Mathematical model of stagnation point flames

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1 Introduction

The aim of this work is to study a mathematical model that describes certain flames known as the stagnation point flames.

Let us start by describing the basic physical phenomenon. We consider a two dimensional jet of combustible mixture which impiges on a heated wall. Upon contact with the wall, the combustible mixture will generally ignite and a flame will form near the surface of the wall, as depicted in figure 1 below.

It is usually assumed that this type of flow is incompressible. Hence the velocity field is the potential field, given by

\[ v_x = -ax, \quad v_y = ay, \quad v_z = 0, \]

where \( a \) is a given positive constant called the ‘strain rate’.

Experiments have shown that for the flame to exist, the strain rate cannot exceed a critical extinction value \( a^* \). More precisely, as \( a \) increases, the stream lines diverge strongly and the reaction zone moves towards the wall until it touches it for \( a = a^* \). Then the flame disappears, and one speaks of extinction. On the contrary, when the strain rate is sufficiently small, there are two positions of the flame front. Only one of them is physically stable, the one for which there is a decrease in temperature as the extinction condition is approached. One may consult [5] or [11] and the references therein for a more complete description of the phenomenon. Formal computations of the conditions of extinction and asymptotic analysis have been made by A.Liñan.
The aim of this paper is to use these models in order to carry out a rigorous mathematical analysis of the stagnation point flames, similar to the one made by H. Berestycki, B. Nicolaenko and B. Scheuerer, [4] for the one-dimensional traveling waves solutions in the premixed flames.

Let us now describe the mathematical framework. The general mathematical description of a combustion phenomenon consists in a coupled system of PDE’s, based on the principles of conservation of energy, mass and momentum. Some simplifying assumptions are made to derive the equations. Firstly, one restricts to the case of simple chemistry, so that there is only one reaction \( X \rightarrow Y \), where \( X \) is the unburnt reactant, and \( Y \) the burnt product. Then, the pressure variation is assumed to be negligible, as are the effects of kinetic energy changes, of thermal diffusion and radiation. For a more precise description of combustion models, see [1], [4], [9] or [15].

One is then left with the equations describing diffusion, reaction and convection. In terms of the nondimensionalized temperature \( T(t, x, y) \) and the concentration of fuel \( X(t, x, y) \), the system is of the following type:

\[
\begin{align*}
\frac{\partial T}{\partial t} - a_x \frac{\partial T}{\partial x} + a_y \frac{\partial T}{\partial y} &= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + g(T)X, \\
\frac{\partial X}{\partial t} - a_x \frac{\partial X}{\partial x} + a_y \frac{\partial X}{\partial y} &= \frac{1}{Le} \left( \frac{\partial^2 X}{\partial x^2} + \frac{\partial^2 X}{\partial y^2} \right) - g(T)X,
\end{align*}
\]

(1.1)
for \( t, x \in \mathbb{R}_+ \) and \( y \in \mathbb{R} \). The constant \( a \) is given by the velocity of the flow. The nonlinear term \( g(T) \) is the kinetic rate of the reaction, derived from the Arrhenius law. In the following, we will assume that \( g \) has an ignition temperature \( \theta \) under which the reaction term is identically zero:

\[
\begin{align*}
g(T) &= 0 & \text{if } T \leq \theta, \\
g(T) &= > 0 & \text{if } T > \theta.
\end{align*}
\]

The constant \( Le \) is the Lewis number of the mixture defined as the ratio of the thermal diffusivity to the molecular diffusivity of the combustible. In addition, the natural conditions on the wall (at \( x = 0 \)), coming from the adiabaticity of the surface, and at infinity are

\[
\begin{align*}
\frac{\partial T}{\partial x}(x = 0) &= \frac{\partial X}{\partial x}(x = 0) = 0, \\
\lim_{x \to +\infty} T(x, y) &= 0, \quad \text{and} \quad \lim_{x \to +\infty} X(x, y) = 1.
\end{align*}
\]

Of particular interest are one dimensional steady solutions to problem (1.1)-(1.2), that is the solutions which do not depend on \( t \) and \( y \). So, in the remaining of this paper, we consider the following system, where \( T = u(x) \) is the temperature, \( X = v(x) \) the concentration of the reactant and \( g \) a locally Lipschitz continuous function satisfying \((G)\), and increasing on \((\theta, \infty)\):

\[
\begin{align*}
u'' + a xv' + vg(u) &= 0, \\
(1/Le) v'' + axv' - vg(u) &= 0, \\
u'(0) &= 0 \quad \text{and} \quad \lim_{x \to +\infty} u(x) = 0, \\
v'(0) &= 0 \quad \text{and} \quad \lim_{x \to +\infty} v(x) = 1.
\end{align*}
\]

In the special case of a unit Lewis number, the system can be reduced to a single differential equation on the temperature \( u \), the concentration of the fuel in nondimensionalized variables being \( 1 - u \):

\[
\begin{align*}
u'' + a xv'u + f(u) &= 0, \\
u'(0) &= 0 \quad \text{and} \quad \lim_{x \to +\infty} u(x) = 0.
\end{align*}
\]

The source term \( f(u) \) is now given by \( f(u) = (1 - u)g(u) \). In the following we will assume that \( f \) is a Lipschitz continuous function satisfying

\[
\begin{align*}f(u) &= 0 & \text{if } u \leq \theta \quad \text{or} \quad u \geq 1, \\
f(u) &= > 0 & \text{if } \theta < u < 1.
\end{align*}
\]

Since in this case, solutions are such that \( 0 < u < 1 \), there is no restriction in extending \( f \) by 0 for \( u \geq 1 \).
Let us mention that there is another physical motivation for equation \((P)\): it can also describe twin flames, that is when symmetrical streams of fuel are sent one against another, see figure 2. The symmetry of the flow allows to look for symmetrical solutions and thus to reduce to the problem in a half-space.

![Figure 2: Twin flames.](image)

The aim of this work is to prove existence and nonexistence results for solutions of \((P)\) and \((S)\) which are consistent with the physical experiments. Our main results are the following

**Theorem 1.1** There exists \(a^*\) such that
- for \(a > a^*\), there is no solution of \((P)\),
- for \(a \in (0, a^*)\), there are at least two solutions of \((P)\).

Here we call solution of \((P)\) any non trivial solution.

**Theorem 1.2** Under the additionnal assumption that \(f\) is concave on \((\theta, 1)\), then there are exactly two solutions of \((P)\) for \(a \in (0, a^*)\), the upper solution being stable and the lower solution unstable.
The results of Theorem 1.2 allow us to prove the existence of solutions of 
(S) when $Le$ is close to 1. The precise statement will be made in section 5.

The paper is organized as follows: firstly, in section 2, we use a shooting method to derive the main properties of solutions of $(P)$. In section 3, we apply a local method developed in [2]-[3] based on a topological degree argument, in order to prove the existence of two solutions for a certain range of $a$: first we consider the problem in a bounded domain, and then we take the infinite domain limit. The next section is devoted to the proof of Theorem 1.2. Then, in section 5, we derive existence of solutions of $(S)$ near $Le = 1$ thanks to the Implicit Function Theorem. Finally, in section 6 we study the singular limit when the activation energy becomes infinite and we illustrate our main Theorems with numerical results in section 7.

\section{Preliminary results}

We consider the following initial value problem

\begin{equation}
\begin{cases}
    u'' + axu' + f(u) = 0, \\
    u(0) = \alpha \quad \text{and} \quad u'(0) = 0.
\end{cases}
\end{equation}

(2.1)

Notice that $\alpha \in (0, 1)$. This solution will be either called $u_\alpha$ or $u_{\alpha, 0}$. Multiplying (2.1) respectively by 1, $u$, $u'$, $e^{ax^2}u'$ and $e^{ax^2}$, and integrating between 0 and $x$, yield the following equalities that will be useful later on:

\begin{equation}
u'(x) + axu(x) - a \int_0^x u(s) \, ds + \int_0^x f(u(s)) \, ds = 0,$
\end{equation}

(2.2)

\begin{equation}u(x)u'(x) - \int_0^x u'^2(s) \, ds + \frac{a}{2} xu^2(x) - \frac{a}{2} \int_0^x u^2(s) \, ds + \int_0^x f(u(s))u(s) \, ds = 0,
\end{equation}

(2.3)

\begin{equation}\frac{1}{2} u'^2(x) + a \int_0^x su'^2(s) \, ds = \int_0^x f(s) \, ds,
\end{equation}

(2.4)

\begin{equation}\frac{1}{2} u'^2(x) + F(u(x)))e^{ax^2} - F(\alpha) = 2a \int_0^x F(u(s))e^{as^2} \, ds,
\end{equation}

(2.5)

\begin{equation}e^{ax^2/2}u'(x) = - \int_0^x f(u(s))e^{as^2/2} \, ds.
\end{equation}

(2.6)

We have defined

\begin{equation}F(t) = \int_0^t f(s) \, ds \quad \text{and} \quad m = F(1).
\end{equation}

(2.7)
Proposition 2.1 If $u$ is a solution of (2.1), then $u$ is decreasing and has a limit at infinity.

