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Existence of Solutions for Compressible Fluid Models of Korteweg Type

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Abstract

The purpose of this work is to prove existence and uniqueness results of suitably smooth solutions for an isothermal model of capillary compressible fluids derived by J.E. Dunn and J. Serrin in [11], which can be used as a phase transition model.

We first study the well-posedness of the model in spaces with critical regularity indices with respect to the scaling of the associated equations. In a functional setting as close as possible to the physical energy spaces, we prove global existence of solutions close to a stable equilibrium, and local in time existence for solutions when the pressure law may present spinodal regions. Uniqueness is also obtained.

Assuming a lower and upper control of the density, we also show the existence of weak solutions in dimension 2 near equilibrium. Finally, referring to the work of Z. Xin in the non-capillary case [22], we describe some blow-up properties of smooth solutions with finite total mass.

1 Introduction

Let us consider a fluid of density $\rho \geq 0$, velocity field $u \in \mathbb{R}^d (d \geq 2)$, entropy density $s$, energy density $e$, and temperature $\theta = (\partial e / \partial s)_{\rho}$. We are interested in the following model of compressible capillary fluid, which can be derived from a Cahn-Hilliard like free energy (see the pioneering work by J.E. Dunn and J. Serrin in [11], and also [1] [6] [12])

\begin{align}
\partial_t \rho + \text{div} (\rho u) &= 0, \\
\partial_t (\rho u) + \text{div} (\rho u \otimes u) &= \text{div} (S + K), \\
\partial_t (\rho (e + \frac{u^2}{2})) + \text{div} (\rho u (e + \frac{u^2}{2})) &= \text{div} (\alpha \nabla \theta) + \text{div} ((S + K) \cdot u),
\end{align}

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where the viscous stress tensor \( S \) and the Korteweg stress tensor \( K \) read as

\[
S_{i,j} = (\lambda \text{div} \, u - P(\rho, \theta)) \delta_{i,j} + 2\mu D(u)_{i,j},
\]

\[
K_{i,j} = \frac{\kappa}{2} \left( \Delta \rho^2 - |\nabla \rho|^2 \right) \delta_{i,j} - \kappa \partial_i \rho \partial_j \rho,
\]

\[
D(u)_{i,j} = (\partial_i u_j + \partial_j u_i)/2
\]

being the strain tensor, and \((\lambda, \mu)\) the constant viscosity coefficients of the fluid. We require that \( \lambda \) and \( \mu \) satisfy \( \mu > 0 \) and \( \lambda + 2\mu > 0 \), which in particular covers the case when \( \lambda \) and \( \mu \) satisfy Stokes’ law \( d\lambda + 2\mu = 0 \). The thermal conduction coefficient \( \alpha \) is a given non negative function of the temperature \( \theta \) and the surface tension coefficient \( \kappa > 0 \) is assumed to be constant. In view of the first principle of thermodynamics, the entropy density \( s \) solves

\[
\partial_t (\rho s) + \text{div} (\rho s u) - \frac{1}{\theta} \left( \text{div}(\alpha \nabla \theta) + K : D(u) + 2\mu D(u) : D(u) + \lambda |\text{div} u|^2 \right).
\]

As a reasonable starting point of our analysis, we consider the scaled Van der Waals equation of state

\[
P(\rho) = a \rho \theta \left( \frac{8}{3} - \frac{3\rho}{\theta} \right),
\]

where \( a \) is a positive constant, and the critical density \( \rho_c \) and temperature \( \theta_c \) are equal to 1. Depending on the fixed temperature \( \theta \), the pressure is a nondecreasing function of the density \( \rho \) or may present decreasing regions (spinodal regions) for some values of \( \rho \), which are thermodynamically unstable. The above equation of state (7) ensures the presence of two basic states, a “liquid” one, and a “gaseous” one. Let us as in [20] put emphasis on the existence of steady solutions connecting a gas phase to a liquid phase through a smoothly varying density profile. When initial conditions involve densities in the unstable (spinodal) region, the two phases are expected to spontaneously separate. For details on the derivation of the above Korteweg like model, we refer to [1] [11] [12] [16].

In what follows, we do not consider thermal fluctuations so that the pressure \( P \) is a function of \( \rho \) only. The corresponding isothermal model which was also considered in [13] [20] then reads as

\[
\partial_t \rho + \text{div}(\rho u) = 0,
\]

\[
\partial_t (\rho u) + \text{div}(\rho u \otimes u) - \mu \Delta u - \kappa \partial_i \rho \partial_j \rho = \text{div} \, K + \rho f,
\]
where $f$ is an exterior forcing term, supplemented with initial conditions

$$
\rho|_{t=0} = \rho_0 \geq 0, \quad \text{and} \quad \rho u|_{t=0} = m_0. \tag{10}
$$

In a bounded domain $\Omega$, we would have to precise the boundary conditions, namely homogeneous Dirichlet conditions for the velocity: $u|_{\partial \Omega} = 0$ and Neumann conditions for the density: $\partial_\nu \rho|_{\partial \Omega} = 0$. In order to simplify the presentation, we will focus on the whole space case $\mathbb{R}^d$ ($d \geq 2$) and study the well-posedness of (8)(9) for initial densities close enough to an equilibrium density $\bar{\rho} > 0$, or at least bounded away from vacuum, which is a major difficulty in most of compressible fluid models.

Before getting into the heart of mathematical results, we first derive the physical energy bounds of the above system in the case $f \equiv 0$ to simplify the presentation. Let $\bar{\rho} > 0$ be a constant reference density, and $\pi$ defined by

$$
\pi(s) = s \left( \int P(z) \frac{dz}{z^2} - \frac{P(\bar{\rho})}{\bar{\rho}} \right), \tag{11}
$$

so that $P(s) = s \pi'(s) - \pi(s)$, $\pi'(\bar{\rho}) = 0$, and

$$
\partial_t \pi(\rho) + \text{div}(u \pi(\rho)) + P(\rho) \text{div} u = 0 \quad \text{in} \quad D'((0,T) \times \mathbb{R}^d). \tag{12}
$$

Notice that $\pi$ is convex as soon as $P$ is non decreasing (since $P'(s) = s \pi''(s)$), which is the case for $\gamma$-type pressure laws. Multiplying the equation of momentum conservation by $u$ and integrating by parts over $\mathbb{R}^d$, we obtain the following energy estimate

$$
\int_{\mathbb{R}^d} \left( \frac{1}{2} \rho u^2 + (\pi(\rho) - \pi(\bar{\rho})) + \frac{\kappa}{2} |\nabla \rho|^2 \right)(t) \, dx + \int_0^t ds \int_{\mathbb{R}^d} (\mu \text{D}(u) : \text{D}(u) + (\lambda + \mu) |\text{div} u|^2) \, dx \leq \int_{\mathbb{R}^d} \left( \frac{|m_0|^2}{2 \rho_0} + (\pi(\rho_0) - \pi(\bar{\rho})) + \frac{\kappa}{2} |\nabla \rho_0|^2 \right) \, dx. \tag{13}
$$

Indeed, in order to compute formally the contribution to energy of the capillary tensor $K$, we observe that

$$
\text{div} K = \kappa \rho \nabla \Delta \rho. \tag{14}
$$

In view of the above expression, this model can be understood as a diffuse interface model, in which surface tension takes place between level sets of the continuously varying density. As a matter of fact, the right hand side (14) can
be rewritten up to a gradient term as the product between $\nabla \rho$ and $\Delta \rho$, which, roughly speaking, respectively represent the normal direction and the curvature of the level sets of the density. As observed for instance in [1], formal analyses show that the sharp interface limit leads to the classical two-fluid problem. We obtain indeed

$$-\int_{\mathbb{R}^d} \mathbf{u} \cdot \text{div} \mathbf{K} \, d\mathbf{x} = \int_{\mathbb{R}^d} \kappa \text{div} (\rho \mathbf{u}) \Delta \rho \, d\mathbf{x} = \kappa \int_{\mathbb{R}^d} \partial_t \nabla \rho \cdot \nabla \rho \, d\mathbf{x} = \kappa \frac{d}{dt} \int_{\mathbb{R}^d} \frac{|\nabla \rho|^2}{2} \, d\mathbf{x}.$$ 

It follows that assuming that the total energy is finite

$$\mathcal{E}_0 = \int_{\mathbb{R}^2} \left( \frac{1}{2} \rho_0 |\mathbf{u}|^2 + (\pi(\rho_0) - \pi(\bar{\rho})) + \frac{\kappa}{2} |\nabla \rho_0|^2 \right) < +\infty, \quad (15)$$

we have the a priori bounds

$$\pi(\rho) - \pi(\bar{\rho}) \quad \text{and} \quad \rho|\mathbf{u}|^2 \in L^\infty(0, \infty; L^1(\mathbb{R}^d)), \quad (16)$$

$$\nabla \rho \in L^\infty(0, \infty; L^2(\mathbb{R}^d))^d, \quad \text{and} \quad \nabla \mathbf{u} \in L^2((0, \infty) \times \mathbb{R}^d)^d. \quad (17)$$

Let us emphasize at this point that the above a priori bounds do not provide any $L^\infty$ control on the density from below or from above. Indeed, even in dimension $d = 2$, $H^1(\mathbb{R}^d)$ functions are not necessarily locally bounded. Thus, vacuum patches are likely to form in the fluid in spite of the presence of capillary forces, which are expected to smooth out the density.

# 2 Mathematical results

We wish to prove existence and uniqueness results of solutions to (8)(9) in functional spaces very close to energy spaces. In the case $\kappa = 0$ and $p(\rho) = a\rho^\gamma$, with $a > 0$ and $\gamma > 1$, P.-L. Lions proved in [17] [18] the global existence of weak solutions “à la Leray” $(\rho, \mathbf{u})$ to (8)(9) for $\gamma \geq 3d/(d+2)$ and initial data $(\rho_0, \mathbf{m}_0)$ such that

$$\pi(\rho_0) - \pi(\bar{\rho}) \quad \text{and} \quad \frac{|\mathbf{m}_0|^2}{\rho_0} \in L^1(\mathbb{R}^d), \quad (18)$$

where we agree that $\mathbf{m}_0 = 0$ on $\{ \mathbf{x} \in \mathbb{R}^d / \rho_0(\mathbf{x}) = 0 \}$. More precisely, he obtains the existence of global weak solutions $(\rho, \mathbf{u})$ to (8)(9)(10) such that

- $\rho - \bar{\rho} \in L^\infty(0, \infty; L^2_\rho(\mathbb{R}^d))$ (where $L^2_\rho(\mathbb{R}^d)$ spaces are Orlicz spaces defined in [18]),

- $\rho \in L^\infty(0, \infty; L^2(\mathbb{R}^d))$ (where $L^2(\mathbb{R}^d)$ spaces are Orlicz spaces defined in [18]),

- $\mathbf{u} \in L^\infty(0, \infty; L^2(\mathbb{R}^d))^d$ (where $L^2(\mathbb{R}^d)^d$ spaces are Orlicz spaces defined in [18]),
\[ \bullet \, u \in L^2(0, \infty; \dot{H}^1(\mathbb{R}^d))^d, \]

with in addition

\[ \bullet \, \rho \in C([0, \infty); L^p_{\text{loc}}(\mathbb{R}^d)) \text{ if } 1 \leq p < \gamma, \]

\[ \bullet \, \rho |u|^2 \in L^\infty(0, \infty; L^1(\mathbb{R}^d)), \rho u \in C([0, \infty); L^{2\gamma/(\gamma+1)}_{\text{loc}}(\mathbb{R}^d) - w), \]

\[ \bullet \, \rho \in L^3_{\text{loc}}([0, \infty) \times \mathbb{R}^d) \text{ where } q = \gamma - 1 + 2\gamma/d. \]

Moreover, the energy inequality (13) holds for almost every \( t \geq 0. \)

Notice that the main difficulty for proving Lions’ theorem consists in strong compactness properties of the density \( \rho \) in \( L^p_{\text{loc}} \) spaces required to pass to the limit in the pressure term \( p(\rho) = \alpha \rho^\gamma. \) In the capillary case \( \kappa > 0, \) more a priori bounds are available for the density, which belongs to \( L^\infty(0, \infty; \dot{H}^1(\mathbb{R}^d)). \) Hence, one can easily pass to the limit in the pressure term. However, in the remaining quadratic terms involving gradients of the density \( \nabla \rho \) (see (5)), we have been unable to pass to the limit.

