary and therefore its adjoint is given by \( U_T^* f = U_T^{-1} f = f \circ T^{-1} \).

**Definition 1.** A coboundary is an element of \( \text{Im}(I - U_T) \). Two functions \( f \) and \( g \) belonging to \( L^2(\mathbb{T}^2) \) are said to be cohomologous if and only if \( f - g \) is a coboundary and this equivalence relation will be denoted \( f \sim g \).
The next proposition describes the elementary properties of what will be, in the limit $\epsilon \to 0$, the diffusion coefficient. It is essentially based on the ergodic and mixing properties of the mapping $T$.

**Proposition 2.**

1) Any function $a \in L^2(T^2)$ which satisfies the relation $a = a \circ T$ is constant and the subspace $\text{Im}(I - U_T)$ is dense in the space of functions $a \in L^2(T^2)$ such that $<a> = 0$ (notice that this space is invariant under $T$).

Let $s > 0$ and $a : T^2 \to \mathbb{R}^d$ belong to the Sobolev class $H^s(T^2)$, with mean value $<a> = 0$

2) One has:

$$D(a) = \frac{1}{2} <a^2> + \sum_{k \geq 1} <a \circ T_k \otimes a> = \frac{1}{2} \lim_{N \to \infty} \left( \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a \circ T_k \right) \geq 0,$$

where the series

$$\sum_{k \geq 1} \|a \circ T_k \otimes a\| < +\infty$$

for any norm $\|\cdot\|$ on $M_d(\mathbb{R})$.

3) Let $\xi \in \mathbb{R}^d$ and $b \in H^s(T^2; \mathbb{R}^d)$ such that $<b> = 0$. If $a \cdot \xi \sim b \cdot \xi$ then $\xi \cdot D(a)\xi = \xi \cdot D(b)\xi$.

4) For any $\xi \in \mathbb{R}^d$, the following properties are equivalent:

i] $D(a)\xi = 0$;

ii] $\xi \cdot D(a)\xi = 0$;

iii] The sequence of functions $f_N \cdot \xi$ of $L^2(T^2)$

$$f_N \cdot \xi = \sum_{k=1}^{N} (a \circ T_k) \cdot \xi$$

is (uniformly with respect to $N$) bounded in $L^2(T^2)$;

iv] The function $a \cdot \xi$ is a coboundary.

The main result of this paper is the following
**Theorem 3.** Let \( s > d/2, \ a : \mathbb{T}^2 \to \mathbb{R}^d \) be in the Sobolev class \( H^s(\mathbb{T}^2) \) with mean value \( \langle a \rangle = 0 \) and let \( \phi \in C^2(\mathbb{R}^d) \) be an initial data. Denote by \( u(t, x) \) the solution of

\[
\partial_t u = h \epsilon \nabla_x \cdot (\nabla_x u) , \quad u(0, x) = \phi(x) .
\]  

(13)

Then the family of functions \( f^\pm \) defined by (7), (8) and (9) converges to \( u(x, t) \) as \( \epsilon \to 0 \) in the following sense: for any \( \tau > 0 \) and any compact \( K \subset \mathbb{R}^d \)

\[
\langle f^\pm (t, x, z, \omega) \rangle \to u(t, x) , \quad C^0(0, \tau) \times K \times \mathbb{T}^2 , \quad \epsilon \to 0 .
\]  

(14a)

\[
f^\pm (t, x, z, \omega) \to u(t, x) , \quad C^0(0, \tau) \times L^\infty(\mathbb{R}^d \times \mathbb{T}^2) .
\]  

(14b)

Furthermore with \( \psi_i (x, t) \) defined by the formula (11) one has:

\[
\| f^\pm - \psi_i \|_{L^\infty(\mathbb{R} \times \mathbb{R}^d \times \mathbb{T}^2)} = O(\epsilon) .
\]  

(15)

The proof of this theorem is tailored on the proof of the Ito formula (cf. for instance [D]); in particular it will be shown that the average of the different products appearing in a Taylor expansion are, in the limit completely decorrelated and therefore converge to the product of the corresponding limiting averages. In the original Ito formula, this point is straightforward since (by construction) the Brownian motion has independent increments. At variance in the present paper, the independence is obtained only in the limit \( \epsilon \to 0 \) and, as will be shown below, is a consequence of the different mixing properties of the map \( T \).

An analogous result has been proven by Denker-Philipp [DP] for suspensions of finite type subshifts under Hölder continuous maps. It would be theoretically possible to reduce the present analysis to this situation by coding the mapping \( T \) with a Markov partition. However, we recall that our goal in the present paper is to give an example of diffusion approximation of a reversible system which is as explicit as possible and uses only elementary techniques. In the present case, Fourier series expansions are used instead of a Markov
partition as a coding of the system.\footnote{While going through the revision of this manuscript, we became aware of an independent contribution by Babovsky [Ba] on the same model. He uses the coding based on a Markov partition instead of the explicit method of the present article, and hence his analysis is a particular case of [DP]. However, he provides a phenomenological derivation of models similar to (3)-(4)-(5) as a linearization of the law of specular reflection at a rough surface. This might add some practical interest for this class of examples.}

The outline of the paper is as follows. In section 3, the basic mixing properties of the map $T$ are given (property (H1) of proposition 5); they are based on the fact that the eigenvalues of the matrix defining the cat map are algebraic of degree 2 and therefore satisfy a diophantine condition. The mixing property (H1) is used to prove proposition 2. Then the decorrelation properties of the process (property (H2)) are first proven in proposition 6. This implies property (H3) of corollary 7, used to estimate the remainder of the Taylor expansion formula. Then proposition 8 and corollary 9 provide all the material for the proof of theorem 3 which belongs to section 4. Section 5 contains final remarks and comments on the numerical simulations.

3.— Proofs of the Mixing Properties and of Proposition 2.

As announced above, the proofs are based on Fourier series techniques. Indeed the properties of the dynamical system induced by the map $T$ on the torus $\mathbb{T}^2$ are most conveniently translated into those of the dynamical system defined in $\mathbb{Z}^2$ (the dual group of $\mathbb{T}^2$) by the iteration of the matrix

$$ M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}. $$

The matrix $M$ is strictly hyperbolic, i.e. has two (distinct) real eigenvalues given by

$$ \lambda_+ = 1 + \theta, \quad \lambda_- = \lambda_+^{-1}, \quad \text{with } \theta = \frac{1 + \sqrt{5}}{2}. \quad (16) $$
and the corresponding eigenvectors (generating the unstable and stable manifolds) are

$$e_+ = \mu \left( \frac{0}{1}, \frac{1}{-\theta} \right), \quad e_- = \mu \left( \frac{1}{0}, \frac{0}{1-\theta} \right).$$

with $\mu = (1+\theta^2)^{-1/2}$. The vectors $(e_+, e_-)$ define an orthonormal basis. In this basis $M$ reduces the diagonal form

$$M \sim \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix},$$

Most of the results that we shall obtain are based on the fact that the eigenvalues of $M$ are algebraic numbers and therefore satisfy a diophantine estimate. We recall this simple fact (originally due to Kronecker) in the following

**Lemma 4.** Let $\theta = \frac{4 + \sqrt{5}}{2}$. Then:

(i) (best rational approximation for $\theta$)

$$\inf_{(p, q) = 1} |\theta - \frac{p}{q}| \geq \frac{1}{(1 + \sqrt{5})q^2}$$

(ii) (diophantine estimate for $\theta$)

$$\inf_{(p, q) \in \mathbb{Z}^2 \setminus \{0\}} |q\theta - p| \geq \frac{1}{(1 + \sqrt{5}) \sup(|p|, |q|)}$$

**Proof.** The minimal polynomial of $\theta$ over $\mathbb{Q}$ is $P(X) = X^2 - X - 1 = (X - \theta)(X + \theta^{-1})$. Let $p$ and $q$ be two integers $(p, q) = 1$. First $P\left(\frac{p}{q}\right) \neq 0$ since $P$ has no rational root, and therefore

$$|P\left(\frac{p}{q}\right)| = \frac{|p^2 - qp - q^2|}{q^2} \geq \frac{1}{q^2}$$

On the other hand

$$|\frac{p}{q} - \theta| = \frac{|P\left(\frac{p}{q}\right)|}{|\frac{p}{q} - \theta + \theta^{-1}|} = \frac{|P\left(\frac{p}{q}\right)|}{|\frac{p}{q} - \theta + \sqrt{5}|} \geq \frac{1}{q^2\left(|\frac{p}{q} - \theta + \sqrt{5}|\right)}.$$

Considering both cases $|\frac{p}{q} - \theta| > 1$ and $|\frac{p}{q} - \theta| < 1$, one obtains for any pair of integers $(p, q)$ with $q \neq 0$

$$|\frac{p}{q} - \theta| \geq \inf \left(1, \frac{1}{(1 + \sqrt{5})q^2}\right)$$

13
which proves point i). Point ii) follows directly from point i).