Proof: Equation (2.1) can be rewritten

$$(e^{ax^2/2}u')' = -e^{ax^2/2}f(u).$$

(2.7)

Since $f(u) \geq 0$, it follows easily from the initial condition that $u'$ is nonpositive. Moreover, if $u$ crosses $\theta$, then there exists a positive constant $\lambda$ such that

$$u'(x) = -\lambda e^{-ax^2/2} \quad \text{when} \quad u \leq \theta.$$ 

So $u$ is decreasing and has a limit at infinity.

Proposition 2.2 For a large, there is no solution of $(P)$.

Proof: Let us assume that $u$ is a solution of $(P)$. We know that $u$ is decreasing and that $\lim_{x \to \infty} u(x) = 0$, therefore there exists $x_\theta$ such that $u(x_\theta) = \theta$. Then

$$u(x) = \lambda \int_x^\infty e^{-as^2/2} ds \quad \text{for} \quad x \geq x_\theta.$$ 

(2.8)

In particular,

$$\theta = \lambda \int_{x_\theta}^\infty e^{-as^2/2} ds = -u'(x_\theta) \int_0^\infty e^{-q_\theta s^2-q^2} ds$$

(2.9)

Using the energy conservation (2.4), we obtain that $|u'(x_\theta)|$ is bounded by $\sqrt{2m}$. So we get from (2.9)

$$\theta \leq \sqrt{2m} \int_0^\infty e^{-as^2/2} ds,$$

and this implies that $a$ is bounded by $m/\theta^2$.

Let us now consider the following subsets of $(\theta, 1)$:

$$A_-(a) = \{ \alpha \in (\theta, 1), \text{ such that } \exists x_0 > 0 \text{ with } u_{a,\alpha}(x_0) = 0 \},$$

$$A_+(a) = \{ \alpha \in (\theta, 1), \text{ such that } \lim_{x \to \infty} u_{a,\alpha}(x) > 0 \},$$

$$A_0(a) = \{ \alpha \in (\theta, 1), \text{ such that } \lim_{x \to \infty} u_{a,\alpha}(x) = 0 \}.$$

Lemma 2.3 $A_-(a)$ is an open subset of $(\theta, 1)$. 

The proof follows from the continuous dependence of \( u_\alpha \) with respect to \( \alpha \) as in [4].

**Lemma 2.4** For a small, \( A_-(a) \) is nonempty.

The proof follows easily from the fact that \( A_-(0) = (\theta, 1) \) and \( \{a, A_-(a) \neq \emptyset\} \) is open.

**Lemma 2.5** For all \( a \), there exists \( \alpha_a \) in \( (\theta, 1) \), such that \( (\theta, \alpha_a) \subset A_+(a) \).

**Proof:** Let \( u_\alpha \) be a solution of (2.1). We are going to prove that \( |u_\alpha(x) - \theta| \) tends to 0 when \( \alpha \) tends to \( \theta \), uniformly in \( x \).

If \( u_\alpha \) remains greater than \( \theta \), the proof is over since \( u_\alpha \) is decreasing. So let us assume that there exists \( x_\theta \) such that \( u_\alpha(x_\theta) = \theta \). As in the proof of Proposition 2.2, we obtain that

\[
|u_\alpha(x) - \theta| \leq |u'_\alpha(x_\theta)| \int_0^\infty e^{-as^2/2} \, ds.
\]

Then, we get from (2.4) that

\[
\lim_{\alpha \to \theta} u'_\alpha(x_\theta)^2 \leq 2 \lim_{\alpha \to \theta} \int_\theta^\alpha f(s) \, ds = 0.
\]

So \( \lim_{\alpha \to \theta} u_\alpha(x) = \theta \) uniformly in \( x \) and \( \alpha \in A_+(a) \) when \( \alpha \) is close to \( \theta \).

**Proposition 2.6** If \( 0 \leq p < a \), then

\[
A_-(a) \subset A_-(p) \quad \text{and} \quad A_+(p) \subset A_+(a), \quad (2.10)
\]

\[
A_0(a) \subset A_-(p) \quad \text{and} \quad A_0(p) \subset A_+(a). \quad (2.11)
\]

**Proof:** Let \( 0 \leq p < a \), let \( u \) be a solution of (2.1) and \( v \) a solution of the same equation with \( p \) instead of \( a \). We are going to prove that \( w = u - v \) is positive; this will imply that the first two inclusions (2.10) are true. Notice that \( w \) satisfies

\[
\begin{cases}
  \ w'' + axw' + c(x)w = (p - a)xv' \geq 0, \\
  w(0) = 0 \quad \text{and} \quad w'(0) = 0,
\end{cases}
\]

where \( c(x) \) is a locally Lipschitz continuous function. Thanks to the Maximum Principle in small domains and the Hopf Lemma, it can easily be seen that \( w > 0 \) in a neighbourhood of 0.
Now we are going to prove that
\[ u'(x) > v'(y) \quad \text{whenever} \quad u(x) = v(y), \] (2.12)
which means that \( w \) remains positive for all \( x \).

Since \( u \) and \( v \) are decreasing, there exist inverse functions \( x(t) \) and \( y(t) \) such that \( u(x(t)) = v(y(t)) = t \). Moreover, in a neighbourhood of \( \alpha \), \( x(t) > y(t) \). Let us perform the following change of variables in (2.1):
\[ t = u(x) \quad \text{and} \quad U(t) = u^2(x). \] (2.13)
Then \( U(t) \) is a solution of
\[ \frac{1}{2} U''(t) = ax(t)\sqrt{U(t)} - f(t). \] (2.14)

Similarly, we define \( V(t) \) corresponding to \( v \). Then, as long as \( u > v \), \( W(t) = V(t) - U(t) \) satisfies
\[ \frac{1}{2} W''(t) < \frac{ax(t)}{\sqrt{V(t)} + \sqrt{U(t)}} W(t) \quad \text{for} \quad t < \alpha, \] (2.15)
\[ W(\alpha) = 0. \] (2.16)

The theory of differential inequalities (see [14]) yields \( W(t) > 0 \) for all \( t \) smaller than \( \alpha \). Thus (2.12) is true, so that \( u \) and \( v \) cannot intersect. Then \( u > v \) on \( \mathbb{R}_+ \) and (2.10) is satisfied.

Now, let us prove (2.11) by contradiction and assume that \( u \) and \( v \) have the same limit \( l = 0 \) at infinity. We know that \( u \) and \( v \) do not intersect and \( u > v \) on \( \mathbb{R}_+ \). We define \( x(t) \) and \( y(t) \) as before and use the same change of variables as in (2.13)-(2.14)-(2.15), except that we define \( W(t) \) for \( t > l \), with the initial condition \( W(l) = 0 \). Then we get from (2.15) that \( W(t) < 0 \) for \( t > l \), which contradicts (2.12) and yields the result.

**Proposition 2.7** For all \( \alpha \) in \((\theta, 1)\), there exists a unique \( a_\alpha \) such that \( u_{a,a_\alpha} \) solves \((P)\). Moreover, \( a_\alpha \) depends continuously on \( \alpha \).

**Proof:** Let \( \alpha \in (\theta, 1) \) be fixed. Since \( A_-(0) = (\theta, 1) \), it follows from the continuous dependence of \( A_-(a) \) with respect to \( a \) that \( \alpha \in A_-(a) \) for \( a \) small enough. Then a proof similar to the one in Proposition 2.2 yields that for \( a \) large enough, \( \alpha \in A_+(a) \). So if we define \( a_\alpha = \sup\{a, \text{ s.t. } \alpha \in A_-(a)\} \), then \( a_\alpha \in (0, \infty) \). We are going to show that \( u_{a,a_\alpha} \) is a solution of \((P)\).
Let $a_n$ be a sequence that tends to $a_\alpha$ such that $\alpha \in A_-(a_n)$. We call $u_n = u_{a_n,a_\alpha}$, and $x_n$, $y_n$ the points where $u_n$ crosses respectively $\theta$ and $0$. Since $f(u) = 0$ when $u \leq \theta$, we get that
\[
u_n(x) = -u'_n(x)e^{\frac{a_n}{2}} \int x^y e^{-\frac{a_n}{2} s^2} ds \quad \text{for} \quad x \geq x_n. \tag{2.17}
\]
In particular, we set $x = x_n$ in (2.17) and use the change of variable $s = x_n + t$ to get
\[
\theta \leq |\nu_n(x_n)| \int_0^\infty e^{-a_n x_n t} dt. \tag{2.18}
\]
We deduce from (2.18) and the conservation of energy (2.4) which gives a bound on $|\nu_n(x_n)|$ that $\theta a_n x_n \leq \sqrt{2m}$. So $x_n$ is bounded and up to the extraction of a subsequence converges to some $\overline{x}$. Let us call $\overline{y} = \lim \sup_{n \to \infty} y_n$. We are going to prove that $\overline{y} = +\infty$. First of all, on $(0, \overline{y})$, the sequence $u_n$ is bounded in $W^{2,p}$ for all finite $p$: indeed, $u_n$ is between 0 and 1, the conservation of energy (2.4) provides a bound on $u_n'$, and the differential equation (2.1) on $u_n''$. So $u_n$ converges uniformly in $C^1(0, \overline{y})$ to some function $u$ which is a solution of (2.1) on $(0, \overline{y})$. Passing to the limit in (2.17) yields
\[
u(x) = -u'(\overline{x})e^{\frac{a_\alpha}{2}} \int x^\overline{y} e^{-a_\alpha s^2} ds \quad \text{for} \quad x \geq \overline{x}. \tag{2.19}
\]
In particular,
\[
\theta = -u' (\overline{x}) e^{\frac{a_\alpha}{2}} \int_\overline{x}^{\overline{y}} e^{-a_\alpha s^2} ds. \tag{2.20}
\]
If $\overline{y}$ is finite, (2.19) and (2.20) imply that $u(\overline{y}) = 0$ and $\alpha \in A_-(a)$. So we infer from the definition of $a_\alpha$ and the fact that $\{a, A_-(a) \neq \emptyset\}$ is open that $\overline{y} = +\infty$. But then (2.20) implies that $\lim_{x \to \infty} u(x) = 0$, so $u$ is a solution of (P) with $u(0) = \alpha$ and $a = a_\alpha$.