Let us mention now that the existence of strong solutions is known since the works by H. Hattori and D. Li [13] [14]. Notice that high order regularity in Sobolev spaces \( H^s \) is required, namely the initial data \((\rho_0, m_0)\) is assumed to belong to \( H^s \times H^{s-1} \) with \( s \geq d/2 + 4. \) Moreover, they considered convex pressure profiles, which cannot cover the case of Van der Waals’ equation of state.

Here we want to investigate the well-posedness of the problem in critical spaces, that is, in spaces which are invariant by the scaling of Korteweg’s system. Recall that such an approach is now classical for incompressible Navier-Stokes equations (see for example [7] and the references enclosed) and yields local well-posedness (or global well-posedness for small data) in spaces with minimal regularity.

Let us explain precisely the scaling of Korteweg’s system. We can easily verify that, if \((\rho, u)\) solves (8)(9), so does \((\rho_\lambda, u_\lambda)\) where

\[ \rho_\lambda(t, x) = \rho(\lambda^2 t, \lambda x) \quad \text{and} \quad u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x), \]

provided the pressure law \( P \) has been changed into \( \lambda^2 P. \)

**Definition 1** We will say that a functional space is critical with respect to the scaling of the equation if the associated norm is invariant under the transformation \((\rho, u) \rightarrow (\rho_\lambda, u_\lambda)\) (up to a constant independent of \( \lambda \)).
This suggests us to choose initial data \((\rho_0, u_0)\) in spaces whose norm is invariant by \((\rho_0, u_0) \mapsto (\rho_0(\lambda), \lambda u_0(\lambda))\).

A natural candidate is the homogeneous Sobolev space \(H^{d/2} \times (H^{d/2-1})^d\), but unfortunately \(\dot{H}^{d/2}\) is not included in \(L^\infty\), so we cannot expect to get \(L^\infty\) control on the density choosing \(\rho_0 \in \dot{H}^{d/2}\). This is why, instead of working with classical homogeneous Sobolev spaces \(\dot{H}^s(\mathbb{R}^d)\), we will consider homogeneous Besov spaces with the same derivative index \(B^s = B^{\frac{s}{d}}_{2,1}(\mathbb{R}^d)\) (for the corresponding definitions, we refer to section 3). One of the nice properties of \(B^s\) spaces for critical exponents \(s\) is that \(B^{d/2}\) is an algebra embedded in \(L^\infty\). This allows to control the density from below and from above, without requiring more regularity on derivatives of \(\rho\).

Since a global in time approach does not seem to be accessible for general data, we will mainly consider the global well-posedness problem for initial data close enough to stable equilibria (section 4). More precisely, we will state the following theorem:

**Theorem 1** Let \(\bar{\rho} > 0\) be such that \(P'(\bar{\rho}) > 0\). Suppose that the initial density fluctuation \(\rho_0 - \bar{\rho}\) belongs to \(B^{d/2} \cap B^{d/2-1}\), that the initial velocity \(u_0\) is in \(B^{d/2-1}\) and that the exterior forcing term \(f\) is in \(L^1(\mathbb{R}^+; B^{d/2-1})^d\). Then there exists a constant \(\eta > 0\) depending only on \(\kappa, \mu, \lambda, \rho, P'(\bar{\rho})\) and \(d\), and such that, if

\[
\|\rho_0 - \bar{\rho}\|_{B^{d/2-1} \cap B^d} + \|u_0\|_{B^{d/2-1}} + \|f\|_{L^1(\mathbb{R}^+; B^{d/2-1})^d} \leq \eta,
\]

then (8)(9)(10) has a unique global solution \((\rho, u)\) such that the density fluctuation \((\rho - \bar{\rho})\) is in \(C(\mathbb{R}^+; B^{d/2-1} \cap B^{d/2}) \cap L^1(\mathbb{R}^+; B^{d/2+1} \cap B^{d/2+2})\) and the velocity \(u\) is in \(C(\mathbb{R}^+; B^{d/2-1})^d \cap L^1(\mathbb{R}^+; B^{d/2+1})^d\).

In section 5, we get a local in time existence result for initial densities bounded away from zero, which does not require any stability assumption on the pressure law, and thus applies to Van der Waals' law. The precise statement reads as follows:

**Theorem 2** Let \(0 < c \leq \bar{\rho}\). Suppose that the forcing term \(f\) belongs to \(L^1_{loc}(\mathbb{R}^+; B^{d/2-1})^d\), that the initial velocity \(u_0\) belongs to \((B^{d/2-1})^d\), and that the initial density \(\rho_0\) satisfies \((\rho_0 - \bar{\rho}) \in B^{d/2}\) and \(\rho_0 \geq c\). Then there exists \(T > 0\) such that (8)(9)(10) has a unique solution \((\rho, u)\) satisfying \((\rho - \bar{\rho})\) is in \(C([0, T]; B^{d/2}) \cap L^1([0, T]; B^{d/2+2})\) and \(u\) is in \(C([0, T]; B^{d/2-1}) \cap L^1([0, T]; B^{d/2+1})\).

In section 6, we show that the problem is still locally well-posed in more general scaling invariant Besov spaces of type \(B^s_{p,1}\) which are not related to energy.
spaces (namely $\rho - \bar{\rho}$ is assumed to be in $B^{d/p}_{p,1}$ and $u_0$ to be in $(B^{d/p-1}_{p,1})^d$). No stability assumption on the pressure is required, but we have to suppose that the density is close to a constant (see theorem 5). Let us observe that working with $p > d$ allows to consider initial velocities in $B^a_{p,1}$ spaces with negative exponents $a$, which is in particular relevant for oscillating initial data.

Finally, we will investigate blow-up properties of smooth solutions without smallness assumptions on the data, like in the work of Z. Xin [22, and study sufficient conditions for the existence of weak solutions close to equilibria in dimension $d = 2$.

**Notation:** In all the paper, $C$ will stand for a “harmless” constant, and we will sometimes use the notation $A \lesssim B$ equivalently to $A \leq CB$.

### 3 Littlewood-Paley theory and Besov spaces

#### 3.1 Littlewood-Paley decomposition

We begin this section by recalling definitions of the homogeneous Littlewood-Paley decomposition, which relies upon a dyadic partition of unity. We can use for instance any $\varphi \in C^\infty(\mathbb{R}^d)$, supported in $\mathcal{C} \defeq \{\xi \in \mathbb{R}^d, 3/4 \leq |\xi| \leq 8/3\}$ such that

$$\sum_{\ell \in \mathbb{S}} \varphi(2^{-\ell}\xi) = 1 \quad \text{if} \quad \xi \neq 0.$$  

Denoting $h = \mathcal{F}^{-1}\varphi$, we then define the dyadic blocks as follows

$$\Delta_\ell u \defeq \varphi(2^{-\ell}D)u = 2^{\ell d} \int_{\mathbb{R}^d} h(2^\ell y)u(x-y)\,dy, \quad \text{and} \quad S_\ell u = \sum_{k \leq \ell-1} \Delta_k u.$$

The formal decomposition

$$u = \sum_{\ell \in \mathbb{S}} \Delta_\ell u \quad \tag{19}$$

is called homogeneous Littlewood-Paley decomposition. Let us observe that the above formal equality does not hold in $\mathcal{S}'(\mathbb{R}^d)$ for two reasons:

i) The right-hand side does not necessarily converge in $\mathcal{S}'(\mathbb{R}^d)$,

ii) Even if it does, the equality is not always true in $\mathcal{S}'(\mathbb{R}^d)$ (consider the case $u = 1$).

Nevertheless, (19) is true modulo polynomials, that is, if $u \in \mathcal{S}'(\mathbb{R}^d)$, then $\sum_{\ell \in \mathbb{S}} \Delta_\ell u$ converges modulo $\mathcal{P}[\mathbb{R}^d]$ and (19) holds in $\mathcal{S}'(\mathbb{R}^d)/\mathcal{P}[\mathbb{R}^d]$ (see [21]).
Furthermore, the above dyadic composition has nice properties of quasi-orthogonality: with our choice of $\varphi$, we have
\[ \Delta_k \Delta_\ell u \equiv 0 \quad \text{if} \quad |k - \ell| \geq 2, \quad \text{and} \quad \Delta_k (S_{\ell-1} u \Delta_\ell u) \equiv 0 \quad \text{if} \quad |k - \ell| \geq 5. \quad (20) \]

### 3.2 Homogeneous Besov spaces

**Definition 2** For $s \in \mathbb{R}$, $p \in [1, +\infty]$, $q \in [1, +\infty]$ and $u \in \mathcal{S}'(\mathbb{R}^d)$, we set
\[
\|u\|_{B^s_{p,q}} \overset{\text{def}}{=} \left( \sum_{\ell \in \mathbb{Z}} 2^{s\ell} \|\Delta_\ell u\|_{L^p}^q \right)^{\frac{1}{q}}.
\]

A complication due to the choice of homogeneous spaces arises at this point. Indeed, $\|\cdot\|_{B^s_{p,q}}$ cannot be a norm on $\{u \in \mathcal{S}'(\mathbb{R}^d), \|u\|_{B^s_{p,q}} < +\infty\}$ because $\|u\|_{B^s_{p,q}} = 0 \iff u$ is a polynomial. This enforces us to adopt the following definition for homogeneous Besov spaces (see [5] for more details):

**Definition 3** Let $s \in \mathbb{R}$, $p \in [1, +\infty]$ and $q \in [1, +\infty]$. Denote $m = [s - d/p]$ if $s - d/p \notin \mathbb{Z}$ or $q > 1$ and $m = s - d/p - 1$ otherwise. If $m < 0$, then we define $B^s_{p,q}(\mathbb{R}^d)$ as
\[
B^s_{p,q} = \{u \in \mathcal{S}'(\mathbb{R}^d), \|u\|_{B^s_{p,q}} < \infty \text{ and } u = \sum_{\ell \in \mathbb{Z}} \Delta_\ell u \text{ in } \mathcal{S}'(\mathbb{R}^d)\}\.
\]

If $m \geq 0$, we denote by $\mathcal{P}_m(\mathbb{R}^d)$ the set of polynomials of degree $\leq m$ and we set
\[
B^s_{p,q} = \{u \in \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}_m(\mathbb{R}^d), \|u\|_{B^s_{p,q}} < \infty \text{ and } u = \sum_{\ell \in \mathbb{Z}} \Delta_\ell u \text{ in } \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}_m(\mathbb{R}^d)\}\.
\]

**Remark 1** The above definition is a natural way of generalizing the homogeneous Sobolev or Hölder spaces: one can show that $B^s_{1,\infty}$ is the homogeneous Hölder space $C^s$ and that $B^s_{2,2}$ is the homogeneous Sobolev space $H^s$.

