A proof of property (RD) for discrete cocompact subgroups of $SL_3(R)$

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In this article we prove that discrete cocompact subgroups of $SL_3(\mathbb{R})$ satisfy property (RD) of Jolissaint (this property was introduced in [Jol85, Jol87, Jol90]). The argument is a very close imitation of the argument of [RRS97]: in this article Ramagge, Robertson and Steger prove a general result (stated below) implying that discrete cocompact subgroups of $SL_3(\mathbb{Q}_p)$ satisfy property (RD). Our result is a special case of a conjecture of Valette which claims that any discrete group acting isometrically, properly and cocompactly either on a Riemannian symmetric space or on an affine building has property (RD) ([FRR93] page 74).

Up to now property (RD) has been proved for free groups by Haagerup in [Haa79], and then for hyperbolic groups by de la Harpe in [dlH88], using [Jol87]. Recently, in [RRS97], Ramagge Robertson and Steger have proved property (RD) for any discrete group acting freely on the vertices of an $A^1 \times A^1$ or $A^2$ building by type-rotating automorphisms and this provided the first example of higher rank groups with property (RD). Our article is just an adaptation of [RRS97] to $SL_3(\mathbb{R})$ and it doesn’t bring any new idea. On the other hand a new idea is needed in order to prove property (RD) for cocompact lattices in Lie groups or $p$-adic groups of rank more than 1 and of type other than $A^2$.

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1 General facts about property (RD)

Let $\Gamma$ be a discrete group. A length $l$ on $\Gamma$ is a function $l : \Gamma \to \mathbb{R}_+$ such that $l(1) = 0$, $l(g^{-1}) = l(g)$ and $l(g_1g_2) \leq l(g_1) + l(g_2)$. We write $\Gamma_r = \{g, l(g) \leq r\}$.

**Definition 1** ([Jol90]) We say that $\Gamma$ satisfies (RD) with respect to a length
l if there is a polynomial $P$ such that, for any $r \in \mathbb{R}_+$, $f_1, f_2 \in \mathcal{C}$ with $\text{supp} f_i \in \Gamma_r$, we have $\|f_1 * f_2\|_{L^2(\Gamma)} \leq P(r) \|f_1\|_{L^2(\Gamma)} \|f_2\|_{L^2(\Gamma)}$.

It is enough to check the inequality for $f_1, f_2 \in \mathcal{C}$.

Let us notice the following fact: if $\Gamma$ satisfies (RD) with respect to a length $l$, there is a polynomial $P$ such that, for any $r \in \mathbb{R}_+$ and for any $f \in \mathcal{C}$ with $\text{supp} f \in \Gamma_r$, one has $\|f\|_{P(\Gamma)} \leq \|f\|_{C^*_r(\Gamma)} \leq P(r) \|f\|_{L^2(\Gamma)}$. This very good estimate for the norm in $C^*_r(\Gamma)$ has an important consequence: $C^*_r(\Gamma)$ has the same K-theory as a much simpler algebra, we introduce now.

**Proposition 2** If $\Gamma$ satisfies (RD) w.r.t. $l$, if $s \in \mathbb{R}_+$ is big enough, the completion $H^s(\Gamma)$ of $\mathcal{C}$ for the norm $\|f\| = \left( \sum_{g \in \Gamma} |f(g)|^2 (1 + l(g))^{2s} \right)^{1/2}$ is a subalgebra of $C^*_r(\Gamma)$ which is dense and closed under holomorphic functional calculus.

In [Jol89] this is proved for the Jolissaint algebra $H^\infty(\Gamma) = \bigcap H^s(\Gamma)$. We give an adaptation of the proof to our case.

**Proof.** Let $P$ be the polynomial in the definition. Take any $s \in \mathbb{R}_+$ such that $s > \text{deg}(P)$.

**a** We prove that $H^s(\Gamma)$ is a subspace of $C^*_r(\Gamma)$. We denote by $\chi_0$ the characteristic function of $\{g, l(g) \in [0, 1]\}$ and for any $n \in \mathbb{N}^*$ we denote by $\chi_n$ the characteristic function of $\{g, l(g) \in [2^{n-1}, 2^n]\}$. For any $f \in \mathcal{C}$, $\|f\|_{C^*_r(\Gamma)} \leq \sum_{n=0}^\infty \|f \chi_n\|_{C^*_r(\Gamma)} \leq \sum_{n=0}^\infty P(2^n) \|f \chi_n\|_{L^2(\Gamma)} \leq C \|f\|_{H^s(\Gamma)}$ with $C = \left( P(1)^2 + \sum_{n=1}^\infty (P(2^n)(1 + 2^{n-1})^{-s}) \right)^{1/2}$ by the Cauchy-Schwarz inequality.

**b** We prove that $H^s(\Gamma)$ is an algebra. For any $f_1, f_2 \in \mathcal{C}$ and for any $g \in \Gamma$ we have

$$\|(f_1 * f_2)(g)(1 + l(g))^s \leq \sum_{g_1, g_2, \text{ s.t. } g_1 + g_2 = g} 2^s |f_1(g_1)||f_2(g_2)|((1 + l(g_1))^s + (1 + l(g_2))^s)$$

and therefore

$$\left\| g \mapsto |(f_1 * f_2)(g)(1 + l(g))^s \right\|_{P(\Gamma)} \leq \left\| g \mapsto \left( \sum_{g_1, g_2, \text{ s.t. } g_1 + g_2 = g} 2^s |f_1(g_1)||f_2(g_2)|((1 + l(g_1))^s + (1 + l(g_2))^s) \right) \right\|_{P(\Gamma)}$$

$$+ \left\| g \mapsto \left( \sum_{g_1, g_2, \text{ s.t. } g_1 + g_2 = g} 2^s |f_1(g_1)||f_2(g_2)|((1 + l(g_1))^s + (1 + l(g_2))^s) \right) \right\|_{L^2(\Gamma)}.$$  

The two terms are analogous and for the first one we have

$$\left\| g \mapsto \left( \sum_{g_1, g_2, \text{ s.t. } g_1 + g_2 = g} 2^s |f_1(g_1)||f_2(g_2)|((1 + l(g_1))^s + (1 + l(g_2))^s) \right) \right\|_{P(\Gamma)} \leq 2^s C \|f_1\|_{H^s(\Gamma)} \|f_2\|_{H^s(\Gamma)}.$$
by part a).

c) Let $t \in [\deg(P), s]$. We first prove two intermediate results.