It follows from (2.10) and (2.11) in the previous proposition that for $a < a_\alpha$, $\alpha \in A_-(a)$ and for $a > a_\alpha$, $\alpha \in A_+(a)$. So we have proved the existence and uniqueness of $\alpha$ such that $u_{a_\alpha,a_\alpha}$ solves (P).

Now let us prove the continuous dependence of $a_\alpha$ with respect to $\alpha$. Assume by contradiction, that there exists a sequence $\alpha_n$ that tends to $\alpha$ such that $a_n = a_{\alpha_n}$ does not tend to $a_\alpha$. We know thanks to Proposition 2.1 that $a_n$ is bounded, so up to the extraction of a subsequence, converges to some $\overline{a} \neq a_\alpha$. Since $A_-(0) = (\theta, 1)$, and $A_-(a)$ is continuous with respect to $a$, it follows that $\overline{a} > 0$. Now if we call $x_n$ the point where $u_n = u_{a_n,a_\alpha}$ crosses $\theta$, we can show, as in the first part of the proof, that $x_n$ is bounded by $\sqrt{2m}/\overline{a}$. The same kind of proof also gives that $u_n$ converges on every compact set to $u$ which is a solution of (2.1) with $a = \overline{a}$. Passing to the limit into (2.17) where $y_n = \infty$, we obtain that $\lim_{x \to \infty} u(x) = 0$, so $u$ is a solution of (P). But there is a unique solution of (P) with $u(0) = \alpha$, so that $\overline{a} = a_\alpha$. 

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**Proposition 2.8** Two solutions of \((P)\) do not intersect.

**Proof:** Let us assume that there are two solutions \(u\) and \(v\) of \((P)\). Then they both satisfy an equality of type (2.8). More precisely, there exists \(x_\theta, y_\theta, \lambda\) and \(\mu\) such that

\[
\begin{align*}
u(x) &= -\lambda \int_x^\infty e^{-as^2/2} ds \quad \text{for} \quad x \geq x_\theta, \\
v(y) &= -\mu \int_y^\infty e^{-as^2/2} ds \quad \text{for} \quad y \geq y_\theta.
\end{align*}
\] (2.21) (2.22)

First of all, notice that \(u\) and \(v\) cannot intersect between 0 and \(\theta\), since equalities (2.21) and (2.22) would imply they coincide everywhere. So let us call \(a\) the first point of intersection when coming from infinity. We can assume that \(u > v\) on \((a, +\infty)\). Then

\[
u(a) = v(a) = \tau \in (\theta, 1) \quad \text{and} \quad v'(a) < u'(a).
\] (2.23)

Since both functions are decreasing, there exist inverse functions \(x(t)\) and \(y(t)\) such that

\[
u(x(t)) = v(y(t)) = t.
\]

We can assume without loss of generality that \(u > v\) on \((\tau, +\infty)\), so that \(x(t) > y(t)\).

Then we use the same change of function as (2.13)-(2.14) and get the differential inequality (2.15) on \(W(t) = V(t) - U(t)\) for \(t\) in \((0, \tau)\). Here the initial condition is \(W(0) = 0\). The same technique yields

\[
|v'(y(t))| \leq |u'(x(t))| \quad \text{when} \quad u(x(t)) = v(y(t)).
\] (2.24)

But for \(t = a\), this contradicts (2.23) so that the two solutions do not intersect. Notice that (2.24) is true for two solutions \(u\) and \(v\) that have the same limit at infinity.

**Remark:** For a solution \(u\) of \((P)\), we can infer from (2.21) that

\[
u(x) \sim \frac{\lambda e^{-ax^2/2}}{ax} \quad \text{when} \quad x \rightarrow +\infty.
\]
3 Existence of two solutions

Let us call $a^* = \sup \{ a, \text{ s. t. } A_-(a) \neq \emptyset \}$. Notice that $a^*$ is finite since Proposition 2.2 gives that $A_+(a) = (\theta, 1)$ for $a$ large enough. Then we derive from Lemma 2.4 that $a^* > 0$ and from Proposition 2.6 that $A_-(a) \neq \emptyset$ for all $a < a^*$. Moreover, for $a > a^*$, $A_-(a)$ is empty, hence, with Proposition 2.6, it means that $A_0(a)$ is empty too. The aim of this section is to prove the existence of two solutions of $(P)$ when $a < a^*$.

We are going to use a local method developed in [3] for the study of semilinear elliptic equations in $\mathbb{R}^n$. The idea consists in studying first the problem on a bounded interval $(0, R)$

$$
\begin{align*}
\begin{cases}
u'' + ax\nu' + f(\nu) = 0 & \text{in } (0, R), \\
\nu'(0) = 0 \quad \text{and} \quad \nu(R) = 0,
\end{cases}
\end{align*}
$$

in order to get the existence of two solutions by a topological degree argument. Then we let $R$ tends to $+\infty$ and check that the limits of the two solutions are different.

**Theorem 3.1** For $a < a^*$, there exists $R_0 > 0$ such that

(i) For $R < R_0$, there exists no non-trivial solution of $(P_R)$.

(ii) For $R \geq R_0$, there exists a maximum solution $\overline{u}_R$ of $(P_R)$. Moreover, if we extend $\overline{u}_R$ by $0$ in $(R, +\infty)$, then

$$
\begin{align*}
\overline{u}_R(x) \leq \overline{u}_{R'}(x) & \quad \forall x \in \mathbb{R}_+, \quad \forall R' \geq R \geq R_0, \\
\overline{u}_R(x) \searrow \overline{u}(x) & \quad \text{when } R \nearrow +\infty,
\end{align*}
$$

where $\overline{u}$ is a solution of $(P)$.

(iii) For $R > R_0$, there exists another solution $u_R$ such that

$$
\exists \xi_R \in (0, R_0), \quad u_R(\xi_R) < \overline{u}_{R_0}(\xi_R).
$$

For a subsequence $R_n$ that tends to $+\infty$, $u_{R_n}$ converges uniformly on every compact set to a solution $u$ of $(P)$, which is different from $\overline{u}$.

The rest of this section is devoted to the proof of Theorem 3.1. Since we have seen that for $a > a^*$, there is no solution of $(P)$, it is clear that Theorem 3.1 implies Theorem 1.1.

3.1 Study of problem $(P_R)$

Let $a < a^*$ be fixed. Then $A_-(a) \neq \emptyset$ so there exists at least one $R_*$ such that $(P_{R_*})$ has a non-trivial solution. We are going to use the same technique as in [2]-[3].
1st step: If \( R \) is small enough, there is no solution of \((P_R)\).

There exists \( K \) such that \( f(t) \leq Kt \) for all \( t \) in \((0, 1)\). Let us assume that \( u \not= 0 \) is a solution of \((P_R)\). Since \( u(R) = 0 \), we obtain thanks to (2.3)

\[
\int_0^R u'^2 \, ds \leq (K - a/2) \int_0^R u^2. 
\]

But if \( R \) is small enough, the smallest eigenvalue of \(-u''\) in \((0, R)\) is bigger than \( K - a/2 \), so that \( u \equiv 0 \).

2nd step: There exists \( R_0 > 0 \) such that for all \( R \geq R_0 \), there exists a maximum solution \( \overline{u}_R \), which is increasing with \( R \).

Let \( I = \{ R \) such that \((P_R)\) has a nontrivial solution\}. We already know that \( I \not= \emptyset \) because \( \lambda_-(a) \not= \emptyset \). If \( R \in I \), let us show that \( R' \in I \) as soon as \( R' \geq R \): we extend \( u_R \) by 0 outside \((0, R)\). Then

\[
u''_R + axu'_R + f(u_R) \geq 0 \quad \text{in} \quad D(0, R').
\]

This is a consequence of Lemma 1.1 in [2]. It means that \( u_R \) is a weak subsolution of \((P_{R'})\). Moreover, \( u \equiv 1 \) is a weak supersolution and \( u_R < 1 \). It implies that there exists a maximum solution \( \overline{u}_R \) of \((P_{R'})\).

Let \( R_0 = \inf \{ R, \ \text{s.t.} \ R \in I \} \). The 1st step gives that \( R_0 > 0 \). For \( R > R_0 \), there exists a maximum solution \( \overline{u}_R \) which is increasing with \( R \).