**Proposition 5.** Let $0 \leq \chi(R)$ be a nonincreasing positive function such that

$$
\lim_{R \to \infty} \chi(R) = 0.
$$

Consider the class of functions:

$$
H_\chi = \{ f \in L^2(T^2) \text{ st. } \sum_{|k_1|,|k_2| > R} |\hat{f}(k)|^2 \leq \chi(R)^2 \|f\|^2 \}.
$$

Then one has the following

i) **“Rate of Mixing”:** For all $(f, g) \in H_\chi$ such that $\langle f \rangle = \langle g \rangle = 0$

$$
|\langle f \circ T^n \cdot g \rangle| \leq \frac{1}{3\pi^2} \|f\|_2 \|g\|_2 \chi \left( C_0 \left( \frac{3+\sqrt{5}}{2} \right)^{n/2} \right), \text{ with } C_0 = \frac{1+\sqrt{5}}{\sqrt{2}}. \tag{22}
$$

ii) **Property (H1): “Exponential mixing”:** For all $s > 0$ and all $f \in H^s(T^2)$ such that $\langle f \rangle = 0$, the self-correlation coefficient defined as

$$
C_f(n) = \langle f \circ T^n \cdot f \rangle
$$

satisfies the decay estimate:

$$
|C_f(n)| \leq C \|f\|^2 e^{-s\alpha n} \text{ with } \alpha = \log \left( \frac{3+\sqrt{5}}{2} \right). \tag{23}
$$

**Proof.** With the Plancherel formula one has, for any pair $(f, g) \in L^2(T^2)$ with mean value $\langle f \rangle = \langle g \rangle = 0$, the formula:

$$
\langle f \circ T^n \cdot g \rangle = \frac{1}{4\pi^2} \sum_{k \neq 0} \hat{f}(M^{-n}k)g(-k) \tag{24}
$$

For any $R > 0$, decompose the above sum into two parts corresponding to $K_R$ and $K_R^c$ with $K_R$ given by

$$
K_R = \{ k \in \mathbb{Z}^2 \text{ s.t. } \sup(|k_1|, |k_2|) \leq R \}.
$$
Since \( g \) belongs to the class \( H_\chi \), the Cauchy-Schwarz inequality yields the estimate:

\[
\left| \sum_{k \in K_n^{-}} f(M^{-n}k)\hat{g}(-k) \right| \leq \|f\|_2 \|g\|_2 \chi(R).
\]  \hspace{1cm} (25)

For \( k \in K_R \), one introduces the decomposition \( k = (k \cdot e_+)e_+ + (k \cdot e_-)e_- \); Kronecker’s estimate (19) shows that

\[
|k \cdot e_-| \geq \theta^{-1}|k|^{-1} \geq (\sqrt{2R\theta})^{-1}
\]

whence

\[
|M^{-n}k| \geq \frac{\lambda^0_+}{\sqrt{2R\theta}}.
\]

Using that \( f \in H_\chi \) and that \( \chi \) is nonincreasing, this implies the estimate:

\[
\left| \sum_{k \in K_R^{-}[0]} f(M^{-n}k)\hat{g}(-k) \right| \leq \|f\|_2 \|g\|_2 \chi \left( \frac{\lambda^0_+}{\sqrt{2R\theta}} \right).
\]  \hspace{1cm} (26)

The relation (22) is obtained by choosing \( R = \lambda^{n/2}_+ \) in (25) and (26). To obtain the exponential rate of mixing (23), one specializes (22) to the case \( f = g \) and uses, the expression \( \chi(R) = R^{-1}(\|f\|_2 \|f\|_2^{-1})^{1/2} \).

**Proof of Proposition 2.** Let \( k \in \mathbb{Z}^2 \setminus 0 \). Since \( \text{Ker} \,(M - \lambda_- I) \) is a line with irrational slope, the orthogonal projection of \( k \) on \( \text{Ker} \,(M - \lambda_- I) \) is not 0 and therefore \( |M^{-n}k| \to +\infty \) as \( n \to +\infty \).

Let \( a \in L^2(T^2) \) and consider its Fourier series:

\[
a(\omega) = \sum_{k \in \mathbb{Z}^2} \tilde{a}(k)e^{ik\omega}
\]

The relation \( a = a \circ T \) shows that for any \( k \in \mathbb{Z}^2 \) and any \( n \) one has

\[
a(k) = \tilde{a}(M^{-n}k).
\]

But \( a \in L^2(T^2) \) since \( a \in L^2(T^2) \): hence for all \( k \neq 0 \) in \( \mathbb{Z}^2 \) \( a(k) = a(M^{-n}k) \to 0 \) as \( n \to +\infty \). Therefore \( \tilde{a} \equiv 0 \) on \( \mathbb{Z}^2 \setminus 0 \), which means that \( a \) is a constant. This proves the
ergodic property \( \text{Ker}(I - U_T) = \mathbb{R} \). The same remark applies to the kernel of the adjoint \((I - U_T)^* = U_T^{-1}(U_T - I)\) in \(L^2(T^2)\) which also is reduced to the constants. Therefore the set of coboundaries is dense in the space of functions \(a \in L^2(T^2)\) such that \(<a> = 0\).

For point 2), start with the formula

\[
\langle a \circ T^k \otimes a \circ T^l \rangle = \langle a \circ T^{k-l} \otimes a \rangle. 
\] (27)

which follows from the invariance of the measure \(d\omega_1 d\omega_2\) under \(T\). Summing with respect to \(m = k - l\) yields

\[
\left( \sum_{k=0}^{N-1} a \circ T^k \right)^{\otimes 2} = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \langle a \circ T^{k-l} \otimes a \rangle = N \langle a^{\otimes 2} \rangle + 2 \sum_{m=1}^{N-1} (N - m) \langle a \circ T^m \otimes a \rangle 
\] (28)

or in other words

\[
\langle a^{\otimes 2} \rangle + 2 \sum_{m=1}^{N-1} \langle a \circ T^m \otimes a \rangle = \left( \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a \circ T^k \right)^{\otimes 2} + \frac{2}{N} \sum_{m=1}^{N-1} m \langle a \circ T^m \otimes a \rangle. 
\] (29)

Since the vector field \(a\) has mean zero and is of Sobolev class \(H^s\) with \(s > 0\), it follows from proposition 5 that the exponential estimate (23) holds; it implies the absolute convergence of the series

\[
\sum_{m \geq 1} \langle a \circ T^m \otimes a \rangle
\]

which appears in the left hand side of (12) (point 2) of proposition 2) and of

\[
\sum_{m \geq 1} m \langle a \circ T^m \otimes a \rangle.
\]

Therefore, the last term of the right hand side of (29) goes to zero, leading to the relation

\[
D(a) = \frac{1}{2} \lim_{N \to \infty} \left\{ \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a \circ T^k \right\}^{\otimes 2} 
\] (30)

and completing the proof of point 2) in the proposition 2.

To prove point 3), write \(a \sim b\) as \(a - b = \phi - \phi \circ T\) and use the relation:

\[
\xi : D(b)\xi = \lim_{N \to \infty} \left\| \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} (\xi : b) \circ T^k \right\|_2^2 = 
\]

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\[
\lim_{N \to \infty} \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} ((\xi \cdot a) \circ T^k + (\xi \cdot \phi) \circ T^k - (\xi \cdot \phi) \circ T^{k+1})^2
\]

\[
\lim_{N \to \infty} \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} (a \circ T^k \cdot \xi)^2 = \xi \cdot D(a) \cdot \xi. 
\] (31)

Since the matrix \(D(a)\) is symmetric and nonnegative, points 4 i) and 4 ii) are equivalent.

