On the stability of boundary layers of incompressible Euler equations

Emmanuel GRENIER

LMENS - 98 - 10
On the stability of boundary layers of incompressible Euler equations

Emmanuel GRENIER*

LMENS - 98 - 10

March 1998

Laboratoire de Mathématiques de l’Ecole Normale Supérieure
45 rue d’Ulm 75230 PARIS Cedex 05
Tel : (33)(1) 44 32 30 00
Adresse électronique : grenier@dmi.ens.fr

* Laboratoire d’Analyse Numérique, CNRS URA 189 Paris VI
Place Jussieu 75005 Paris
On the stability of boundary layers
of incompressible Euler equations

E. Grenier

Abstract
In this paper we investigate the stability and instability of boundary layers of incompressible Euler equations.
Key words: fluid mechanics, Euler equations, stability, boundary layers
AMS Classification: 76E05, 76E10, 76E15.

1 Introduction
In this paper we study the stability of boundary layer type solutions for the 2D incompressible Euler equations
\[ \partial_t u + (u \nabla) u + \nabla p = 0, \tag{1} \]
\[ \nabla \cdot u = 0. \tag{2} \]
There are two main motivations for this study. First the Prandtl boundary layers which appear in the inviscid limit of the Navier Stokes equations have a size $\sqrt{\nu}$ where $\nu$ is the viscosity. Recent works [14], [15], [6] have shown that when the size of the layer is of order of the viscosity the layer is completely dominated by viscous effects: if the layer is small enough it is stable, else it can be unstable (and is unstable indeed in some cases [14]). The size of Prandtl layer being much larger that $\nu$, the viscosity is no longer able to stabilize by itself the layer and we have to find a stabilizing effect elsewhere, namely in the corresponding inviscid equations, in the Euler equations. Therefore we have first to investigate the stability of boundary layer type solutions for the Euler equations, which is moreover a classical approach in Fluid Mechanics, see [7], [20], [22]. The idea is the following: we physically know that the viscosity is not sufficient to control Prandtl layer and that an eventual stabilization mechanism has to be found in Euler equations. Hence we first ignore the viscosity and study the stability of...
inviscid boundary layers, before trying to add viscous effects (which is a real
difficulty: as noticed by Rayleigh, viscosity can have a destabilizing effect).
The methods of this paper will be applied to Prandtl layer in [13] (with weaker
results). The second motivation is the study of the semigeostrophic asymptotic,
which arises in the study of frontogenesis in Meteorology, following Hoskins [16],
which is a limit more complicated than that studied here, but which contains it
as a particular case.

So let us turn to the stability of a solution

$$u^\eta = (u_1^\eta(t, x, y, \frac{y}{\eta}), u_2^\eta(t, x, y, \frac{y}{\eta}))$$

of $(1, 2)$ on $\mathbb{T}_x \times \mathbb{R}_y^+$. It will be clear in the proof that the geometry of the domain
is unimportant in the proof of the stability Theorems, which can be extended to
$\mathbb{R}_x \times \mathbb{R}_y^+$, to smooth exterior or interior domains. The dimension $2$ is however

The study of the stability of profiles of the form $(0, u_2(t, x, y))$ has a long
story and began with Rayleigh (1880) who proved that a necessary condition
for instability is that $u_2$ should have an inflection point. It was then sharpened
by Fjortoft, Tollmien, Lin (see the books of P.G. Drazin and W.H. Reid [7],
L. Landau and E. Lifchitz [18] and H. Schlichting [22]). Mathematically the
problem has been investigated by V.I. Arnold who gives sufficient conditions for
the nonlinear stability of stationary solutions of $(1, 2)$. Using the functionnal

$$\int |\nabla \phi|^2 + \frac{\nabla \psi}{\nabla \Delta \psi} |\Delta \phi|^2$$

where $\phi$ is the stream function of the perturbation and $\psi$ the stream function
of the stationary flow, he in particular proves that if this quadratic form is
positive definite, or if

$$\int |\nabla \phi|^2 + \max(\frac{\nabla \psi}{\nabla \Delta \psi}) |\Delta \phi|^2$$

is negative definite, then the flow is stable.

Notice however that this criterium does not apply in our case since $\nabla \psi/\nabla \Delta \psi$
is neither positive nor negative. Moreover we would like to handle nonstationary
solutions. The prize to pay is that we get stability over times of order $O(1)$
and not global stability.

The paper is divided in two parts. In the first one we prove stability results
using an energy method when basically there is no inflection point in the profile
(the conditions are a little more technical). Namely let us assume

$$H)$$

there exists $\phi_\eta(t, x, y) > 0$ and $C > 0, \alpha > 0$ such that

$$\phi_\eta = \exp(-2\alpha y)$$

for $y \geq 1$, (3)

$$\left| \frac{\partial \phi_\eta}{\phi_\eta} \right|_{L^\infty} + \left| \frac{u_1^\eta \partial_2 \phi_\eta}{\phi_\eta} \right|_{L^\infty} + \left| \frac{u_2^\eta \partial_y \phi_\eta}{\phi_\eta} \right|_{L^\infty} \leq C,$$ (4)
\[ |\partial_{xx}^2 u_2^\eta|_{L^\infty} + |\partial_{xy}^2 u_1^\eta|_{L^\infty} + |\partial_{yy}^2 u_2^\eta|_{L^\infty} \leq C \sqrt{\phi_\eta}, \quad (5) \]
\[ \left| \frac{\partial_{yy} u_1^\eta}{\phi_\eta} + 1 \right| \leq \frac{C}{\sqrt{\phi_\eta}} \quad (6) \]

and

\[ C_1 \exp(-2\alpha y) + C_1 \frac{\mu}{\eta^2} \exp(-y/\eta) \leq \phi_\eta \leq C_2 \exp(-2\alpha y) + C_2 \frac{\mu}{\eta^2} \exp(-y/\eta). \quad (7) \]

Most of these assumptions are natural if \( u^\eta \) has a boundary layer and use in particular the incompressibility condition. For instance, as the layer is in the \( y \) direction, we enforce the \( x \) and \( t \) derivatives to be bounded: \( |\partial_x u_1^\eta|_{L^\infty} \leq C \) and \( |\partial_\eta \phi_\eta| + |\partial_x \phi_\eta| \leq C|\phi_\eta| \) are natural. By incompressibility condition this leads to \( |\partial_y u_2^\eta|_{L^\infty} \leq C \) and \( |u_2^\eta| \leq C \eta \), hence, in the boundary layer, \( |u_2^\eta \partial_y \phi_\eta| \leq C \eta |\partial_y \phi_\eta| \leq C \phi_\eta \). The bound (4) is therefore natural and similarly for (5). The main assumption is in fact (6) which essentially says \( \phi_\eta = -\partial_{yy} u_1^\eta \) in the boundary layer. Assumption (7) could be replaced by other decreasing properties. Notice that under assumption (H), there is no inflection point in the boundary layer. (H) is however more strict since we enforce asymptotic behavior (7) and spatial regularity (which are classical requirements in boundary layer theory).

Let us first consider the linear equation

\[ \partial_t v + (u^\eta, \nabla) v + (v, \nabla) u^\eta + \nabla p = w, \quad (8) \]
\[ \nabla \cdot v = 0. \quad (9) \]

Let

\[ I(t) = \int \frac{\| \text{curl} \, v \|^2}{\phi_\eta} dxdy. \]

**Theorem 1.1 (linear stability under assumption (H)).**

Under assumption (H) there exists \( C |t| \geq 0 \) depending only on \( u^\eta \) such that

\[ \partial_t I(t) \leq C(t) I(t) + \int \frac{\| \text{curl} \, w \|^2}{\phi_\eta} \]

for every solution \( v \) of (8,9).

Let us turn to the nonlinear equations

\[ \partial_t v + (u^\eta, \nabla) v + (v, \nabla) u^\eta + (v, \nabla) v + \nabla p = w, \quad (10) \]
\[ \nabla \cdot v = 0. \quad (11) \]

Let us assume moreover (H') which consists of (12) and (13):

\[ |\partial_\alpha \partial_\beta u_2^\eta| \leq \frac{C}{\eta^3} \exp(-y/\eta) + C \exp(-\alpha y) \quad \text{for} \quad \alpha + \beta \leq \sigma, \quad (12) \]
\[ |\sqrt{\phi_0} \partial_x u_2^0|_{L^\infty} + \frac{\partial_y u_1^0}{\sqrt{\phi_0}}|_{L^\infty} \leq C. \]  

(13)

Let
\[ \|v\|^2 = \sum_{|\alpha|+|\beta| \leq s} \eta^{8s+8\beta} \int \frac{\partial_\alpha^* \partial_\beta^* \text{curl } v^2}{\phi_0^{1+\beta}}. \]

**Theorem 1.2** (nonlinear stability under \((H)\) and \((H')\)).

Let \(s \geq 0\). Let \(\sigma\) be large enough. Under assumption \((H)\) and \((H')\) there exists \(C(t) \geq 0\) depending only on \(u^0\) such that
\[ \partial_t \|v\|^2 \leq C(t) \|v\|^2 + \frac{C(t)}{\eta^{16s}} \|v\|^2 + \|v\|^2. \]

This estimate is not uniform in \(\eta\) but is sufficient to justify asymptotic expansions since the large factor \(\eta^{-16s}\) is compensated by the cube \(\|v\|^2\) (see for instance [14]).

As an application of Theorems 1.1 and 1.2 we will justify asymptotic expansions of the boundary layer. Namely

**Theorem 1.3** (asymptotic expansion under \((H)\))

Let \(N\) be an integer, large enough. Let \(u_0^0\) be a given sequence of functions of the form
\[ u_0^0(x, y) = \sum_{j=0}^N \eta^j u_{j, \text{int}}^0(x, y) + \sum_{j=0}^N \eta^j u_{j, b}^0(x, \frac{y}{\eta}), \]  

(14)

where \(u_{j, \text{int}}^0\) and \(u_{j, b}^0\) are in \(H^s\), for every \(s\), the functions \(u_{j, b}^0\) being rapidly decreasing in their second variable. Let us assume that \((H)\) and \((H')\) hold true at \(t = 0\). Then there exists \(T > 0\) independent on \(\eta\), a constant \(\mu\) independent on \(N\) and functions \(u_{j, \text{int}}(t, x, y)\) and \(u_{j, b}(t, x, y)\) in \(L^\infty([0, T], H^s)\) (for every \(s\)), \(u_{j, b}\) being moreover rapidly decreasing in its last variable, with
\[ u_{j, \text{int}}(0, x, y) = u_{j, \text{int}}^0, \quad \text{and} \quad u_{j, b}(0, x, y) = u_{j, b}^0, \]

such that the solution \(u^0\) of Euler equations with initial value \(u_0^0\) satisfies
\[ u^0(t, x, y) = \sum_{j=0}^N \eta^j u_{j, \text{int}}(t, x, y) + \sum_{j=0}^N \eta^j u_{j, b}(t, x, \frac{y}{\eta}) + \eta^{N-\mu} \mathcal{R}^0(t, x, y) \]  

(15)

on \([0, T]\), with
\[ \|\mathcal{R}^0\|_{L^\infty([0, T], H^s)} \leq C_s \eta^{-2s} \]  

(16)

for \(0 < \eta \leq 1\) and for every \(s\).