In the sequel, we will use only Besov spaces $B^s_{p,q}$ with $q = 1$ and we will denote them by $B^s_p$ or even by $B^s$ if there is no ambiguity on the index $p$.  

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3.3 Basic properties of Besov spaces

Proposition 1 The following properties hold:

i) Density: if \( p < +\infty \), then \( C_0^\infty \) is dense in \( B_p^s \).

ii) Derivation: there exists a universal constant \( C \) such that

\[
C^{-1} \| u \|_{B_p^s} \leq \| \nabla u \|_{B_p^{s-1}} \leq C \| u \|_{B_p^s}.
\]

ii') Fractional derivation: let \( \Lambda \overset{\text{def}}{=} \sqrt{-\Delta} \) and \( \sigma \in \mathbb{R} \). Then the operator \( \Lambda^\sigma \) is an isomorphism from \( B_p^s \) to \( B_p^{s-\sigma} \).

iii) Sobolev embeddings: if \( p_1 < p_2 \) then \( B_p^{s_1} \hookrightarrow B_p^{s_2} \)

iv) Algebraic properties: for \( s > 0 \), \( B_p^s \cap L^\infty \) is an algebra.

v) Interpolation: \( (B_p^{s_1}, B_p^{s_2})_{\theta, 1} = B_p^{\theta s_1 + (1-\theta)s_2} \).

In section 6, we will make extensive use of the space \( B_p^{d/p} \). Note that, if \( p < +\infty \), then \( B_p^{d/p} \) is an algebra included in the space \( C_0 \) of continuous functions which tend to 0 at infinity. Note also that \( B_p^{d/p} \times (B_p^{d/p-1})^d \) is invariant by the scaling of Korteweg’s system.

In sections 4 and 5, we will focus on the case \( p = 2 \). Note that the following inclusion chain

\[
B_2^{d/2} \hookrightarrow \dot{H}^{d/2} = B_2^{d/2} \hookrightarrow B_2^{d/2}.
\]

shows us that \( \dot{H}^{d/2} \) is very close to \( B_2^{d/2} \). But \( B_2^{d/2} \) has two additional nice properties: to be an algebra and to be a subset of \( C_0 \).

3.4 Besov-Chemin-Lerner spaces

The study of non stationary PDE’s usually requires spaces of type \( L_p^r(T; X) \overset{\text{def}}{=} L^r(0, T; X) \) for appropriate Banach spaces \( X \). In our case, we expect \( X \) to be a Besov space, so that it is natural to localize the equations through Littlewood-Paley decomposition. We then get estimates for each dyadic block and perform integration in time. But, in doing so, we obtain bounds in spaces which are not of type \( L^r(0, T; B_p^d) \) except if \( r = 1 \). This approach was initiated in [9] and naturally leads to the following definitions:

Definition 4 Let \( (\rho, p) \in [1, +\infty]^2 \), \( T \in [0, +\infty] \) and \( s \in \mathbb{R} \). We set

\[
\| u \|_{L_T^p(B_p^s)} \overset{\text{def}}{=} \sum_{\ell \in \mathbb{Z}} 2^{\ell s} \left( \int_0^T \| \Delta_{\ell} u(t) \|_{L_p}^\rho \ dt \right)^{1/\rho}.
\]

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Noticing that Minkowski’s inequality yields \( \| u \|_{L^p_T(B^p)} \leq \| u \|_{\widetilde{L}^p_T(B^p)} \), we define \( \widetilde{L}^p_T(B^p) \) spaces as follows

\[
\widetilde{L}^p_T(B^p) \overset{df}{=} \{ u \in L^p_T(B^p), \| u \|_{\widetilde{L}^p_T(B^p)} < +\infty \}.
\]

Let us observe that \( L^p_T(B^p) = \widetilde{L}^p_T(B^p) \) but that the embedding \( \widetilde{L}^p_T(B^p) \subset L^p_T(B^p) \) is strict if \( \rho > 1 \).

We will denote by \( \widetilde{C}_T(B^p) \) the subset of functions \( u \) belonging to \( \widetilde{L}^p_T(B^p) \cap C_T(B^p) \) such that for all finite \( t \in [0,T] \),

\[
\lim_{t_1, t_2 \to t} \sum_{\ell \in \mathbb{Z}} 2^{\ell s} \sup_{[t_1, t_2]} \| \Delta_t u(\tau) \|_{L^p} = \| u(t) \|_{B^p}.
\]

We will often use the following interpolation property

\[
\| u \|_{\widetilde{L}^p_T(B^p)} \leq \| u \|_{L^p_T(B^p)}^{\frac{s}{\rho_1}} \| u \|_{\widetilde{L}^p_T(B^p)}^{\frac{s}{\rho_2}} \text{ with } \frac{1}{\rho} = \frac{s}{\rho_1} + \frac{1-s}{\rho_2} \text{ and } s = s_1 + (1-s) s_2,
\]

and the following embeddings

\( \widetilde{L}^p_T(B^{d/p}) \hookrightarrow L^p_T(C^0) \) and \( \widetilde{C}_T(B^{d/p}) \hookrightarrow C([0,T] \times \mathbb{R}^d) \).

The \( \widetilde{L}^p_T(B^p) \) spaces suit particularly well to the study of smoothing properties of the heat equation. In [7], J.-Y. Chemin proves the following proposition

**Proposition 2** Let \( p \in [1, +\infty], 1 \leq \rho_2 \leq \rho_1 \leq +\infty \), and \( u \) solving

\[
\begin{aligned}
&\partial_t u - \nu \Delta u = f, \\
&u_{t=0} = u_0.
\end{aligned}
\]

Then there exists \( C > 0 \) depending only on \( d, \nu, \rho_1 \) and \( \rho_2 \) such that

\[
\| u \|_{L^p_T(B^{d/p})} \leq C \| u_{0} \|_{B^p} + C \| f \|_{L^p_T(B^{d-2/n \rho_2})}.
\]

In sections 4, 5 and 6, we will point out similar smoothing properties for the linearized Korteweg system.

Let us now state properties of \( \widetilde{L}^p_T(B^p) \) spaces with respect to the product.
Proposition 3 If $s > 0$, $1/\rho_2 + 1/\rho_3 = 1/\rho_1 + 1/\rho_4 = 1/\rho \leq 1$, $u \in \tilde{L}^\rho_T (B^s_p)$ and $v \in \tilde{L}^\rho_T (B^s_p)$ then $uv \in \tilde{L}^\rho_T (B^s_p)$ and
\[
\|uv\|_{\tilde{L}^\rho_T (B^s_p)} \lesssim \|u\|_{\tilde{L}^\rho_T (L^\infty)} \|v\|_{\tilde{L}^\rho_T (B^s_p)} + \|v\|_{\tilde{L}^\rho_T (L^\infty)} \|u\|_{\tilde{L}^\rho_T (B^s_p)}.
\]

If $s_1, s_2 \leq d/p$, $s_1 + s_2 > 0$, $1/\rho_1 + 1/\rho_2 = 1/\rho \leq 1$, $u \in \tilde{L}^\rho_T (B^{s_1}_p)$ and $v \in \tilde{L}^\rho_T (B^{s_2}_p)$ then $uv \in \tilde{L}^\rho_T (B^{s_1+s_2-d/p}_p)$ and
\[
\|uv\|_{\tilde{L}^\rho_T (B^{s_1+s_2-d/p}_p)} \lesssim \|u\|_{\tilde{L}^\rho_T (B^{s_1}_p)} \|v\|_{\tilde{L}^\rho_T (B^{s_2}_p)}.
\]

The proof is a straightforward adaptation of the corresponding results for usual homogeneous Besov spaces (see [8]). We finally need a composition lemma in $\tilde{L}^\rho_T (B^s_p)$ spaces.

Lemma 1 Let $s > 0$, $p \in [1, +\infty]$ and $u \in \tilde{L}^\rho_T (B^s_p) \cap L^\infty_T (L^\infty)$.

i) Let $F \in W^{[s]+2,\infty}_{loc} (\mathbb{R}^d)$ such that $F(0) = 0$. Then $F(u) \in \tilde{L}^\rho_T (B^s_p)$. More precisely, there exists a function $C_0$ depending only on $s$, $p$, $d$ and $F$ such that
\[
\|F(u)\|_{\tilde{L}^\rho_T (B^s_p)} \leq C_0 \left(\|u\|_{L^\infty_T (L^\infty)}\right) \|u\|_{\tilde{L}^\rho_T (B^s_p)}.
\]

ii) If $v$ also belongs to $\tilde{L}^\rho_T (B^s_p) \cap L^\infty_T (L^\infty)$ and $G \in W^{[s]+3,\infty}_{loc} (\mathbb{R}^d)$, then $G(v) - G(u)$ belongs to $\tilde{L}^\rho_T (B^s_p)$ and there exists a function $C$ depending only on $s$, $p$, $d$ and $G$, and such that
\[
\|G(v) - G(u)\|_{\tilde{L}^\rho_T (B^s_p)} \leq C \left(\|u\|_{L^\infty_T (L^\infty)}, \|v\|_{L^\infty_T (L^\infty)}\right) \\
\times \left(\|v - u\|_{L^\infty_T (B^s_p)} (\|u\|_{L^\infty_T (L^\infty)} + \|v\|_{L^\infty_T (L^\infty)}) + \|v - u\|_{L^\infty_T (L^\infty)} (\|u\|_{\tilde{L}^\rho_T (B^s_p)} + \|v\|_{\tilde{L}^\rho_T (B^s_p)})\right).
\]

Proof: For i), one just has to use the proof of [2] and replace $L^2$ norms with $L^p$ norms. For ii), we use the following identity
\[
G(v) - G(u) = (v - u) \int_0^1 H(u + \tau (v - u)) d\tau + G'(0)(v - u),
\]
where $H(w) = G'(w) - G'(0)$, and we conclude using i) and proposition 3.
4 Global solutions near equilibrium

In this section, we want to prove global existence and uniqueness results of suitably smooth solutions to the Korteweg system (8)(9) in functional spaces of type $L^s_t(B^s_{r_t})$ very close to the physical energy spaces. Given a reference density $\rho$ such that the stability condition $P'(\bar{\rho}) > 0$ is satisfied, we introduce the density fluctuation $q = (\rho - \bar{\rho})/\bar{\rho}$ and the scaled momentum $m = \rho u/\bar{\rho}$. We also define the scaled viscosity coefficients $\bar{\mu} = \mu/\bar{\rho}$ and $\bar{\lambda} = \lambda/\bar{\rho}$ and the scaled surface tension coefficient $\bar{\kappa} = \rho \kappa$. Assuming that the density $\rho$ is bounded away from zero, we rewrite the Korteweg system (8)(9) as follows

$$\partial_t q + \text{div} \ m = 0,$$

$$\partial_t m - \bar{\mu} \Delta m - (\bar{\lambda} + \bar{\rho}) \text{div} \ m - \bar{\kappa} \nabla \Delta q + P'(\bar{\rho}) \nabla q = G(q, m) + f,$$  \hspace{1cm} (22)

$$(q, m)|_{t=0} = (q_0, m_0),$$  \hspace{1cm} (23)

where we define $G = G_1 + G_2 + G_3 + G_4 + G_5$ by

$$G_1(q, m) = -\text{div} \left( \frac{m \otimes m}{1 + q} \right),$$

$$G_2(q, m) = -\nabla H(q),$$

$$G_3(q, m) = -\bar{\mu} \Delta \left( \frac{q m}{1 + q} \right) - (\bar{\lambda} + \bar{\rho}) \text{div} \left( \frac{q m}{1 + q} \right),$$

$$G_4(q, m) = \frac{\bar{\kappa}}{2} \nabla \left( \Delta q^2 - |\nabla q|^2 \right) - \bar{\kappa} \text{div} (\nabla q \otimes \nabla q) = \bar{\kappa} q \nabla \Delta q,$$

and

$$G_5(q, m) = f_q,$$

$H$ being defined by $H(q) = (P(\bar{\rho}(1 + q)) - P(\bar{\rho}) - P'(\bar{\rho}) q \bar{\rho})/\bar{\rho}$.