$\alpha)$ $H^t(\Gamma)$ is an algebra by part b) and $H^s(\Gamma)$ is stable under holomorphic functional calculus in $H^t(\Gamma)$. The proof is as follows. Let $f \in H^s(\Gamma)$. We have to prove that

$$\lim_{n \to \infty} \|f^n\|_H^{1/n} = \lim_{n \to \infty} \|f^n\|_{H^t(\Gamma)},$$

For any $g \in \Gamma$ we have

$$|f^n(g)| \leq \sum_{g_1, \ldots, g_n \in \Gamma, g_1 \cdots g_n = g} |f(g_1)| \cdots |f(g_n)|$$

and if $g = g_1 \cdots g_n$, $(1 + l(g))^{s-t} \leq n^{s-t}((1 + l(g_1))^{s-t} + \cdots + (1 + l(g_n))^{s-t})$. Therefore

$$\|f^n\|_{H^t(\Gamma)} = \|f \mapsto (1 + l(g))^{s-t}f^n(g)\|_{H^t(\Gamma)} \leq n^{s-t+1}C^n \|f\|_{H^t(\Gamma)} \|f\|_{H^s(\Gamma)},$$

where $C^n$ is a constant such that $\|f_1f_2\|_{H^t(\Gamma)} \leq C^n \|f_1\|_{H^t(\Gamma)} \|f_2\|_{H^t(\Gamma)}$ for any $f_1, f_2 \in H^s(\Gamma)$. When $n$ goes to infinity we get $\lim_{n \to \infty} \|f^n\|_{H^t(\Gamma)}^{1/n} \leq C^n \|f\|_{H^t(\Gamma)}$ and the results easily follows by putting $f^p$ instead of $f$ in this inequality and making $p$ go to infinity.

$\beta)$ For any $f \in H^s(\Gamma)$ we have $\|f\|_{H^t(\Gamma)} \leq \|f\|_{H^s(\Gamma)}^{1/n} \|f\|_{H^t(\Gamma)}^{1/n}$ by Hölder’s inequality.

Now let $f \in H^s(\Gamma)$. We have to prove that $f$ has the same spectral radius in $H^s(\Gamma)$ and $C^*_{\Gamma}(\Gamma)$. If $\rho_{H^t(\Gamma)}(f) = 0$ this is obvious because $\rho_{H^s(\Gamma)}(f) \leq \rho_{H^t(\Gamma)}(f)$. Otherwise we have $\|f^n\|_{C^*_{\Gamma}(\Gamma)} \geq \|f^n\|_{H^t(\Gamma)} \geq \|f^n\|_{H^s(\Gamma)}^{1/n} \|f^n\|_{H^t(\Gamma)}^{1/n}$ and the result follows.

2 Analytical part of the proof

In this section we consider a discrete metric space $(X, d)$ and a discrete group $\Gamma$ acting freely and isometrically on $X$, and we introduce the groupoid $\mathcal{G} = X \times_\Gamma X$ such that $\mathcal{G}^{(0)} = \Gamma \backslash X$ and $\mathcal{G}^{(1)} = \Gamma \backslash X^2$ and we define $\mathcal{G}_r = \{[x, y] \in \mathcal{G}, d(x, y) \leq r\}$ for any $r \in \mathbb{R}_+$ and $\|f\|_{\mathcal{G}} = \left(\sum_{g \in \mathcal{G}} |f(g)|^2\right)^{1/2}$ for any $f \in \mathcal{G}$.

We say that $X$ and $\Gamma$ satisfy the property $P(X, \Gamma)$ if there is a polynomial $P$ such that for any $r \in \mathbb{R}_+$, $f_1, f_2 \in \mathbb{R}_+ \mathcal{G}$ with $\text{supp} f_1 \in \mathcal{G}_r$, one has $\|f_1 * f_2\|_{P(\mathcal{G})} \leq P(r) \|f_1\|_{P(\mathcal{G})} \|f_2\|_{P(\mathcal{G})}$. 

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Proposition 3 If $P(X, \Gamma)$ holds then $\Gamma$ satisfies (RD) w.r.t. the length $l(g) = d(x_0, gx_0)$ for any $x_0 \in X$.

Proof. For any $g_1, g_2 \in \Gamma$, $[x_0, g_1x_0] \circ [x_0, g_2x_0] = [x_0, g_1g_2x_0]$. Let us define $T : \mathbb{C}\Gamma \to \mathbb{C}G$ by $T(f)([x, y]) = 0$ if $x \notin \Gamma x_0$ or $y \notin \Gamma x_0$ and $T(f)([x_0, gx_0]) = f(g)$. For any $f \in \mathbb{C}\Gamma$, $\|f\|_{\mathbb{C}\Gamma} = \|T(f)\|_{\mathbb{C}G}$ and for $f_1, f_2 \in \mathbb{C}\Gamma$, $T(f_1 \star T(f_2) = T(f_1) \star T(f_2)$. The result follows.

Definition 4 Let $(Z, d)$ be a metric space and $\delta > 0$. For any points $x_i \in Z$ we say that $x_1 \ldots x_n$ is a $\delta$-path if $d(x_1, x_2) + \cdots + d(x_{n-1}, x_n) \leq d(x_1, x_n) + \delta$ and that $x_1 x_2 x_3$ is a $\delta$-thin triangle if there exists $t \in Z$ s.t. $x_1tx_2, x_2tx_3$ and $x_3tx_1$ are $\delta$-paths. We say that $(Z, d)$ satisfies (H$_\delta$) if there exists a polynomial $P$ s.t. for any $r \in \mathbb{R}^+$, $x, y \in Z$, one has

$$\#\{t \in Z, xty \delta$-path, $d(x, t) \leq r\} \leq P(r).$$

Proposition 5 Let $\delta > 0$. If $(X, d)$ satisfies (H$_\delta$), there exists a polynomial $P$ s.t. for any $r \in \mathbb{R}^+$, $f_1, f_2, f_3 \in \mathbb{R}^G$, with $\text{supp}(f_1) \subset \mathbb{R}^G$, one has

$$\sum_{(x_1, x_2, x_3) \in \Gamma \setminus X^3, x_1x_2x_3 \delta$-thin} f_1([x_2, x_3])f_2([x_3, x_1])f_3([x_1, x_2]) \leq P(r)\|f_1\|_{\mathbb{C}(G)}\|f_2\|_{\mathbb{C}(G)}\|f_3\|_{\mathbb{C}(G)}.$$

This proposition implies the result of [dIH88] : hyperbolic groups satisfy property (RD).