To prove that 4) ii) implies 4 iii), write (29) in the form

\[
\left\| \sum_{k=0}^{N-1} (\xi \cdot a) \circ T^k \right\|_2^2 = N\xi \cdot D(a)\xi - 2N \sum_{m \geq N} \langle (\xi \cdot a) \circ T^m(\xi \cdot a) \rangle - 2 \sum_{m=1}^{N-1} m((\xi \cdot a) \circ T^m(\xi \cdot a)) .
\] (32)

We recall that, thanks to the exponential decay estimate (23), the series

\[
\sum_{m \geq 1} |m((\xi \cdot a) \circ T^m(\xi \cdot a))| < +\infty
\]

and

\[
\sum_{m \geq N} |((\xi \cdot a) \circ T^m(\xi \cdot a))| = o\left(\frac{1}{N}\right). \tag{33}
\]

Hence, if one assumes that \(\xi \cdot D(a)\xi = 0\), one has

\[
\left\| \sum_{k=0}^{N-1} (\xi \cdot a) \circ T^k \right\|_2^2
\]

\[
\leq 2 \sup_{N \geq 1} \left( N \sum_{m \geq N} |((\xi \cdot a) \circ T^m(\xi \cdot a))| \right) + 2 \sum_{m \geq 1} m((\xi \cdot a) \circ T^m(\xi \cdot a)) | < +\infty, \tag{34}
\]

Finally, to prove that iii) implies iv) observe that the sequence \(f_N \cdot \xi\) is, in any case, the “formal inverse” for the equation

\[
a \cdot \xi = g^\xi - g^\xi \circ T. \tag{35}
\]

Since the sequence

\[
f_N \cdot \xi = \sum_{k=1}^{N} a \circ T^k \cdot \xi
\]

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is bounded in $L^2(T^2)$, one can consider $f$, one of its weak $L^2(T^2)$ limit points and, using lemma 4, one has, for any function $\phi \in C^\infty(T^2)$

$$\langle (f_N - f_N \circ T)\phi \rangle = \langle \xi \cdot a\phi \rangle = -\langle \xi \cdot a \circ T^{N+1} \phi \rangle \to 0, \text{ for } N \to \infty$$

(36)

which shows that $f - f \circ T = a \cdot \xi$. //

As we said above, the proof of theorem 2 follows in many respects the proof of the Ito formula. Therefore it will be important to study the decorrelation of events occurring in two separate intervals of time, uniformly with respect to the size of these intervals, and under the only assumption that their mutual distance is large enough. This is dealt with in the next proposition. What we prove is a property similar to the “Weak Bernoulli property” as introduced by Ornstein-Weiss (cf. [Bo]). The main difference with the classical formulation as in [Bo] is the use of trigonometric polynomials instead of indicator functions of the elements of a partition.

**Proposition 6.** The transformation $T$ has the following

**Property (H2).** There exist two constants $\beta_0 > 0$ and $\beta_1$ such that for all $l, m \in \mathbb{N}$, $U \subset \{n, ..., n + l\}$, $V \subset \{n, ..., n + m\}$ and for all pair of trigonometric polynomials $P, Q$, of degree less than $R$, one has, for all $n \geq \beta_0 \log R + \beta_1$

$$\langle \prod_{k \in U} P \circ T^{-k} \prod_{k \in V} Q \circ T^k \rangle - \langle \prod_{k \in U} P \circ T^{-k} \rangle \langle \prod_{k \in V} Q \circ T^k \rangle = 0$$

(37)

**Proof.** This proof follows that of proposition 5. The method is similar to the one used by Katznelson [Ka], but here a more precise result is needed and proven. Observe that it is enough to study expressions of the following type:

$$\frac{1}{4\pi^2} \int_{T^2} \exp \left( i\omega \cdot \sum_{k \in U} M^{-k} \xi_k \right) \exp \left( i\omega \cdot \sum_{k \in V} M^k \eta_k \right) d\omega$$

$$- \left( \frac{1}{4\pi^2} \int_{T^2} \exp \left( i\omega \cdot \sum_{k \in U} M^{-k} \xi_k \right) d\omega \right) \left( \frac{1}{4\pi^2} \int_{T^2} \exp \left( i\omega \cdot \sum_{k \in V} M^k \eta_k \right) d\omega \right)$$

(38)
with \( \xi_k \in K_R, \forall n \leq k \leq n + l \) and \( \eta_k \in K_R \forall n \leq k \leq n + m \) (where, as in the proof of the proposition 5, \( K_R = \{ k \in \mathbb{Z} \} \) s.t. \( \sup(|k_1|, |k_2|) \leq R \}).

With the notations

\[
X_U^- = \sum_{k \in U} M^{-k} \xi_k, \quad X_V^+ = \sum_{k \in V} M^k \eta_k,
\]

the relation (37) is equivalent to the following assertion:

**There exist** \( \beta_0 > 0 \) **and** \( \beta_1 \) **such that:**

\[
X_U^- + X_V^+ = 0, \quad X_U^- = 0, \quad \text{and} \quad X_V^+ = 0, \quad \forall n > \beta_0 \log R + \beta_1, \quad \forall l, m \in \mathbb{N}.
\]

Denote by \( I = Re_+ \) and \( S = Re_- \) the unstable and stable manifolds of \( M \) acting on \( \mathbb{R}^2 \). Let \( \xi \in K_R \); one has

\[
|X_U^- \cdot e_+| \leq \sum_{k \in U} |M^{-n} \xi_k \cdot e_+| \leq \sum_{k \in U} |\xi_k \cdot M^{-n} e_+| = \sum_{k \in U} \lambda_k |\xi_k \cdot M^{-n} e_+|.
\]

Proceeding likewise with \( X_V^+ \), for all \( \xi \) and \( \eta \) in \( K_R \) one has

\[
|X_U^- \cdot e_+| \leq \sqrt{2} R \frac{\lambda^n}{1 - \lambda_-}, \quad |X_V^+ \cdot e_-| \leq \sqrt{2} R \frac{\lambda^n}{1 - \lambda_-},
\]

This implies that \( X_U^- \) belongs to a neighborhood \( S^n_R \) of \( S \) while \( X_V^+ \) belongs to a neighborhood \( I^n_R \) of \( I \) given by the formulas:

\[
I^n_R = \{ X \in \mathbb{R}^2 \} \text{ s.t. } |X \cdot e_-| \leq \sqrt{2} R \frac{\lambda^n}{1 - \lambda_-},
\]

\[
S^n_R = \{ X \in \mathbb{R}^2 \} \text{ s.t. } |X \cdot e_+| \leq \sqrt{2} R \frac{\lambda^n}{1 - \lambda_-},
\]
Since \( X_U^- + X_V^+ = 0 \), both \( X_U^- \) and \( X_V^+ \) belong to \( K_{R,n} = S_R^+ \cap I_R^+ \) (cf. fig 2) which for an \( n \) greater than a given value \( N_0 \) is contained in \( K_{R,n} \). Whenever they are not both equal to zero, the diophantine estimate (19) of lemma 4 can be used to give:

\[
|X_U^- \cdot \epsilon_+| \geq \frac{1}{(1 + \sqrt{3})R}, \quad \text{or} \quad |X_V^+ \cdot \epsilon_-| \geq \frac{1}{(1 + \sqrt{3})R}. \tag{42}
\]
For $n > (2 \log R + \log(\sqrt{2}(1 + \sqrt{5}))/\log \lambda_+$, one has
\[
\frac{1}{(1 + \sqrt{5}) R} > \sqrt{2} R \frac{\chi_+}{1 - \chi_+}.
\]
(43)

Therefore, if $n$ is greater than
\[
\sup \left( a_0, \frac{2 \log R + \log(\sqrt{2}(1 + \sqrt{5}))}{\log \lambda_+} \right),
\]
$X_U^-$ and $X_U^+$ are both equal to zero. //

**Corollary 7.** Let $0 \leq \chi(R)$ be a nonincreasing positive function going to 0 as $R$ tends to infinity, such that
\[
\chi(R) = O((1/ \log R)^6) \quad \text{for} \quad R \to +\infty
\]
(44)

Consider the class of functions defined by:
\[
W_{\chi} = H_{\chi} \cap \{ f \in L^\infty(T^2) \text{ s.t. } \frac{1}{4\pi} \sum_{\text{sup}(|k_1|,|k_2|) > R} |\hat{f}(k)| \leq \|f\|_{\infty} \chi(R), \forall R > 0 \} \quad (45)
\]

Then one has

**Property (H3)** For all $f \in W_{\chi}$ such that $\langle f \rangle = 0$
\[
\frac{1}{\sqrt{N}} \sum_{k=0}^{N} f \circ T^k
\]
is uniformly bounded (with respect to $N$) in $L^4(T^2)$.

**Proof.** The proof follows the same line as the one indicated by Ratner [Ra] when the mapping $T$ is replaced by any $K$ system. For the sake of being complete, we give a proof based only on property (H2) (which is more restrictive than assuming the K property).