Therefore the boundary layer has a completely regular behavior if initially there is no inflection point in it (more precisely if \((H)\) holds true). This fact is physically well known. Similar theorems have been proved for the study of the inviscid limit of parabolic equations, in the noncharacteristic case in [14] and
in the totally characteristic case in [12]. Notice that here the boundary layer is stable and survives over times of order $O(1)$ whereas there is no dissipation mechanism and no viscosity. The layer is therefore purely inertial. Totally characteristic hyperbolic systems as noticed in the last section of [12] have a similar behavior: in this latest case boundary layers can survive because of a particular algebraic property of the coefficients of the system. The stabilization effect is here much more complex. We refer to the beautiful mechanism suggested by Lin, as described for instance in [11] for physical insights.

In the second part of the paper we prove an instability result for a particular profile having an inflection point. Namely we first recall

**Theorem 1.4 (Rayleigh, 1894, revisited).**
There exists a sequence of stationary solutions $u^0$ of (1.2) and a solution $v^0$ of (8.9) such that

$$\|v^0\|_{L^2} = C_1 \exp(C_2 t)$$

(17)

for some positive constants $C_1$ and $C_2$.

We then precise Theorem 1.4 in the following fully nonlinear instability result

**Theorem 1.5 (Nonlinear instability).**
For every $N$ and $s$ arbitrary large there exists two solutions $u^0$ and $v^0$ of (1.2), $u^0$ being moreover stationary and smooth and having a boundary layer type behavior

$$|\partial_x^\alpha \partial_y^\beta u^0| \leq \frac{C_{\alpha, \beta}}{\eta^s} \exp(-y/\eta)$$

for every $\alpha$ and $\beta \in \mathbb{N}$, such that

$$\|u^0(.,.) - v^0(.,.)\|_{H^s} \leq \eta^N$$

and

$$\|u^0(T_\eta, .) - v^0(T_\eta, .)\|_{L^\infty} \geq \sigma > 0$$

for some constant $\sigma$ independent on $\eta$, and for some sequence $T_\eta \to 0$ as $\eta \to 0$. As a consequence, Theorem 1.3 is false without assumption (H) at $t = 0$.

We also have

$$\|u^0(T_\eta) - v^0(T_\eta)\|_{L^2} \geq \sigma \eta^{1/2},$$

the $\eta^{1/2}$ factor coming from the small size of the boundary layer. The proof uses elementary tools and relies on upper bounds on the growth of solutions of inviscid Orr Sommerfeld equations, proved in section 7.1 by O.D.E. arguments.

The situation is therefore highly chaotic when there are inflection points in the boundary layer profile: the smallest error on the initial data (measured in Sobolev spaces) leads to order one errors on the solution, even in very short times, of order $o(1)$. We conjecture that the same profile $u^0$ would be stable over times of order $O(1)$ if we consider analytic perturbations instead of perturbations with Sobolev regularity, since it will be clear in the proof that the instability
comes from eigenmodes which are highly oscillatory in the $x$ direction, with period $\eta$, these eigenmodes being typically “killed” by any reasonable analytic type space (they are damped by a factor $\exp(-C/\eta)$). This remark is coherent and makes the link with the work [4] on the inviscid limit of Navier Stokes equations in an analytic framework.

Notice that the stability results are valid only in two space dimension, whereas Theorems 1.4 and 1.5 hold in fact in any space dimension.

After rescaling space and time by an $\eta$ factor as explained in section 6, Theorem 1.5 gives an example of stationary solution of 2D Euler equations, in $C^\infty(T \times \mathbb{R}_+)$, which is linearly and nonlinearly unstable. Unstable stationary solutions have been recently constructed in the periodic case by S. Friedlander, W. Strauss and M. Vishik in [9] in the spirit of [10] (however as stated they only proved instability in $H^s$ with $s > 2$, whereas here we go to $L^\infty$ a really physically relevant space).

Acknowledgments

The author is indebted to Y. Brenier [3], F. Merle for many interesting and helpful discussions on Prandtl type equations, in particular concerning section 5.1. The author would also like to thank W. E, O. Guès and J. Rauch for motivating discussions on Prandtl equations.

2 Preliminaries

2.1 Vorticity equation

Let $\zeta$ be the two dimensionnal vorticity. Nonlinear 2D Euler equations (1,2) are equivalent to

$$\frac{\partial}{\partial t} \zeta + (u, \nabla) \zeta = 0$$

with $\text{curl } u = \zeta$ and $\text{div } u = 0$. The linearized version of (18) is

$$\frac{\partial}{\partial t} \Theta + (u_l, \nabla) \Theta + (v, \nabla) \Theta_t = 0$$

where

$$\text{curl } v = \Theta, \quad \text{div } v = 0, \quad (20)$$

$$\text{curl } u_l = \Theta_l, \quad \text{div } u_l = 0, \quad (21)$$

which is equivalent to (8,9).

2.2 Green functions

Let us compute in this section the Green function $G_k(z_0, z)$ of

$$\frac{\partial^2}{\partial z^2} f - k^2 f = \delta_{z_0}$$

with boundary conditions

$$f = 0 \text{ for } z = 0 \text{ and } f \to 0 \text{ as } z \to +\infty. \quad (23)$$
We get
\[ G_k(z_0, z) = \frac{1}{2|k|} \exp(-|k|z) \left( \exp(|k|z) - \exp(-|k|z) \right) \quad \text{for } z \leq z_0 \]
and
\[ G_k(z_0, z) = \frac{1}{2|k|} \exp(-|k|z_0) \left( \exp(|k|z_0) - \exp(-|k|z_0) \right) \quad \text{for } z \geq z_0. \]

Notice that
\[ |G_k(z_0, z)| \leq \frac{1}{2|k|} \]
for every \( z_0 \) and \( z \), that \( G_k \) is symmetric
\[ G_k(z, z') = G_k(z', z) \quad (24) \]
for every \( z \) and \( z' \) and that \( G_k \) is increasing on \([0, z_0]\) and decreasing on \([z_0, +\infty]\).

For \( z \leq z_0 \),
\[ \exp(|k|z) - \exp(-|k|z) = \exp(|k|z)(1 - \exp(-2|k|z)) \]
is bounded by \( 2|k|z \exp(|k|z) \), hence
\[ |G_k(z_0, z)| \leq z \exp(|k|(z - z_0)) \leq z. \]
Similarly, for \( z \geq z_0 \),
\[ |G_k(z_0, z)| \leq z_0, \]
hence in both cases
\[ |G_k(z_0, z)| \leq \inf(z_0, z, \frac{1}{2|k|}). \quad (25) \]

### 2.3 Weighted inequalities

**Lemma 2.1** Let \( \phi \) be a weight such that for \( 0 \leq y \leq 1 \),
\[ C_1 + C_1 \frac{\mu}{\eta^2} \exp(-y/\eta) \leq \phi \leq C_2 + C_2 \frac{\mu}{\eta^2} \exp(-y/\eta). \quad (26) \]

Let \( \Psi \) be supported in \( 0 \leq y \leq 1 \), and let
\[ (v_1, v_2) = \nabla^I \Psi, \quad \Delta \Psi = \Theta. \quad (27) \]

Then
\[ \|v_1\|_{L^2(T \times [0,1])}^2 + \|v_2\|_{L^2(T \times [0,1])}^2 \leq C \int \frac{\Theta^2}{\phi}. \quad (28) \]
the constant \( C \) depending only on \( \phi \).
Proof

Let us take the Fourier transforms of $\Theta$ and $\Psi$ in the $x$ variable only. We have

$$\Theta(t, x, y) = \sum_{k=-\infty}^{+\infty} \Theta_k(t, y) \exp(ikx)$$

and

$$\Psi(t, x, y) = \sum_{k=-\infty}^{+\infty} \Psi_k(t, y) \exp(ikx).$$

Using (27) we get

$$(\partial_{yy} - k^2) \Psi_k(t, y) = \Theta_k(t, y)$$

which can be solved using the Green function $G_k(y, y')$. Namely for $k \neq 0$

$$\Psi_k(t, y) = \int_0^{+\infty} dy' G_k(y', y) \Theta_k(t, y').$$

(29)

Hence for $k \neq 0$,

$$\int_0^1 |\Psi_k(t, y)|^2 dy \leq \left( \int_0^{+\infty} \frac{|\Theta_k|^2}{\phi} \right) \left( \int_0^1 dy \int_0^y \phi(y') G_k^2(y', y) dy' \right)$$

$$+ \int_0^1 dy \int_0^1 \phi(y') G_k^2(y', y) dy'. $$

But

$$\frac{\mu}{4k^2\eta^2} \int_0^{+\infty} dy \int_0^y e^{-y'/\eta} e^{-2ky} (e^{2ky'} - 2 + e^{-2ky'}) = \frac{\mu\eta k}{(1 + 2\eta k)(1 + 4\eta k)k^2},$$

$$\frac{\mu}{4k^2\eta^2} \int_0^{+\infty} dy \int_y^{+\infty} e^{-y'/\eta} e^{-2ky} (e^{2ky} - 2 + e^{-2ky}) = \frac{2\mu\eta^2 k^2}{(1 + 2\eta k)^2(1 + 4\eta k)k^2},$$

$$\frac{1}{4k^2} \int_0^1 dy \int_0^y e^{-2ky} (e^{2ky} - 2 + e^{-2ky}) = \frac{4k + 8ke^{-2k} + 4e^{-2k} + e^{-4k}}{32k^4} - \frac{5}{32k^4},$$

these three quantities being bounded by $C/k^2$, and similarly

$$\frac{1}{4k^2} \int_0^1 dy \int_y^{+\infty} e^{-2ky} (e^{2ky} - 2 + e^{-2ky}) \leq \frac{C}{k^2}.$$

Therefore

$$\int_0^1 |\Psi_k(t, y)|^2 \leq \frac{C}{k^2} \int_0^{+\infty} \frac{|\Theta_k|^2}{\phi}.$$

But

$$v_2(t, x, y) = - \sum_{k=-\infty}^{+\infty} ik \Psi_k(t, y) \exp(ikx)$$

8
therefore
\[ \|v_2\|_{L^2}^2 = \sum_{k=-\infty}^{+\infty} k^2\|\Psi_k\|_{L^2}^2 \leq C \sum_{k=-\infty}^{+\infty} \int_0^{+\infty} \frac{|\Theta_k|^2}{\phi} \leq C \int \frac{|\Theta|^2}{\phi} \]

which proves half of the Lemma (the case \( k = 0 \) being straightforward).