The following lemma is obvious.

Lemma 6 If $H_1, H_2, H_3$ are Hilbert spaces, and $T_1 \in L(H_3, H_2), T_2 \in L(H_1, H_3), T_3 \in L(H_2, H_1)$ have finite Hilbert-Schmidt norms, $|\text{Tr}(T_1T_2T_3)| \leq \|T_1\|_{HS}\|T_2\|_{HS}\|T_3\|_{HS}$.

Proof of the proposition 5. We have

$$\sum_{(x_1, x_2, x_3, t) \in \Gamma \setminus X^4, x_1tx_2x_3x_1 \delta$-paths} f_1([x_2, x_3])f_2([x_3, x_1])f_3([x_1, x_2]),$$

Note that if $x_2tx_3$ is a $\delta$-path and $d(x_2, x_3) \leq r$, then $d(x_2, t) \leq r + \delta$ and $d(x_3, t) \leq r + \delta$.

Let $H_1, H_2, H_3 \subset L^2(\Gamma \setminus X^2)$ be defined by $H_1 = L^2(\Gamma \setminus X^2), H_2 = L^2(\{(t, x) \in \Gamma \setminus X^2, d(x_2, t) \leq r + \delta\})$ and $H_3 = L^2(\{(t, x) \in \Gamma \setminus X^2, d(x_3, t) \leq r + \delta\})$, and let $T_1 \in L(H_3, H_2)$ be the operator defined as a matrix by
\( T_{1,[t,x_2],[t',x_3]} = f_1([x_2,x_3]) \) if \( t, t' \) are in the same \( \Gamma \)-orbit (in this case we suppose \( t = t' \)) and if \( x_2tx_3 \) is a \( \delta \)-path

and otherwise the coefficient is 0,

and let \( T_2 \) and \( T_3 \) be defined in the same way. We have

\[
\sum_{(x_1,x_2,x_3,t) \in \Gamma \setminus X^4, \atop x_1tx_2,x_3t_3tx_3 \text{ are in the same } \Gamma \text{-orbit}} f_1([x_2,x_3])f_2([x_3,x_1])f_3([x_1,x_2]) = Tr(T_1T_2T_3),
\]

but

\[
\|T_1\|_{H(S)}^2 = \sum_{(x_2,t,x_3) \in \Gamma \setminus X^3, \atop x_1tx_2 \text{ is a } \delta \text{-path} \atop d(x_2,t) \leq r + \delta} |f_1([x_2,x_3])|^2 \leq P(r + \delta)\|f_1\|_{L^2(\mathcal{G})}^2
\]

by \((H_\delta)\) and in the same way \( \|T_2\|_{H(S)} \leq \sqrt{P(r + \delta)}\|f_2\|_{L^2(\mathcal{G})} \) and \( \|T_3\|_{H(S)} \leq \sqrt{P(r + \delta)}\|f_3\|_{L^2(\mathcal{G})} \). The proposition follows.

Let \( \delta > 0 \). We say that \( X \) and \( \Gamma \) satisfy \((K_\delta)\) if there exist \( k \in \mathbb{N} \) and \( \Gamma \)-invariant subsets \( \mathcal{T}_1, \ldots, \mathcal{T}_k \) of \( X^3 \) such that

- \((K_\delta a)\) there exists \( C_1 \in \mathbb{R}_+ \) such that for any \( (x_1,x_2,x_3) \in X^3 \), there exist \( i \in \{1,\ldots,k\} \) and \( (t_1,t_2,t_3) \in \mathcal{T}_i \) such that

\[
\max(d(t_1,t_2),d(t_2,t_3),d(t_3,t_1)) \leq C_1\left(\min(d(x_1,x_2),d(x_2,x_3),d(x_3,x_1)) + \delta\right),
\]

and \( x_1t_1t_2x_2, x_2t_2t_3x_3, x_3t_3t_1x_1 \) are \( \delta \)-paths,

- \((K_\delta b)\) for any \( i \in \{1,\ldots,k\} \) and \( t_1,t_2,t_3,t'_3 \in X \), if \( (t_1,t_2,t_3) \) and \( (t_1,t_2,t'_3) \) are in \( \mathcal{T}_i \) then the triangles \( t_1t_3t'_3 \) and \( t_2t_3t'_3 \) are \( \delta \)-thin.

**Theorem 7** If, for some \( \delta > 0 \), \( X \) and \( \Gamma \) satisfy \((H_\delta)\) and \((K_\delta)\) then \( P(X,\Gamma) \) holds and therefore \( \Gamma \) satisfies property \((RD)\).