First observe that
\[
\langle | \sum_{k=0}^{N} f \circ T^k |^4 \rangle = \sum_{0 \leq k, k_2, k_3, k_4 \leq N} \langle f \circ T^{k_1} f \circ T^{k_2} f \circ T^{k_3} f \circ T^{k_4} \rangle
\]
\[
= 4! \sum_{0 \leq k_1 \leq k_2 \leq k_3 \leq k_4 \leq N} \langle f \circ T^{k_1} f \circ T^{k_2} f \circ T^{k_3} f \circ T^{k_4} \rangle.
\]
(46)
and introduce the following sets of indices:

\[ A = \{(k_1, k_2, k_3, k_4) \text{ s.t. } 0 \leq k_1 \leq k_2 \leq k_3 \leq k_4 \leq N, \; \sup_{2 \leq i \leq 4} |k_i - k_{i-1}| \leq N^{1/3}\}, \]

\[ B = \{(k_1, k_2, k_3, k_4) \text{ s.t. } 0 \leq k_1 \leq k_2 \leq k_3 \leq k_4 \leq N, \; k_2 - k_1 > N^{1/3}\}, \]

\[ C = \{(k_1, k_2, k_3, k_4) \text{ s.t. } 0 \leq k_1 \leq k_2 \leq k_3 \leq k_4 \leq N, \; k_3 - k_2 > N^{1/3}\}, \]

\[ D = \{(k_1, k_2, k_3, k_4) \text{ s.t. } 0 \leq k_1 \leq k_2 \leq k_3 \leq k_4 \leq N, \; k_4 - k_3 > N^{1/3}\}. \]

One has

\[ A \cap B = A \cap C = A \cap D = \emptyset \quad \text{and} \quad 0, 1, \ldots, N^d = A \cup B \cup C \cup D \quad (47) \]

with the subsets \( B, C, D \) having non empty intersection. From the relation (47) one deduce the estimate:

\[
\begin{aligned}
&\left\| \sum_{k=0}^{N} f \circ T^k \right\|_4 \leq 4! \sum_{(k_1, k_2, k_3, k_4) \in A} \left| (f \circ T^{k_1} f \circ T^{k_2} f \circ T^{k_3} f \circ T^{k_4}) \right| \\
&\quad + 4! \sum_{(k_1, k_2, k_3, k_4) \in B} \left| (f \circ T^{k_1} f \circ T^{k_2} f \circ T^{k_3} f \circ T^{k_4}) \right| \\
&\quad + 4! \sum_{(k_1, k_2, k_3, k_4) \in C} \left| (f \circ T^{k_1} f \circ T^{k_2} f \circ T^{k_3} f \circ T^{k_4}) \right| \\
&\quad + 4! \sum_{(k_1, k_2, k_3, k_4) \in D} \left| (f \circ T^{k_1} f \circ T^{k_2} f \circ T^{k_3} f \circ T^{k_4}) \right|.
\end{aligned}
\]  

(48)

To prove the corollary we show that all the terms which appears in (48) are uniformly bounded with respect to \( N^2 \). This will be done in three steps: step 1: terms of (48) with support in \( A \); step 2: terms with support in \( B \) or \( D \) (the proof being similar in these two cases); and step 3: terms with support in \( C \). For steps 2 and 3 the Fourier expansion of \( f \) truncated at degree \( R \) will be used; it is denoted by \( P_R(f) \) and since \( f \in W_\chi \) one has:

\[ |f(\omega) - P_R(f)(\omega)| \leq \chi(R)||f||_\infty. \]  

(50)
Step 1. Summation with support in $A$.

Observe that $\|A\| \leq 3!N(N^{1/3} + 1)^3$; this implies the estimate:

$$\sum_{(k_1, k_2, k_3, k_4) \in A} |(f \circ T^{k_1} f \circ T^{k_2} f \circ T^{k_3} f \circ T^{k_4})| \leq 3!N(N^{1/3} + 1)^3 \|f\|_\infty^3.$$  (51)

Step 2. Summation with support in $B$ and $D$.

As already noticed, the proof is similar for these two terms and therefore only the sum with support in $B$ is considered.

Since $f$ belongs to $W_\chi$, estimate (50) shows that

$$\sum_{(k_1, k_2, k_3, k_4) \in B} |(f \circ T^{k_1} f \circ T^{k_2} f \circ T^{k_3} f \circ T^{k_4})|$$

$$\leq \sum_{(k_1, k_2, k_3, k_4) \in B} |(P_R(f) \circ T^{k_1} P_R(f) \circ T^{k_2} P_R(f) \circ T^{k_3} P_R(f) \circ T^{k_4})|$$

$$+ 4N^3\|f\|_\infty^3 \chi(R)(\sup(\chi(R), 1))^3.$$  (52)

To use property (II2), write:

$$\langle P_R(f) \circ T^{k_1} P_R(f) \circ T^{k_2} P_R(f) \circ T^{k_3} P_R(f) \circ T^{k_4} \rangle =$$

$$\langle P_R(f) \circ T^{k_1 - \kappa} P_R(f) \circ T^{k_2 - \kappa} P_R(f) \circ T^{k_3 - \kappa} P_R(f) \circ T^{k_4 - \kappa} \rangle$$  (53)

In (53) choose $\kappa = \left\lfloor \frac{k_1 + k_4}{2} \right\rfloor$, notice that $P_R(f)$ is of mean value zero (because $f$ is of mean value zero) and apply proposition 6 to the sets

$$U = \{\kappa - k_1\}, \quad V = \{k_2 - \kappa, k_3 - \kappa, k_4 - \kappa\}$$

It shows that, for

$$R = \exp\left(\frac{N^{1/3} - 2\beta_0}{2\beta_0}\right)$$

and for any $(k_1, k_2, k_3, k_4) \in B$, one has

$$\langle P_R(f) \circ T^{k_1} P_R(f) \circ T^{k_2} P_R(f) \circ T^{k_3} P_R(f) \circ T^{k_4} \rangle = 0.$$  (54)
Therefore with \( \chi(R) = O((1/\log R)^\delta) \) for \( R \to +\infty \), the estimate

\[
\sum_{(k_1, k_2, k_3, k_4) \in B} |(f \circ T^{k_1} f \circ T^{k_2} f \circ T^{k_3} f \circ T^{k_4})| \leq 4N^4\|f\|_\infty^4 \chi\left(\exp\left(\frac{N^{1/3} - 2\beta_1}{2\beta_0}\right)\right) = O(N^3)
\]

is deduced from (52).

**Step 3. Summation with support in \( C \).** As in step 2 write:

\[
\sum_{(k_1, k_2, k_3, k_4) \in C} |(f \circ T^{k_1} f \circ T^{k_2} f \circ T^{k_3} f \circ T^{k_4})| \leq \sum_{(k_1, k_2, k_3, k_4) \in C} \|(P_{R}(f) \circ T^{k_1} P_{R}(f) \circ T^{k_2} P_{R}(f) \circ T^{k_3} P_{R}(f) \circ T^{k_4})| \leq 4N^4\|f\|_\infty^4 \chi(R)(\sup(\chi(R), 1))^3.
\]

Then for any \((k_1, k_2, k_3, k_4) \in C\), apply proposition 6 to the sets

\[
U = \{\kappa - k_1, \kappa - k_2\}, \quad \text{and} \quad V = \{k_3 - \kappa, k_4 - \kappa\}
\]

with \( \kappa = \left[\frac{k_1 + k_2}{2}\right] \). It shows that, for

\[
R = \exp\left(\frac{N^{1/3} - 2\beta_1}{2\beta_0}\right)
\]

one has:

\[
(P_{R}(f) \circ T^{k_1} P_{R}(f) \circ T^{k_2} P_{R}(f) \circ T^{k_3} P_{R}(f) \circ T^{k_4}) = (P_{R}(f) \circ T^{k_1} P_{R}(f) \circ T^{k_2} P_{R}(f) \circ T^{k_3} P_{R}(f) \circ T^{k_4}).
\]

Therefore

\[
\sum_{(k_1, k_2, k_3, k_4) \in C} |(f \circ T^{k_1} f \circ T^{k_2} f \circ T^{k_3} f \circ T^{k_4})| \leq \sum_{(k_1, k_2, k_3, k_4) \in C} \|(P_{R}(f) \circ T^{k_1} P_{R}(f) \circ T^{k_2} P_{R}(f) \circ T^{k_3} P_{R}(f) \circ T^{k_4})| + 4N^4\|f\|_\infty^4 \chi(R).
\]

\[\text{(58)}\]
On the other hand:

\[
\sum_{(k_1, k_2, k_3, k_4) \in \mathcal{C}} ||P_R(f) \circ T^{k_1} P_R(f) \circ T^{k_2} (P_R(f) \circ T^{k_3} P_R(f) \circ T^{k_4})||^2 \leq \sum_{0 \leq k_1, k_2, k_3, k_4 \leq N} ||P_R(f) \circ T^{k_1} P_R(f) \circ T^{k_2}|| \cdot ||P_R(f) \circ T^{k_3} P_R(f) \circ T^{k_4}||^2
\]

\[
= \left( \sum_{0 \leq k_1, k_2 \leq N} ||P_R(f) \circ T^{k_1} P_R(f) \circ T^{k_2}|| \right)^2.
\]  

(59)

Since \( f \) belongs to the class \( W_X \subset H_X \), \( P_R(f) \) also belongs to the class \( H_X \) and the estimate (22) of proposition 5 can be used to give

\[
||P_R(f) \circ T^{k_1} P_R(f) \circ T^{k_2}|| \leq \frac{1}{2\pi} ||f||_2^2 \chi(C_\lambda \lambda^{(k_2-k_1)/2})
\]  

(60)