Now, let us show that \( R_0 \in I \). We call \( \overline{u}_{R_0} = \lim_{R \to R_0} \overline{u}_R \). The limit is well defined since, as \( R \) decreases to \( R_0 \), \( \overline{u}_R \) is a decreasing sequence of positive functions. Moreover, \( \overline{u}_R(0) \in (\theta, 1) \), and by (2.4), \( |\overline{u}_R'| \leq \sqrt{2m} \). So when passing to the limit, we obtain that \( \overline{u}_{R_0} \) is a nontrivial solution of \((P_{R_0})\).

Thus \( I \) is the interval \([R_0, +\infty)\).

3rd step: There exists a second solution \( u_R \) satisfying (3.3) when \( R > R_0 \).

Let \( R > R_0 \). We know that \( \overline{u}_{R_0} > 0 \) in \((0, R_0)\) and that \( \overline{u}_{R_0} \) extended by 0 is a nonnegative, nontrivial, strict subsolution of \((P_R)\) in \((0, R)\). On the other hand, \( u \equiv 1 \) is a strict supersolution. Then, since \( f \equiv 0 \) on \((0, \theta)\), a general result of Rabinowitz [10] based on a topological degree argument, (or more precisely an extension proved in [2] and used in [3]), gives the existence of another positive solution \( u_R \) such that there exists \( \xi_R \), with \( u_R(\xi_R) < \overline{u}_{R_0}(\xi_R) \).

Notice that \( \xi_R < R_0 \) since \( \overline{u}_{R_0} \equiv 0 \) on \((R_0, +\infty)\).

Let us now recall this topological degree argument. We introduce the operators \( F_t, 0 \leq t \leq 1 \), as follows: for \( v \in C^1_{+}(0, R) = \{ v \in C^1(0, R), v'(0) = 0 \) and \( v(R) = 0 \} \), we define \( w = F_t v \) as the solution of

\[
\begin{cases}
  w'' + (t\bar{a} + (1-t)a)xw' - \mu w = -f(v) - \mu v & \text{in} \quad (0, R), \\
  w'(0) = 0 \quad \text{and} \quad w(R) = 0,
\end{cases}
\]

(3.4)
where \( \mu = \sup_{[\theta, 1]} |f'(s)| \) and \( \tilde{a} > a^* \) is fixed arbitrarily. It is well-known that \( F_t \) is a compact operator from \( (t, v) \in [0, 1] \times C_0^1(0, R) \) into \( C_0^1(0, R) \). We call \( L \) the constant such that \( w = F_tv \) satisfies \( \|w\|_{C^1} < L \) as soon as \( \|v\|_{L^\infty} < 1 \). Moreover, since \( f(s) + \mu s \) is nondecreasing in \( (0, 1) \), \( F_t \) is order preserving for \( v \) in \([0, 1], \) i.e. if \( 0 \leq v \leq \tilde{v} \leq 1 \), then \( F_tv \leq F_t\tilde{v} \). The problem is now to find a fixed point of \( F_0 \), which crosses \( \overline{\nu}_{R_0} \).

To that effect, let us define the following open sets in \( C_0^1(0, R) \):

\[
\mathcal{B} = \{ v \in C_0^1(0, R), \|v\|_{C^1} < M, \ v > 0 \text{ in } (0, R), \ v(0) \in (\theta, 1), \ v'(R) < 0 \},
\]

\[
\mathcal{O} = \{ v \in \mathcal{B}, \ v > \overline{\nu}_{R_0} \text{ in } (0, R) \},
\]

where \( M = 1 + \max(\sqrt{2m}, L) \). This choice of \( M \) will become clear later in the course of the proof.

We are going to prove that \( d(I - F_0, \mathcal{B}, 0) = 0 \) and \( d(I - F_0, \mathcal{O}, 0) = +1 \), where \( d \) denotes the Leray-Schauder degree for compact perturbations of the identity. Then

\[
d(I - F_0, \mathcal{B} \setminus \overline{\mathcal{O}}, 0) = d(I - F_0, \mathcal{B}, 0) - d(I - F_0, \mathcal{O}, 0) = -1,
\]

and we can conclude to the existence of a solution \( u_R \) of \( (P_R) \) which is not in \( \overline{\mathcal{O}} \), and thus will have the requested properties.

(i): proof of \( d(I - F_t, \mathcal{B}, 0) = 0 \).
Assume that there exists \( v \) in \( \overline{\mathcal{B}} \) and \( t \) in \([0, 1]\) such that \( v = F_tv \). Then,

\[
v'' + (t\tilde{a} + (1 - t)a)xv' + f(v) = 0,
\]

so that, \( v \) is decreasing, \( 0 < v < 1 \) in \((0, R)\), and \( |v'| \leq \sqrt{2m} \) by (2.4). The choice of \( M \) gives that \( \|v\|_{C^1} < M \). Moreover \( v(0) > \theta \) and \( v'(R) < 0 \) by Cauchy-Lipschitz, since \( v \) is not constant. So in fact \( v \in \mathcal{B} \), and \( I - F_t \) is never 0 on \( \partial\mathcal{B} \). Then \( d(I - F_t, \mathcal{B}, 0) \) is well defined and homotopy invariant. But, since \( F_t \) cannot have any fixed point as \( \tilde{a} > a^* \), we get

\[
d(I - F_0, \mathcal{B}, 0) = d(I - F_1, \mathcal{B}, 0) = 0.
\]

(ii): proof of \( d(I - F_0, \mathcal{O}, 0) = +1 \).
We already know that \( \overline{\nu}_R \) is in \( \mathcal{O} \). Let us define

\[
G_t = tF_0 + (1 - t)\overline{\nu}_R, \quad t \in [0, 1].
\]

If we now suppose that \( v \) in \( \overline{\mathcal{O}} \) is such that \( v = G_tv \), then \( \|v\|_{C^1} < M \): indeed, \( \|F_0v\|_{C^1} < L \) and \( \|\overline{\nu}_R\|_{C^1} < M \). Recall that \( F_0 \) is order preserving in \( \mathcal{O} \) by
the definition of $\mu$. By convexity, we have in fact $v \in \mathcal{O}$, and $d(I - G_t, \mathcal{O}, 0)$ is well defined and homotopy invariant. As $\overline{\nu}_R$ is in $\mathcal{O}$, we get
\[ d(I - F_0, \mathcal{O}, 0) = d(I - \overline{\nu}_R, \mathcal{O}, 0) = +1. \]

The argument yields a solution $u_R$ in $B \setminus \overline{\mathcal{O}}$, so it satisfies the required property (3.3).

### 3.2 Passage to the limit

In this part, we call $u_R$ a solution of $(P_R)$ extended by 0 on $(R, +\infty)$. We know that for each $R$, $u_R$ is decreasing and $0 \leq u_R \leq 1$. Moreover (2.4) gives that $|u'_R| \leq \sqrt{2m}$. So, up to a subsequence, $u_R$ converges in $C^2_{\text{loc}}(\mathbb{R}_+)$ to $u$, which is a solution of
\[
\begin{cases}
  u'' + axu' + f(u) = 0 & \text{in } (0, +\infty), \\
  u'(0) = 0.
\end{cases}
\]

Moreover, we know that for $\overline{\nu}_R$, the convergence is monotone since $\overline{\nu}_R$ is an increasing sequence.

We are going to prove that $\lim_{x \to +\infty} u(x) = 0$. For each $R$, since $u_R(0) > \theta$ and $u_R(R) = 0$, there exists $x_R$ such that $u_R(x_R) = \theta$.

**1st step: $x_R$ is bounded.**

For $x \geq x_R$,
\[
u'_R(x) = u'_R(x_R) e^{ax_R^2/2} e^{-ax^2/2}.
\]

Integrating between $x_R$ and $R$ gives
\[
\theta = -u'_R(x_R) \int_{x_R}^{R} e^{ax_R^2/2} e^{-as^2/2} ds.
\]

We write $s = x_R + u$ and get
\[
\theta \leq \sup |u'_R| \int_{0}^{+\infty} e^{-ax_R} du.
\]

Hence $a\theta x_R \leq \sqrt{2m}$.

**2nd step: The limit $u$ is a solution of $(P)$.**

Up to the extraction of a subsequence, $x_R$ tends to $x_\infty$. Let $x > x_\infty$ be fixed. We let $R$ tend to infinity in (3.5) and (3.6), and obtain
\[
u'(x) = u'(x_\infty) e^{ax_\infty^2/2} e^{-ax^2/2},
\]
Comparing both equations implies \( \lim_{x \to +\infty} u(x) = 0 \). We also derive that \( u(0) \in (\theta, 1) \) so that \( u \) is non trivial.

Now let us call \( u \) and \( \bar{\theta} \) the limits of \( u_R \) and \( \bar{u}_R \). We derive from the previous steps that \( u \) and \( \bar{\theta} \) are solutions of \((P)\). We only need to know that \( u \) and \( \bar{\theta} \) are different. This comes from \((3.3)\) since \( \bar{\theta}_R \) is increasing with \( R \).

4 Exact number of solutions

The aim of this section is to prove Theorem 1.2.