**Proof of the theorem.** Let \( \mathcal{G} = X \times_\Gamma X \), and \( f_1,f_2,f_3 \in \mathbb{R}_+\mathcal{G} \) with \( \text{supp}(f_i) \subset \mathcal{G}_r \). We have
\[
\sum_{(x_1,x_2,x_3) \in X^3} f_1([x_2,x_3])f_2([x_3,x_1])f_3([x_1,x_2])
\]
\[
\leq \sum_{i=1}^{k} \left( \sum_{(x_1,x_2,x_3) \in X^3, (t_1,t_2,t_3) \in T_i} f_1([x_2,x_3])f_2([x_3,x_1])f_3([x_1,x_2]) \right)
\]
\[
\leq \sum_{i=1}^{k} \left( \sum_{(t_1,t_2,t_3) \in \Gamma \setminus T_i, \max(t_1,t_2,t_3) \leq r + \delta} h_1([t_2,t_3])h_2([t_3,t_1])h_3([t_1,t_2]) \right),
\]
where \( h_1 \in \mathbb{R}_+ \mathcal{G} \) is defined by
\[
h_1([t_2,t_3]) = \left( \sum_{x \in X^2, d(x_2,x_3) \leq r + \delta} f_1([x_2,x_3])^2 \right)^{1/2}
\]
if \( d(t_2,t_3) \leq C_1(r + \delta) \) and \( h_1([t_2,t_3]) = 0 \) otherwise and \( h_2, h_3 \in \mathbb{R}_+ \mathcal{G} \) are defined by similar expressions. The last inequality comes from lemma 6 with \( H_1 = H_2 = H_3 = l^2(X) \) and \( T_1 \) with coefficient \( T_1_{x_2,x_3} = f_1([x_2,x_3]) \) if \( x_2x_3x \) is a \( \delta \)-path and \( d(x_2,t_2) \leq r + \delta \) and 0 otherwise and with \( T_2 \) and \( T_3 \) defined in the same way. But
\[
\|h_1\|^2_{\mathcal{P}} \leq \sum_{(x_2,t_2,t_3) \in \Gamma \setminus X^4, x_2x_3x \delta\text{-path, } d(x_2,t_2) \leq r + \delta} f_1([x_2,x_3])^2 \leq P(r + \delta)P(C_1(r + \delta))\|f_1\|^2_{\mathcal{P}}
\]
and the same inequality holds for \( h_2 \) and \( h_3 \).

Fix \( i \in \{1, \ldots, k\} \) and replace \( C_1(r + \delta) \) by \( r \). It remains to show that there is a polynomial \( P \) s.t. for any \( r \in \mathbb{R}_+ \) and \( h_1, h_2, h_3 \in \mathbb{R}_+ \mathcal{G} \), with support in \( \mathcal{G}_r \), we have
\[
\sum_{(t_1,t_2,t_3) \in \Gamma \setminus T_i} h_1([t_2,t_3])h_2([t_3,t_1])h_3([t_1,t_2]) \leq P(r)\|h_1\|_{\mathcal{P}}\|h_2\|_{\mathcal{P}}\|h_3\|_{\mathcal{P}}.
\]
But the sum is equal to \( \langle h_1 \ast_T h_2, \hat{h}_3 \rangle \) for some partial convolution along \( T_i \), where \( \hat{h}_3([x, y]) = \tilde{h}_3([y, x]). \) We compute

\[
\sum_{(t_1, t_2, t_3, t'_3) \in \Gamma \setminus X^4, (t_1, t_2, t_3) \in T_i, (t_1, t_2, t'_3) \in T_i} \langle h_1 \ast_T h_2, h_1 \ast_T h_2 \rangle = h_1([t_2, t_3]) h_2([t_3, t_1]) \tilde{h}_1([t_2, t'_3]) \tilde{h}_2([t'_3, t_1]).
\]

By (H2b) the triangle \( t_1 t_3 t'_3 \) and \( t_2 t_3 t'_3 \) are \( \delta \)-thin. By proposition 5 there is a polynomial \( P \) with

\[
\begin{align*}
\bigg\| [t_3, t'_3] &\mapsto \sum_{t_1 \in X, t_2, t_3, t'_3 \text{ \( \delta \)-thin}} h_2([t_3, t_1]) \tilde{h}_2([t'_3, t_1]) \bigg\|_{L^2(\hat{G})} \leq P(r) \| h_2 \|^2_{L^2(\hat{G})}, \\
\bigg\| [t_3, t'_3] &\mapsto \sum_{t_2 \in X, t_2, t_3, t'_3 \text{ \( \delta \)-thin}} h_1([t_2, t_3]) \tilde{h}_1([t_2, t'_3]) \bigg\|_{L^2(\hat{G})} \leq P(r) \| h_1 \|^2_{L^2(\hat{G})},
\end{align*}
\]

and the theorem follows.

### 3 Geometrical part of the proof

#### 3.1 The case of \( \tilde{A}^2 \)-buildings

The following theorem is easily deducible from the arguments of [RRS97].

**Theorem 8** *(extracted from [RRS97])* If \( X \) is a free \( \Gamma \)-space and \( Z \) is the set of vertices of some \( \tilde{A}^2 \)-building on which \( \Gamma \) acts by type rotating automorphisms, and \( d \) is the graph-theoretic distance on the 1-skeleton, and \( \theta : X \to Z \) is a surjective \( \Gamma \)-equivariant map such that \( \sup_{z \in Z} \#(\theta^{-1}(z)) < +\infty \), and \( X \) is equipped with the distance \( \theta^*(d) \), then \( X \) and \( \Gamma \) satisfy (H0) and (K0) with \( k = 2 \); \( T_1 \cup T_2 \) the set of \( (t_1, t_2, t_3) \in X^3 \) such that \( \theta(t_1) \theta(t_2) \theta(t_3) \) is an equilateral triangle in some apartment, and \( (t_1, t_2, t_3) \in T_1 \) if \((\theta(t_1), \theta(t_2)) \) is of shape \((p,0), p \in \mathbb{N}, \) and \((t_1, t_2, t_3) \in T_2 \) if \((\theta(t_1), \theta(t_2)) \) is of shape \((0,p), p \in \mathbb{N}^*). Consequently \( \Gamma \) satisfies property (RD).

We thus obtain a very slight improvement of the result of [RRS97].

**Corollary 9** Any discrete group \( \Gamma \) acting on the set \( X \) of vertices of an \( \tilde{A}^2 \)-building by type-rotating automorphisms satisfies property (RD), provided that \( \sup_{x \in X} \# \{ g \in \Gamma, gx = x \} < +\infty \).
3.2 The case of $SL_3(R)$

Now consider $G = SL_3(\mathbb{R})$, $K = SO_3(\mathbb{R})$, $A = \left\{ \begin{pmatrix} e^{\alpha_1} & 0 & 0 \\ 0 & e^{\alpha_2} & 0 \\ 0 & 0 & e^{\alpha_3} \end{pmatrix} , \alpha_1 + \alpha_2 + \alpha_3 = 0 \right\}$ and $A^+ = \left\{ \begin{pmatrix} e^{\alpha_1} & 0 & 0 \\ 0 & e^{\alpha_2} & 0 \\ 0 & 0 & e^{\alpha_3} \end{pmatrix} , \alpha_1 \geq \alpha_2 \geq \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 = 0 \right\}$.