The sum \( \sum_{n \geq 0} \chi(C_\lambda \lambda^{n/2}) \) converges (by assumption: see (44)) and therefore:

\[
\left( \sum_{0 \leq k_1, k_2 \leq N} ||P_R(f) \circ T^{k_1} P_R(f) \circ T^{k_2}|| \right)^2 \leq \left( \frac{1}{2\pi} ||f||_2^2 \right)^2 \sum_{0 \leq k_1, k_2 \leq N} \chi(C_\lambda \lambda^{(k_2-k_1)/2}) = O(N^2)
\]  

(61)

With (56), (57), (58) and (61) one obtains, for

\[
R = \exp\left( \frac{N^{1/3} - 2\beta_1}{2\beta_0} \right),
\]

the formula

\[
\sum_{(k_1, k_2, k_3, k_4) \in \mathcal{C}} ||f \circ T^{k_1} f \circ T^{k_2} f \circ T^{k_3} f \circ T^{k_4}||^2 \leq 4N^4 ||f||_2^4 \chi(\exp(\frac{N^{1/3} - 2\beta_1}{2\beta_0})) + O(N^2) \leq CN^2.
\]

which concludes the proof of step 3 and of corollary 7. //

To further extend the decorrelation properties, this section is concluded with proposition 8 involving functions \( f \in H^s(\mathbb{R}^d) \) and corollary 9 which allows to consider smooth functions with subquadratic growth at infinity.
**Proposition 8.** Assume that the vector field \(a\) is smooth enough (say \(a \in C^3(T^2, \mathbb{R}^2)\)) and satisfies \(< a > = 0\). Then for any pair of functions \(f\) and \(g \in H^s(\mathbb{R}^d)\) with \(d/2 < s\) one has:

\[
\left| f \left( \sum_{k=0}^{[\epsilon t/\epsilon^2]} a \circ T^k \right) g \left( \sum_{l=[(t+\tau)/\epsilon^2]}^{[\epsilon t/\epsilon^2]} a \circ T^l \right) - (f \left( \sum_{k=0}^{[\epsilon t/\epsilon^2]} a \circ T^k \right) g \left( \sum_{l=[(t+\tau)/\epsilon^2]}^{[\epsilon t/\epsilon^2]} a \circ T^l \right) \right|
\]

\[
\leq \|f\|_s \|g\|_s \left( C \frac{\epsilon^{-d/2}}{s-d/2} + \epsilon^{-d} H(K, \delta, \epsilon) \right) \tag{62}
\]

with \(H(K, \delta, \epsilon)\) given by:

\[
H(K, \delta, \epsilon) = \frac{C''}{\epsilon^2} \exp \left( -\frac{C' \delta}{\epsilon^2} \right) \exp \left( \frac{C''}{\epsilon^2} \exp \left( -\frac{C' \delta}{\epsilon^2} \right) \right) \tag{63}
\]

In (62) and (63) \(C, C'\) and \(C''\) denote some constants independent of \(\epsilon, \delta\) and \(t, t+\tau\) in the bounded interval \([0, K]\).

**Proof.** First represent \(f\) and \(g\) in terms of their Fourier transforms:

\[
f(x) = \int_{\mathbb{R}^d} e^{i\xi \cdot x} \hat{f}(\xi) \frac{d\xi}{(2\pi)^d}, \quad g(x) = \int_{\mathbb{R}^d} e^{i\eta \cdot x} \hat{g}(\eta) \frac{d\eta}{(2\pi)^d} \tag{64}
\]

and observe that since \(f\) and \(g\) belong to \(H^s(\mathbb{R}^d)\) the above integrals can, in the sequel, be replaced by

\[
f_{\epsilon}(x) = \int_{\mathbb{R}^d \cap \{ |\xi| \leq 1 \}} e^{i\epsilon \xi \cdot x} \hat{f}(\xi) \frac{d\xi}{2\pi}, \quad g_{\epsilon}(x) = \int_{\mathbb{R}^d \cap \{ |\xi| \leq 1 \}} e^{i\epsilon \eta \cdot x} \hat{g}(\eta) \frac{d\eta}{2\pi}. \tag{65}
\]

This truncation introduces in the proof an error of the order of \(\|f\|_s \|g\| \frac{\epsilon^{-d/2}}{s-d/2} \) which correspond to the first term in the right hand side of (62) and reduces the proof of the proposition 8 to estimating the expression:

\[
\epsilon^{-d} \sup_{|\xi| \leq 1, |\eta| \leq 1} \left| \prod_{k=0}^{[\epsilon t/\epsilon^2]} \exp(i\xi \cdot \epsilon ha \circ T^k) \prod_{l=[(t+\tau)/\epsilon^2]}^{[\epsilon t/\epsilon^2]} \exp(i\eta \cdot \epsilon ha \circ T^l) \right|
\]

\[
- \left( \prod_{k=0}^{[\epsilon t/\epsilon^2]} \exp(i\xi \cdot \epsilon ha \circ T^k) \right) \left( \prod_{l=[(t+\tau)/\epsilon^2]}^{[\epsilon t/\epsilon^2]} \exp(i\eta \cdot \epsilon ha \circ T^l) \right). \tag{66}
\]
However proposition 6 (property (H2) or relation (37) ) cannot be directly applied to the above formula because the functions \( \exp(i \xi \cdot \varepsilon a) \) and \( \exp(i \eta \cdot \varepsilon a) \) are not trigonometric polynomials in the variable \( \omega \). Therefore these functions have to be approximated by their truncated Fourier series. In order to do so it is convenient to introduce the following notations:

\[
\theta_{\xi}(\omega) = \exp(i \xi \cdot \varepsilon a(\omega)), \quad P_R(\theta_{\xi}) = \sum_{|k| \leq R} \hat{\theta}_{\xi}(k) \exp(ik \cdot \omega)
\]

and \( A_{\xi,R} = \sum_{|k| \geq R} |\hat{\theta}_{\xi}(k)| \).

Assuming that \( a \in C^3(T^2, \mathbb{R}^2) \), the quantities \( A_{\xi,R} \) are finite and one has:

\[
\| P_R(\theta_{\xi}) - \theta_{\xi} \|_{\infty} \leq \sum_{|k| \geq R} |\hat{\theta}_{\xi}(k)| = A_{\xi,R},
\]

\[
\| P_R(\theta_{\xi}) - 1 \|_{\infty} \leq \| P_R(\theta_{\xi}) - \theta_{\xi} \|_{\infty} \leq A_{\xi,R}, \quad \| P_R(\theta_{\xi}) \|_{\infty} \leq (1 + A_{\xi,R}).
\]

and

\[
\forall \sigma, \quad 0 < \sigma < 1 \quad A_R = \sup_{|k| \leq 1} A_{\xi,R} \leq \frac{C}{(1 - \sigma)^{R^1 - \sigma}}.
\]

Going back to (66), one has the inequality:

\[
\| \prod_{k=0}^{[\varepsilon t^2 \hbar]} \theta_{\xi} \circ T^k \prod_{i=[t+\varepsilon t^2 \hbar]}^{[\varepsilon t^2 \hbar]} \theta_{\eta} \circ T^k \rangle - \langle \prod_{k=0}^{[\varepsilon t^2 \hbar]} \theta_{\xi} \circ T^k \prod_{i=[t+\varepsilon t^2 \hbar]}^{[\varepsilon t^2 \hbar]} \theta_{\eta} \circ T^k \rangle \|
\]

\[
\leq \| \prod_{k=0}^{[\varepsilon t^2 \hbar]} P_R(\theta_{\xi}) \circ T^k \prod_{i=[t+\varepsilon t^2 \hbar]}^{[\varepsilon t^2 \hbar]} P_R(\theta_{\eta}) \circ T^k \rangle - \langle \prod_{k=0}^{[\varepsilon t^2 \hbar]} P_R(\theta_{\xi}) \circ T^k \prod_{i=[t+\varepsilon t^2 \hbar]}^{[\varepsilon t^2 \hbar]} P_R(\theta_{\eta}) \circ T^k \rangle \|
\]

\[
- \| \prod_{k=0}^{[\varepsilon t^2 \hbar]} P_R(\theta_{\xi}) \circ T^k \langle \prod_{i=[t+\varepsilon t^2 \hbar]}^{[\varepsilon t^2 \hbar]} P_R(\theta_{\eta}) \circ T^k \rangle \|
\]

\[
+ \| \prod_{k=0}^{[\varepsilon t^2 \hbar]} \theta_{\xi} \circ T^k \prod_{i=[t+\varepsilon t^2 \hbar]}^{[\varepsilon t^2 \hbar]} \theta_{\eta} \circ T^k \rangle - \langle \prod_{k=0}^{[\varepsilon t^2 \hbar]} \theta_{\xi} \circ T^k \prod_{i=[t+\varepsilon t^2 \hbar]}^{[\varepsilon t^2 \hbar]} \theta_{\eta} \circ T^k \rangle \|
\]

\[
- \| \prod_{k=0}^{[\varepsilon t^2 \hbar]} P_R(\theta_{\xi}) \circ T^k \prod_{i=[t+\varepsilon t^2 \hbar]}^{[\varepsilon t^2 \hbar]} P_R(\theta_{\eta}) \circ T^k \rangle - \langle \prod_{k=0}^{[\varepsilon t^2 \hbar]} P_R(\theta_{\xi}) \circ T^k \prod_{i=[t+\varepsilon t^2 \hbar]}^{[\varepsilon t^2 \hbar]} P_R(\theta_{\eta}) \circ T^k \rangle \|
\]
which leads to the function $H$.