In section 4.1, we study the linearized system around $(q, m) = (0, 0)$, which turns out to have the same smoothing properties as the heat equation. Finally, we prove in section 4.2 our main global theorem, estimating the right-hand side $G$ of (22) in terms of suitable norms of $(q, m)$. Notice that in section 4 and 5, $B^s_{r_t}$ will stand for $B^s_{2,1}$.

4.1 Estimates for the linearized system

This section is devoted to the linearized isothermal system of Korteweg type around $(q, m) = (0, 0)$. This system reads

$$\begin{cases}
\partial_t q + \text{div} \ m = F, \\
\partial_t m - \bar{\mu} \Delta m - (\bar{\lambda} + \bar{\rho}) \text{div} \ m - \bar{\kappa} \nabla \Delta q + \beta \nabla q = G.
\end{cases}$$  \hspace{1cm} (LNSK1)
The term $\beta \nabla q$ corresponds to the linearized pressure (that is $\beta = P'(\tilde{p}) > 0$). Our purpose is to prove estimates for \((LNSK/1)\) in Besov spaces closely related to energy spaces. We get:

**Proposition 4** Let $s \in \mathbb{R}$, $1 \leq \rho_1 \leq \rho \leq +\infty$ and $T \in [0, +\infty]$. If $(q_0, m_0) \in (B^s \cap B^{s-1}) \times (B^{s-1})^d$ and $(F, G) \in L_T^1((B^{s-2+2/\rho_1} \cap B^{s-3+2/\rho_1}) \times (B^{s-3+2/\rho_1}))$ then the linear system \((LNSK/1)\) has a unique solution $(q, m) \in \tilde{C}_T((B^s \cap B^{s-1}) \times (B^{s-1})^d) \cap L_T^1((B^{s-2+1+2/\rho_1} \cap B^{s-1+2/\rho_1}) \times (B^{s-1+2/\rho_1})^d)$. Moreover, there exists a constant $C$ depending only on $\rho_1$, $\tilde{p}$, $\tilde{\lambda}$, $\tilde{\kappa}$ and $\beta$ such that the following inequality holds:

\[
\|q\|_{\tilde{L}_T^p(B^{s+2/\rho_1} \cap B^{s+1+2/\rho_1})} + \|m\|_{\tilde{L}_T^p(B^{s-1+2/\rho_1})} \leq C \left(\|q_0\|_{B^s \cap B^{s-1}} + \|m_0\|_{B^{s-1}} + \|f\|_{\tilde{L}_T^p(B^{s-2+1+2/\rho_1} \cap B^{s-1+2/\rho_1})} + \|G\|_{\tilde{L}_T^p(B^{s-1+2/\rho_1})}\right).
\]

**Proof:** Denote by $W(t)$ the semi-group associated to \((LNSK/1)\). According to Duhamel’s formula,

\[
\begin{pmatrix}
  q(t) \\
  m(t)
\end{pmatrix} = W(t) \begin{pmatrix}
  q_0 \\
  m_0
\end{pmatrix} + \int_0^t W(t-s) \begin{pmatrix}
  F(s) \\
  G(s)
\end{pmatrix} ds. \quad (24)
\]

Let us first consider the case $F \equiv 0$ and $G \equiv 0$ and denote $(q_\ell(t), m_\ell(t)) = W(t)(\Delta q_0, \Delta m_0)^\ell$. Then, we have the following lemma

**Lemma 2** There exists two positive constants $c$ and $C$ depending only on $\tilde{\lambda}$, $\tilde{p}$, $\tilde{\kappa}$ and $\beta$ such that for all $\ell \in \mathbb{Z}$,

\[
\|m_\ell(t)\|_{L^2} + \|\nabla q_\ell(t)\|_{L^2} + \|q_\ell(t)\|_{L^2} \leq C e^{-ct^2 \ell} \left(\|\Delta q_0\|_{L^2} + \|\Delta m_0\|_{L^2} + \|\nabla \Delta q_0\|_{L^2} + \|\Delta \Delta q_0\|_{L^2}\right).
\]

**Proof of lemma 2:** We apply the operator $\Delta_\ell$ to \((LNSK/1)\) in the case $F \equiv G \equiv 0$ and get

\[
\begin{align*}
\partial_\ell q_\ell + \text{div } m_\ell &= 0, \quad (25) \\
\partial_\ell m_\ell - \tilde{p} \Delta m_\ell - (\tilde{\lambda} + \tilde{p}) \text{div } m_\ell - \tilde{\kappa} \nabla q_\ell - \beta \nabla q_\ell &= 0. \quad (26)
\end{align*}
\]

In view of equation (25), integrations by parts yield

\[
\int_{\mathbb{R}^d} \Delta q_\ell \text{div } m_\ell \, dx = -\frac{1}{2} \frac{d}{dt} \|\nabla q_\ell\|_{L^2}^2 \quad \text{and} \quad \int_{\mathbb{R}^d} q_\ell \text{div } m_\ell \, dx = -\frac{1}{2} \frac{d}{dt} \|q_\ell\|_{L^2}^2.
\]
Thus, taking scalar product of (26) with $m_t$, we get
\[
\frac{1}{2} \frac{d}{dt} \left( \|m_t\|_{L^2}^2 + \beta \|q_t\|_{L^2}^2 + \bar{\kappa} \|\nabla q_t\|_{L^2}^2 \right) + \bar{\mu} \|\nabla m_t\|_{L^2}^2 + (\bar{\lambda} + \bar{\mu}) \|\text{div} m_t\|_{L^2}^2 = 0. \tag{27}
\]

In order to obtain a second energy estimate, we take the scalar product of $m_t$ with the gradient of $m_t$, which yields
\[
\int_{\mathbb{R}^d} m_t \cdot \partial_t \nabla q_t \, dx - \|\text{div} m_t\|_{L^2}^2 = 0. \tag{28}
\]
Taking the scalar product of (26) with $\nabla q_t$, we obtain
\[
\int_{\mathbb{R}^d} \nabla q_t \cdot \partial_t m_t \, dx + \bar{\kappa} \|\Delta q_t\|_{L^2}^2 + \beta \|\nabla q_t\|_{L^2}^2 \leq C \|\nabla m_t\|_{L^2} \left\| \nabla^2 q_t \right\|_{L^2}. \tag{29}
\]
Summing (28) and (29), we deduce
\[
\frac{d}{dt} \left( \int_{\mathbb{R}^d} m_t \cdot \nabla q_t \, dx \right) + \frac{\bar{\kappa}}{2} \left\| \nabla^2 q_t \right\|_{L^2}^2 + \beta \|q_t\|_{L^2} = C \|\nabla m_t\|_{L^2}^2. \tag{30}
\]
Let $\alpha > 0$ be a constant to be chosen later and denote
\[
h_t^2 = \|m_t\|_{L^2}^2 + \bar{\kappa} \|\nabla q_t\|_{L^2}^2 + \beta \|q_t\|_{L^2}^2 + 2\alpha \int_{\mathbb{R}^d} m_t \cdot \nabla q_t \, dx.
\]
As a result, from (30) and (27), we derive for some positive constant $C_0$
\[
\frac{1}{2} \frac{d}{dt} h_t^2 + C_0 \left( \|\nabla m_t\|_{L^2}^2 + \alpha \left\| \nabla^2 q_t \right\|_{L^2}^2 + \alpha \|\nabla q_t\|_{L^2}^2 \right) \leq C \alpha \|\nabla m_t\|_{L^2}^2. \tag{31}
\]
Now choosing $\alpha$ suitably small, we deduce that
\[
\frac{1}{\delta} h_t^2 \leq \|m_t\|_{L^2}^2 + \bar{\kappa} \|\nabla q_t\|_{L^2}^2 + \beta \|q_t\|_{L^2}^2 \leq \delta h_t^2, \tag{32}
\]
for some positive $\delta$. Thus, there exists a constant $c > 0$ such that
\[
\frac{1}{2} \frac{d}{dt} h_t^2 + c2^t h_t^2 \leq 0,
\]
so that the proof of lemma 2 is complete. \[\Box\]

**Proof of proposition 4 (continued):** In view of lemma 2 and formula (24), we have
\[
\|\Delta_t m(t)\|_{L^2} + \|\nabla \Delta_t q(t)\|_{L^2} + \|\Delta_t q(t)\|_{L^2} \leq C e^{-\sigma_2 t} \left( \|\Delta_t m_0\|_{L^2} + \|\nabla \Delta_t q_0\|_{L^2} + \|\Delta_t q_0\|_{L^2} \right)
\]
\[
+ \int_0^t e^{-\sigma_2 \tau} \left( \|\Delta_t G(\tau)\|_{L^2} + \|\nabla \Delta_t F(\tau)\|_{L^2} + \|\Delta_t F(\tau)\|_{L^2} \right) \, d\tau,
\]
so routine computations yield proposition 4. \[\Box\]
4.2 Global existence and uniqueness

Let us first introduce functional spaces needed in the main global existence result. We will prove existence in the space

\[ E = \left( \tilde{C}(B^{\frac{d}{2}} \cap B^{\frac{d}{2}}) \cap L^1(B^{\frac{d}{2}} \cap B^{\frac{d}{2}+1}) \right)^d, \]

and uniqueness in the larger space

\[ \tilde{E} = \tilde{C}(B^{\frac{d}{4}} \times (B^{\frac{d}{4}-1})^d) \cap L^2(B^{\frac{d}{2}} \times (B^{\frac{d}{2}})^d). \]

We denote by \( \| \cdot \|_{E} \) and \( \| \cdot \|_{\tilde{E}} \) the corresponding norms. Since \( B^{\frac{d}{2}} \) is a Banach space, it is easy to verify that \( E \) and \( \tilde{E} \) are also Banach spaces. We now turn to our main global existence theorem

**Theorem 3** Let \( \tilde{\rho} > 0 \) be such that \( P'(\tilde{\rho}) > 0 \). Suppose that the initial density fluctuation \( q_0 \) belongs to \( B^{\frac{d}{2}} \cap B^{\frac{d}{2}+1} \), that the initial momentum \( m_0 \) is in \( B^{\frac{d}{2}-1} \) and that the forcing term \( f \) is in \( L^1(\mathbb{R}^+; B^{\frac{d}{2}-1})^d \). Then there exists a constant \( \eta > 0 \) depending only on \( \tilde{\kappa}, \tilde{\rho}, \tilde{\lambda}, \tilde{\rho}, P'(\tilde{\rho}) \) such that, if

\[ \|q_0\|_{B^{\frac{d}{4}-1} \cap B^{\frac{d}{4}}} + \|m_0\|_{B^{\frac{d}{4}-1}} + \|f\|_{L^1(B^{\frac{d}{4}-1})} \leq \eta \]

then (21)(22)(23) has a unique global solution \( (q, m) \) in \( \tilde{E} \). In addition, \( (q, m) \) belongs to \( E \).