Let us now bound \( v_1 \). For this we need to bound \( \partial_y \Psi_k \). For \( k \neq 0 \),
\[
\partial_y \Psi_k(t,y) = \int_0^{+\infty} dy' \partial_y G_k(y',y) \Theta_k(t,y'),
\]

hence
\[
\int_0^1 |\partial_y \Psi_k|^2 dy \leq \left( \int_0^{+\infty} \frac{|\Theta_k|^2}{\phi} \right) \int_0^1 dy \int_0^1 dy' |\partial_y G_k(y',y)|^2 \phi(y').
\]

But
\[
\mu \int_0^{+\infty} dy \int_0^y dy' \frac{e^{-2ky}}{4} (e^{2ky} - 2 + e^{-2ky}) e^{-y/\eta} = \frac{\mu k \eta}{8k^2 \eta^2 + 6k \eta + 1},
\]

\[
\mu \int_0^{+\infty} dy \int_0^y dy' \frac{e^{-2ky}}{4} (e^{2ky} + 2 + e^{-2ky}) e^{-y/\eta} = \frac{\mu (2k^2 \eta^2 + 4k \eta + 1)}{(1 + 2k \eta)^2 (1 + 4k \eta)}
\]

\[
\int_0^1 dy \int_0^y dy' e^{-2ky} (e^{2ky} - 2 + e^{-2ky}) = \frac{4k + 8ke^{-2k} + 4e^{-2k} + 1}{8k^2} - \frac{5}{8k^2},
\]

these three quantities being bounded by some constant \( C \) independent on \( k \), and
\[
\int_0^1 dy \int_0^y dy' e^{-2ky} (e^{2ky} + 2 + e^{-2ky}) \leq C,
\]

therefore
\[
\int_0^{+\infty} |\partial_y \Psi_k(t,y)|^2 \leq C \left( \int \frac{|\Theta_k|^2}{\phi} \right).
\]

It remains to handle the case \( k = 0 \). As \( \partial_y \Psi \) is supported in \( y \in [0,1] \),
\[
\partial_y \Psi_0(y) = - \int_0^1 \partial_{yy} \Psi_0,
\]

\[
|\partial_y \Psi_0(y)|^2 \leq C \int_0^1 \frac{|\partial_{yy} \Psi_0|^2}{\phi} \int_0^1 (1 + \frac{\mu}{\eta^2} \exp(-y'/\eta)) dy' \leq C \int_0^1 \frac{|\partial_{yy} \Psi_0|^2}{\phi} (2 + \frac{\mu}{\eta} \exp(-y/\eta)).
\]

Hence
\[
\int_0^1 |\partial_y \Psi_0(y)|^2 \leq C \int_0^1 \frac{|\partial_{yy} \Psi_0|^2}{\phi},
\]

which ends the proof of the Lemma.

To handle large \( y \) we have to put a weight on \( v_1 \) and \( v_2 \). Let us prove
Lemma 2.2 Let $\alpha > 0$ and let $\phi$ be a weight such that

$$C_1 \exp(-2\alpha y) + C_1 \frac{\mu}{\eta^2} \exp(-y/\eta) \leq \phi \leq C_2 \exp(-2\alpha y) + C_2 \frac{\mu}{\eta^2} \exp(-y/\eta) \quad (31)$$

then the solution $(v_1, v_2)$ of

$$(v_1, v_2) = \nabla^+ \Psi, \quad \Delta \Psi = \Theta \quad (32)$$

satisfies

$$\|v_1\|_{L^2(T \times \mathbb{R}_+)}^2 + \|v_2\|_{L^2(T \times \mathbb{R}_+)}^2 \leq C_0 \int \frac{\Theta^2}{\phi} \quad (33)$$

Proof

Let $\chi$ be a smooth decreasing positive function with support in $0 \leq y \leq 1$ and which equals 1 for $y \leq 1/2$. Splitting $\Psi$ in $\chi \Psi$ and $(1 - \chi) \Psi$ and using Lemma 2.1, we are lead to prove (33) for $\Psi$ and $\Theta$ with support in $y \geq 1/2$. Repeating the proof of the previous section we have to bound, for $k \neq 0$,

$$\int_0^{+\infty} dy \int_0^y dy' e^{-2\alpha y} G_k^2(y', y)dy' dy = \int_0^{+\infty} dy \int_0^y dy' e^{-2\alpha y'} - 2k y (e^{ky} - e^{-ky})^2$$

$$= \frac{k}{2\alpha(2k^2 + 3\alpha k + \alpha^2)} \leq C,$$

$$\int_0^{+\infty} dy \int_y^{+\infty} dy' e^{-2\alpha y} G_k^2(y', y)dy' dy = \int_0^{+\infty} dy \int_y^{+\infty} dy' e^{-2\alpha y'} - 2k y (e^{ky} - e^{-ky})^2$$

$$= \frac{k^2}{2\alpha(2k^2 + 3\alpha k + \alpha^2)(\alpha + k)} \leq C$$

and

$$\int_0^{+\infty} dy \int_0^y dy' e^{-2\alpha y} |\partial_y G_k(y', y)|^2 dy' dy = \int_0^{+\infty} dy \int_0^y dy' e^{-2\alpha y'} - 2ky (e^{ky} - e^{-ky})^2$$

$$\leq C,$$

$$\int_0^{+\infty} dy \int_y^{+\infty} dy' e^{-2\alpha y} |\partial_y G_k(y', y)|^2 dy' dy = \int_0^{+\infty} dy \int_y^{+\infty} e^{-2\alpha y'} - 2ky (e^{ky} + e^{-ky})^2$$

$$= \frac{1}{2} \frac{k^2 + 4\alpha k + 2\alpha^2}{(\alpha + k)(2k^2 + 3\alpha k + \alpha^2)\alpha} \leq C.$$
and
\[ \int_0^{+\infty} |\partial_y \Psi_0|^2 \leq C \int \frac{\Theta_0^2}{\phi}. \]
which ends the proof of the Lemma. \qed

3 Linear stability: proof of Theorem 1.1

Let us prove Theorem 1.1. Let \( \Theta = \text{curl} \, v \) and let
\[ I(t) = \int \frac{\Theta^2(t, x, y)}{\phi(t, y)} \, dx \, dy \]
where we dropped the index \( \eta \). We have
\[ \partial_t I = 2 \int \frac{\Theta \partial_t \Theta}{\phi} - \int \frac{\partial^2 \Theta^2}{\phi} \partial_t \phi. \]
Using (H) we get
\[ |\int \frac{\Theta^2}{\phi^2} \partial_t \phi| \leq CI(t). \]
But
\[ \int \frac{\Theta \partial_t \Theta}{\phi} = -\int \frac{u_1 \Theta \partial_x \Theta}{\phi} - \int \frac{u_2 \Theta \partial_y \Theta}{\phi} - \int \frac{v_1 \partial_y \Theta \partial_x \Theta}{\phi} - \int \frac{v_2 \partial_y \Theta \partial_x \Theta}{\phi} + \int \frac{\Theta \text{curl} \, w}{\phi}. \]
First
\[ -\int \frac{u_1 \Theta \partial_x \Theta}{\phi} = \int \frac{u_1 \partial_x \Theta^2}{\phi} = \int \frac{\partial_x u_1 \Theta^2}{\phi} - \int \frac{u_1 \partial_x \phi \Theta^2}{\phi^2}. \]
and using (H),
\[ \left| \int \frac{u_1 \partial_x \phi \Theta^2}{\phi^2} \right| \leq C \int \frac{\Theta^2}{\phi}. \]
Next
\[ -\int \frac{u_2 \Theta \partial_y \Theta}{\phi} = -\int \frac{u_2 \partial_y \Theta^2}{\phi} = \int \frac{\partial_y u_2 \Theta^2}{\phi} - \int \frac{u_2 \partial_y \phi \Theta^2}{\phi^2}. \]
and using (H)
\[ \left| \int \frac{u_2 \partial_y \phi \Theta^2}{\phi^2} \right| \leq C \int \frac{\Theta^2}{\phi}. \]
Moreover, by incompressibility condition,
\[ \int \frac{\partial_x u_1 \Theta^2}{\phi^2} + \int \frac{\partial_y u_2 \Theta^2}{\phi^2} = 0. \]
Now
\[ \int \frac{v_1 \partial_x \Theta_1}{\phi} = \int \frac{v_1 \partial_x \Theta_1 u_1}{\phi} - \int \frac{v_1 \partial_x \Theta_2 u_2}{\phi}. \]
Notice that $|\partial_{xx}^2 u_2| \leq C \sqrt{\phi}$, therefore
\[
\left| \int v_1 \frac{\partial_{xx}^2 u_2}{\phi} \right| \leq \|v_1\|_{L^2} \left( \int \frac{\Theta^2}{\phi} \right)^{1/2} \leq C \int \frac{\Theta^2}{\phi},
\]
where we used Lemma 2.2. Moreover,
\[
\left| \frac{\partial_{xy}^2 u_1}{\phi} \right| \leq \frac{C}{\sqrt{\phi}}
\]
hence
\[
\left| \int v_1 \frac{\partial_{xy}^2 u_1}{\phi} \right| \leq C \|v_1\|_{L^2} \left( \int \frac{\Theta^2}{\phi} \right)^{1/2} \leq C \int \frac{\Theta^2}{\phi},
\]
where we used Lemma 2.2. Next
\[
\int v_2 \frac{\partial_{xy}^2 u_1}{\phi} \Theta = \int v_2 \frac{\partial_{xy}^2 u_1}{\phi} \Theta - \int v_2 \frac{\partial_{xy}^2 u_2}{\phi} \Theta.
\]
Let us introduce $\Psi$ the stream function such that $(v_1, v_2) = \nabla \Psi$ and $\Delta \Psi = \Theta$. We recall that, by (H),
\[
\left| \frac{\partial_{xy}^2 u_1}{\phi} + 1 \right| \leq \frac{C}{\sqrt{\phi}}
\]
We have
\[
\int v_2 \Theta = -\int \partial_x \Psi \Delta \Psi = \int \partial_y \nabla \Psi \Delta \Psi = 0
\]
since $\partial_y \Psi = 0$ when $y = 0$. Therefore it remains to bound
\[
\int \frac{|v_2|^2}{\sqrt{\phi}} \leq \left( \int |v_2|^2 \right)^{1/2} \left( \int \frac{|\Theta|^2}{\phi} \right)^{1/2} \leq C \int \frac{\Theta^2}{\phi},
\]
using Lemma 2.2. Notice that by (H)
\[
\left| \frac{\partial_{xy}^2 u_2}{\phi} \right| \leq \frac{C}{\sqrt{\phi}}
\]
hence
\[
\left| \int v_2 \frac{\partial_{xy}^2 u_2}{\phi} \Theta \right| \leq C \int \frac{\Theta^2}{\phi},
\]
where we used Lemma 2.2. Next
\[
\left| \int \frac{\Theta \text{ curl } w}{\phi} \right| \leq \left( \int \frac{\Theta^2}{\phi} \right)^{1/2} \left( \int \frac{\text{ curl } w^2}{\phi} \right)^{1/2}.
\]
Summing up all these estimates we get
\[
\partial_t I(t) \leq C I(t) + \int \frac{\text{ curl } w^2}{\phi}
\]
which ends the proof of Theorem 1.1. $\square$
4 Nonlinear stability: proof of Theorem 1.2