Then for any positive constant $M$,

$$
\left\langle \prod_{k=0}^{[c/\epsilon^2 h]} \theta_{\epsilon^2 h} \circ T^k \right\rangle \prod_{l=[c(\epsilon+\delta)/\epsilon^2 h]}^{[c(\epsilon+\tau)/\epsilon^2 h]} \theta_{\epsilon\eta} \circ T^k

- \left\langle \prod_{k=0}^{[c/\epsilon^2 h]} P_R(\theta_{\epsilon^2 h} \circ T^k) \right\rangle \prod_{l=[c(\epsilon+\delta)/\epsilon^2 h]}^{[c(\epsilon+\tau)/\epsilon^2 h]} P_R(\theta_{\epsilon\eta} \circ T^k) \right].

\right) (70)

The property (H3) can be used for the first term of the right hand side of (70) which turns out to be zero for

$$
R = \exp \left( \frac{1}{\beta_0} (|c(t + \delta)/\epsilon^2 h| - [c/\epsilon^2 h] - \beta_1) \right).

\right) (71)

The estimates (68) and (69) can be used to control the second and third term of (70), with $|\epsilon^2 h| \leq 1$ and $|\epsilon\eta| \leq 1$ their sum is bounded by:

$$
2A_R \left[ \frac{c(t + \tau)}{\epsilon^2 h} \right] (1 + A_R) \left[ \frac{c(t + \tau)}{\epsilon^2 h} \right] \leq 2A_R \left[ \frac{c(t + \tau)}{\epsilon^2 h} \right] \exp \left( A_R \left[ \frac{c(t + \tau)}{\epsilon^2 h} \right] \right).

\right) (72)

With (69) and the choice of $R$ given by (71), the right hand side of (72) satisfies the following estimate:

$$
A_R \left[ \frac{c(t + \tau)}{\epsilon^2 h} \right] \exp \left( A_R \left[ \frac{c(t + \tau)}{\epsilon^2 h} \right] \right) \leq C^\prime \exp \left( -\frac{C^\prime \delta}{\epsilon^2} \right) \exp \left( C^\prime \exp \left( -\frac{C^\prime \delta}{\epsilon^2} \right) \right) \right) (73)

which leads to the function $H(K, \delta, \epsilon)$ and completes the proof. //

**Corollary 9.** Assume that the vector field $a$ satisfies the assumptions of proposition 8.

Let $f \in C^\infty_0(\mathbb{R}^d)$ and $g \in C^\infty(\mathbb{R}^d)$ with subquadratic growth at infinity as follows

$$
|g(x)| + \sum_{1 \leq [d/1]} |\nabla_x g(x)| \leq C_g(1 + |x|^2)

\right) (74)

Then for any positive constant $M$,

$$
|\left\langle \prod_{k=0}^{[c/\epsilon^2 h]} a \circ T^k \right\rangle g(\left\langle \prod_{l=[c(\epsilon+\tau)/\epsilon^2 h]}^{[c(\epsilon+\tau)/\epsilon^2 h]} a \circ T^l \right\rangle) - \left\langle \prod_{k=0}^{[c/\epsilon^2 h]} a \circ T^k \right\rangle g(\left\langle \prod_{l=[c(\epsilon+\tau)/\epsilon^2 h]}^{[c(\epsilon+\tau)/\epsilon^2 h]} a \circ T^l \right\rangle) |\right.

\leq 2\|f\|_\infty C_g \sup_{N \geq 0} \|\left( \frac{1}{\sqrt{N}} \sum_{k=0}^{N} a \circ T^k \right) \|^2 \left( \frac{\sqrt{\tau - \delta} + 2\epsilon^2}{M} \right)^2

\right) (28)
\[ C_d \|f\|_s M^{\frac{s-d}{d}} \left( C_1 \frac{e^{-d/2}}{s-d/2} + e^{-d} H(K, \delta, \epsilon) \right) \]  

(75)

where \( s \) is chosen equal to \([d/2+1]\), \( \|f\|_s \) represent the \( H^s \) Sobolev norm of \( f \) and \( H(K, \delta, \epsilon) \) is the function defined by (63). As before, the constants are independent of \( t, \tau, \epsilon \) and \( \delta \).

**Proof.** Introduce a cutoff function \( \chi_M \in C^\infty(\mathbb{R}^2) \) with the following properties:

\[ \forall x, \ 0 \leq \chi_M \leq 1, \ \chi_M(x) = 1 \text{ if } |x| \leq M, \ \chi_M(x) = 0 \text{ for } |x| \geq 2M \]  

(76)

and denote by \( g_M \) the function \( g \chi_M \). From the formula \( |g - g_M| \leq |g|_{1 \leq M} \) one deduces the estimate:

\[ \left| \langle f(\epsilon h \sum_{k=0}^{[\epsilon t/\epsilon^2 h]} a \circ T^k) \rangle - \langle f(\epsilon h \sum_{l=\lfloor \epsilon (t+\tau) / \epsilon^2 h \rfloor}^{[\epsilon (t+\tau) / \epsilon^2 h]} a \circ T^l) \rangle \right| \]

\[ \leq \langle f(\epsilon h \sum_{k=0}^{[\epsilon t/\epsilon^2 h]} a \circ T^k) \rangle \langle \epsilon h \sum_{l=\lfloor \epsilon (t+\tau) / \epsilon^2 h \rfloor}^{[\epsilon (t+\tau) / \epsilon^2 h]} a \circ T^l) \rangle \]

\[ + \left( \int |g(\epsilon h \sum_{l=\lfloor \epsilon (t+\tau) / \epsilon^2 h \rfloor}^{[\epsilon (t+\tau) / \epsilon^2 h]} a \circ T^l(\omega))| d\omega \right) \]  

(77)

Corollary 7 shows that

\[ \text{mes} \{ \omega \text{ s.t. } |\epsilon h \sum_{l=\lfloor \epsilon (t+\tau) / \epsilon^2 h \rfloor}^{[\epsilon (t+\tau) / \epsilon^2 h]} a \circ T^l(\omega)| \geq M \} \]

\[ \leq \sup_{N \geq 0} \left\| \frac{1}{\sqrt{N}} \sum_{k=0}^{N} a \circ T^k \right\|_4^4 \left( \frac{\sqrt{\tau - \delta + 2k^2}}{M} \right)^4 \]  

(78)

Using the Cauchy-Schwarz inequality and the subquadratic growth of \( g \) (74), this implies

\[ \int \left| \epsilon h \sum_{l=\lfloor \epsilon (t+\tau) / \epsilon^2 h \rfloor}^{[\epsilon (t+\tau) / \epsilon^2 h]} a \circ T^l(\omega)| \geq M \right| \]  

(79)
$\leq \sup_{N>0} \left\| \left( \frac{1}{\sqrt{N}} \sum_{k=0}^{N} a \circ T^k \right) \right\|_3^2 \left( \frac{\sqrt{\tau - \delta + 2\epsilon^2}}{M} \right)^2. \tag{79}$

Therefore, with estimate (62), one concludes that

$$
\begin{align*}
&\|f(\epsilon h \sum_{k=0}^{[\epsilon t/\epsilon^2 h]} a \circ T^k) g(\epsilon h \sum_{t=[\epsilon t/\epsilon^2 h]}^{[\epsilon (t+\tau)/\epsilon^2 h]} a \circ T^t) - f(\epsilon h \sum_{k=0}^{[\epsilon t/\epsilon^2 h]} a \circ T^k) g(\epsilon h \sum_{t=[\epsilon t/\epsilon^2 h]}^{[\epsilon (t+\tau)/\epsilon^2 h]} a \circ T^t)\|
\leq 2\|f\|_{1,0} \sup_{N>0} \left\| \left( \frac{1}{\sqrt{N}} \sum_{k=0}^{N} a \circ T^k \right) \right\|_3^2 \left( \frac{\sqrt{\tau - \delta + 2\epsilon^2}}{M} \right)^2 \\
&\quad + \|f\| \|g_M\|_2 \left( C_\epsilon \frac{\epsilon - \sigma + \epsilon^2}{\epsilon - \sigma + \epsilon^2} + \epsilon^d H(K, \delta, \epsilon) \right) \tag{80}
\end{align*}
$$

Since (74) shows that $\|g_M\|_2$ is bounded by $C_\epsilon M^{\frac{d_1}{2} + 1}$, (80) completes the proof of corollary 9. //

4.— Proof of Theorem 2.