Let us fix \( a < a^* \) and let \( u_\alpha \) be a solution of \((2.1)\). In the following, we will call \( x_\theta(\alpha) \) the point where \( u_\alpha \) crosses \( \theta \), when it exists, and \( l(\alpha) = \lim_{x \to +\infty} u_\alpha(x) \), which exists by Proposition 2.1. The behaviour of \( l(\alpha) \) is obtained through a careful study of the function \( z_\alpha(x) = \partial u_\alpha(x)/\partial \alpha \), and the use of Sturm’s Comparison Theorem (see Appendix Lemma A1). Note that \( z_\alpha \) satisfies

\[
\begin{align*}
\frac{d^2}{dx^2} z_\alpha + axz_\alpha' + f'(u)z_\alpha &= 0, \\
z_\alpha(0) &= 1 \quad \text{and} \quad z_\alpha'(0) = 0.
\end{align*}
\]

4.1 Study of \( x_\theta(\alpha) \)

We are able to give a precise description of the curve \( x_\theta(\alpha) \), which will enable us to understand the behaviour of \( l(\alpha) \) in the next section.

**Proposition 4.1** The curve \( x_\theta(\alpha) \) is defined for \( \alpha \in (\alpha_1, \alpha_2) \) \((\theta \leq \alpha_1 < \alpha_2 \leq 1)\) and is at least \( C^2 \). Moreover, there exists \( \tilde{\alpha} \) such that \( x_\theta(\alpha) \) is decreasing on \((\alpha_1, \tilde{\alpha})\) and increasing on \((\tilde{\alpha}, \alpha_2)\).

**Remark:** We believe that \( \alpha_1 = \theta \) and \( \alpha_2 = 1 \), so in particular that there cannot exist any solution which stays between \( \theta \) and 1. But since this is not useful for the rest of the analysis, we will not be more precise about it. In the following, for \( \alpha \in (\theta, \alpha_1) \cup (\alpha_2, 1) \), we will set \( x_\theta = \infty \).

The proof of the Proposition requires several Lemmas.

**Lemma 4.2** Consider the problem \((P_{R,0})\)

\[
\begin{align*}
\begin{cases}
  u'' + axu' + f(u) &= 0 \quad \text{in} \quad (0, R), \\
u'(0) &= 0 \quad \text{and} \quad u(R) = \theta.
\end{cases}
\end{align*}
\]

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Then there is a minimal radius $R_{\text{min}}$ such that for $R < R_{\text{min}}$, there is no solution and for $R > R_{\text{min}}$, there is a maximal solution and at most two solutions.

**Proof:** The existence of $R_{\text{min}}$ follows easily from the fact that $f(t) \leq Kt$ as in the proof of Theorem 3.1, 1st step. Then notice that a solution of $(P_{R,\theta})$ extended by $\theta$ is a subsolution of $(P_{R',\theta})$ with $R' > R$, and 1 is a supersolution. So as in the proof of Theorem 3.1 2nd step, we infer that there exists a maximal solution of $(P_{R,\theta})$ for $R > R_{\text{min}}$.

Now let us prove that there are at most two solutions. Assume that for some $R$ there are 3 solutions $u_1, u_2, u_3$. A proof similar to the one of Proposition 2.8 yields that these solutions do not intersect so that $w_1 = u_1 - u_2 > 0$ and $w_2 = u_2 - u_3 > 0$. Now $w_i$ satisfies an equation of the type

$$
\begin{cases}
(e^{ax^2/2}w_i)' + e^{ax^2/2}c_i w_i = 0 \quad \text{in } (0, R), \\
w_i(0) = 0 \quad \text{and} \quad w_i(R) = 0,
\end{cases}
$$

where $c_i(x) = (f(u_i) - f(u_{i+1}))/ (u_i - u_{i+1})$. The concavity of $f$ implies that $c_1(x) < c_2(x)$. So if we multiply the equation for $w_1$ by $w_2$, integrate between 0 and $R$, and substract the equation for $w_2$ multiplied by $w_1$, we get

$$
\int_0^R e^{ax^2/2}(c_1(x) - c_2(x))w_1(x)w_2(x) \, dx = 0,
$$

which is a contradiction to $c_1(x) < c_2(x)$.

**Lemma 4.3** There exists $\tilde{\alpha}$ in $(\theta, 1)$ such that

(i) if $\alpha \in (\tilde{\alpha}, 1)$, $z_\alpha(x) > 0$ in $(0, x_\theta)$,

(ii) if $\alpha \in (\theta, \tilde{\alpha})$, $z_\alpha(x) > 0$ in $(0, x_0(\alpha))$ and $z_\alpha(x) < 0$ in $(x_0(\alpha), x_\theta(\alpha))$.

Moreover $x_0(\alpha)$ is increasing with $\alpha$.

**Proof:** Let $\mathcal{A}_+ = \{\alpha, \text{ s.t. } z_\alpha(x) > 0, \forall x \in (0, x_\theta(\alpha))\}$ and $\tilde{\alpha} = \inf_{\alpha \in (\theta, 1)} \mathcal{A}_+$. Notice that $\mathcal{A}_+$ is nonempty since $\alpha \in \mathcal{A}_+$ as soon as $u_\alpha$ is the maximal solution of $(P_{R,\theta})$ for some $R$. Now we are going to prove that $(\tilde{\alpha}, 1) \subset \mathcal{A}_+$.

This follows from

**Claim 1:** $z_\alpha(x)$ is an increasing function of $\alpha$, when $\alpha \in \mathcal{A}_+$.

Indeed, let $w_\alpha = \partial z_\alpha(x)/\partial \alpha$. Then, since $f$ is concave, $w_\alpha$ satisfies

$$
\begin{cases}
w_\alpha'' + axw_\alpha' + f'(u)w_\alpha > 0, \quad \forall x \in (0, x_\theta), \\
w_\alpha(0) = 0 \quad \text{and} \quad w_\alpha'(0) = 0.
\end{cases}
$$

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Moreover, \( w_\alpha \) is positive for \( x \) close to 0. So we can apply Lemma A1 on \((0, x_\theta)\) till \( w_\alpha \) is positive, and get that

\[
\frac{z'_\alpha(x)}{z_\alpha(x)} \leq \frac{w'_\alpha(x)}{w_\alpha(x)}.
\]  

(4.4)

It means that \( w_\alpha(x) \) does not cross zero before \( z_\alpha(x) \), so \( w_\alpha(x) > 0 \) on \((0, x_\theta)\) and the claim is proved.

It is straightforward that \( \alpha > \theta \), otherwise it would mean that \( \forall \alpha \in (\theta, 1) \) \( u_\alpha \) does not cross \( \theta \), which is not the case. So \( \forall \alpha \in (\theta, \alpha) \), \( z_\alpha \) crosses zero. Since \( z_\alpha(0) = 1 \), we can define the first zero of \( z_\alpha \), called \( x_0(\alpha) \) and we have \( z'_\alpha(x_0(\alpha)) < 0 \). So by the Implicit Function Theorem and the proof of Claim 1, we get that \( w_\alpha(x_0(\alpha)) > 0 \) and \( x_0(\alpha) > 0 \).

Claim 2: \( z_\alpha(x) < 0 \) for \( x > x_0(\alpha) \).

This follows again from Lemma A1: we compare \( z_\alpha(x) \) with \( v_\alpha(x) = -v'_\alpha(x) \), which is a solution of

\[
\begin{cases}
  v''_\alpha + axv'_\alpha + f'(u)v_\alpha \leq 0, & v_\alpha \geq 0, \quad \text{in} \quad (x_0, x_\theta), \\
  v_\alpha(0) = 0 \quad \text{and} \quad v'_\alpha(0) = f(\alpha).
\end{cases}
\]

(4.5)

Since \( v'_\alpha(x_0)/v_\alpha(x_0) < z'_\alpha(x_0)/z_\alpha(x_0) \), it implies that

\[
\frac{v'_\alpha(x)}{v_\alpha(x)} \leq \frac{z'_\alpha(x)}{z_\alpha(x)} \quad \forall x > x_0.
\]

(4.6)

In particular, \( v_\alpha(x) > 0 \) for \( x > 0 \), so \( z_\alpha(x) \) does not cross zero again and the Lemma is proved.

Proof of Proposition 4.1: The continuity of \( x_\theta(\alpha) \) follows from the continuous dependence of \( u_\alpha \) with respect to \( \alpha \). Then the Implicit Function Theorem yields that \( x_\theta(\alpha) \) is at least \( C^2 \) and we have in particular

\[
z_\alpha(x_\theta) + u'_\alpha(x_\theta)x_\theta'(\alpha) = 0.
\]

(4.7)

So we conclude with Lemma 4.3 since \( u'(x_\theta) < 0 \).