4.1 First order derivatives

Let us turn to the control of first order derivatives of (8,9). Let

\[ I_1(t) = \int \frac{\partial_x \Theta}{\phi} + \int \frac{\partial_y \Theta}{\phi^2}. \]

**Lemma 4.1** There exists \( C \) such that

\[ \partial_t I_1(t) \leq CI_1(t) + (C\eta^{-8} + C)I(t). \]

**Proof**

We have

\[ \partial_t (\partial_x \Theta) + u_1 \partial_x (\partial_x \Theta) + u_2 \partial_y (\partial_x \Theta) + (\partial_x v_1) \partial_x \Theta_t + (\partial_x v_2) \partial_y \Theta_t \]
\[ + (\partial_x u_1) \partial_x \Theta + (\partial_x u_2) \partial_y \Theta + v_1 \partial_{xx} \Theta_t + v_2 \partial_{xy} \Theta_t = 0, \]
\[ \partial_t (\partial_y \Theta) + u_1 \partial_x (\partial_y \Theta) + u_2 \partial_y (\partial_y \Theta) + (\partial_y v_1) \partial_x \Theta_t + (\partial_y v_2) \partial_y \Theta_t \]
\[ + (\partial_y u_1) \partial_x \Theta + (\partial_y u_2) \partial_y \Theta + v_1 \partial_{xy} \Theta_t + v_2 \partial_{yy} \Theta_t = 0. \]

Many terms of (35) and (36) can be seen as source terms. Namely let

\[ S_1 = (\partial_x v_1) \partial_x \Theta_t + (\partial_x v_2) \partial_y \Theta_t + v_1 \partial_{xx} \Theta_t + v_2 \partial_{xy} \Theta_t \]

and

\[ S_2 = (\partial_y v_1) \partial_x \Theta_t + (\partial_y v_2) \partial_y \Theta_t + v_1 \partial_{xy} \Theta_t + v_2 \partial_{yy} \Theta_t. \]

Let us first bound \( S_1 \) and \( S_2 \). As \( \text{div} (v_1, v_2) = 0 \) we have

\[ \|\nabla v_1\|_{L^2}^2 + \|\nabla v_2\|_{L^2}^2 \leq C \|\text{curl} (v_1, v_2)\|_{L^2}^2 \leq C \eta^{-2} \int \frac{|\Theta|^2}{\phi}, \]

hence, using (H') and Lemma 2.2,

\[ \int \frac{S_1^2}{\phi} + \int \frac{S_2^2}{\phi} \leq (C\eta^{-8} + C) \int \frac{|\Theta|^2}{\phi}. \]

Now

\[ = \int \frac{u_1 (\partial_x \Theta)^2}{2\phi} \partial_x \phi + \int \frac{u_2 (\partial_x \Theta)^2}{2\phi} \partial_y \phi - \int \frac{\partial_x u_1 |\partial_x \Theta|^2}{2\phi} - \int \frac{\partial_y u_1 |\partial_x \Theta|^2}{2\phi} \]

which, using (H), is bounded by

\[ C \int \frac{|\partial_x \Theta|^2}{\phi}. \]
and similarly for the terms involving $\partial_y \Theta$.

Moreover,

$$\left| \int \frac{\partial_z u_1 (\partial_x \Theta)^2}{\phi} + \int \frac{\partial_z u_2 \partial_x \Theta \partial_y \Theta}{\phi} \right| \leq CI_1(t)$$

using

$$|\partial_x u_1| + \sqrt{\phi} |\partial_x u_2| \leq C,$$

and similarly

$$\left| \int \frac{\partial_z u_1 \partial_x \Theta \partial_y \Theta}{\phi^2} + \int \frac{\partial_y u_2 (\partial_y \Theta)^2}{\phi^2} \right| \leq CI_1(t)$$

using

$$\left| \frac{\partial_x u_1}{\sqrt{\phi}} \right| + |\partial_y u_2| \leq C$$

which ends the proof of the Lemma. \hfill \Box

### 4.2 Higher order derivatives

Let

$$I_n(t) = \sum_{\alpha, \beta = 0}^{n} \int \frac{|\partial_x^\alpha \partial_y^\beta \Theta|^2}{\phi^{\alpha+\beta+1}}.$$

As in the previous section, we have

**Lemma 4.2** There exists $C$ such that

$$\partial_t I_n(t) \leq \sum_{i=0}^{n} (C \eta^{-8(n-i)} + C) I_i(t). \quad (37)$$

As a Corollary,

**Lemma 4.3** The solution $v$ of linearized Euler equations (8,9) satisfies for suitable large enough,

$$\partial_t \|v\| \leq C(t) \|v\|_s$$

with $C(t)$ independent on $\eta$.

### 4.3 Nonlinear stability result

In addition to the terms already bounded in Lemmas 4.1 and 4.2 we have to bound

$$\int \frac{\partial_x^\alpha \partial_y^\beta \partial_x^\sigma v_1 \partial_x \Theta}{\phi^{\alpha+\beta+\sigma}} + \int \frac{\partial_x^\alpha \partial_y^\beta \partial_x^\sigma v_2 \partial_y \Theta}{\phi^{\alpha+\beta+\sigma}}, \quad (38)$$

which is a sum of terms of the form

$$J_{\alpha', \beta'} = \int \frac{\partial_x^\alpha \partial_y^\beta \partial_x^\sigma v_1 \partial_x^{\alpha-\beta'} \partial_y^{\beta'} \Theta}{\phi^{\alpha+\beta+\sigma}} + \int \frac{\partial_x^\alpha \partial_y^\beta \partial_x^\sigma v_2 \partial_x^{\alpha-\beta'} \partial_y^{\beta'} \Theta}{\phi^{\alpha+\beta+\sigma}}$$
for $0 \leq \alpha' \leq \alpha$ and $0 \leq \beta' \leq \beta$.

For $\alpha' = \beta' = 0$,

$$J_{0,0} = \int \frac{v_1}{\phi^{1+\beta}} \frac{\partial_x^2 \partial_y^2 \Theta}{2} \partial_x + \int \frac{v_2}{\phi^{1+\beta}} \frac{\partial_y^2 \partial_y^2 \Theta}{2} \partial_y$$

$$= (1 + \beta) \int \frac{v_1}{\phi^{1+\beta}} \frac{\partial_x \phi}{\phi} \frac{(\partial_x^2 \partial_y^2 \Theta)^2}{2} + (1 + \beta) \int \frac{v_2}{\phi^{1+\beta}} \frac{\partial_y \phi}{\phi} \frac{(\partial_y^2 \partial_y^2 \Theta)^2}{2}$$

since $(v_1, v_2)$ is divergence free.

$$\leq \frac{C}{\eta^s} (\|v_1\|_{L^\infty(\Gamma \times [0,1])} + \|v_2\|_{L^\infty(\Gamma \times [0,1])}) \int \frac{(\partial_x^2 \partial_y^2 \Theta)^2}{\phi^{1+\beta}} \partial_x$$

For $\alpha' + \beta' \geq 1$ and for $s$ large enough, either $\alpha' + \beta' = \alpha + \beta = 2 - \alpha' - \beta'$ is less than $s - 3$. If $\alpha' + \beta' \leq s - 3$ we use

$$\|\partial_{x}^{\alpha'} \partial_{y}^{\beta'} v_1\|_{L^\infty} + \|\partial_{x}^{\alpha'} \partial_{y}^{\beta'} v_2\|_{L^\infty} \leq \frac{C}{\eta^s} \|v\|_s$$

to get

$$|J_{\alpha',\beta'}| \leq \frac{C}{\eta^s} \|v\|_{L^\infty} \int \frac{\partial_x^2 \partial_y^2 \Theta}{\phi^{1+\beta/2+\beta'} / 2} \frac{\alpha - \alpha' + 1}{\phi^{1+\beta/2+\beta'}}$$

For $\alpha + \beta = 2 - \alpha' - \beta' \geq s - 3$ we use

$$\|\partial_x^{\alpha - 1} \partial_y^{\beta - 1} \Theta\|_{L^\infty} \leq \frac{C}{\eta^s} \|v\|_s$$

and get

$$|J_{\alpha',\beta'}| \leq \frac{C}{\eta^s} \|v\|_{L^\infty} \int \frac{\partial_x^{\alpha - 1} \partial_y^{\beta - 1} \Theta}{\phi^{1+\beta/2+\beta'}} \frac{\alpha - \alpha' + 1}{\phi^{1+\beta/2+\beta'}} \|v\|_{L^\infty} \leq \frac{C}{\eta^s} \|v\|_s,$$

which ends the proof of Theorem 1.2.

\begin{flushright}
\square
\end{flushright}

5 Asymptotic expansion : Theorem 1.3

5.1 Inviscid Prandtl equations

Classical scalings in boundary layers lead to the study of

$$\partial_t u_1 + u_1 \partial_x u_1 + u_2 \partial_y u_1 = f(t, x), \quad \text{(39)}$$
$$\partial_x u_1 + \partial_y u_2 = 0 \quad \text{(40)}$$
$$u_2 = 0 \quad \text{for} \quad y = 0 \quad \text{(41)}$$

in the half space $y \geq 0$, where $f$ is a given function which depends only on the $x$ variable. This system can be seen as inviscid Prandtl equations.

\textbf{Proposition 5.1} Let $u_1^0$ and $u_2^0$ be given $C^s$ functions with $s$ large enough, satisfying (40), (41), and let $f \in L^\infty([0, T^*], C^s(\mathbb{T}))$ (with $T^* > 0$). There exists $0 < T \leq T^*$ and solutions $u_1$ and $u_2$ in $L^\infty([0, T], C^{s-1})$ of (39,40,41) with initial data $u_1^0$ and $u_2^0$. 

15
Proof

System (39,40) has a very deep structure (see [3] for other formulations, and in particular kinetic formulations). Let us extend $u_1$ and $u_2$ for $x \in \mathbb{T}$ to $x \in \mathbb{R}$ by periodicity, and let us introduce the characteristics $X(t, x, y)$ and $Y(t, x, y)$ defined by

$$
\partial_t X(t, x, y) = u_1(t, X(t, x, y), Y(t, x, y)),
$$
$$
\partial_t Y(t, x, y) = u_2(t, X(t, x, y), Y(t, x, y)),
$$
with $X(0, x, y) = x$ and $Y(0, x, y) = y$ and let

$$
\tilde{u}_1(t, x, y) = u_1(t, X(t, x, y), Y(t, x, y)).
$$

Then (39,40,41) can be rewritten

$$
\partial_t \tilde{u}_1 = f(t, X),
$$
$$
\partial_t X = \tilde{u}_1,
$$
$$
\partial_x X \partial_y Y - \partial_y X \partial_x Y = 1,
$$
$$
Y(t, x, 0) = 0,
$$
equation (44) being the incompressibility condition (40). Equations (42) and (43) are straightforward to solve and we get

$$
\tilde{u}_1 \in L^\infty(C^s), \quad X - x \in L^\infty(C^s).
$$

By definition of the initial conditions on $X$ and $Y$, (44) can be solved in small time and we get $Y \in L^\infty(C^{s-1})$. Notice the lost of one derivative. Going back to the genuine variables $x, y$ gives the Proposition. \hfill \Box

Remarks

This Proposition holds in any space dimension. Notice the lost of one derivative in the estimate. This is a crucial point, probably the main difficulty of (viscous) Prandtl equations. The proof fails when $f$ depends on $y$! Notice that if $f = 0$ the proof is completely “geometric” : $X$ and $u_1$ are given explicitly, and $Y$ can easily be deduced from the incompressibility condition. In general there is no global smooth solution since (44) can only be solved in small time. It is easy to construct explicit examples of solutions which blow up at a particular time. We refer to [8] for evidence of blow up for (viscous) Prandtl equations.