We equip $G/K$ with the distance $d(x, y) = \log \|x^{-1}y\| + \log \|y^{-1}x\|$. We remark that $d(x, y) = \rho \log a$ if $x^{-1}y \in KaK$ with $a$ in $A^+$, and $\rho$ is defined by $\rho(\alpha_1, \alpha_2, \alpha_3) = \alpha_1 - \alpha_3$.

For any $t \in \mathbb{R}$, $x, y, z \in G/K$, we say that $(x, y)$ is of shape $(t, 0)$ if $x^{-1}y \in Ke^{-\frac{t}{3}} \begin{pmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} K$ and that $(x, y, z)$ is an equilateral triangle of oriented size $t$ if there exists $g \in G$ such that

$$x = gK, \ y = ge^{-\frac{t}{3}} \begin{pmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} K \text{ and } z = ge^{-\frac{2t}{3}} \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{pmatrix} K.$$

If $t \in \mathbb{R}_-$ and $(x, y)$ is of shape $(t, 0)$ we say also that $(x, y)$ is of shape $(0, -t)$: in this way our terminology completely agrees with [RRS97]. For any $\delta_0 > 0$ we define

$$\mathcal{T}_{1, \delta_0} = \{(t_1, t_2, t_3) \in (G/K)^3, \exists t \in \mathbb{R}_+, \exists (s_1, s_2, s_3) \in (G/K)^3, s_1s_2s_3 \text{ an equilateral triangle of oriented size } t, d(s_1, t_1) \leq \delta_0, d(s_2, t_2) \leq \delta_0, d(s_3, t_3) \leq \delta_0 \}$$

and $\mathcal{T}_{2, \delta_0} = \{(t_1, t_2, t_3) \in (G/K)^3, \exists t \in \mathbb{R}_-, \exists (s_1, s_2, s_3) \in (G/K)^3, s_1s_2s_3 \text{ an equilateral triangle of oriented size } t, d(s_1, t_1) \leq \delta_0, d(s_2, t_2) \leq \delta_0, d(s_3, t_3) \leq \delta_0 \}$.

**Theorem 10** Let $\Gamma$ be a discrete subgroup of $G = SL_3(\mathbb{R})$, $Z$ a $\Gamma$-invariant discrete subspace of $G/K$, and $r \in \mathbb{R}_+$ such that the two following conditions are fulfilled:

- (I1) $\bigcup_{x \in Z} B(x, r) = G/K$
- (I2) for any $R \in \mathbb{R}_+$ $\sup_{x \in G/K} \#(B(x, R) \cap Z)$ is finite.

Let $X$ be a free $\Gamma$-space and $\theta : X \to Z$ a $\Gamma$-equivariant map such that $\sup_{z \in Z} \#(\theta^{-1}(z)) < +\infty$, and equip $X$ with the distance $\theta^*(d)$.

Then $X$ and $\Gamma$ satisfy $(H_8)$ and $(K_6)$ for some $\delta > 0$ and with $k = 2$ and $\mathcal{T}_1 = \theta^{-1}(\mathcal{T}_{1,r}) = \{(t_1, t_2, t_3) \in X^3, (\theta(t_1), \theta(t_2), \theta(t_3)) \in \mathcal{T}_{1,r}\}$ and $\mathcal{T}_2 = \theta^{-1}(\mathcal{T}_{2,r})$. Consequently $\Gamma$ satisfies property (RD).
If $\Gamma$ is a discrete cocompact subgroup of $SL_3(\mathbb{R})$, every $\Gamma$-orbit $Z$ in $G/K$ satisfies (11) and (12), and we can choose $X = \Gamma$ and $\theta$ obvious, and therefore $\Gamma$ satisfies (RD).

**Proof of the theorem. a)** We first prove that $X$ satisfies property $(H_\delta)$ for any $\delta > 0$. This part of the argument works for any linear connected semi-simple Lie group.

We recall some notations from chapter 5 of [Kna86]. Let $G$ be a linear connected semi-simple Lie group, $K$ a maximal compact subgroup, $g = t \oplus p$ the decomposition associated to the Cartan involution, $a$ a maximal abelian subspace of $p$, $\Sigma$ the set of restricted roots, $g_{\lambda}$ the root space associated to $\lambda \in \Sigma$, $\Sigma^+$ a choice of a set of positive roots, $a^+ = \{ H \in a, \lambda(H) > 0 \}$ for all $\lambda \in \Sigma^+$, $A = \exp(a)$, $A^+ = \exp(a^+)$, and $\rho = \frac{1}{2} \sum_{\lambda \in \Sigma^+} (\dim g_{\lambda}) \lambda$. We have

$$G = K A^+ K \text{ and } dg = \prod_{\lambda \in \Sigma^+} (\sinh(\lambda(\log a)))^{\dim g_{\lambda}} dk d\lambda$$

is the integral formula corresponding to this decomposition. If $\omega_1, \ldots, \omega_k$ are the fundamental weights, we have $\rho = n_1 \omega_1 + \cdots + n_k \omega_k$ for some positive integers $n_1, \ldots, n_k$ depending on the multiplicities of the roots. We introduce the non-symmetric function $d_i$ on $(G/K)^2$: if $x, y \in G/K$ and $x^{-1}y = KaK$ with $a \in A^+$, $d_i(x, y) = \omega_i(\log a)$. Since $G$ admits a representation of highest weight a multiple of $\omega_i$, if we choose an hermitian metric on this representation compatible with the Cartan involution on $G$, for any $x, y \in G/K$, $d_i(x, y)$ is a fraction of the log of the norm of the image by this representation of any antecedent of $x^{-1}y$ in $G$. Therefore, for any $x, y, z \in G/K$, $d_i(x, z) \leq d_i(x, y) + d_i(y, z)$. Let $a_1, \ldots, a_k \in \mathbb{R}_+$, and consider the non-symmetric function $d_\alpha(x, y) = \sum_{i=1}^k a_i d_i(x, y)$. Up to a constant there is a unique $G$-invariant element of volume on $G/K$.