4.2 Study of \( l(\alpha) \)

The behaviour of \( x_\theta(\alpha) \) will allow us to deduce properties of \( l(\alpha) \). Recall that

\[
u'_\alpha(x) = u'_\alpha(x_\theta)e^{ax^2/2}e^{-ax^2/2} \quad \text{for} \quad x > x_\theta,
\]

(4.8)

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since \(f(u_\alpha(x)) = 0\) for \(x > x_\theta\). In particular

\[
I(\alpha) = \theta + u'_{\alpha}(x_\theta)e^{ax^2_\theta/2} \int_{x_\theta}^{\infty} e^{-as^2/2} \, ds.
\]  

(4.9)

So we deduce from Proposition 4.1 that \(I(\alpha)\) is a \(C^2\) function of \(\alpha\). Moreover, if we differentiate (4.8) with respect to \(\alpha\) and compare it with (4.9), we get easily that

\[
l'(\alpha) = \lim_{x \to \infty} z_\alpha(x) \quad \text{and} \quad l''(\alpha) = \lim_{x \to \infty} \frac{\partial z_\alpha}{\partial \alpha}(x).
\]  

(4.10)

So the main point in this section is to study the sign of \(z_\alpha(x)\). We have the analogous of Lemma 4.3 in \(\mathbb{R}_+\):

**Lemma 4.4** There exists \(\alpha_0\) in \((\theta, 1]\) such that

(i) for \(\alpha \in (\alpha_0, 1)\), \(z_\alpha(x) > 0\) and \(\partial z_\alpha(x)/\partial \alpha > 0\) in \(\mathbb{R}_+\),

(ii) for \(\alpha \in (\theta, \alpha_0)\), \(z_\alpha(x) > 0\) in \((0, x_0(\alpha))\), \(z_\alpha(x) < 0\) in \((x_0(\alpha), \infty)\) and \(\lim_{x \to \infty} z_\alpha(x) < 0\). Moreover, \(x'_0(\alpha) > 0\).

**Proof:** Let us first notice that

\[
z''_\alpha + axz'_\alpha = 0 \quad \text{for} \quad x \geq x_\theta.
\]  

(4.11)

so that \(z'_\alpha(x) = \lambda e^{-ax^2/2}\) for some \(\lambda \in \mathbb{R}\).

(i) Let us call \(\mathcal{A}_+ = \{\alpha, \text{ s.t. } z_\alpha(x) > 0, \forall x > 0\} \) and \(\alpha_0 = \inf_{\alpha \in (\theta, 1]} \mathcal{A}_+\). We know from Lemma 4.3 that \(\tilde{\alpha} \leq \alpha_0 \leq 1\). Let us fix \(\alpha \in \mathcal{A}_+\). For \(x \geq x_\theta\), \(w_\alpha(x) = \partial z_\alpha(x)/\partial \alpha\) satisfies the same equation as \(z_\alpha(x)\). So a proof similar to that of Lemma 4.3 using Sturm’s Comparison Theorem yields that

\[
\frac{z'_\alpha(x)}{z_\alpha(x)} < \frac{u'_\alpha(x)}{u_\alpha(x)} \quad \forall x > 0.
\]

In particular, \(w_\alpha(x)\) remains positive in \(\mathbb{R}_+\) and \(\mathcal{A}_+ = (\alpha_0, 1]\).

(ii) Let us fix \(\alpha\) so that \(z_\alpha\) crosses zero at \(x_0(\alpha)\) for the first time. For \(x \geq x_\theta\), \(v_\alpha(x) = -u'_\alpha(x)\) is a supersolution of (4.11). So again with Sturm’s Comparison Theorem, as in Lemma 4.3, we get

\[
\frac{v'_\alpha(x)}{v_\alpha(x)} \leq \frac{z'_\alpha(x)}{z_\alpha(x)} \quad \forall x > x_\theta(\alpha).
\]  

(4.12)

In particular, since \(v_\alpha > 0\) for \(x > 0\), it means that \(z_\alpha\) does not cross zero again.
Now let us assume that $\lim_{x \to \infty} z_\alpha(x) = 0$. We know that $z'_\alpha(x) = \lambda e^{-ax^2/2}$ for $x \geq x_0$, so

$$z_\alpha(x) = -\lambda \int_x^\infty e^{-as^2/2} \, ds \quad \text{for} \quad x \geq x_0.$$  

An integration by parts on $z_\alpha(x)$ yields easily that $z_\alpha(x) > -z'_\alpha(x)/ax$ for $x \geq x_0$. This contradicts (4.12) since $v'_\alpha(x)/v_\alpha(x) = -ax$ for $x \geq x_0$. Hence we deduce that $\lim_{x \to \infty} z_\alpha(x) < 0$.

Finally, $x'_\alpha(\alpha) > 0$ follows from the Implicit Function Theorem because $z'_\alpha(x_0(\alpha)) < 0$.

Proposition 4.5 We have that $l(\alpha)$ is at least a $C^2$ function of $\alpha$. Moreover, there exists $\alpha_0$ in $(\theta, 1)$ such that

(i) $l'(\alpha) \geq 0$ and $l''(\alpha) \geq 0$ for $\alpha \in (\alpha_0, 1)$,

(ii) $l'(\alpha) < 0$ for $\alpha \in (\theta, \alpha_0)$.

Proof: The proof follows easily from Lemma 4.4 since we have seen that $l'(\alpha) = \lim_{x \to \infty} z_\alpha(x)$ and $l''(\alpha) = \lim_{x \to \infty} z'_\alpha(x)$. We only need to prove that $\alpha_0 < 1$. But if $\alpha_0 = 1$, it means that $l'(\alpha) < 0$ for $\alpha \in (\theta, 1)$ and this contradicts the fact that there are at least two solutions of $(P)$.

Corollary 4.6 There are exactly two solutions of $(P)$. For the upper solution $l'(\alpha) > 0$, and for the lower solution $l'(\alpha) < 0$.

4.3 Stability

Theorem 4.7 The upper solution $\overline{u}$ of $(P)$ is stable and the lower solution $\underline{u}$ is unstable, that is if one considers the time dependent problem

$$\begin{align*}
\begin{cases}
u_t = u_{xx} + axu_x + f(u), & x > 0, \quad t > 0, \\
u_x(t, 0) = 0 & \text{and} \quad \lim_{x \to +\infty} u(t, x) = 0 \quad \forall t > 0, \\
u(0, x) = u_0(x)
\end{cases} 
\end{align*}$$

$$(P_t)$$

(i) given any $\varepsilon > 0$, there exists $\eta$ such that if $\|u_0(x) - \overline{u}(x)\|_{C^1(\mathbb{R}_+)} < \eta$, then the corresponding solution $u(t, x)$ of $(P_t)$ will satisfy

$$\|u(t, x) - \overline{u}(x)\|_{C^1(\mathbb{R}_+)} < \varepsilon \quad \forall t > 0.$$  

(ii) there are initial data $u_0$ arbitrarily close to $\underline{u}$ such that the corresponding solution $u(t, x)$ does not remain close to $\underline{u}$.
Proof: Recall that for the stationary problem (2.1) the solutions are ordered. More precisely, if $|\alpha - \overline{u}(0)|$ is small, then $u_\alpha(x)$ is an increasing function of $\alpha$, whereas if $|\alpha - \underline{u}(0)|$ is small, then $u_\alpha(x)$ crosses $\underline{u}(x)$ exactly once.

(i) If $\alpha > \overline{u}(0)$, then $u_\alpha$ is a supersolution of $(P)$ and hence of $(P_t)$. Now if $u_0(x) < u_\alpha(x) \forall x > 0$ then the Maximum Principle for parabolic equations (see [8] or [13]) yields that the corresponding solution $u(t, x)$ of $(P_t)$ satisfies

$$u(t, x) < u_\alpha(x) \quad \forall x > 0, \forall t > 0.$$ 

Similarly if $\alpha$ is close to $\overline{u}(0)$ with $\alpha < \overline{u}(0)$, and if $u_0(x) > u_\alpha(x) \forall x > 0$ then $u(t, x) > u_\alpha(x) \forall x > 0, \forall t > 0$. This proves (i) since $u_\alpha(x)$ is a continuous function of $\alpha$, uniformly in $x > 0$.

(ii) In the case $\alpha > \underline{u}(0)$, then $u_\alpha$ is, this time, a subsolution of $(P)$ and hence of $(P_t)$. So the solution of $(P_t)$ with $u_0(x) = u_\alpha(x)$ satisfies $u(t, x) > u_\alpha(x) \forall t > 0$. In particular $u(t, x)$ is never close to $\underline{u}$. If, on the contrary, $\alpha < \underline{u}(0)$, then $u_\alpha$ is a supersolution and the inequalities are reversed, but the conclusion remains the same.

Remark: Let us call $\overline{u} = \overline{u}(0)$ for the upper solution and $\underline{u} = \underline{u}(0)$ for the lower solution. Then Proposition 2.6 implies that $\overline{u}$ is a decreasing function of $\alpha$ and $\underline{u}$ an increasing function of $\alpha$. So the stable solution is, as in the physical experiments, the one for which there is a decrease of temperature as extinction is reached.