5.2 Linearized inviscid Prandtl equations

Let us turn to the study of

$$
\partial_t v_1 + u_1 \partial_x v_1 + u_2 \partial_y v_1 + v_1 \partial_x u_1 + v_2 \partial_y u_1 = 0,
$$
$$
\partial_x v_1 + \partial_y v_2 = 0,
$$

\(v_2 = 0 \quad \text{for} \quad y = 0\) \hspace{1cm} (48)

\(v_1\) and \(v_2\) being given at \(t = 0\), and where \((u_1, u_2)\) is a solution of (39,40) with some force \(f\).

**Proposition 5.2** Let \(v_1^0\) and \(v_2^0\) be given \(C^s\) functions with \(s\) large enough, and let \(u_1, u_2\) be a solution of (39,40,41) on \([0, T]\) with initial data in \(C^s\) and force term \(f \in L^\infty(C^s)\). Then there exists solutions \(v_1\) and \(v_2\) in \(L^\infty([0, T], C^{s-2})\) of (46,47,48) with initial data \(v_1^0\) and \(v_2^0\).

**Proof**

Let \(\delta > 0\) and let \(u_1'\) and \(u_2'\) be the solution of (39,40) with the same force \(f\) and with initial data
\[ u_1'(0, x, y) = u_1(0, x, y) + \delta v_1^0, \quad u_2'(0, x, y) = u_2(0, x, y) + \delta v_2^0. \]

Let \(X'\) and \(Y'\) be the characteristics associated to \(u_1'\) and \(u_2'\). We have
\[ \partial_t \tilde{u}_1 = f(t, X), \quad \partial_t \tilde{u}_1' = f(t, X'), \]
\[ \partial_t X = \tilde{u}_1, \quad \partial_t X' = \tilde{u}_1'. \]

Therefore
\[ \partial_t (|\tilde{u}_1 - \tilde{u}_1'| + |X - X'|) \leq (1 + |f|_{L^1}) (|\tilde{u}_1 - \tilde{u}_1'| + |X - X'|) \]
which leads to
\[ |\tilde{u}_1 - \tilde{u}_1'| + |X - X'| \leq C\delta \exp((1 + |f|_{L^1}) t) \]
(|\(f|_{L^1}\) denoting the supremum of \(f\')). Similarly,
\[ \|\tilde{u}_1 - \tilde{u}_1'\|_{C^s} + \|X - X'\|_{C^s} \leq \delta C \exp(Ct), \]
and
\[ \|Y - Y'\|_{C^{s-1}} \leq \delta C \exp(Ct) \]
for \(t \leq T\) where \(T\) is independent on \(\delta\). Going back to \(u_1, u_2\) and \(u_1', u_2'\) we get
\[ \|u_1 - u_1'\|_{L^\infty([0, T], C^{s-1})} + \|u_2 - u_2'\|_{L^\infty([0, T], C^{s-1})} \leq C\delta. \]

Let now
\[ v_1^\delta = \frac{u_1' - u_1}{\delta}, \quad v_2^\delta = \frac{u_2' - u_2}{\delta}. \]

We have \(v_1^\delta(0) = v_1^0\) and \(v_2^\delta(0) = v_2^0\), and using a compactness argument, \(v_1^\delta\) and \(v_2^\delta\) converge as \(\delta\) goes to 0 to functions \(v_1\) and \(v_2\) in \(L^\infty([0, T], C^{s-2})\). Passing to the limit in the equations on \(v_1^\delta\) and \(v_2^\delta\)
\[ \partial_t v_1^\delta + u_1 \partial_x v_1^\delta + v_1^\delta \partial_x u_1 + u_2 \partial_y v_2^\delta + u_2 \partial_y v_1^\delta + \frac{1}{2} \partial_x v_1^\delta + \frac{1}{2} \partial_y v_2^\delta = 0, \]
\[ \partial_x v_1^\delta + \partial_y v_2^\delta = 0 \]
gives (46,47). \(\mathbb{Q.E.D.}\)
Remarks

The proof is in fact more elaborate than the proof of existence for nonlinear inviscid Prandtl equations. This proof opens many interesting questions: first it is crucial that \(u_1\) and \(u_2\) are solutions of (39,40) (notice that \(f\) must be independent on \(y\), therefore we can not define \(f\) by (39) even if \((u_1, u_2)\) is divergence free). Moreover there is again a lost of regularity in the solution, at \(t = 0\): is it possible to get this result with a “classical” energy method?

5.3 Construction of an approximate solution

Notice that the pressure \(p\) also has an asymptotic expansion, namely

\[
p = \sum_{j=0}^{N} \eta^j p_{j,\text{int}}(t, x, y) + \sum_{j=0}^{N} \eta^j p_{j,\text{b}}(t, x, \frac{y}{\eta}).
\]  

(49)

As usual in boundary layer theory we will get \(p_{0,\text{b}} = 0\).

Putting the Ansatz (15) in incompressible Euler equations we get that \(u_{0,\text{int}}\) satisfies Euler equations

\[
\partial_t u_{0,\text{int}} + (u_{0,\text{int}}, \nabla) u_{0,\text{int}} = -\nabla p_{0,\text{int}},
\]

(50)

\[
\nabla \cdot u_{0,\text{int}} = 0
\]

(51)

with boundary conditions

\[
u_{0,\text{int},2} = 0
\]

(52)

where \(u_{0,\text{int},2}\) denotes the second component of \(u_{0,\text{int}}\). By standard results there exists \(T_1 > 0\) and a solution \(u_{0,\text{int}}\) in \(L^\infty([0, T_1], H^s)\) to (50,51,52) (for every \(s\)).

Let

\[
u_{0,b}(t, x, Y) = \left( \begin{array}{c} u_{0,b,1}(t, x, Y) \\ \eta u_{0,b,2}(t, x, Y) \end{array} \right)
\]

where \(Y\) is the fast variable \(Y = y/\eta\). Notice the \(\eta\) factor in front of \(u_{0,b,2}\) which comes from incompressibility condition. This term could be rejected in \(u_{1,b}\), but the construction would then be more awkward. Let us derive the equation on \(u_{0,b}\). Putting (15) in Euler equations we get, up to terms of order \(\eta\),

\[
\partial_t(u_{0,\text{int},1} + u_{0,b,1}) + (u_{0,\text{int},1} + u_{0,b,1})\partial_x(u_{0,\text{int},1} + u_{0,b,1}) = \partial_x p_{0,\text{int}} + \partial_x p_{0,b}
\]

(53)

\[
+(u_{0,\text{int},2} + \eta u_{0,b,2})\partial_y(u_{0,\text{int},1} + u_{0,b,1}) = \partial_y p_{0,\text{int}} + \partial_y p_{0,b}
\]

and

\[
\partial_t(u_{0,\text{int},2} + \eta u_{0,b,2}) + (u_{0,\text{int},1} + u_{0,b,1})\partial_x(u_{0,\text{int},2} + \eta u_{0,b,2}) = \partial_x p_{0,\text{int}} + \partial_x p_{0,b}
\]

(54)

\[
+(u_{0,\text{int},2} + \eta u_{0,b,2})\partial_y(u_{0,\text{int},1} + \eta u_{0,b,1}) = \partial_y p_{0,\text{int}} + \partial_y p_{0,b}
\]

Making the change of variables \(Y = y/\eta\) we get from (54)

\[p_{0,b} = 0\]
(as usual the pressure does not change in the boundary layer at first order). Moreover \((u_{0 \text{,int}, 2} + \eta u_{0 \text{,b}, 2} \partial y u_{0 \text{,int}, 1})\) is of order \(\eta\) for \(Y\) bounded, and can therefore be forgotten. On the other side

\[
(u_{0 \text{,int}, 2} + \eta u_{0 \text{,b}, 2}) \partial y u_{0 \text{,b}, 1} = \left(\frac{u_{0 \text{,int}, 2}}{\eta} + u_{0 \text{,b}, 2}\right) \partial y u_{0 \text{,b}, 1}
\]

and

\[
\frac{u_{0 \text{,int}, 2}}{\eta} = Y \partial y u_{0 \text{,int}, 2}(t, x, 0) + O(\eta),
\]

therefore up to terms of order \(\eta\), (53) can be rewritten

\[
\partial_t(u_{0 \text{,int}, 1}(t, x, 0) + u_{0 \text{,b}, 1}) + (u_{0 \text{,int}, 1}(t, x, 0) + u_{0 \text{,b}, 1}) \partial_x(u_{0 \text{,int}, 1}(t, x, 0) + u_{0 \text{,b}, 1})
\]

\[
+ (Y \partial_y u_{0 \text{,int}, 2}(t, x, 0) + u_{0 \text{,b}, 2}) \partial_y u_{0 \text{,b}, 1} = \partial_x p_{0 \text{,int}}(t, x, 0)
\]

(55)

Setting

\[
\bar{u}_{0,1}(t, x, Y) = u_{0 \text{,int}, 1}(t, x, 0) + u_{0 \text{,b}, 1}(t, x, Y)
\]

and

\[
\bar{u}_{0,2}(t, x, Y) = Y \partial_y u_{0 \text{,int}, 2}(t, x, 0) + u_{0 \text{,b}, 2}(t, x, Y)
\]

we get

\[
\partial_t \bar{u}_{0,1} + \bar{u}_{0,1} \partial_x \bar{u}_{0,1} + \bar{u}_{0,2} \partial_y \bar{u}_{0,1} = \partial_x p_{0 \text{,int}}(t, x, 0)
\]

(56)

\[
\partial_x \bar{u}_{0,1} + \partial_y \bar{u}_{0,2} = 0
\]

(57)

\[
\bar{u}_{0,2} = 0 \quad \text{at} \quad Y = 0.
\]

(58)

Using Proposition 5.1 there exists solutions \((\bar{u}_{0,1}, \bar{u}_{0,2})\) of (56,57,58) on a time interval \([0, T]\) for some \(T \leq T_1\). We then recover \(u_{0,1,1}\) and \(u_{0,1,2}\) which are in \(L^\infty([0, T], C^s)\) for every \(s\). It is easy to prove that if \(u_{0,1}\) is initially rapidly decreasing in \(Y\) it remains so on \([0, T]\).