**Lemma 11** For any $\delta > 0$ there is a polynomial $P$ such that for any $x, y \in G/K$,

$$\text{Vol}\{ t \in G/K, d_\alpha(x, t) + d_\alpha(t, y) \leq d_\alpha(x, y) + \delta \text{ and } d_\alpha(x, t) \leq r \} \leq P(r).$$

The lemma is false if some $a_i$ is 0: in this case the best estimate for the volume grows exponentially in $r$.

**Proof of the lemma.** We denote by $d$ the following distance on $G/K$:

$$d(x, y) = \sum_{i=1}^k n_i d_i(x, y) = \rho \log(a) \text{ if } x^{-1}y = KaK \text{ with } a \in A^+.$$ 

Denote by 1 the origin in $G/K$. We may suppose $x = 1$. There exists some constant $C_0$ depending on $\alpha$ such that the conditions $d_\alpha(1, t) + d_\alpha(t, y) \leq d_\alpha(1, y) + \delta$ and $d_\alpha(1, t) \leq r$ imply $d_i(1, t) + d_i(t, y) \leq d_i(1, y) + C_0 \delta$ for any
apply lemma /1/1 with $xty$ where $i$ and $d(1, t) \leq C_0r$. Because of (1) there exists some constant $C_1 \in \mathbb{R}^*_+$ such that

$$\text{Vol}\{z \in G/K, \exists k \in K, d(y, k z) \leq 1\} \geq C_1 e^{2d(1,y)}.$$

Therefore

$$\text{Vol}\left\{(t, z) \in (G/K)^2, \forall i, d_i(1, t) + d_i(t, z) \leq d_i(1, y) + C_0\delta + 1, d(1, t) \leq C_0r\right\} \geq C_1 e^{2d(1,y)}\text{Vol}\left\{t \in G/K, \forall i, d_i(1, t) + d_i(t, y) \leq d_i(1, y) + C_0\delta, d(1, t) \leq C_0r\right\}$$

because $\left\{t \in (G/K), \forall i, d_i(1, t) + d_i(t, z) \leq d_i(1, y) + C_0\delta + 1, d(1, t) \leq C_0r\right\} \supset \left\{t \in (G/K), \forall i, d_i(1, t) + d_i(t, y) \leq d_i(1, y) + C_0\delta, d(1, t) \leq C_0r\right\}$ if $d(y, z) \leq 1$

and $\text{Vol}\left\{t \in (G/K), \forall i, d_i(1, t) + d_i(t, z) \leq d_i(1, y) + C_0\delta + 1, d(1, t) \leq C_0r\right\}$ depends only on $K\mathbb{z}$ in $K\backslash G/K$. The following fact comes from (1) : there exists a constant $C_2$ such that

$$\text{for any } a_1, \ldots, a_k \in \mathbb{R}^+, \text{ Vol}\{u \in G/K, \forall i, d_i(1, u) \leq a_i\} \leq C_2 e^{2\sum n_i a_i}.$$

Now fix $t \in G/K$ such that $d_i(1, t) \leq C_0r$. We have

$$\text{Vol}\left\{z \in (G/K), \forall i, d_i(1, t) + d_i(t, z) \leq d_i(1, y) + C_0\delta + 1\right\} = \text{Vol}\left\{z \in (G/K), \forall i, d_i(t, z) \leq d_i(1, y) + C_0\delta + 1 - d_i(1, t)\right\} \leq C_2 \exp \left(2d(1, y) + 2(C_0\delta + 1)(\sum_{i=1}^{k} n_i) - 2d(1, t)\right),$$

and $\int_{t \in G/K, d(1, t) \leq C_0r} e^{-2d(1,t)} dt \leq P(r)$ for some polynomial $P$, because of (1). The lemma now results from Fubini’s lemma.

We now come back to the notations of the theorem 10. We fix $\delta > 0$ and we explain why $X$ satisfies $(H_\delta)$. Of course it is enough to prove that $(Z, d)$ satisfies $(H_\delta)$. By (12) there exists $N \in \mathbb{N}$ such that $\#(B(x, 1) \cap Z) \leq N$ for any $x \in G/K$. For any $r \in \mathbb{R}^+, x, y \in G/K$,

$$\#\{t \in Z, txy \text{-} \delta\text{-path}, d(x, t) \leq r\} \leq \frac{N}{V} \text{Vol}\{t' \in G/K, xt' y (\delta + 2)\text{-path}, d(x, t') \leq r + 1\},$$

where $V = \text{Vol}\{z \in G/K, d(1, z) \leq 1\}$, because if $d(t, t') \leq 1, d(x, t) \leq r$ and $xty$ is a $\delta$-path, then $xt'y$ is a $(\delta + 2)$-path and $d(x, t') \leq r + 1$. It remains to apply lemma 11 with $d_\alpha = d$.
b) We now prove that \(X\) and \(\Gamma\) satisfy \((K_\delta)\) with \(k = 2\), \(T_1\) and \(T_2\) as in the theorem 10 and if \(\delta\) is big enough. This part of the proof doesn’t work for Lie groups other than \(SL_2(\mathbb{R})\).

The following lemma is analogous to the study of the foldings in [RRS97]. We call a flat in \(G/K\) any subset of \(G/K\) equal to \(gAK\), for some \(g \in G\). We now study the distance of some fixed point of \(G/K\) to the points of a fixed flat in \(G/K\). Up to left translation by an element of \(G\), we may suppose that the flat is \(AK\). We prove the following result: there is some \(\delta > 0\) such that for any \(x \in G/K\) there exist \(y, y_2, y_3\) in \(G/K\) such that, for any \(a \in A\), \(xy(aK)\) is a \(\delta\)-path, and such that \(yy_2y_3\) is an equilateral triangle, \(y_2\) and \(y_3\) are at distance less than \(\delta\) from \(AK\) and for any \(a \in A\) there exists some point \(z\) on the side \(y_2y_3\) of the triangle \(yy_2y_3\) such that \(yz(aK)\) is a \(\delta\)-path.