**Proof:** Let us denote \( (q_L, m_L) \) the “free” solution of the linearized system

\[ \begin{pmatrix} q_L(t) \\ m_L(t) \end{pmatrix} = W(t) \begin{pmatrix} q_0 \\ m_0 \end{pmatrix} + \int_0^t W(t-s) \begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds. \]

We define the functional \( \Psi_{q_L,m_L} \) in a neighbourhood of 0 in \( E \) by

\[ \Psi_{q_L,m_L}(\tilde{q}, \tilde{m}) = \int_0^t W(t-s) \begin{pmatrix} 0 \\ G(q_L + \tilde{q}, m_L + \tilde{m})(s) \end{pmatrix} ds. \quad (33) \]

To prove the existence part of the theorem, we just have to show that \( \Psi_{q_L,m_L} \) has a fixed point in \( E \).
First step: stability of $B(0, R)$.

We start by proving that the ball $B(0, R)$ of $E$ is stable under $\Psi_{qL, mL}$ provided $R$ is small enough. Denote $q = qL + \tilde{q}$ and $m = mL + \tilde{m}$. According to proposition 4, we have

$$\|(qL, mL)\|_E \leq C\eta,$$
(34)

$$\|\Psi_{qL, mL}(\tilde{q}, \tilde{m})\|_E \leq C\|G(q, m)\|_{L^1(B_{1/4}^d)}.$$
(35)

Making the assumption

$$\|q\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)} \leq \frac{1}{2},$$
\((H)\)

and using proposition 3 and lemma 1, we deduce the following estimates:

$$\|G_1(q, m)\|_{L^1(B_{1/4}^d)} \leq C\|m\|_{L^1(B_{1/4}^d)}\|m\|_{L^\infty(B_{1/4}^d)}(1 + \|q\|_{L^\infty(B_{1/4}^d)}),$$

$$\|G_2(q, m)\|_{L^1(B_{1/4}^d)} \leq C\|q\|_{L^1(B_{1/4}^d)}\|q\|_{L^\infty(B_{1/4}^d)} + \|q\|_{L^\infty(B_{1/4}^d)}\|m\|_{L^1(B_{1/4}^d)}$$

$$\|G_3(q, m)\|_{L^1(B_{1/4}^d)} \leq C\|q\|_{L^1(B_{1/4}^d)}\|q\|_{L^\infty(B_{1/4}^d)} + \|q\|_{L^\infty(B_{1/4}^d)}\|m\|_{L^1(B_{1/4}^d)}$$

$$\|G_4(q, m)\|_{L^1(B_{1/4}^d)} \leq C\|q\|_{L^1(B_{1/4}^d)}\|q\|_{L^\infty(B_{1/4}^d)}$$

$$\|G_5(q, m)\|_{L^1(B_{1/4}^d)} \leq C\|q\|_{L^1(B_{1/4}^d)}\|q\|_{L^\infty(B_{1/4}^d)}$$

since by interpolation, we have

$$\|Z\|_{L^2(B^d)} \leq \|Z\|_{L^\infty(B_{1/4}^d)}\|Z\|_{L^1(B_{1/4}^d)}.$$
Second step: Contraction properties.

We consider two elements \((\tilde{q}_1, \tilde{m}_1)\) and \((\tilde{q}_2, \tilde{m}_2)\) in \(B(0, R)\), and denote \(q_1 = q_L + \tilde{q}_1\), \(m_1 = m_L + \tilde{m}_1\), \(q_2 = q_L + \tilde{q}_2\) and \(m_2 = m_L + \tilde{m}_2\). According to (33) and to proposition 4, we have

\[
\|\Psi_{q_1, m_1}(\tilde{q}_1, \tilde{m}_1) - \Psi_{q_2, m_2}(\tilde{q}_2, \tilde{m}_2)\|_E \leq C\|G(q_2, m_2) - G(q_1, m_1)\|_{L^1(B^{4-1})}. \tag{38}
\]

Under assumption \((\mathcal{H})\) for \(q_1\) and \(q_2\), we obtain estimates for \(G(q_2, m_2) - G(q_1, m_1)\). Indeed, we just have to apply proposition 3 and lemma 1 to

\[
G_1(q_2, m_2) - G_1(q_1, m_1) = \text{div} \left( m_1 \otimes m_1 \left( \frac{q_2}{1 + q_2} - \frac{q_1}{1 + q_1} \right) - \frac{m_2}{1 + q_2} \frac{q_1}{1 + q_1} \right) \]

\[
G_2(q_2, m_2) - G_2(q_1, m_1) = -\nabla \left( (q_2 - q_1) \tilde{H}(q_2) + q_1 (\tilde{H}(q_2) - \tilde{H}(q_1)) \right) \]

\[
G_3(q_2, m_2) - G_3(q_1, m_1) = -\left( \bar{\mu} \Delta + (\bar{\lambda} + \bar{\mu}) \text{div} \right) \left( m_2 - m_1 \right) \frac{q_2}{1 + q_2} \]

\[
+ m_1 \left( \frac{q_2}{1 + q_2} - \frac{q_1}{1 + q_1} \right) \]

\[
G_4(q_2, m_2) - G_4(q_1, m_1) = \kappa(q_2 - q_1) \nabla \Delta q_2 + \kappa q_1 \nabla \Delta (q_2 - q_1), \]

\[
G_5(q_2, m_2) - G_5(q_1, m_1) = f(q_2 - q_1). \]

This leads to the following inequality

\[
\|\Psi_{q_1, m_1}(\tilde{q}_1, \tilde{m}_1) - \Psi_{q_2, m_2}(\tilde{q}_2, \tilde{m}_2)\|_E \leq C\|\tilde{q}_2 - \tilde{q}_1, \tilde{m}_2 - \tilde{m}_1\|_E \times \left( \|q_1\|_{L^1(B^{4-1})} + \|q_2\|_{L^1(B^{4-1})} + 2\|q_L\|_{L^1(B^{4-1})} + \|f\|_{L^1(B^{4-1})} \right).
\]

Now, if \((R, \eta)\) satisfies (37) (for a greater constant \(C\) if needed), we deduce

\[
\|\Psi_{q_1, m_1}(\tilde{q}_1, \tilde{m}_1) - \Psi_{q_2, m_2}(\tilde{q}_2, \tilde{m}_2)\|_E \leq \frac{4}{3}\|\tilde{q}_2 - \tilde{q}_1, \tilde{m}_2 - \tilde{m}_1\|_E
\]

and the proof of the existence part of theorem 3 is achieved.

Notice that in view of (34) and (37), \((q, m)\) satisfies

\[
\|q\|_{L^\infty(\mathbb{R} \times (0, T_\eta))} \leq \frac{2}{5} \quad \text{and} \quad \|q\|_{L^\infty(B^{4/5})} \leq \frac{2}{5} (5C).
\]

The solution \((q, m)\) obviously belongs to the space \(\tilde{E}^2_\infty\) defined in section 6.2 (we have \(\tilde{E} = \tilde{E}^2_\infty\)). Changing \(C\) into a greater constant if necessary, we therefore can apply lemma 4 to get uniqueness in \(\tilde{E}\).
5 Local solutions away from vacuum

In this section, we want to show that we do not need smallness conditions on \((\rho - \bar{\rho}, \mathbf{m})\) in \(B^{d/2} \times (B^{d/2-1})^d\) to have local well-posedness for the Korteweg system. We just need the initial density to be bounded away from zero. Let us emphasize also that we do not require the pressure to be nondecreasing in terms of the density. The only assumption is that \(P\) is a suitably smooth function of \(\rho\) \((P \in W^{[\frac{d}{2}]+3,\infty}_{\text{loc}}\) is enough).

It is convenient to rewrite (8)(9) in terms of \(q\) and \(\mathbf{m}\) as follows using the same scaled coefficients as in section 4

\[
\partial_t q + \text{div} \mathbf{m} = 0, \tag{39}
\]

\[
\partial_t \mathbf{m} - \bar{\rho} \text{div}\left(\frac{\nabla \mathbf{m}}{1+q}\right) - (\bar{\lambda} + \bar{\rho}) \nabla\left(\frac{\text{div} \mathbf{m}}{1+q}\right) - \bar{\kappa} \nabla \left((1+q)\Delta q\right) = \Gamma(q, \mathbf{m}) + \mathbf{f}, \tag{40}
\]

where \(\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5\) is defined by

\[
\begin{align*}
\Gamma_1(q, \mathbf{m}) &= -\nabla \left( P(\bar{\rho}(1+q)) - P(\bar{\rho}) \right) / \bar{\rho}, \\
\Gamma_2(q, \mathbf{m}) &= -\text{div}\left( \frac{\mathbf{m} \otimes \mathbf{m}}{1+q} \right), \\
\Gamma_3(q, \mathbf{m}) &= \bar{\rho} \text{div} \left( \mathbf{m} \otimes \nabla \left( \frac{1}{1+q} \right) \right) + (\bar{\lambda} + \bar{\rho}) \nabla \left( \mathbf{m} \cdot \nabla \left( \frac{1}{1+q} \right) \right), \\
\Gamma_4(q, \mathbf{m}) &= -\bar{\kappa} \text{div} \left( \nabla q \otimes \nabla q \right) - \frac{\bar{\kappa}}{2} \nabla |\nabla q|^2, \\
\Gamma_5(q, \mathbf{m}) &= qf.
\end{align*}
\]

The proof of our local existence theorem relies upon the study of the linearized system around \((1+q, 0)\) for a given \(q\) such that \(1+q\) is bounded away from zero, whereas the first order linearized pressure term is dropped. The purpose of section 5.1 is to derive estimates for such a system. Well-posedness for (39)(40) is obtained in section 5.2 through an iterative method.

5.1 Estimates for the linearized system

We now study the following linearized system

\[
\begin{cases}
\partial_t q + \text{div} \mathbf{m} = F, \\
\partial_t \mathbf{m} - \bar{\rho} \text{div}(a \nabla \mathbf{m}) - (\bar{\lambda} + \bar{\rho}) \nabla(\text{div} \mathbf{m}) - \bar{\kappa} \nabla(b \Delta q) = \mathbf{G},
\end{cases}
\tag{LNSK2}
\]

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where \( \mathbf{m} \) is a vector field in \( \mathbb{R}^d \), and \( a, b \) are scalar functions, bounded and bounded away from zero

\[
0 < c_1 \leq a \leq M_1 < +\infty, \quad 0 < c_2 \leq b \leq M_2 < +\infty \quad \text{on} \quad [0, T].
\]

Our purpose is to prove estimates for (LNSK2) in Besov spaces closely related to energy spaces. We obtain