5 Solutions of $(S)$ for $Le$ close to 1

We want to solve $(S)$ for $Le$ close to 1 using the Implicit Function Theorem near a solution of $(P)$. Let us define the Banach space

$$E = \{h \in C^1(\mathbb{R}_+), \text{ s.t. } e^{ax^2/2}h(x) \in L^\infty(\mathbb{R}_+)\}$$

with the norm

$$\|h\|_E = \|h'(x)\|_{L^\infty(\mathbb{R}_+)} + \|e^{ax^2/2}h(x)\|_{L^\infty(\mathbb{R}_+)}.$$ 

Theorem 5.1 Let $u$ be a solution of $(P)$ with $a < a^*$. Assume that $g$ satisfies $(G)$ and either

(i) $(1-u)g(u)$ is concave on $(\theta, 1)$, or
Then there are constants \( \varepsilon_1, \varepsilon_u, \varepsilon_v \), such that for \( |Le - 1| < \varepsilon_1 \), there exists a unique solution \((\tilde{u}, \tilde{v})\) of \((S)\) with

\[
\|\tilde{u} - u\|_E < \varepsilon_u \quad \text{and} \quad \|\tilde{v} - (1 - u)\|_E < \varepsilon_v.
\]

**Remark:** Notice that if \( g \) is for instance concave increasing then \((i)\) is true. Moreover \((i)\) always implies \((ii)\) by Corollary 4.6.

We are first going to prove a result for the linearized operator of \((P)\) around \( u \), which will enable us to apply the Implicit Function Theorem.

**Proposition 5.2** Consider the linearized problem

\[
\begin{cases}
  h'' + axh' + f'(u)h = \xi, \\
  h'(0) = 0 \quad \text{and} \quad \lim_{x \to +\infty} h(x) = 0.
\end{cases}
\]  
\((L_\xi)\)

For all \( \xi \) in \( E \), there exists a unique solution \( h \) of \((L_\xi)\). Moreover \( \|h\|_E \leq C\|\xi\|_E \), where \( C \) only depends on \( u \).

**Proof:** We know that there exists \( x_0 \) such that \( u(x) < \theta \) for \( x > x_0 \). Then in particular \( f'(u(x)) = 0 \) and

\[
h(x) = b_\xi(x) + \lambda c(x) \quad \text{for} \quad x > x_0,
\]  
\((5.1)\)

where \( \lambda \) is a real to be determined by the equation on \((0, x_0)\) and

\[
\begin{align*}
b_\xi(x) &= -\int_{x}^{\infty} \left( \int_{0}^{s} \xi(t)e^{at^2/2} dt \right)e^{-as^2/2} ds, \quad (5.2) \\
c(x) &= \int_{x}^{\infty} e^{-as^2/2} ds. \quad (5.3)
\end{align*}
\]

The idea is to match the solution on \((x_0, \infty)\) given by \((5.1)\) with a solution of \((L_\xi)\) on \((0, x_0)\). Since we want \( h \) to be \( C^1 \), we need

\[
\begin{align*}
h(x_0) &= b_\xi(x_0) + \lambda c(x_0), \\
h'(x_0) &= b_\xi'(x_0) + \lambda c'(x_0).
\end{align*}
\]

So that \( h \) solves

\[
\begin{cases}
  h'' + axh' + f'(u)h = \xi \quad \text{on} \quad (0, x_0) \\
  h'(0) = 0 \quad \text{and} \quad h'(x_0) - \gamma h(x_0) = \beta_\xi,
\end{cases}
\]  
\((5.4)\)

where \( \beta_\xi = (c(x_0)b_\xi'(x_0) - c'(x_0)b_\xi(x_0))/c(x_0) \) and \( \gamma = c'(x_0)/c(x_0) \). Recall that on \((0, x_0)\), \( z_\alpha(x) = \partial u(x)/\partial \alpha \) is a solution of \((4.1)\) Moreover, because of
hypothesis \((i)\) and Corollary 4.6, or because of hypothesis \((ii)\), we have that 
\[ \lim_{x \to \infty} z_\alpha(x) \neq 0, \] 
so as a consequence \(z_\alpha'(x_\theta) - \gamma z_\alpha(x_\theta) \neq 0\). Now we can apply Fredholm theory for problem \((5.4)\) to conclude that given any \(\xi\) in \(E\) (and hence any \(\beta_\xi\)), there exists a unique solution \(h\) of \((5.4)\) and we have

\[ \|h\|_{W^{1,\infty}(0,x_\theta)} \leq C\|\xi\|_{W^{1,\infty}(0,x_\theta)} \quad (5.5) \]

where \(C\) only depends on \(u\). In particular when we match this solution on \((0,x_\theta)\) with \((5.1)\) on \((x_\theta,\infty)\), this gives the existence and uniqueness of \(\lambda\) in \((5.1)\). Hence there is existence and uniqueness of the solution of \((L_\xi)\). Moreover, we get from \((5.2)-(5.3)\)

\[ e^{ax^2/2}|h(x)| \leq \frac{\|\xi\|_E}{a} + \frac{\lambda}{ax} \quad \text{for} \quad x > x_\theta, \]

\[ |h'(x)| \leq \|\xi\|_E e^{-ax^2/2} + \lambda e^{-ax^2/2} \quad \text{for} \quad x > x_\theta. \]

Notice that \((5.5)\) yields a bound on \(\lambda\) so we have the proof of the proposition.

**Proof of Theorem 5.1:** We want to apply the Implicit Function Theorem around a solution \((u,v)\) of \((S)\) with \(Le = 1\). Recall that such \(u\) is a solution of \((P)\) and \(v = 1 - u\). So we define the linearized operator of \((S)\) around \((u,v)\): for any \((\eta,\zeta)\) in \(E\), we want to solve

\[
\begin{cases}
  h'' + axh' + kg(u) + hv'g(u) = \eta, \\
  \left(1/Le\right)k'' + axk' - kg(u) - hv'g(u) = \zeta, \\
  h'(0) = 0 \quad \text{and} \quad \lim_{x \to +\infty} h(x) = 0, \\
  k'(0) = 0 \quad \text{and} \quad \lim_{x \to +\infty} k(x) = 0.
\end{cases} \quad (S_{\eta,\zeta})
\]

and estimate \(\|h\|_E\) and \(\|k\|_E\). Notice that if we add both equations, we get

\[ h'(x) + k'(x) = e^{-ax^2/2} \int_0^x (\eta(s) + \zeta(s)) e^{as^2/2} \, ds. \]

Then

\[ k(x) = -h(x) - \int_x^\infty e^{-at^2/2} \int_0^t (\eta(s) + \zeta(s)) e^{as^2/2} \, ds \, dt, \quad (5.6) \]

and \(h\) is a solution of \((L_\xi)\) with

\[ \xi(x) = \eta(x) + g(u(x)) \int_x^\infty e^{-at^2/2} \int_0^t (\eta(s) + \zeta(s)) e^{as^2/2} \, ds \, dt. \]

Notice that

\[ |e^{ax^2/2}\xi(x)| \leq \|\eta\|_E + \left\{(\|\eta\|_E + \|\zeta\|_E) \sup_{0,1} g\right\} \]

\[ \quad \sup_{0,1} \]
so that $\xi \in E$ and $\|\xi\|_E \leq C(\|\eta\|_E + \|\zeta\|_E)$. Now Proposition 5.2 gives the existence and uniqueness of the solution $h$ of $(L_\xi)$ with the bound $\|h\|_E \leq C(\|\eta\|_E + \|\zeta\|_E)$. Then we use (5.6) to conclude that there exists a unique solution $(h, k)$ of $(S_{\eta, \zeta})$ and

$$\|h\|_E + \|k\|_E \leq C(\|\eta\|_E + \|\zeta\|_E).$$

It means that the linearized operator $T$ at $Le = 1$ defined by $T(\eta, \zeta) = (h, k)$ where $(h, k)$ is the unique solution of $(S_{\eta, \zeta})$ is a continuous bijective mapping from $E$ to $E$ so this allows us to apply the Implicit Function Theorem around $(u, v)$ and get the desired conclusion.

### 6 High activation energy

In this section, we assume that the source term $f$ depends on a parameter $\varepsilon$ and we let $f = f_\varepsilon$. We call $u_\varepsilon$ a solution of

$$\begin{cases}
  u''_\varepsilon + axu'_\varepsilon + f_\varepsilon(u_\varepsilon) = 0, \\
  u'_\varepsilon(0) = 0 \quad \text{and} \quad \lim_{x \to +\infty} u_\varepsilon(x) = 0.
\end{cases}$$

(P$_\varepsilon$)

We know that $u_\varepsilon$ exists when $a < a^*_\varepsilon$ by Theorem 1.1. In the following, we assume

(i) $\lim_{\varepsilon \to 0} \theta_\varepsilon = 1$, where $\theta_\varepsilon$ is the ignition temperature of $f_\varepsilon$,

(ii) $\lim_{\varepsilon \to 0} m_\varepsilon = m > 0$, where $m_\varepsilon = F_\varepsilon(1) = \int_0^1 f_\varepsilon(s) \, ds$.

A typical example is the function

$$f_\varepsilon(u) = \frac{1}{\varepsilon} f(1 - \frac{1 - u}{\varepsilon})$$

where $f$ is a fixed function with an ignition temperature $\theta$. In scaled variables, the parameter $\varepsilon$ represents the inverse of the activation energy $E$, given by the Arrhenius law. The aim of this section is to derive an asymptotic limit as $E$ becomes infinite.