Denote by \(W_0\) the subgroup of \(G\) whose elements are \(Id, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\) and \(\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\).

**Lemma 12** For some \(\delta > 0\) the following is true. For any \(x \in G/K\) there exist \(y \in G/K, t \in \mathbb{R}_+, m \in W_0A\) such that

- \(xy(aK)\) is a \(\delta\)-path for any \(a \in A\),
- \(d(y, m\begin{pmatrix} e^{s_1} & 0 & 0 \\ 0 & e^{s_2} & 0 \\ 0 & 0 & e^{s_3} \end{pmatrix}K) - t - \max(|s_2 - s_3|, s_1 - s_2 - t, s_1 - s_3 - t, s_2 - s_1, s_3 - s_1)\) \(\leq \delta\) for any \(s_1, s_2, s_3 \in \mathbb{R}\), \(s_1 + s_2 + s_3 = 0\),
- there exists \(h_2 \in G\) such that \(d(h_2\begin{pmatrix} e^{s_1} & 0 & 0 \\ 0 & e^{s_2} & 0 \\ 0 & 0 & e^{s_3} \end{pmatrix}K, m\begin{pmatrix} e^{s_1} & 0 & 0 \\ 0 & e^{s_2} & 0 \\ 0 & 0 & e^{s_3} \end{pmatrix}K) \leq \delta\) if \(s_1 + s_2 + s_3 = 0\) and \(s_2 \geq s_3\) and \(d(h_2e^{-\frac{\theta}{3}}\begin{pmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^t \end{pmatrix}K, y) \leq \delta\),
- there exists \(h_3 \in G\) such that \(d(h_3\begin{pmatrix} e^{s_1} & 0 & 0 \\ 0 & e^{s_2} & 0 \\ 0 & 0 & e^{s_3} \end{pmatrix}K, m\begin{pmatrix} e^{s_1} & 0 & 0 \\ 0 & e^{s_2} & 0 \\ 0 & 0 & e^{s_3} \end{pmatrix}K) \leq \delta\) if \(s_1 + s_2 + s_3 = 0\) and \(s_2 \leq s_3\) and \(d(h_3e^{-\frac{\theta}{3}}\begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{pmatrix}K, y) \leq \delta\).

We can notice that the second condition is in fact a consequence of the two last ones (for a different \(\delta\)).
Proof of the lemma. Up to left translation by some element \( m \in W_0 A \) we may suppose that \( x = \lambda \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} K \), with \( \lambda \in \mathbb{R}_+^*, \ e_1, e_2, e_3 \in \mathbb{R}^3, \ ||e_1|| = ||e_2|| = ||e_3|| = 1 \) and \( ||e_2 \wedge e_3|| \leq \min(||e_1 \wedge e_2||, ||e_1 \wedge e_3||) \). We have

\[
||e_1 \wedge e_2|| - ||e_1 \wedge e_3|| \leq \min(||e_1 \wedge (e_2 - e_3)||, ||e_1 \wedge (e_2 + e_3)||) \\
\leq \min(||e_2 - e_3||, ||e_2 + e_3||) \leq \sqrt{2}||e_2 \wedge e_3||.
\]

Take \( t = \log \left( \frac{||e_1 \wedge e_2||}{||e_2 \wedge e_3||} \right) \). By the last inequalities \( |t - \log(\frac{||e_1 \wedge e_2||}{||e_2 \wedge e_3||})| \leq \log(1 + \sqrt{2}) \).

For \( a = \begin{pmatrix} e^{s_1} & 0 & 0 \\ 0 & e^{s_2} & 0 \\ 0 & 0 & e^{s_3} \end{pmatrix} \), with \( s_1 + s_2 + s_3 = 0 \), we have

\[
d(aK, x) = \log \left\| a^{-1} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \right\| + \log \left\| \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}^{-1} a \right\|
\]

and

\[
\left| \log \left\| a^{-1} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \right\| - \max(-s_1, -s_2, -s_3) \right| \leq \log 3
\]

and since

\[
\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}^{-1} = \det(e_1, e_2, e_3)^{-1} \begin{pmatrix} e_2 \wedge e_3 \\ e_3 \wedge e_1 \\ e_1 \wedge e_2 \end{pmatrix}
\]

we have

\[
\left| \log \left\| \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} a \right\| + \log ||e_1 \wedge e_2 \wedge e_3|| \\
- \max(s_1 + \log ||e_2 \wedge e_3||, s_2 + \log ||e_1 \wedge e_3||, s_3 + \log ||e_1 \wedge e_2||) \right| \leq \log 3.
\]

Therefore

\[
d(aK, x) - \max(-s_1, -s_2, -s_3) - \max(s_1 - t, s_2, s_3) \\
+ \log ||e_1 \wedge e_2 \wedge e_3|| - \log ||e_1 \wedge e_2|| \leq \delta_0
\]

for some numerical constant \( \delta_0 \).
Now consider \( y = e^{t} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & e^{-t} \end{pmatrix} K \). By applying the last argument to \( y \) instead of \( x \), we obtain

\[
|d(aK, y) - \max(-s_1, -s_2, -s_3) - \max(s_1 - t, s_2, s_3) - t| \leq \delta_1
\]

for some numerical constant \( \delta_1 \) (different from \( \delta_0 \) because one has to normalize \( (0, 1, e^{-t}) \)). But

\[
d(x, y) = \log \left\| \begin{pmatrix} e_1 \\ e_2 \\ e^t(e_3 - e_2) \end{pmatrix} \right\| + \log \left\| \begin{pmatrix} e_1 \\ e_2 \\ e^t(e_3 - e_2) \end{pmatrix} \right\|^{-1}
\]

because \( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & e^{-t} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -e^t & e^t \end{pmatrix} \). Therefore

\[
|d(x, y) - \log \|e_1 \wedge e_2\| + t + \log \|e_1 \wedge e_2 \wedge e_3\| | \leq \delta_2
\]