**Proposition 5** Let \( 1 \leq r_1 \leq r \leq +\infty, \quad (q_0, \mathbf{m}_0) \in B^{d/2} \times (B^{d/2-1})^d \) and \((F, G) \in \tilde{L}^{r_1}_T (B^{d/2-2+2/r_1} \times (B^{d/2-3+2/r_1})^d) \). Suppose that \( \nabla b \) and \( \nabla a \) belong to \( \tilde{L}^2_T (B^{d/2}) \), \( \partial_t b \in L^1(0, T; L^\infty) \). Let \((q, \mathbf{m}) \in \tilde{L}^r_T (B^{d/2+2/r} \times (B^{d/2-1+2/r})^d) \cap \tilde{L}^\infty_T (B^{d/2+1} \times (B^{d/2})^d) \) be a solution of the system (LNSK2). Then there exists a constant \( C \) depending only on \( r, r_1, \bar{\lambda}, \bar{\mu}, \bar{\kappa}, c_1, c_2, M_1, \) and \( M_2 \) such that the following inequality holds:

\[
\| (\nabla q, \mathbf{m}) \|_{\tilde{L}^r_T (B^{\frac{d}{2}-\frac{1}{r}+\frac{2}{q}})} \left( 1 - C \| \nabla b \|_{L^\infty_T (L^\infty)} \right) \leq C \left( \| (\nabla q_0, \mathbf{m}_0) \|_{B^{\frac{d}{2}-1}} + \| (\nabla F, G) \|_{L^3_T (B^{\frac{d}{2}-\frac{1}{r}+\frac{2}{q}})} + \| \partial_t b \|_{L^1_T (L^\infty)} \| \nabla q \|_{L^\infty_T (B^{\frac{d}{2}-1})} + \| \nabla (q, \mathbf{m}) \|_{L^2_T (B^{\frac{d}{2}})} \left( \| \nabla b \|_{L^2_T (B^{\frac{d}{2}})} + \| \nabla a \|_{L^2_T (B^{\frac{d}{2}})} \right) \right).
\]

**Proof:** It is just a matter of showing appropriate estimates for \( \Delta_t q \) and \( \Delta_t \mathbf{m} \). Denoting \( q_\ell = \Delta_t q, \mathbf{m}_\ell = \Delta_t \mathbf{m}, F_\ell = \Delta_t F, G_\ell = \Delta_t G \), and applying \( \Delta_t \) to (LNSK2), we get

\[
\partial_t q_\ell + \text{div} \mathbf{m}_\ell = F_\ell, \quad \partial_t \mathbf{m}_\ell - \bar{\mu} \text{div} (a \nabla \mathbf{m}_\ell) - (\bar{\lambda} + \bar{\mu}) \nabla (a \text{div} \mathbf{m}_\ell) - \bar{\kappa} \nabla (b \Delta q_\ell) = G_\ell + \mathbf{R}_\ell, \tag{42}
\]

where \( \mathbf{R}_\ell = -\bar{\mu} \text{div} ([a, \Delta_t \nabla] \mathbf{m}) - (\bar{\lambda} + \bar{\mu}) \nabla ([a, \Delta_t \text{div}] \mathbf{m}) - \bar{\kappa} \nabla ([b, \Delta_t \text{div}] \nabla q) \).

Using integrations by parts and equation (42), we obtain

\[
- \int_{\mathbb{R}^d} \mathbf{m}_\ell \nabla (b \Delta q_\ell) \, dx = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} b |\nabla q_\ell|^2 \, dx - \int_{\mathbb{R}^d} \left( \text{div} \mathbf{m}_\ell (\nabla q_\ell \cdot \nabla b) + \frac{|\nabla q_\ell|^2}{2} \partial_t b + b \nabla q_\ell \cdot \nabla F_\ell \right) \, dx.
\]

We now take the scalar product of (43) with \( \mathbf{m}_\ell \) and use the previous identity, which yields

\[
\frac{1}{2} \frac{d}{dt} \left( \| \mathbf{m}_\ell \|^2_{L^2} + \bar{\kappa} \int_{\mathbb{R}^d} b |\nabla q_\ell|^2 \, dx \right) + \int_{\mathbb{R}^d} \left( \bar{\mu} |\nabla \mathbf{m}_\ell|^2 + (\bar{\lambda} + \bar{\mu}) |\text{div} \mathbf{m}_\ell|^2 \right) \, dx
\]
In order to obtain a second estimate, we take the scalar product of the gradient of \((G_\ell + R_\ell)\) with \(m_\ell\), the scalar product of \((G_\ell)\) with \(q_\ell\), and sum both inequalities. We obtain

\[
\frac{d}{dt} \int_{\mathbb{R}^d} \nabla q_\ell \cdot m_\ell \, dx + \int_{\mathbb{R}^d} \nabla \cdot m_\ell \, dx = \int_{\mathbb{R}^d} \left( \left( G_\ell + R_\ell \right) \cdot \nabla q_\ell + |\nabla m_\ell|^2 \right)
\]

\[
+ m_\ell \cdot \nabla F_\ell - \mu a \nabla q_\ell \cdot \nabla^2 q_\ell - \left( \lambda + \bar{\mu} \right) a \Delta q_\ell \nabla m_\ell \right) \, dx
\]  

(45)

Let \(\alpha > 0\) be suitably small, and define

\[
k_\ell^2 \overset{\text{def}}{=} \|m_\ell\|_{L^2}^2 + \int_{\mathbb{R}^d} \left( \bar{\mu} \nabla q_\ell \cdot m_\ell \right) \, dx \quad \text{and} \quad \bar{\mu} = \inf(\bar{\mu}, \lambda + 2\bar{\mu}).
\]

Using (44), (45) and (41), we deduce that when

\[
\alpha \leq \frac{\bar{\mu}}{2} \left( \frac{M_1(\bar{\mu} + [\lambda + \bar{\mu}]^2)}{2c_2 \bar{\mu}} + 1 \right) \leq 1,
\]

we have

\[
\frac{1}{2} \frac{d}{dt} k_\ell^2 + \frac{1}{2} \int_{\mathbb{R}^d} \left( a \nabla \cdot m_\ell \right)_{L^2}^2 + \alpha \bar{\mu} \left| \nabla q_\ell \right|_{L^2}^2 \right) \, dx
\]

\[
\leq \left( \|G_\ell\|_{L^2} + \|R_\ell\|_{L^2} \right) \left( \|m_\ell\|_{L^2} + \|m_\ell\|_{L^2} \right) + \left( \|F_\ell\|_{L^2} + \|m_\ell\|_{L^2} + \|q_\ell\|_{L^2} \right)
\]

\[
+ \frac{1}{2} \|\partial \cdot b\|_{L^\infty} \|\nabla q_\ell\|_{L^2}^2 + \bar{\mu} \|\nabla b\|_{L^\infty} \|\nabla q_\ell\|_{L^2} \|\nabla m_\ell\|_{L^2} \cdot
\]

(46)

Using (41), we clearly have for \(\alpha > 0\) small enough

\[
\frac{1}{2} k_\ell^2 \leq \|m_\ell\|_{L^2}^2 + \bar{\mu} \int \left| \nabla q_\ell \right|^2 \leq \frac{3}{2} k_\ell^2,
\]

(47)

so that using the above energy estimate, the spectral localisation of \((\nabla q_\ell, m_\ell)\) and (41), we get for some positive \(K\)

\[
\frac{1}{2} \frac{d}{dt} k_\ell^2 + K^2 k_\ell^2 \leq C k_\ell \left( \|G_\ell\|_{L^2} + \|\nabla F_\ell\|_{L^2} \right) + \frac{1}{2} \|\nabla q_\ell\|_{L^2}^2 + \frac{1}{2} \|\partial \cdot b\|_{L^\infty} \|\nabla q_\ell\|_{L^2}^2
\]

\[
+ 2^t k_\ell^2 \|\nabla b\|_{L^\infty} \cdot
\]

(48)
Integrating with respect to time yields
\[
 k_t(t) \leq e^{-K_2t} k_t(0) + C \int_0^t e^{-K_2(t-\tau)} \left( \| \partial_t b(\tau) \|_{L^\infty} \right. \\
 + \left. \| \nabla F(\tau) \|_{L^2} + \| G(\tau) \|_{L^2} + \| R(\tau) \|_{L^2} + 2t k_t(\tau) \| \nabla b(\tau) \|_{L^\infty} \right) d\tau. \quad (49)
\]
Using convolution inequalities, we easily get
\[
 \| k_t \|_{L^r([0,T])} \leq C \left( 2^{-\frac{m}{2}} k_t(0) + 2^{-2t \left( 1 + \frac{1}{\gamma} - \frac{1}{2} \right)} \| (\nabla F_t, G_t) \|_{L^\frac{2}{1}([0,T])} + 2^{-\frac{m}{2}} \| R_t \|_{L^\frac{2}{1}([0,T])} \\
 + 2^{-\frac{m}{2}} \| \nabla q \|_{L^\frac{2}{2}([0,T])} \| \partial_t b \|_{L^\frac{m}{2}(L^\infty)} + \| \nabla b \|_{L^\frac{m}{2}(L^\infty)} \right) \| k_t \|_{L^r([0,T])}.
\]
We first use (47) and (41) to infer that there exists a constant $C > 0$ such that
\[
 C^{-1} k_t \leq \| \nabla q \|_{L^2} + \| m \|_{L^2} \leq C k_t,
\]
then multiply both sides of (50) by $2^{\left( \frac{d}{2} - 1 - \frac{m}{2} \right)}$ and sum over $Z$, which yields
\[
 \| (\nabla q, m) \|_{L^\frac{2}{1} \left( B^\frac{d}{2}(-1+\frac{1}{2}) \right)} \left( 1 - C \| \nabla b \|_{L^\frac{2}{1}(L^\infty)} \right) \leq \| (\nabla q_0, m_0) \|_{B^\frac{d}{2}}
 + \| (\nabla F, G) \|_{L^\frac{2}{1} \left( B^\frac{d}{2}(-1+\frac{1}{2}) \right)} + \| \nabla q \|_{L^\frac{2}{2} \left( B^\frac{d}{2}(-1) \right)} \| \partial_t b \|_{L^\frac{m}{2}(L^\infty)} + \sum_{q \in \mathbb{Z}} 2^{\left( \frac{d}{2} - 1 \right)} \| R_t \|_{L^\frac{2}{1}(L^2)}.
\]
We then obtain the desired inequality thanks to lemma 5 in the appendix. \[\square\]

**Remark 2** When $r_1 = 1$, estimates of proposition 5 clearly enable us to prove the existence and uniqueness of a solution $(q, m)$ to (LNSK2) in the space $L^1_T \left( B^{d/2+2} \times \left( B^{d/2+1} \right)^d \right) \cap \tilde{C}_T \left( B^{d/2} \times \left( B^{d/2-1} \right)^d \right)$ as long as
\[
 \| \nabla q \|_{L^\frac{2}{2} \left( B^{d/2} \right)} + \| \nabla b \|_{L^\frac{2}{2} \left( B^{d/2} \right)} + \| \partial_t b \|_{L^\frac{m}{2}(L^\infty)} \leq C^{-1}.
\]
This stems from a basic duality method.

**5.2 Local existence and uniqueness for the Korteweg system**

First we define the functional spaces needed in our local existence and uniqueness theorem. We will prove existence in the space
\[
 F_T \overset{\text{def}}{=} L^1_T \left( B^{d/2+2} \times \left( B^{d/2+1} \right)^d \right) \cap \tilde{C}_T \left( B^{d/2} \times \left( B^{d/2-1} \right)^d \right)
\]
endowed with the norm
\[
\| (q, m) \|_{\mathcal{F}_T} = \| q \|_{L^1_T(B^{d/2+1})} + \| q \|_{L^\infty_T(\mathbb{R}^d)} + \| m \|_{L^1_T(B^{d/2+1})} + \| m \|_{L^\infty_T(\mathbb{R}^{d-1})},
\]
and uniqueness in the larger space
\[
\tilde{F}_T \overset{\text{def}}{=} \tilde{L}^2_T(B^{d/2+1} \times (B^{d/2})^d) \cap \tilde{C}_T(B^{d/2} \times (B^{d/2-1})^d)
\]
endowed with the norm
\[
\| (q, m) \|_{\tilde{F}_T} = \| q \|_{L^1_T(B^{d/2+1})} + \| q \|_{L^\infty_T(\mathbb{R}^{d/2})} + \| m \|_{L^2_T(B^{d/2+1})} + \| m \|_{L^\infty_T(\mathbb{R}^{d/2})},
\]

**Theorem 4** Let \( 0 < \bar{c} \leq \bar{p} \). Suppose that the exterior forcing term \( f \) belongs to \((L^1_T(B^{d/2+1-1}))^d\), that the initial momentum \( m_0 \) belongs to \((B^{d/2+1-1})^d\), and that the initial density \( \rho_0 \) satisfies \((\rho_0 - \tilde{\rho}) \in B^{d/2} \) and \( \rho_0 \geq c \). Then, there exists \( T > 0 \) such that the system (39)-(40) with initial data \(((\rho_0 - \tilde{\rho})/\bar{\rho}, m_0)\) has a unique solution \((q, m)\) in \( \tilde{F}_T \). In addition, \((q, m)\) belongs to \( F_T \).