As it was said in the introduction the proof of theorem 2 is inspired by the proof the Ito formula for the Brownian motion. Therefore the starting point is the Taylor formula at order three for the increment.

$$
\begin{align*}
&\langle \psi_t(t + \tau, x, \cdot, \cdot) \rangle - \langle \psi_t(t, x, \cdot, \cdot) \rangle \\
&= \langle \nabla \phi(x - \epsilon h \sum_{k=0}^{[\epsilon t/\epsilon^2 h]} a(T^k \omega)) \rangle \\
&\quad + \frac{1}{2} \langle \nabla^2 \phi(x - \epsilon h \sum_{k=0}^{[\epsilon t/\epsilon^2 h]} a(T^k \omega)) : \left( \epsilon h \sum_{k=[\epsilon t/\epsilon^2 h]+1}^{[\epsilon (t+\tau)/\epsilon^2 h]} a(T^k \omega) \right) \rangle \tag{81}
\end{align*}
$$

The analysis of the limit for $\epsilon \to 0$ in the above expression will be done in 3 steps and $C$ will be used to denote any constant independant of $\epsilon$ and $\tau$. 30
Using corollary /7/, the following bound is deduced from /8/:

\[ \langle \langle ch \sum_{k=\lceil ct/\epsilon^2 h \rceil} a(T^k \omega) \rangle \rangle \leq \langle \langle ch \sum_{k=0} a(T^k \omega) \rangle \rangle + O((ch)^3) \]  

or, using Hölder’s inequality with \( N = \lceil c\tau/\epsilon^2 h \rceil \)

\[ \langle \langle ch \sum_{k=\lceil ct/\epsilon^2 h \rceil} a(T^k \omega) \rangle \rangle \leq \sqrt{2\pi}(ch\sqrt{N})^3 \left( \frac{1}{\sqrt{N}} \sum_{k=0}^N a(T^k \omega) \right)^4 + O((ch)^3) \]  

Using corollary 7, the following bound is deduced from (83)

\[ \limsup_{\epsilon \to 0} \left( \langle ch \sum_{k=\lceil ct/\epsilon^2 h \rceil} a(T^k \omega) \rangle \right)^3 \leq C\tau^{3/2}(\sqrt{ch})^{3/2} \]  

**Step 2. Decorrelation in (81).** Since the treatment of the linear term is simpler than the treatment of the quadratic term (but follows the same lines), only the latter, will be considered in detail. A “small” positive time \( \delta \) is introduced and one has:

\[ \langle \langle \nabla^2_\epsilon \phi(x - ch \sum_{k=0} a(T^k \omega)) : (ch \sum_{k=\lceil \epsilon t \rceil} a(T^k \omega)) \rangle \rangle \leq \langle \langle \nabla^2_\epsilon \phi(x - ch \sum_{k=0} a(T^k \omega)) : (ch \sum_{k=\lceil \epsilon t \rceil} a(T^k \omega)) \rangle \rangle \]

\[ \leq \langle \langle \nabla^2_\epsilon \phi(x - ch \sum_{k=0} a(T^k \omega)) : (ch \sum_{k=\lceil \epsilon t \rceil} a(T^k \omega)) \rangle \rangle \]

\[ + 2\langle \langle \nabla^2_\epsilon \phi(x - ch \sum_{k=0} a(T^k \omega)) : (ch \sum_{k=\lceil \epsilon t \rceil} a(T^k \omega)) \rangle \rangle + O((ch)^3) \]  

The first term of the right hand side of (85) is bounded by:

\[ \|\nabla^2_\epsilon \phi\|_\infty \|\langle ch \sum_{k=\lceil \epsilon t \rceil} a(T^k \omega) \rangle\|_2^2 \]  

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\[ \leq \| \nabla^2 \phi \|_\infty 2 \pi h \sup_{N>0} \left\| \left( \frac{1}{\sqrt{N}} \sum_{k=0}^{N} a \circ T^k \right) \right\|_{2}^2 (\delta + 2\epsilon^2). \] (86)

Similarly the second term is bounded by:
\[ \| \nabla^2 \phi \|_\infty 2 \pi h \sup_{N>0} \left\| \left( \frac{1}{\sqrt{N}} \sum_{k=0}^{N} a \circ T^k \right) \right\|_{2}^2 \sqrt{(\tau + 2\epsilon^2)(\delta + 2\epsilon^2)}. \] (87)

The only remaining term is
\[ \langle \nabla^2 \phi(x - \frac{\epsilon h}{c} \sum_{k=0}^{[c/\epsilon^2 h]} a(T^k \omega)) : \left( \frac{\epsilon h}{c} \sum_{k=[c/\epsilon^2 h]+1}^{[c/(1+t)/\epsilon^2 h]} a(T^k \omega) \right)^{\otimes 2} \rangle. \]

Here corollary 9 is used with \( g(x) = x^{\otimes 2} \) leading to the following estimate:
\[ \langle \nabla^2 \phi(x - \frac{\epsilon h}{c} \sum_{k=0}^{[c/\epsilon^2 h]} a(T^k \omega)) : \left( \frac{\epsilon h}{c} \sum_{k=[c/\epsilon^2 h]+1}^{[c/(1+t)/\epsilon^2 h]} a(T^k \omega) \right)^{\otimes 2} \rangle \]
\[ - \langle \nabla^2 \phi(x - \frac{\epsilon h}{c} \sum_{k=0}^{[c/\epsilon^2 h]} a(T^k \omega)) : \left( \frac{\epsilon h}{c} \sum_{k=[c/\epsilon^2 h]+1}^{[c/(1+t)/\epsilon^2 h]} a(T^k \omega) \right)^{\otimes 2} \rangle \]
\[ \leq \| \nabla^2 \phi \|_\infty 2 \pi h \sup_{N>0} \left\| \left( \frac{1}{\sqrt{N}} \sum_{k=0}^{N} a \circ T^k \right) \right\|_{2}^2 (\delta + 2\epsilon^2) \]
\[ + \| \nabla^2 \phi \|_\infty 2 \pi h \sup_{N>0} \left\| \left( \frac{1}{\sqrt{N}} \sum_{k=0}^{N} a \circ T^k \right) \right\|_{2}^2 \sqrt{(\tau + 2\epsilon^2)(\delta + 2\epsilon^2)} \]
\[ + \| \nabla^2 \phi \|_\infty \sup_{N>0} \left\| \left( \frac{1}{\sqrt{N}} \sum_{k=0}^{N} a \circ T^k \right) \right\|_{2}^2 \delta \]
\[ + 2 \| \nabla^2 \phi \|_\infty \sup_{N>0} \left\| \left( \frac{1}{\sqrt{N}} \sum_{k=0}^{N} a \circ T^k \right) \right\|_{2}^2 \left( \frac{\sqrt{\tau - \delta + 2\epsilon^2}}{M} \right)^2 \]
\[ + C \| \nabla^2 \phi \|_s M^{\frac{d+2}{s}} (C_1 \frac{e^{s-d/2}}{2} + e^{-d} H(K, \delta, \epsilon)). \] (88)

For a given \( \tau \in (0, 1) \), and \( 0 < \epsilon < \tau \) one uses the special form of the function \( H(K, \delta, \epsilon) \);

it shows that, by choosing
\[ \delta = \epsilon \text{ and } M = \epsilon^{-\frac{2(2-d)}{d(2+d)}}, \]

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one has
\[
\limsup_{t \to 0} \left\{ (\nabla^2 \phi(x) - \frac{\epsilon h}{c^2} \sum_{k=0}^{[ct/c^2 h]} a(T^k \omega)) : (\frac{\epsilon h}{c^2} \sum_{k=[ct/c^2 h]+1}^{[ct/c^2 h+1]} a(T^k \omega)) \right\} = 0.
\]

**Step 3. Weak and strong limits.** Denote by \( u_\epsilon(t, x) \) the family of functions \( (\psi_\epsilon(t, x, \cdot)) \).
Starting with (81), using the estimate for the remainder (84), the \( L^\infty \) bounds on \( \nabla \phi \) and \( \nabla^2 \phi \), the inequality
\[
\# \{ k \text{ s.t. } [ct/c^2 h] + 1 \leq k \leq [c(t + \tau)/c^2 h] \leq [c\tau/c^3 h], \quad (90)
\]
and property (H3) from corollary 7, we arrive at the (uniform in \( \tau \in [0, 1] \) and \( \epsilon \in [0, 1] \)) bound:
\[
[u_t(t + \tau, x) - u_t(t, x, \cdot)] \leq C\sqrt{\tau}. \quad (91)
\]