for some numerical constant \( \delta_2 \) because \( \log \|e_1 \wedge e_2\| \) and \( \log \|e_2 \wedge (e^t(e_3 - e_2))\| \) are equal, and we have seen that this implies that \( \log \|e_1 \wedge (e^t(e_3 - e_2))\| \leq \log \|e_1 \wedge e_2\| + \log(1 + \sqrt{2}) \). Thus the first two assertions are proved. Now we take \( h_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \) and \( h_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \). We check that

\[
h_3e^{-\frac{2t}{3}} \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{pmatrix} K = y \text{ and that } d(h_2e^{-\frac{2t}{3}} \begin{pmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^t \end{pmatrix} K, y) \text{ is bounded by a numerical constant, because}
\]

\[
y^{-1}h_2e^{-\frac{2t}{3}} \begin{pmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^t \end{pmatrix} K = K \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -e^t & e^t \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 1 \\ 0 & 0 & 1 \end{pmatrix} K = K \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 1 \\ 0 & -1 & 0 \end{pmatrix} K.
\]

Finally it is obvious that \( d(h_2 \begin{pmatrix} e^{s_1} & 0 & 0 \\ 0 & e^{s_2} & 0 \\ 0 & 0 & e^{s_3} \end{pmatrix} K, \begin{pmatrix} e^{s_1} & 0 & 0 \\ 0 & e^{s_2} & 0 \\ 0 & 0 & e^{s_3} \end{pmatrix} K) \leq \delta \) if \( s_1 + s_2 + s_3 = 0 \) and \( s_2 \geq s_3 \), and if \( \delta \) is big enough. The last condition for \( h_3 \) is similar.

Now consider \( x_1, x_2, x_3 \in G/K \). Take \( x = x_1 \) and choose any flat containing \( x_2 \) and \( x_3 \). After a small discussion of the possible positions of \( x_2 \) and \( x_3 \) in this flat (a similar discussion occurs in [RRS97]) we obtain the following lemma which immediately implies that property \( (K_\delta) \) holds with \( k = 2 \) and \( T_1 \) and \( T_2 \) as in the theorem 10 if \( \delta \) is big enough.
Lemma 13 For some $\delta > 0$ the following is true. For any $x_1, x_2, x_3 \in G/K$ there exist $t_1, t_2, t_3 \in G/K$ such that $x_1 t_1 t_2 x_2, x_2 t_2 t_3 x_3$ and $x_3 t_3 t_1 x_1$ are $\delta$-paths and $t_1 t_2 t_3$ is an equilateral triangle.

c) Now we prove that property $(K_{\delta b})$ holds with $k = 2$ and $T_1$ and $T_2$ as in the theorem if $\delta$ is big enough.

This results from the following lemma, applied to $\delta_0 = 2r$.

Lemma 14 For any $\delta_0 > 0$ there exists $\delta > 0$ such that the following is true. For any $s, t \in \mathbb{R}$ of the same sign, and $x_1, y_1, x_2, z_2 \in G/K$ with $d(x_1, x_2) \leq \delta_0$, $(x_1, y_1)$ of shape $(s, 0)$ and $(x_2, z_2)$ of shape $(t, 0)$, then $x_1 y_1 z_2$ is a $\delta$-thin triangle.

Proof of the lemma. By symmetry with respect to 1 in $G/K$ we may suppose $s, t \in \mathbb{R}_+$. Up to left translation by an element of $G$ we may suppose $x_1 = 1$ and $y_1 = e^{-\frac{s}{3}} \begin{pmatrix} e^s & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} K$. We have $z_2 = h e^{-\frac{t}{3}} \begin{pmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} K$ with $h \in G$ such that $d(1, hK) \leq \delta_0$. Write $h = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix}$ and $h^{-1} = \begin{pmatrix} h_{11}' & h_{12}' & h_{13}' \\ h_{21}' & h_{22}' & h_{23}' \\ h_{31}' & h_{32}' & h_{33}' \end{pmatrix}$. By Cramer’s formula there exists a constant $\delta_1$ depending only on $\delta_0$ such that $|\log(\max(|h_{21}|, |h_{31}|)) - \log(\max(|h_{21}'|, |h_{31}'|))| \leq \delta_1$.

Take $r = \min(s, t, -\log(\max(|h_{21}|, |h_{31}|)))$ and $u = e^{-\frac{s}{3}} \begin{pmatrix} e^r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} K$. Then $x_1 uy_1, y_1 u z_2$ and $z_2 u x_1$ are $\delta$-paths for some $\delta$ depending only on $\delta_0$. Indeed $d(x_1, u) = r$, $d(u, y_1) = s - r$, and since

$$u^{-1} z_2 = K e^{\frac{r}{3}} \begin{pmatrix} e^{-r} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} h \begin{pmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} K$$

$$= K e^{\frac{r}{3}} \begin{pmatrix} e^{t-r} h_{11} & e^{t-r} h_{12} & e^{t-r} h_{13} \\ e^{t} h_{21} & h_{22} & h_{23} \\ e^{t} h_{31} & h_{32} & h_{33} \end{pmatrix} K$$
and \( z_2^{-1} u = K e^{\frac{i\pi}{3}} \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} h^{-1} \begin{pmatrix} e^r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} K \\
= K e^{\frac{i\pi}{3}} \begin{pmatrix} e^{-t}h'_{11} & e^{-t}h'_{12} & e^{-t}h'_{13} \\ e^r h'_{21} & h'_{22} & h'_{23} \\ e^r h'_{31} & h'_{32} & h'_{33} \end{pmatrix} K, \)

we see that \( |d(u, z_2) - (t - r)| \leq \delta_2 \) and \( |d(y_1, z_2) - (t - r) - (s - r)| \leq \delta_2 \) by the same computation, where \( \delta_2 \) depends only on \( \delta_0 \).

The lemmas 12 and 14 are more intuitive if one considers quadratic forms on \( \mathbb{R}^3 \) instead of elements of \( SL_3(\mathbb{R}) \) but it is more difficult to write correct proofs in this way.

### References