**Proof:** the existence part of the theorem is proved by an iterative method. We define a sequence \(((q^n, m^n))_{n \in \mathbb{N}}\) as follows: the first term \((q^0, m^0)\) is taken to be the solution of the heat equation
\[
\partial_t \left( \begin{array}{c} q^0 \\ m^0 \end{array} \right) - \Delta \left( \begin{array}{c} q^0 \\ m^0 \end{array} \right) = \left( \begin{array}{c} 0 \\ f \end{array} \right), \quad \left( \begin{array}{c} q^0 \\ m^0 \end{array} \right)_{|t=0} = \left( \begin{array}{c} q_0 \\ m_0 \end{array} \right),
\]
with \( q_0 = (\rho_0 - \tilde{\rho})/\bar{\rho} \). Assuming that \((q^n, m^n)\) belongs to \( F_T \), we then define \( q^{n+1} = q^n + \bar{q}^{n+1} \) and \( m^{n+1} = m^n + \bar{m}^{n+1} \) with \((\bar{q}^{n+1}, \bar{m}^{n+1})\) solution of the following linear system
\[
\begin{cases}
\partial_t \bar{q}^{n+1} + \text{div} \bar{m}^{n+1} = \Delta q^n - \text{div} m^n, \\
\partial_t \bar{m}^{n+1} - \bar{\mu} \text{div} \left( \nabla \bar{m}^{n+1} \right) - (\bar{\lambda} + \bar{\mu}) \nabla \left( \frac{\text{div} \bar{m}^{n+1}}{1 + q^n} \right) \\
\bar{q}^{n+1}_{|t=0} = 0, \quad \bar{m}^{n+1}_{|t=0} = 0,
\end{cases}
\]
where
\[
H^0(q^n, m^n) = -\Delta m^n + \bar{\mu} \text{div} \left( \frac{\nabla m^n}{1 + q^n} \right) + (\bar{\lambda} + \bar{\mu}) \nabla \left( \frac{\text{div} m^n}{1 + q^n} \right) + \bar{\kappa} \nabla ((1 + q^n) \Delta q^n).
\]
First step: uniform bounds in $F_T$

We want now to show that $(q^n, m^n)$ is uniformly bounded in $F_T$. Denote $E_0 = \|q_0\|_{B^{2/3}} + \|m_0\|_{B^{2/3}-1}$. Let $\epsilon \in (0, 1)$. In view of proposition 2, we can choose $T \in ]0, \epsilon]$ such that

$$\|f\|_{L^1_T(B^{4/3-1})} \leq \epsilon,$$
$$\|q^0\|_{L^1_T(B^{4/3})} + \|m^0\|_{L^1_T(B^{4/3+1})} \leq \epsilon,$$
$$\|q^0\|_{\tilde{L}^\infty_T(B^{4/3})} + \|m^0\|_{\tilde{L}^\infty_T(B^{4/3-1})} \leq C(E_0 + 1).$$

We are going to show that if $\epsilon$ is chosen suitably small, we have for all $n \in \mathbb{N}$,

$$\|(\tilde{q}^n, \tilde{m}^n)\|_{F_T} \leq \sqrt{\epsilon}.$$  \hfill (P$_n$)

Since $\tilde{q}^0 = 0$ and $\tilde{m}^0 = 0$, (P$_0$) is obviously true. Suppose that (P$_n$) is fulfilled and that $\sqrt{\epsilon}$ is less than $c/(4C_1\tilde{p})$ (where $C_1$ is the norm of the embedding $B^{d/2} \hookrightarrow L^\infty$). From the fact that $q^n(t) - q_0 = -\int_0^t \div m^n(\tau) \, d\tau$ and (H$_\epsilon$), we gather

$$\|q^n - q_0\|_{L^\infty([0,T] \times \mathbb{R}^d)} \leq C_1(\|\div m^0\|_{L^1_T(B^{\frac{4}{3}})} + \|\div \tilde{m}^n\|_{L^1_T(B^{\frac{4}{3}})}),$$
$$\leq C_1(\epsilon + \sqrt{\epsilon}),$$
$$\leq c/2\tilde{p}.$$

We thus have

$$\frac{c}{2\tilde{p}} - 1 \leq q^n \leq \frac{\|q_0\|_{L^\infty}}{\tilde{p}} \quad \text{on} \quad [0, T],$$

which entails

$$\left\| \frac{1}{1 + q^n} - 1 \right\|_{\tilde{L}^\infty_T(B^{\frac{4}{3}})} \leq C\|q^n\|_{\tilde{L}^\infty_T(B^{\frac{4}{3}})},$$

$$\left\| \nabla \left( \frac{1}{1 + q^n} \right) \right\|_{L^2_T(B^{\frac{4}{3}})} \leq C\|\nabla q^n\|_{\tilde{L}^2_T(B^{\frac{4}{3}})} \left( 1 + \|q^n\|_{\tilde{L}^\infty_T(B^{\frac{4}{3}})} \right).$$

Indeed, this is an immediate consequence of the identities

$$\frac{1}{1 + q^n} - 1 = -\frac{q^n}{1 + q^n} \quad \text{and} \quad \nabla \left( \frac{1}{1 + q^n} \right) = -\nabla q^n + \frac{(q^n)^2 + 2q^n}{(1 + q^n)^2} \nabla q^n,$$

combined with lemma 1. We now apply proposition 5 to (52), recalling that $\partial_t \rho^n = \div m^n$. This yields

$$\|(\tilde{q}^n, \tilde{m}^{n+1})\|_{F_T} \left( 1 - C \left( \|\nabla q^n\|_{\tilde{L}^2_T(B^{\frac{4}{3}})} + \|\nabla \left( \frac{1}{1 + q^n} \right) \|_{\tilde{L}^2_T(B^{\frac{4}{3}})} + \|\div m^n\|_{L^1_T(B^{\frac{4}{3}})} \right) \right) \leq \sqrt{\epsilon},$$
\[ \leq C \left( \| \Gamma q^n, m^n \|_{L^1_T(B^{q-1})} + \| H[q^n, m^n] \|_{L^1_T(B^{q-1})} + \| \Delta q^n \|_{L^1_T(B^{q-1})} + \| \text{div} m^n \|_{L^1_T(B^{q-1})} \right). \tag{56} \]

Next, we use proposition 3, lemma 1, (53), (54) and (55) to estimate the right-hand side of (56). The following bounds hold:

\[
\begin{align*}
\| \Gamma_1(q^n, m^n) \|_{L^1_T(B^{q-1})} & \leq C T \| q^n \|_{L^\infty_T(B^{q-1})}, \\
\| \Gamma_2(q^n, m^n) \|_{L^1_T(B^{q-1})} & \leq C \| m^n \|_{L^2_T(B^{q-1})} \left( 1 + \| q^n \|_{L^\infty_T(B^{q-1})} \right), \\
\| \Gamma_3(q^n, m^n) \|_{L^1_T(B^{q-1})} & \leq C \| m^n \|_{L^2_T(B^{q-1})} \| \nabla q^n \|_{L^2_T(B^{q-1})} \left( 1 + \| q^n \|_{L^\infty_T(B^{q-1})} \right), \\
\| \Gamma_4(q^n, m^n) \|_{L^1_T(B^{q-1})} & \leq C \| \nabla q^n \|_{L^2_T(B^{q-1})}^2, \\
\| \Gamma_5(q^n, m^n) \|_{L^1_T(B^{q-1})} & \leq C \| q^n \|_{L^\infty_T(B^{q-1})} \| f \|_{L^1_T(B^{q-1})}.
\end{align*}
\]

Therefore, using again (55), the above computations, (56) and \((P_n)\), we deduce that

\[ \| (\tilde{q}^{n+1}, \tilde{m}^{n+1}) \|_{F_T(1 - C \sqrt{\epsilon} (E_0 + 1)^{3/2})} \leq C \epsilon (1 + E_0)^2. \]

Choosing \( \epsilon \leq (4C^2(1 + E_0)^3)^{-1} \), this implies \((P_{n+1})\). The sequence \((q^n, m^n)_{n \in \mathbb{N}}\) is therefore bounded in the space \(F_T\). Moreover, (53) holds for all \(n \in \mathbb{N}\).

**Second step: convergence of the sequence in \(F_T\).**

Next, we are going to show that \((q^n, m^n)\) converges strongly in \(F_T\) to a solution \((q, m)\) of (39) (40). We denote \(\delta q^n = q^{n+1} - q^n\) and \(\delta m^n = m^{n+1} - m^n\). According to (51) and (52), we have

\[
\left\{ \begin{array}{l}
\partial_t \delta q^n + \text{div} \, \delta m^n = 0, \\
\partial_t \delta m^n - \bar{\rho} \text{div} \left( \frac{\nabla \delta m^n}{1 + q^n} \right) - (\lambda + \bar{\mu}) \text{div} \left( \frac{\delta m^n}{1 + q^n} \right) - \bar{\kappa} \nabla ((1 + q^n) \Delta \delta q^n) \\
\quad \quad = \Gamma(\rho^n, m^n) - \Gamma(\rho^{n-1}, m^{n-1}) + H(q^{n-1}, m^{n-1}, q^n, m^n), \\
\delta q^n |_{t=0} = 0, \quad \delta m^n |_{t=0} = 0,
\end{array} \right. \tag{57}
\]
with
\[
H(q^{n-1}, m^{n-1}, q^n, m^n) = -\bar{\mu} \text{div} \left( \frac{\delta q^{n-1} \nabla m^n}{(1+q^n)(1+q^{n-1})} \right) \\
- (\bar{\lambda} + \bar{\mu}) \nabla \left( \frac{\delta q^{n-1} \text{div} m^n}{(1+q^n)(1+q^{n-1})} \right) + \bar{\kappa} \nabla (\delta q^{n-1} \Delta q^n).
\]

We keep the same assumptions on \(\epsilon\) as in the first step. According to proposition 5, (53), (\(P_n\)) and (55), we thus get
\[
\| (\delta q^n, \delta m^n) \|_{L^q} \leq C \left( \| \Gamma (\rho^n, m^n) - \Gamma (\rho^{n-1}, m^{n-1}) \|_{L^q (B \frac{d-1}{d})} + \| H(q^{n-1}, m^{n-1}, q^n, m^n) \|_{L^q (B \frac{d}{d}-1)} \right),
\]