It follows from Ascoli’s theorem that the family \( u_\epsilon \) is relatively compact in \( C^0([0, \tau]; w^* - L^\infty(\mathbb{R}^d)) \). Let \( u \) be a limit point of this family and rename as usual, \( f^\pm, \psi_\epsilon \) and \( u_\epsilon \) the corresponding subfamilies, with \( u_\epsilon \) converging to \( u \). Starting from the “Ito” formula (82), using (85) and (89) one obtains:
\[
[u_t(t + \tau, x) - u_t(t, x)] = \lim_{\epsilon \to 0} \{ u_\epsilon(t + \tau, x) - u_\epsilon(t, x) \}
\]
\[
= \lim_{\epsilon \to 0} \{ \frac{1}{2} \nabla^2 u_\epsilon(t, x) : \lim_{\epsilon \to 0} (\frac{\epsilon h}{c^2} \sum_{k=[ct/c^2 h]+1}^{[ct/c^2 h+1]} a(T^k \omega)) \} + O(\tau^{4/3}) \quad (92)
\]
or with (12) (cf. point 2 of proposition 2.)
\[
u(t + \tau, x) - u(t, x) = \frac{1}{2} h \tau D(a) : \nabla^2 u(t, x) + O(\tau^{4/3}). \quad (93)
\]
Dividing (93) by \( \tau \) and letting \( \tau \) go to zero, one sees that \( u \) solves the initial value problem
for the diffusion equation:
\[
\frac{\partial u}{\partial t} - \frac{1}{2} D(a) : \nabla^2 u, \quad u(x, 0) = \phi(x) \quad (94)
\]
The solution of (94) being unique, it follows that the whole families
\[
\langle f_{\epsilon}^+(t, x, \omega) \rangle \text{ or } u_{\epsilon}(x, t) = \langle \psi_{\epsilon}(t, x, \omega) \rangle = \langle \phi(x - \epsilon h \sum_{k=0}^{[\epsilon T / \epsilon h]} a(T^k \omega)) \rangle
\]
converge in \(C^0([0, \tau]; w^* - L^\infty(\mathbb{R}^d))\) to \(u(t, x)\).

Observing that the problem (7)-(10) is translation invariant in the variable \(x \in \mathbb{R}^d\)
and using the regularity of \(\phi \in C^2(\mathbb{R}^d)\), one can see that the family of functions \(u_{\epsilon}(t, x)\)
satisfies the (uniform in \(\epsilon\)) estimate:
\[
\|u_{\epsilon}(t, x)\|_{L^\infty(\mathbb{R}^d)} \leq C \quad (95)
\]
Acoli’s theorem, (91) and (95) show that for any \(\tau > 0\) and any compact \(K \subset \mathbb{R}^d\), the
sequence \(u_{\epsilon}(t, x)\) converges to \(u(t, x)\) in \(C([0, \tau] \times K)\). This argument completes the proof
of the strong convergence of the averages (14a).

Finally let \(f\) be in the \(w^* \) closure in \(L^\infty(\mathbb{R}_t^+ \times \mathbb{R}_x^d \times \mathbb{T}_2)\) of the family \(\psi_{\epsilon}(t, x, \omega)\). One
deduces from (11) that \(f\) is invariant under the action of \(T\)
\[
f(t, x, \omega) = f(t, x, T \omega) \quad (96)
\]
The ergodicity property (proposition 2 point 1)) implies that \(f\) is independent of \(\omega\) and
therefore coincide with the function \(u(t, x)\): this demonstrates the convergence (14b) and
concludes the proof of theorem 3. //

5. — Final Remarks and Numerical Experiments..

The model (3), (4a), (4b) and (5) defines a global broken hamiltonian flow which
induces an isometry on \(L^p(\mathbb{R}^d \times \mathbb{T}^2)\) for all \(1 \leq p \leq \infty\). In particular the quantity
\[
\|f_{\epsilon}^+(t, \cdot, \cdot)\|_2 + \|f_{\epsilon}^-(t, \cdot, \cdot)\|_2
\]
is conserved for any \(\epsilon > 0\) at variance with the quantity
\[
\|u(t, \cdot)\|_2.
\]
Therefore the type of convergence which is given in the theorem 3 (strong
for the average and weak for the solution itself) is optimal with the exception of the trivial case where \( D(a) = 0 \) which correspond to no diffusion. In this situation the solution exhibits an initial layer near \( t = 0 \) which is taken care of by the time scaling and the solution \( f^x_t(x, t, z\omega) \) converges strongly to its initial value \( \phi(x) \).

The paradox of deriving a well posed irreversible problem for \( t > 0 \) from a reversible problem can be explained from the following facts.

1) The scaling has been done with the a priori choice of considering the solution for large positive times;

2) It gives the correct approximation at the order \( \epsilon \) of local averages of the solution in term of the local averages of its initial data; the dependence on \( \omega \) is lost in the approximation. In some sense, this decay of information can be estimated by the decay of the \( L^2 \) norm of \( u \) which, for the diffusion equation, is the linearized version of the classical entropy.

The relevance of the above remarks depends on the analysis of the strict positivity of the diffusion matrix \( D(a) \). In fact it results from proposition 2 that \( D(a) = 0 \) for \( a \) varying in a dense subset of \( L^2(T^2)/\mathbb{R} \) (isomorphic to the space of mean zero functions in \( L^2(T^2) \)). Furthermore the diffusion is degenerate in the directions \( \xi \) for which the “excursion length”:

\[
    f_N \cdot \xi = \sum_{k=1}^{N} (a \circ T^k) \cdot \xi
\]

is (uniformly with respect to \( N \)) bounded in \( L^2(T^2) \), an observation which turns out to be in agreement with the intuition.

Although the space of coboundaries is dense in the subspace of \( L^2(T^2) \) consisting of mean zero functions, it is not \( L^2 \) closed. In other words, it is possible to find smooth functions of \( L^2(T^2) \) not being coboundaries. For example, observe that for \( d = 1 \) and \( a(\omega_1, \omega_2) = \cos \omega_1 \) one has \( D(a) = \frac{1}{4} \).

More generally it would be extremely useful to have an explicit expression of \( D(a) \).
For $d = 1$ it can be easily obtained in term of the Fibonacci sequence $F_n$

\[ F_0 = 0, \ F_1 = 1, \ F_n = F_{n-1} + F_{n-2} \]  

(97)

Observe that the minimal polynomial of the matrix $M$ is $X^2 - 3X + 1$ and introduce the matrix $P - I$ which has for minimal polynomial $X^2 - X - 1$, then on one hand one has

\[ M = P + I = P^2 \]  

(98)

and on the other hand

\[ P^n = P^{n-1} + P^{n-2} \]  

(99)

and this implies the formula:

\[ M^n = \begin{pmatrix} F_{2n+1} & F_{2n} \\ F_{2n} & F_{2n-1} \end{pmatrix} \]

Therefore, with the diffusion matrix given by (12) one has, for any function $f : \mathbb{T}^2 \to \mathbb{R}$ in $H^s$ with $s > 0$:

\[ D(f) - \frac{1}{2}(f^2) = \sum_{n \geq 1} \langle f \circ T^n \cdot f \rangle = \frac{1}{4\pi^2} \sum_{n \geq 1} \sum_{k \in \mathbb{Z}^2 - \{0\}} \hat{f}(M^n k) \hat{f}(-k) \]

\[ = \frac{1}{4\pi^2} \sum_{k \in \mathbb{Z}^2 - \{0\}} \sum_{n \geq 1} \hat{f}(F_{2n+2}k_1 + F_{2n+1}k_2, F_{2n+1}k_1 + F_{2n-1}k_2) \hat{f}(-k_1, -k_2) \]  

(100)

In particular, since the Fibonacci sequence is rapidly increasing this will provide and exact formula (involving a small number of non zero terms ) for $D(f)$ whenever $f$ is a trigonometric polynomial.

The numerical experiment were done by the third author. They intend to illustrate the difference between the diffusive and the non diffusive case. In two space variables the trajectories of 128 particles over 1000 interactions with the upper and lower boundary has been obtained. The diffusive case (see fig. 3) corresponds to horizontal velocity field:

\[ a(\omega) = (a_1(\omega), a_2(\omega)) = (\cos \omega_1, \cos \omega_2) \]  

(101)
As shown on fig. 4 the trajectory of a single particle is in general ergodic, however notice that even in this case some exceptional trajectories are not ergodic. This will be the case for any particle starting with a velocity $a(\omega_0)$ with $\omega_0$ any periodic point for the mapping $T$. For instance on fig. 4 is plotted the path of a single particle driven by the flow given by (101) with initial velocity:

$$a(\omega_0), \quad \omega_0 = (0, \pi/2)$$

Observe the relation $\omega_n = T^3 \omega_0$ which correspond to the behavior of the particle.

Fig. 5 is devoted to the simulation (128 particles and 1000 collisions) of the non diffusive case with a vector field given by the formula:

$$a'(\omega) = a(T\omega) - a(T)$$

First there is an initial layer then the process stabilizes to a stationary state.

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