Let us turn to first order terms. First \(u_{1,\text{int}}\) satisfies

\[
\partial_t u_{1,\text{int}} + (u_{0,\text{int}} \cdot \nabla) u_{1,\text{int}} + (u_{1,\text{int}} \cdot \nabla) u_{0,\text{int}} = \nabla p_{1,\text{int}},
\]

(59)

\[
\nabla \cdot u_{1,\text{int}} = 0,
\]

(60)

\[
u_{1,\text{int}, 2}(t, x, 0) = 0.
\]

(61)

By classical arguments there exists a solution \(u_{1,\text{int}}\) in \(L^\infty([0, T], H^s)\) for every \(s\). Higher order terms can be handled as previously. We will not detail them. Let \(s\) and \(N\) be arbitrarily large. By constructing high order terms we get an approximate solution \(v^{app}\) which satisfies (1) up to \(\eta^N R_N^s\) where \(\|R_N^s\|_{H^s} \leq C\eta^{-s}\), and which satisfies (2).

The proof of Theorem 1.3 starting from this approximate solution \(v^{app}\) of high order \(N\) and using Theorem 1.2 is straightforward and standart (see [12], [14], [6]).
6 Linear instability: proof of Theorem 1.4

The construction is classical (see for instance [11], [18]) and can be traced back to Lord Rayleigh [21]. We will however recall it since it is the first step of the next section. First we rescale time and space according to

\[ t = \eta t' \quad \text{and} \quad x = \eta x', \ y = \eta y' \quad (62) \]

and observe that equations (8,9) are invariant under this change of variables. Let \( \tilde{v}_t \) be a given profile and let

\[ u^v = \left( \begin{array}{c} \tilde{v}_t(1,y) \\ 0 \end{array} \right). \]

We are lead to look for an exponentially increasing eigenmode of

\[ \partial_t v + \tilde{v}_t \partial_x v + v_2 \partial_y \left( \begin{array}{c} \tilde{v}_t \\ 0 \end{array} \right) + \nabla p = 0, \quad (63) \]

\[ \nabla . v = 0. \quad (64) \]

Following the classical construction [11], we look for \( v \) of the form

\[ v = \left( \begin{array}{c} \Psi'(y) \exp ik(x - ct) \\ -ik \Psi(y) \exp ik(x - ct) \end{array} \right), \quad (65) \]

where \( \Psi(y) \exp ik(x - ct) \) plays the role of a current function. Taking the curl of (63,64) we obtain the inviscid Orr Sommerfeld equation

\[ (\tilde{v}_t - c) \left( \partial_{yy}^2 - k^2 \right) \Psi - \Psi \partial_{yy}^2 \tilde{v}_t = 0 \quad (66) \]

with boundary conditions

\[ \Psi = 0 \quad \text{on} \quad y = 0, \quad \Psi \to 0 \quad \text{as} \quad y \to +\infty. \quad (67) \]

As noticed in [18], the resolution of (66,67) is straightforward when \( \tilde{v}_t \) is piece-wise linear. So let us consider \( \tilde{v}_t \) defined by

\[ \tilde{v}_t(y) = \begin{cases} \alpha y & \text{for} \quad y \leq 1, \\ \beta y + (\alpha - \beta) & \text{for} \quad 1 \leq y \leq 1 + \gamma, \\ \alpha + \beta \gamma & \text{for} \quad y \geq 1 + \gamma. \end{cases} \quad (68) \]

As \( \partial_{yy}^2 \tilde{v}_t \) is a sum of two Dirac masses, one in \( 1 \) and another in \( 1 + \gamma \),

\[ (\partial_{yy}^2 - k^2) \Psi = \sigma_1 \delta_1 + \sigma_2 \delta_{1+\gamma}. \]

Hence

\[ (\tilde{v}_t(1) - c) \sigma_1 = \Psi(1)(\beta - \alpha) = \sigma_1(\beta - \alpha) G(1,1) + \sigma_2(\beta - \alpha) G(1 + \gamma, 1) \]
and

$$(\tilde{v}(1 + \gamma) - c)\sigma_2 = -\Psi(2)\beta = -\sigma_1\beta G(1, 1 + \gamma) - \sigma_2\beta G(1 + \gamma, 1 + \gamma).$$

Therefore $c$ is an eigenvalue of the matrix

$$\mathcal{M} = \begin{pmatrix} \tilde{v}(1) - (\beta - \alpha) G(1, 1) & -(\beta - \alpha) G(1 + \gamma, 1) \\ \beta G(1, 1 + \gamma) & \tilde{v}(1 + \gamma) + \beta G(1 + \gamma, 1 + \gamma) \end{pmatrix}. \quad (69)$$

This matrix is completely explicit, and depends on $\alpha$, $\beta$, $\gamma$ and $k$. It has two eigenvalues $c_+(k)$ and $c_-(k)$ depending on $\alpha$, $\beta$ and $\gamma$. The main property is that $c_+$ and $c_-$ are real excepted for particular values of $k$ if $\beta > \alpha$ (which corresponds to “inflection point” on $\tilde{v}$). A typical picture of $\Re c_+$ is given by figure 1. When the imaginary part of $c_+$ is strictly positive, the corresponding eigenvector of $\mathcal{M}$ leads to an exponentially increasing eigenmode $v$, which ends the proof of the Proposition. \qed

7 Nonlinear instability : Theorem 1.5

7.1 Growth of solutions of the inviscid Orr Sommerfeld equation

Let us begin by majorations on solutions of the inviscid Orr Sommerfeld equations.

Figure 1: $\Re c_+$ for $\alpha = 1/2$ and $\beta = 1$
Proposition 7.1 Let \( \tilde{v}_t \) be a smooth increasing function with
\[
\tilde{v}_t(y) = \begin{cases} 
\alpha y & \text{for } y \leq 1 - 2\delta \\
\beta y + (\alpha - \beta) & \text{for } 1 + 2\delta \leq y \leq 1 + \gamma - 2\delta \\
\alpha + \beta \gamma & \text{for } 1 + \gamma + 2\delta \leq y 
\end{cases}
\]
such that \( \partial_{yy}^2 \tilde{v}_t \) has a constant sign on \([1 - 2\delta, 1 + 2\delta]\) and \([1 + \gamma - 2\delta, 1 + \gamma + 2\delta]\) and such that
\[
|\partial_y^n \tilde{v}_t| \leq C_n \delta^{-n}
\]
for every \( n \geq 0 \). Let \( N \) be large enough and \( \delta \) small enough. Let \( \Psi \) be the solution of
\[
(\partial_t + ik\tilde{v}_t)(\partial_{yy}^2 - k^2)\Psi - ik\Psi\partial_{yy}^2 \tilde{v}_t = \tilde{\Phi} \tag{70}
\]
with boundary conditions
\[
\Psi = 0 \text{ on } y = 0, \quad \Psi \to 0 \text{ as } y \to +\infty \tag{71}
\]
with source term \( \tilde{\Phi} \) satisfying
\[
|\partial_y^n \tilde{\Phi}| \leq C_n \exp(-D_0y) \exp(D_1t) \tag{72}
\]
for \( 0 \leq n \leq N \) and initial data \( \Phi^0 \) such that
\[
|\partial_y^n \Phi^0| \leq C_n \exp(-D_0y) \tag{73}
\]
for \( 0 \leq n \leq N \) (with \( D_0 > 0 \) and \( D_1 > 0 \)). Then \( \Psi \) satisfies
\[
|\partial_y^n \Psi(t, y)| \leq C_{n, N} \exp(D_1t) \exp(-y \inf(D_0, k)) \tag{74}
\]
for \( 0 \leq n \leq N/2 \), provided \( D_0 \neq k \) and \( D_1 > |3c_+|k \).

Proof

We split the proof in several parts.

7.1.1 First step : study of \( \Psi_1 \)

Let \( I_1 = [1 - 2\delta, 1 + 2\delta] \) and \( I_2 = [1 + \gamma - 2\delta, 1 + \gamma + 2\delta] \). As \( \partial_{yy}^2 \tilde{v}_t \) vanishes outside \( I_1 \cup I_2 \) which are small intervals, we will in a first step ignore the term \( \Psi\partial_{yy}^2 \tilde{v}_t \). Therefore let \( \Psi_1 \) be the solution of
\[
(\partial_t + ik\tilde{v}_t)\Theta_1 = \tilde{\Phi} \tag{75}
\]
with initial data \( \Psi_1(0, y) = \Phi^0(y) \), where \( \Theta_1 = (\partial_{yy}^2 - k^2)\Psi_1 \).

Lemma 7.2 For \( 0 \leq n \leq N - 2 \), there exist constants \( C_n \) such that
\[
|\partial_y^n \Psi_1(t, y)| \leq \frac{C_n}{\delta^n} \exp(D_1t) \exp(-y \inf(D_0, k)).
\]
Proof of Lemma 7.2

We have
\[ \Theta_1(t, y) = \exp(-ik \int_0^t \tilde{\nu}_1(\tau, y) d\tau) \Theta_1(0, y) \]
\[ + \int_0^t \tilde{\Psi}(\tau, y) \exp(-ik \int_{\tau}^t \tilde{\nu}_1(\tau', y) d\tau') = \Theta_{1,1}(t, y) + \Theta_{1,2}(t, y). \]

But
\[ |\partial_y^n \Theta_{1,1}| \leq C \left( \frac{1 + t}{\delta} \right)^n \exp(-D_0 y) \leq \frac{C}{\delta^n} \exp(D_1 t) \exp(-D_0 y), \]
and
\[ |\partial_y^n \Theta_{1,2}| \leq C \int_0^t \exp(D_1 \tau) \left( \frac{1 + t - \tau}{\delta} \right)^n \exp(-D_0 y) \]
\[ \leq C e^{D_1 t} \left( \frac{1 + t'}{\delta} \right)^n \exp(-D_0 y) \exp(D_1 t). \]
Therefore
\[ |\partial_y^n \Theta_1| \leq \frac{C}{\delta^n} \exp(-D_0 y) \exp(D_1 t). \]

Using now
\[ \Psi(t, y) = \int_0^{+\infty} G(y', y) \Theta_1(t, y') dy' \]
we end the proof of Lemma 7.2.

7.1.2 Second step: study of \( \Psi_2 \)

Let us now turn to \( \Psi_2 = \Psi - \Psi_1 \) which satisfies
\[ (\partial_t + ik\tilde{\nu}_1)\Psi_2 - i k \Psi_2 \partial_{yy}^2 \tilde{\nu}_1 = ik \Psi \partial_{yy}^2 \tilde{\nu}_1, \]
(76)
\[ \Theta_2(0, y) = 0, \]
(77)
\[ \Psi_2(t, 0) = 0, \quad \Psi_2 \to 0 \quad \text{as} \quad y \to +\infty \]
(78)

where
\[ \Theta_2 = (\partial_{yy}^2 - k^2)\Psi_2. \]

The ideas of the majoration of \( \Psi_2 \) are the following: there are two areas where vorticity \( \Theta_2 \) can be created, namely \( I_1 \) and \( I_2 \) (where the flow \( \tilde{\nu}_1 \) is not linear).
Outside \( I_1 \cup I_2 \), the vorticity \( \Theta_2 \) vanishes and \( \Psi_2 \) has a completely known behavior. Moreover as \( I_1 \cup I_2 \) is small, we can use perturbation techniques to go back to the case “\( \delta = 0 \)” (where \( \partial_y \Psi_2 \) has jumps at \( y = 1 \) and \( y = 1 + \gamma \)) which can be explicitly solved analytically.