**Proposition 6.1** If $u_\varepsilon$ is a solution of $(P_\varepsilon)$, with $a = a_\varepsilon$ chosen arbitrarily less then $a^*_\varepsilon$, and $x_\varepsilon$ is the point where $u_\varepsilon(x_\varepsilon) = \theta_\varepsilon$, then we have the following estimates

$$\limsup_{\varepsilon \to 0} |u'_\varepsilon(x)| \leq \sqrt{2m} \quad \forall x \in \mathbb{R}_+, \quad (6.2)$$

$$\limsup_{\varepsilon \to 0} a_\varepsilon \leq m\pi, \quad (6.3)$$

$$\limsup_{\varepsilon \to 0} a_\varepsilon x_\varepsilon \leq \sqrt{2m}. \quad (6.4)$$
Proof: We use the energy estimate (2.4) to get
\[ |u'_\varepsilon(x)| \leq \sqrt{2m_\varepsilon} \quad \forall x \in \mathbb{R}_+, \]
which yields the first inequality. Then, we apply (2.9) to our problem and get
\[ \theta_\varepsilon = |u'_\varepsilon(x_\varepsilon)| \int_0^\infty e^{-a_\varepsilon(s+\varepsilon^2/2)} \, ds. \quad (6.5) \]
It allows us to derive
\[ \theta_\varepsilon \leq \sqrt{2m_\varepsilon} \int_0^\infty e^{-a_\varepsilon s^2/2} \, ds \quad \text{and} \quad \theta_\varepsilon \leq \sqrt{2m_\varepsilon} \int_0^\infty e^{-a_\varepsilon s} \, ds, \]
and we have the result.

Now let \( \overline{a} = \lim \sup_{\varepsilon \to 0} a^*_\varepsilon \). We get from (6.3) that \( \overline{a} \leq m\overline{a} \). The numerical computations presented in the next section show that \( \overline{a} > 0 \), but we are not able to prove it at the moment. In the following, we will assume that \( \overline{a} > 0 \).

**Theorem 6.2** If \( \overline{a} > 0 \), let a be fixed lower then \( \overline{a} \) and let \( u_\varepsilon \) be a solution of \( (P_\varepsilon) \) with \( x_\varepsilon \) the point where \( u_\varepsilon(x_\varepsilon) = \theta_\varepsilon \). Then, up to the extraction of a subsequence, \( x_\varepsilon \) converges to some \( \overline{x} \geq 0 \) and \( u_\varepsilon \) converges to \( u_\overline{x} \) determined by
\[
\begin{align*}
  u_\overline{x}(x) &= 1 \quad \text{if} \quad x \leq \overline{x}, \\
  u_\overline{x}(x) &= \frac{\int_{x-\overline{x}}^{\infty} e^{-a(s+\overline{x}^2/2)} \, ds}{\int_0^\infty e^{-a(s^2+\overline{x}^2/2)} \, ds} \quad \text{if} \quad x \geq \overline{x}.
\end{align*}
\]

Proof: The sequence \( u_\varepsilon \) is between 0 and 1. Moreover, (6.2) gives an upper bound on \( u'_\varepsilon \). Then, up to the extraction of a subsequence, we have that \( u_\varepsilon \) tends to \( u \) uniformly in \( C^{0,\alpha} \) on every compact set. Moreover, (6.4) gives that \( x_\varepsilon \) tends to some \( \overline{x} \geq 0 \). Now, we want to pass to the limit in (2.4). An integration by part yields
\[
0 \geq \int_0^x s u'_\varepsilon(s) \, ds \geq x(u(x) - u(0)),
\]
and with the help of (6.2), we get
\[
\int_0^{x_\varepsilon} s u'^2_\varepsilon(s) \, ds \leq C(1 - \theta_\varepsilon).
\]
So we let \( \varepsilon \) tend to 0 in (2.4) and obtain
\[
\lim_{\varepsilon \to 0} \frac{1}{2} u'^2_\varepsilon(x_\varepsilon) = \lim_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon(0)) = m_0 \leq m. \quad (6.6)
\]
Because of (6.5), we see that \( m_0 > 0 \) (so \( u_\varepsilon(0) \) is close enough to 1), and we can pass to the limit in (6.5) to get, with the help of (6.6)

\[
1 = \sqrt{2m_0} \int_0^\infty e^{-a(s\varepsilon^s/2)} ds.
\]

(6.7)

Since \( f_\varepsilon(u_\varepsilon) = 0 \) when \( u_\varepsilon \leq \theta_\varepsilon \), we know that

\[
u_\varepsilon(x) = -u_\varepsilon'(x_\varepsilon) \int_{x-x_\varepsilon}^\infty e^{-a(sx_\varepsilon+s^2/2)} ds \quad \text{for } x \geq x_\varepsilon.
\]

(6.8)

So the limit function is defined by

\[
u(x) = 1 \quad \text{if } x \leq \overline{x},
\]

\[
u(x) = \sqrt{2m_0} \int_{x-\overline{x}}^\infty e^{-a(s\overline{x}^s/2)} ds \quad x \geq \overline{x},
\]

where \( m_0 \) is given by (6.7). Notice that \( \overline{x} \) depends on \( a \) and the family of solutions \( u_\varepsilon \) (that is the maximal solution or another one).

7 Numerical results

In this section, we illustrate the results presented in Theorems 1.1, 1.2, 3.1 and 6.2 by numerical computations. We choose \( f \) with an ignition temperature \( \theta = 0.5 \), as depicted in figure 3, and \( f_\varepsilon \) given by (6.1).

![Figure 3: The function \( f(u) \) with ignition temperature \( \theta = 0.5 \)](image)

In order to compute the solutions of \( (P) \), we use a shooting method. We solve the ordinary differential equation (2.1) for \( \alpha \in (\theta, 1) \) with a discretisation scheme. The results are shown on figure 4 for the special case \( a = 2.5 \). We study the behaviour of solutions as \( \alpha \) increases from \( \theta \) to 1.
Figure 4: Solutions of (2.1) with $a = 2.5$ obtained by a shooting technique
1. for $\alpha \in (\theta, 0.78)$, the solutions remain positive on $\mathbb{R}_+$, that is $\alpha \in A_+(a)$.

2. for $\alpha \in (0.78, 0.948)$, the solutions cross zero at a point $R(\alpha)$. We notice that, as $\alpha$ increases, $R(\alpha)$ first decreases from infinity to $R_0 = 0.86$ and then increases from $R_0$ to infinity.

3. for $\alpha \in (0.948, 1)$, the solutions remain positive on $\mathbb{R}_+$ and $\alpha \in A_+(a)$ again.

So in this case, $A_-(a) = (0.78, 0.948)$ and $A_+(a) = (\theta, 0.78) \cup (0.948, 1)$.

In particular, we notice that $A_-(a)$ is connected and there are exactly two solutions of $(P)$, for $\alpha = 0.78$ and 0.948.

Now we let $a$ vary and use again the shooting method to determine the solutions of $(P)$. We compute $a^* = 3.099$ and check that for $a > a^*$, there are no solutions of $(P)$ (all solutions of (2.1) are in $A_+(a)$). For $a < a^*$, $A_-(a)$ is connected and there are exactly two solutions of $(P)$: $\overline{\alpha}_a$ and $\underline{u}_a$ with $\underline{u}_a < \overline{\alpha}_a$. Moreover, as $a$ increases, $\overline{\alpha}(a) = \overline{\alpha}_a(0)$ decreases and $\underline{\alpha}(a) = \underline{u}_a(0)$ increases.

We have drawn on figure 5 the curves $\overline{\alpha}(a)$ and $\underline{\alpha}(a)$.

In the case of high activation energy, we fix $a = 2.5$. We see that there are two solutions for all $\varepsilon$, so that $\overline{\alpha} > 2.5$. We call $\overline{x} = \lim_{\varepsilon \to 0} x_\varepsilon$ as in Theorem 6.2. Our numerical computations (see figure 6) show that for the lower solution $\overline{x} = 0$, and for the upper solution $\overline{x} = 0.2$. Moreover, when $a$ tends to 0, the value of $\overline{x}$ for the upper solution tends to infinity, and when $a$ tends to $\overline{\alpha}$, $\overline{x}$ tends to 0.
Figure 5: The curve $\bar{\alpha}(\theta)$ and $\underline{\alpha}(\theta)$ for solutions of $(P)$

Figure 6: The convergence of $u_\varepsilon$ as $\varepsilon$ tends to 0 for the upper and lower solutions
Appendix

For the sake of completeness, we recall here Sturm’s Comparison Theorem (see [7] for instance).

**Lemma A1.** Let $U$ and $V$ be two solutions of

\begin{align}
U'' + a(x)U'(x) + b(x)U(x) &\geq 0, \quad U(x) > 0 \quad \in (\mu, \nu), \quad (7.1) \\
V'' + a(x)V'(x) + b(x)V(x) &\leq 0, \quad V(x) > 0 \quad \in (\mu, \nu), \quad (7.2)
\end{align}

where $a(x)$ and $b(x)$ are locally integrable. Suppose that

$$\frac{V'(\mu)}{V(\mu)} \leq \frac{U'(\mu)}{U(\mu)}. \quad (7.3)$$

Then

$$\frac{V'(x)}{V(x)} \leq \frac{U'(x)}{U(x)} \quad \forall x \in (\mu, \nu). \quad (7.4)$$

Equality in (7.4) can occur only if $U \equiv V$ in $(\mu, x)$. If either $\mu$ or $\nu$ is a zero of $U$ or $V$, then the fractions are interpreted as $\infty$.

**References**