More precisely, as \( \partial_{yy}^2 \tilde{\nu}_1 \) vanishes outside \( I_1 \cup I_2 \),
\[ \Theta_2(t, y) = 0 \quad \text{if} \quad y \notin I_1 \cup I_2. \]

23
Let
\[ \sigma_1(t) = \int_{I_1} \Theta_2(t, y) dy \quad \text{and} \quad \sigma_2(t) = \int_{I_2} \Theta_2(t, y) dy. \]

Integrating (76) over \( I_1 \) and \( I_2 \) gives
\[ \begin{aligned}
\{ \tilde{\nu}_1(1)\sigma_1 - ik^{-1}\partial_t \sigma_1 - \sigma_1(\beta - \alpha)G_k(1, 1) - \sigma_2(\beta - \alpha)G_k(1 + \gamma, 1) = \phi_1 \\
\tilde{\nu}_1(1 + \gamma)\sigma_2 - ik^{-1}\partial_t \sigma_2 + \sigma_1\beta G_k(1, 1 + \gamma) + \sigma_2\beta G_k(1 + \gamma, 1 + \gamma) = \phi_2,
\end{aligned} \]

where \( \phi_1 \) and \( \phi_2 \) will be considered as perturbative terms,
\[ \phi_1 = \phi_3 + \phi_4, \quad \text{and} \quad \phi_2 = \phi_5 + \phi_6, \]

\( \phi_3 \) and \( \phi_6 \) being the errors made by approximating the effects of \( \Theta_2 \) restricted to \( I_1 \) and \( I_2 \),
\[ \begin{aligned}
\phi_3 &= -\int_{I_1} (\tilde{\nu}_1(y) - \tilde{\nu}_1(1))\Theta_2(t, y) dy \\
+ \int_{I_1} \partial_{yy}^2 \tilde{\nu}_1 (\Psi_2(t, y) - \sigma_1 G_k(1, 1) - \sigma_2 G_k(1 + \gamma, 1)) dz, \\
\phi_5 &= -\int_{I_2} (\tilde{\nu}_1(y) - \tilde{\nu}_1(1))\Theta_2(t, y) dy \\
+ \int_{I_2} \partial_{yy}^2 \tilde{\nu}_1 (\Psi_2(t, y) - \sigma_1 G_k(1, 1 + \gamma) - \sigma_2 G_k(1 + \gamma, 1 + \gamma)) dy,
\end{aligned} \]

and \( \phi_4 \) and \( \phi_6 \) being the terms induced by \( \Psi_1 \)
\[ \begin{aligned}
\phi_4 &= \int_{I_1} \Psi_1 \partial_{yy}^2 \tilde{\nu}_1, \\
\phi_6 &= \int_{I_2} \Psi_1 \partial_{yy}^2 \tilde{\nu}_1.
\end{aligned} \]

We have
\[ |\phi_4| + |\phi_6| \leq CC_0 \exp(D_1 t). \] (80)

### 7.1.3 Error in the coupling \( I_1/I_2 \)

**Lemma 7.3** There exists \( \bar{C} \) independent on \( \delta \) such that
\[ |\phi_3| + |\phi_5| \leq \bar{C} \delta \left( \int_{I_1} |\Theta_2| + \int_{I_2} |\Theta_2| \right) \] (81)

and
\[ \begin{aligned}
\int_{I_1} |\Theta_2(t, y)| + \int_{I_2} |\Theta_2(t, y)| &\leq \bar{C} \int_0^t \left( \exp(D_1 \tau) + |\sigma_1|(\tau) + |\sigma_2|(\tau) \right. \\
&\left. + \delta \int_{I_1} |\Theta_2(\tau, y') dy' + \delta \int_{I_2} |\Theta_2(\tau, y') dy' \right) d\tau.
\end{aligned} \] (82)
Proof

We immediately have

$$|\int_{I_1} (\tilde{v}_1(y) - \tilde{v}_1(1)) \Theta_2(t, y) dy| \leq \tilde{C} \delta \int_{I_1} |\Theta_2|,$$

$\tilde{C}$ being independent on $\delta$, and similarly on $I_2$. Now

$$\Psi_2(t, y) = \int_{I_1 \cup I_2} dy' \Theta_2(t, y') G_k(y', y). \quad (83)$$

If $y \in I_1$,

$$\int_{I_1} dy' \Theta_2(t, y') G_k(y', y) = \int_{I_1} dy' \Theta_2(t, y') G_k(1, 1)$$

$$+ \int_{I_1} dy' \Theta_2(t, y') (G_k(y', y) - G_k(1, 1)).$$

Using then

$$|G_k(y', y) - G_k(1, 1)| \leq \tilde{C} \delta$$

with $\tilde{C}$ independent on $\delta$ we get

$$\left| \int_{I_1} dy' \Theta_2(t, y') G_k(y', y) - G_k(1, 1) \right| \leq \tilde{C} \delta \int_{I_1} |\Theta_2|,$$

hence

$$\left| \Psi_2(t, y) - G_k(1, 1) \sigma_1 - G_k(1 + \gamma, 1) \sigma_2 \right| \leq \tilde{C} \delta \left( \int_{I_1} |\Theta_2| + \int_{I_2} |\Theta_2| \right), \quad (84)$$

which leads to (81).

Moreover, (84) gives

$$\sup_{y \in I_1 \cup I_2} |\Psi_2| \leq \tilde{C} (|\sigma_1| + |\sigma_2|) + \tilde{C} \delta \left( \int_{I_1} |\Theta_2| + \int_{I_2} |\Theta_2| \right),$$

where $\tilde{C}$ is independent on $\delta$, therefore using (76) we get on $I_1 \cup I_2$

$$|\partial_t + i k \tilde{v}_1 \Theta_2| \leq \tilde{C} |\partial^2 \tilde{v}_1| \sup_{I_1 \cup I_2} (|\Psi_1| + |\Psi_2|).$$

As $\Theta_2(0, y) = 0$,

$$|\Theta_2(t, y)| \leq \tilde{C} \int_0^t d\tau |\partial^2 \tilde{v}_1| \sup_{I_1 \cup I_2} (|\Psi_1| + |\Psi_2|).$$

The bound (82) is then straightforward. \qed
7.1.4 Conclusion: $L^\infty$ bounds

Equation (79) can be rewritten

$$
\partial_t \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = -ik\mathcal{M} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} + ik \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{85}
$$

with

$$
\sigma_1(0) = \sigma_2(0) = 0,
$$

where $\mathcal{M}$ is given by (69). Let $A$ and $B$ be two large constants, to be fixed later. Let $T_e$ be the largest time, such that for $0 \leq t \leq T_e$,

$$
\int_{I_1} |\Theta_1| + \int_{I_2} |\Theta_2| \leq A \exp(D_1 t) \tag{86}
$$

and

$$
|\sigma_1| + |\sigma_2| \leq B \exp(D_1 t). \tag{87}
$$

For $A$ and $B$ large enough, $T_e$ exists and is strictly positive. We want to prove that for a suitable choice of $A$ and $B$, $T_e = +\infty$, which would imply that (86) and (87) are global in time. Let us assume by contradiction that $T_e < +\infty$.

Using (80) and (81) we get

$$
|\phi_1| + |\phi_2| \leq CC_0 \exp(D_2 t) + \tilde{C} A\delta \exp(D_1 t),
$$

therefore on $[0, T_e]$, using (85) and $D_1 > |3c_+|k$,

$$
|\sigma_1| + |\sigma_2| \leq C + CA\delta(\exp(|3c_+|t) + \exp(D_1 t))
$$

which is strictly less than $B \exp(D_1 t)$ provided

$$
C < \frac{B}{4}, \quad CA\delta < \frac{B}{4}, \tag{88}
$$

that is provided

$$
\tilde{C} < A, \quad \tilde{C} A < B \tag{89}
$$

where $\tilde{C}$ is a constant independent on $\delta, A, B$ and $T_e$.

Moreover, using (82)

$$
\int_{I_1} |\Theta_2| + \int_{I_2} |\Theta_2| \leq C \exp(D_1 t) + CB \exp(D_1 t) + C\delta A \exp(D_1 t)
$$

which is bounded by $A \exp(D_1 t)$ if $\delta$ is small enough ($C\delta < 1/3$) and if

$$
\tilde{C} < A, \quad \tilde{C} B < A, \tag{90}
$$

for some constant $\tilde{C}$. For $\delta$ small enough it is possible to choose $A$ and $B$ such that (89) and (90) both hold.
For such a choice of $A$ and $B$, at $t = T_e$,
\[ \int_{I_1} |\Theta_1| + \int_{I_2} |\Theta_2| < A \exp(D_1 t) \]
and
\[ |\sigma_1| + |\sigma_2| < B \exp(D_1 t) \]
which is in contradiction with the definition of $T_e$. Hence $T_e = +\infty$ : (86) and (87) are therefore global in time. Using (83) this leads to
\[ |\Psi_2| \leq C \exp(D_1 t) \exp(-kz). \]

7.1.5 Higher order bounds
Using (75) and (76),
\[ |\partial_t \sigma_1| + |\partial_x \sigma_2| + |\partial_t \int_{I_1} |\Theta_2|| + |\partial_t \int_{I_2} |\Theta_2|| \leq C \exp(D_1 t). \]
and therefore
\[ |\partial_t \Psi_2| \leq C \exp(D_1 t) \exp(-kz) \]
With (76) we can then bound
\[ (\ddot{v}_l + \frac{1}{ik} \partial_t) \partial^2_{yy} \Psi_2 \]
and hence $\partial^2_{yy} \Psi_2$ and $\partial_t \partial^2_{yy} \Psi_2$. Repeating this manipulation, we get (74) for every $n \leq N/2$. \hfill \square

7.2 Instability near a smooth profile
In section 6 we have fulfilled the construction of an exponentially increasing eigenmode of linearized Euler equations near a profile $u^0$ which was only continuous and smooth by parts. In this section we want to make a similar construction for a smooth profile $u^0$, regularized version of the linear by parts one. Let us fix $\alpha < 1$ and $\beta = 1$. Let us fix $k$ such that
\[ |\Im c_+(k)|k > 0 \]  
(91)
and such that
\[ |\Im c_+(jk)|jk = 0 \]  
(92)
for $j = 2, 3, \ldots$. Let us take for instance $\alpha = 1/2$, $\beta = 1$ and $k = 0.6$ (see figure 1). Let now $\Xi$ be a smooth positive function, with support in $[-1, 1]$ and $\int \Xi = 1$. Let
\[ \ddot{v}_\mu = \ddot{v}_l \ast \frac{1}{\mu} \Xi(\frac{\cdot}{\mu}), \]
for $0 < \mu < \inf(1/2, \gamma/2)$ and let $\ddot{v}_0 = \ddot{v}_l$. Notice that $\ddot{v}_\mu = \ddot{v}_l$ for $\mu \in [0, 1 - 2\mu] \cup [1 + 2\mu, 1 + \gamma - 2\mu] \cup [1 + \gamma + 2\mu, +\infty]$ and satisfies assumptions of Proposition 7.1.
Proposition 7.4 Let $\nu > 0$ be small. For $\mu$ small enough there exists a solution $v_0(t, y)$ of the linearized Euler equations around $(\tilde{v}_\mu(t, y), 0)$ of the form

$$v_0 = \begin{pmatrix} \Psi_0'(y) \exp ik(x - c_0 t) \\ -ik \Psi_0(y) \exp ik(x - c_0 t) \end{pmatrix}$$

(93)

with

$$|c_0 k - c_+ (k)| \leq \nu.$$

Proof

We have to prove that for $\mu$ small enough there exists $c_0$ near $c_+(k)$ and a solution $v_0$ of (66). We will consider this problem as a shooting problem and apply an implicit function Theorem. Let us associate to $\phi_{\mu, c_0}$ the solution $\phi_{\mu, c_0}$ of

$$\frac{d^2}{dz^2} \phi_{\mu, c_0} = k^2 \phi_{\mu, c_0} + \phi_{\mu, c_0} \frac{\partial^2}{\partial y^2} \psi_{\mu} - c_0,$$

$$\phi_{\mu, c_0}(0) = 0,$$

$$\frac{d}{dz} \phi_{\mu, c_0}(0) = 1.$$

Notice that $\phi_{\mu, c_0} \to 0$ as $z \to +\infty$ if and only if

$$\frac{d}{dz} \phi_{\mu, c_0}(2) = -k \phi_{\mu, c_0}(2).$$

Therefore let

$$F(\mu, c_0) = \frac{d}{dz} \phi_{\mu, c_0}(2) + k \phi_{\mu, c_0}(2).$$

There exists a solution $v_0$ of (66) with parameter $\mu$ and $c_0$ if and only if $F(\mu, c_0) = 0$. By definition of $c_+$,

$$F(0, c_+) = 0.$$

The function $F$ is smooth in both variables and

$$\frac{dF}{dc}(0, c_+) \neq 0$$

since $c_+$ is a simple root of $F(0, c) = 0$, therefore the application of the implicit function Theorem ends the proof.

7.3 Construction of an approximate solution

Let us turn to the proof of Theorem 1.5. The first step is to construct an approximate solution. Let $\nu$ be small enough such that $|2k \Im c_0| > |k \Im c_+(k)|$, and let $u^0 = (\tilde{u}_\mu, 0)$. Notice that $u^0$ is a stationary solution of (1.2). We will build $v^0$ starting from $u^0$ and using several times (8,9) and the estimates on
this system given in section 7.1 in order to construct a very precise approximate solution to (1.2). The idea is to start from the unstable mode \( v_0 \) described in the last section and to add corrective terms in order to get an approximate solution up to times of order \( \log \eta^{-1} \).

**Proposition 7.5** For every \( N > 0 \) and every \( M > 0 \), there exist \( N \) functions \( v^{(1)}, \ldots, v^{(N)} \) (with \( v^{(1)} = \Re v_0 \) given in the preceding paragraph), of the form

\[
v^{(j)} = \sum_{\alpha=1}^{N_\alpha} v^{j, \alpha}
\]

for some integers \( N_\alpha \), with

\[
v^{j, \alpha} = \Re \left( \Psi^{j, \alpha}_k(t, y) \exp ik^{j, \alpha}x - i k^{j, \alpha} \Psi^{j, \alpha}_k(t, y) \exp ik^{j, \alpha}x \right)
\]

for some functions \( \Psi^{j, \alpha}_k \) satisfying for all \( 1 \leq j \leq N \), for all \( \alpha \) and for all \( 0 \leq n \leq N \),

\[
|\partial_y^n \Psi^{j, \alpha}_k(t, y)| \leq D^{(j, \alpha)}_n \exp(jk|3\alpha|t - k\alpha y),
\]

(with \( k_0 < k \)) such that

\[
v^\text{app} = u^n + \sum_{j=1}^{N} \eta M j v^{(j)}
\]

is an approximate solution of Euler equations in the following sense

\[
\partial_t v^\text{app} + (v^\text{app} \cdot \nabla)v^\text{app} + \nabla p = \eta M N R^{\eta},
\]

\[
\text{div } v^\text{app} = 0,
\]

\[
v_2^\text{app} = 0 \quad \text{at} \quad z = 0
\]

where

\[
\left\| R^{\eta} \right\|_{L^2} \leq C \exp(Nk|3\alpha|t)
\]

uniformly for \( 0 < \eta \leq 1 \).

**Proof**

To get the equations on the \( v^{(j)} \) we replace \( v^\text{app} \) by its expression in (98) and equals terms of order \( \eta M j \). Therefore we study

\[
\partial_t v^{(j)} + (u^n \cdot \nabla)v^{(j)} + (v^{(j)} \cdot \nabla)u^n + \nabla p = R^{\eta, j},
\]

\[
\text{div } v^{(j)} = 0,
\]

\[
v_2^{(j)} = 0 \quad \text{at} \quad z = 0
\]

29
where $\mathcal{R}^{n,j}$ is given by the $v^{(j)}$ for $j' < j$ and is a sum of terms of the form $(v^{(j_1)} \nabla)v^{(j_2)}$ for $j_1 < j$, $j_2 < j$ and $j = j_1 + j_2$. But the $L^2$ projection on divergence free vector fields of $(v^{(j_1)} \nabla)v^{(j_2)}$ is a sum of terms of the form

$$
\mathcal{R}_{j_1,j_2} = \Re \left( \Psi'(t,y) \exp i(k^{j_1,\alpha_1} + k^{j_2,\alpha_2})x \right.
- \left. i(k^{j_1,\alpha_1} + k^{j_2,\alpha_2}) \Psi(t,y) \exp i(k^{j_1,\alpha_1} + k^{j_2,\alpha_2})x \right)
$$

for some function $\Psi$, with $\Psi(t,0) = 0$,

$$
|\partial_y^n \Psi(t,y)| \leq CC_n \exp(jk|3c_0|t) \exp(-k_0y)
$$

and $|k^{j_1,\alpha_1} + k^{j_2,\alpha_2}| \leq jk$ if we assume (96) for $j_1 < j$ and $j_2 < j$. But $|jk|3c_0| \geq [2k|3c_0| - |k|3c_+(k)|$ (provided $\nu$ is small enough), hence using (92), $|jk|3c_0| > |k^{j_1,\alpha_1} + k^{j_2,\alpha_2}|\|3c_+(k^{j_1,\alpha_1} + k^{j_2,\alpha_2})|$. Proposition 7.1 then gives that the solution $v^{j_1,j_2}$ of (102.103.104) is of the form

$$
\begin{pmatrix}
\Psi_{j_1,j_2}'(t,y) \exp i(k^{j_1,\alpha_1} + k^{j_2,\alpha_2})x \\
-ik \Psi_{j_1,j_2}(t,y) \exp i(k^{j_1,\alpha_1} + k^{j_2,\alpha_2})x
\end{pmatrix}
$$

with

$$
|\partial_y^n \Psi_{j_1,j_2}(t,z)| \leq CC_n \exp(jk|3c_0|t) \exp(-k_0y).
$$

We will not detail more the proof. \hfill \Box

### 7.4 Proof of instability

Let $N$ such that $Nk|3c_0| > 3$. We have

$$
|\nabla v^{app}||_{L^\infty(x,y)} \leq 1 + \sum_{j=1}^{N} C_j \eta^{M_j} \exp(jk|3c_0|t).
$$

Let

$$
T_0^{\eta} = \frac{M}{k|3c_0|} \ln \frac{1}{\eta}.
$$

We have at time $t = T_0^{\eta} - \tau$

$$
|\nabla v^{app}||_{L^\infty(x,y)} \leq 1 + \sum_{j=1}^{N} C_j \exp(-jk|3c_0|\tau) \leq 2
$$

for $\tau \geq \tau_0$ with $\tau_0$ independent on $\eta$.

Let $v^0$ the solution of Euler equations with initial data $v^{app}(0)$. Let $w^0 = v^0 - v^{app}$ which satisfies

$$
\partial_t w^0 + (v^{app} \nabla)w^0 + (w^0, \nabla)v^{app} + (w^0, \nabla)w^0 + \nabla p = -\eta^{MN} \mathcal{R}^{n}.
$$

Multiplying by $w^0$ and integrating, using $\div v^{app} = \div w^0 = 0$ and $|\nabla v^{app}| \leq 2$ for $t \leq T_0^{\eta} - \tau_0$, we get

$$
\frac{1}{2} \partial_t \int |w^0|^2 \leq 3 \int |w^0|^2 + C\eta^{2MN} \exp(2Nk|3c_0|t).
$$
For $N$ large enough, we then get
\[\int |w^\eta|^2 \leq C \eta^{2MN} \exp(2Nk|3c_0|t).\]

In particular at $t = T_0^N - \tau$ with $\tau \geq \tau_0$,
\[\int |w^\eta|^2 \leq C \exp(-2\tau Nk|3c_0|).\] (105)

But
\[\|v^{app} - u^\eta\|_{L^2} \geq C_0 \eta^M \exp(k|3c_0|t) - \sum_{j=2}^{N} C_j \eta^{M_j} \exp(jk|3c_0|t)\]

for some non negative constant $C_0$, therefore at $t = T_0^N - \tau$,
\[\|v^{app} - u^\eta\|_{L^2} \geq C_0 \exp(-k|3c_0|\tau) - \sum_{j=2}^{N} C_j \exp(-jk|3c_0|\tau) \geq \frac{C_0}{2} \exp(-k|3c_0|\tau)\] (106)

for $\tau \geq \tau_1$ with $\tau_1 \geq \tau_0$ independent on $\eta$. Combining (105) and (106) gives at $t = T_0^N - \tau$
\[\|u^\eta - v^\eta\|_{L^2} \geq \frac{C_0}{2} \exp(-k|3c_0|\tau) - C \exp(-2\tau Nk|3c_0|) \geq \frac{C_0}{4} \exp(-k|3c_0|\tau)\]

for $\tau \geq \tau_2$ with $\tau_2 \geq \tau_1$ independent on $\eta$. We therefore get at $t = T_0^N - \tau_2$
\[\|u^\eta - v^\eta\|_{L^\infty} \geq \sigma\]

with $\sigma$ independent on $\eta$ which ends the proof of Theorem 1.5 after scaling back by (62). \qed

References


31