Denote \(\delta \Gamma^n :\equiv \Gamma_i (q^n, m^n) - \Gamma_i (q^{n-1}, m^{n-1})\). Thanks to proposition 3, to lemma 1, (53), (54) and (55), we obtain the following estimates:
\[
\| \delta \Gamma^1 \|_{L^q (B \frac{d-1}{d})} \leq T \left( 1 + \| q^n \|_{L^q (B \frac{d}{d})} + \| q^{n-1} \|_{L^q (B \frac{d}{d})} \right) \| \delta q^{n-1} \|_{L^q (B \frac{d}{d})},
\]
\[
\| \delta \Gamma^2 \|_{L^q (B \frac{d-1}{d})} \leq (1 + \| q^n \|_{L^q (B \frac{d}{d})}) \left( \| m^{n-1} \|_{L^q (B \frac{d}{d})} + \| m^n \|_{L^q (B \frac{d}{d})} \right) \| \delta m^{n-1} \|_{L^q (B \frac{d}{d})} + \| m^{n-1} \|_{L^q (B \frac{d}{d})} \left( 1 + \| q^{n-1} \|_{L^q (B \frac{d}{d})} \right) \left( 1 + \| q^n \|_{L^q (B \frac{d}{d})} \right) \| \delta q^{n-1} \|_{L^q (B \frac{d}{d})},
\]
\[
\| \delta \Gamma^3 \|_{L^q (B \frac{d-1}{d})} \leq \| \nabla q^n \|_{L^q (B \frac{d}{d})} \left( 1 + \| q^n \|_{L^q (B \frac{d}{d})} \right) \| \delta m^{n-1} \|_{L^q (B \frac{d}{d})} + \| m^{n-1} \|_{L^q (B \frac{d}{d})} \left( \| \nabla q^{n-1} \|_{L^q (B \frac{d}{d})} + \| \nabla q^n \|_{L^q (B \frac{d}{d})} \right) \| \delta q^{n-1} \|_{L^q (B \frac{d}{d})} + \| m^{n-1} \|_{L^q (B \frac{d}{d})} \| \nabla q^{n-1} \|_{L^q (B \frac{d}{d})},
\]
\[
\| \delta \Gamma^4 \|_{L^q (B \frac{d-1}{d})} \leq \| \nabla \delta q^{n-1} \|_{L^q (B \frac{d}{d})} \left( \| q^{n-1} \|_{L^q (B \frac{d}{d})} + \| q^n \|_{L^q (B \frac{d}{d})} \right),
\]
\[
\| \delta \Gamma^5 \|_{L^q (B \frac{d-1}{d})} \leq \| f \|_{L^q (B \frac{d}{d})} \| \delta q^{n-1} \|_{L^q (B \frac{d}{d})},
\]
\[
\| \frac{\delta q^{n-1} \nabla m^n}{(1+q^n)(1+q^{n-1})} \|_{L^q (B \frac{d}{d})} \leq \left( 1 + \| q^n \|_{L^q (B \frac{d}{d})} \right) \left( 1 + \| q^{n-1} \|_{L^q (B \frac{d}{d})} \right) \times \| \nabla m^n \|_{L^q (B \frac{d}{d})} \| \delta q^{n-1} \|_{L^q (B \frac{d}{d})},
\]

25
\[
\| dq^{n+1} - dq^n \|_{L^1_t(B^{d/2})} \lesssim \| dq^n \|_{L^1_t(B^{d/2})} \| dq^{n+1} \|_{L^\infty_t(B^{d/2})}.
\]

Using \( (\mathcal{H}_t) \) and \( (\mathcal{P}_{\text{m}}) \) in the above computations, then (58), we thus get
\[
\| (\delta q^n, \delta m^n) \|_{F_T} \leq C \sqrt{\epsilon} (E_0 + 1)^3.
\]

Now, if we choose an \( \epsilon \) such that \( \epsilon \leq (4C^2)^{-1}(E_0 + 1)^{-6} \) holds, \((q_n, m_n)_{n \in \mathbb{N}} \) is clearly a Cauchy sequence and thus converges in \( F_T \) to a limit \((q, m) \) which satisfies (53). The verification that the limit is solution of (39)(40) in the sense of distributions is a straightforward application of proposition 3. \( \square \)

**Third step: uniqueness in \( \tilde{F}_T \)**

Consider the solution \((q, m)\) built in the previous part and suppose that \((q', m') \in \tilde{L}_T^2(B^{d/2+1} \times (B^{d/2}) \cap \tilde{C}_T(B^{d/2} \times (B^{d/2-1})^d) \) also solves (39)(40) with initial data \((q_0, m_0)\). Denote \( \delta q = q' - q \) and \( \delta m = m' - m \). We have

\[
\begin{cases}
\partial_t \delta q + \text{div} \delta m = 0, \\
\partial_t \delta m - \bar{\mu} \text{div} \left( \frac{\nabla \delta m}{1+q} \right) - (\bar{\lambda} + \bar{\mu}) \nabla \left( \frac{\text{div} \delta m}{1+q} \right) - \bar{\pi} \nabla (\text{div} (1+q) \Delta q)
= \Gamma(q', m') - \Gamma(q, m) - H(q', m', q, m),
\end{cases}
\tag{59}
\]

Let \( T^* \in [0, T] \) be the greatest time such that (53) is satisfied by \( q' \) on \([0, T^*] \). Continuity for \( q' \) in \( C([0, T]; B^{d/2}) \) implies that \( 0 < T^* \leq T \).

We now apply proposition 5 to (59) using bounds of step 1 and (53) for \( q \). Unlike in the second step, our assumptions on \((q', m')\) only provide us with bounds for \( H \) in \( \tilde{L}_{T^*}^2(B^{d/2-2}) \). This leads to the following estimate:

\[
\| (\delta q, \delta m) \|_{\tilde{F}_{T^*}} \lesssim \| \Gamma(q', m') - \Gamma(q, m) \|_{L_{T^*}^1(B^{d/2-1})} + \| H(q', m', q, m) \|_{\tilde{L}_{T^*}^2(B^{d/2-2})}
\]

Using the same estimates as in step 2 for \( \| \Gamma(q', m') - \Gamma(q, m) \|_{L_{T^*}^1(B^{d/2-1})} \), and the fact that

\[
\| H(q', m', q, m) \|_{\tilde{L}_{T^*}^2(B^{d/2-2})} \lesssim (1 + \| q \|_{\tilde{L}^\infty_{T^*}(B^{d/2})})(1 + \| q' \|_{\tilde{L}^\infty_{T^*}(B^{d/2})})
\times \left( \| m' \|_{\tilde{L}^2_{T^*}(B^{d/2})} + \| \nabla q' \|_{\tilde{L}^2_{T^*}(B^{d/2})} \right),
\]

26
we finally gather
\[ \| (\delta q, \delta m) \|_{\tilde{F}^*_{T'}} \leq (1 + \| q \|_{L_{q,T}^\infty(\mathbb{B}^2_0)}) (1 + \| q' \|_{L_{q',T}^\infty(\mathbb{B}^2_0)}) (T^* + \| f \|_{L_{f,T}^1(\mathbb{B}^2_0-1)}) \]
\[ + K \left( \| m \|_{L_{m,T}^2(\mathbb{B}^2_1)} + \| \nabla q \|_{L_{q',T}^2(\mathbb{B}^2_1)} + \| m' \|_{L_{m',T}^2(\mathbb{B}^2_1)} + \| \nabla q' \|_{L_{q'',T}^2(\mathbb{B}^2_1)} \right) \| (\delta q, \delta m) \|_{\tilde{F}^*_{T'}} , \]
with \( K(z) = z + z^2 \). This obviously entails \( \delta q \equiv 0 \) and \( \delta m \equiv 0 \) on a suitably small interval \([0, T']\) with \( T' > 0 \).

Using the same arguments as for the proof of lemma 4, we can now conclude that the two solutions coincide on the whole interval \([0, T']\).

\[ \square \]

6 Local strong solutions near equilibrium

In this section, we want to show that local well-posedness for Korteweg system when the density is close to a constant also holds in spaces of type \( B^p_q \) with \( p \neq 2 \), that is, in spaces which are not related to physical energy spaces. Recall that this approach was extensively used for the study of incompressible Navier-Stokes equations (see [7] and the references enclosed).

This viewpoint enables us to get well-posedness even if the initial velocity belongs to a space \( B^p_q \) such that the regularity index \( s \) is negative, which in particular is relevant for oscillating initial velocities.

To avoid tedious discussions about the definition of \( qf \), we suppose from now on, that \( f \equiv 0 \) (see remark 3 at the end of the section). Using the same notations as in section 4, the Korteweg system rewrites as follows:

\[ \partial_t q + \text{div } m = 0, \] (60)
\[ \partial_t m - \bar{\mu} \Delta m - (\bar{\lambda} + \bar{\mu})\nabla \text{div } m - \bar{\kappa} \nabla \Delta q = G(q, m), \] (61)
\[ (q, m)|_{t=0} = (q_0, m_0), \] (62)

where we define \( G = G_1 + G_2 + G_3 + G_4 + G_5 \) by

\[ G_1(q, m) = -\nabla \left( P(\bar{\rho}(1 + q)) - P(\bar{\rho}) \right) / \bar{\rho}, \]
\[ G_2(q, m) = -\text{div } \left( \frac{m \otimes m}{1 + q} \right), \]
\[ G_3(q, m) = -\frac{\bar{\kappa}}{2} \nabla (|\nabla q|^2) - \bar{\kappa} \text{div } (q \nabla^2 q), \]

and \( G_4(q, m) = -\bar{\mu} \text{div } \left( \nabla \left( \frac{q}{1 + q} \right) \otimes m + \frac{q}{1 + q} \nabla m \right) - (\bar{\lambda} + \bar{\mu}) \nabla \left( \nabla \left( \frac{q}{1 + q} \right) \cdot m + \frac{q}{1 + q} \text{div } m \right). \)
Let us emphasize that no stability assumption on the pressure law is required: we just have to assume that $P$ is suitably smooth ($P \in W^{[d/\rho]+3,\infty}_{loc}$ is enough).

In section 6.1 we study the linearized system around $(0,0)$ where we drop the first order linearized pressure term. Local well-posedness for $(q_0,m_0)$ is then proved in 6.2 through a fixed-point argument.

### 6.1 Estimates for the linearized pressure-less system

This section is devoted to the proof of estimates for the following linear system:

\[
\begin{align*}
\partial_t q + \text{div} m &= F, \\
\partial_t m - \bar{\mu} \Delta m - (\bar{\lambda} + \bar{\mu}) \text{div} m - \bar{\kappa} \Delta q &= \mathbf{G}.
\end{align*}
\]

\[ (LNSK3) \]

The main result of this section is the following proposition:

**Proposition 6** Let $s \in \mathbb{R}$, $p \in [1, +\infty]$, $1 \leq \rho_1 \leq +\infty$ and $T \in [0, +\infty]$. If $(q_0, m_0) \in B^s_p \times (B^{s-1}_p)^d$ and $(F, \mathbf{G}) \in \tilde{L}^p_T (B^{s-2+1/\rho_1}_p \times (B^{s-3+2/\rho_1}_p)^d)$ then the linear system (LNSK3) has a unique solution $(q, m) \in \tilde{C}_T (B^s_p \times (B^{s-1}_p)^d) \cap \tilde{L}^p_T (B^{s+2/\rho_1}_p \times (B^{s+1+2/\rho_1}_p)^d)$. Moreover, for all $\rho \in [\rho_1, +\infty]$, there exists a constant $C$ depending only on $\bar{\mu}, \bar{\lambda}, \bar{\kappa}, \rho, \rho_1$ and $d$ such that the following inequality holds:

\[
\|q\|_{\tilde{L}^p_T (B^{s+2/\rho_1}_p)} + \|m\|_{\tilde{L}^p_T (B^{s-1+2/\rho_1}_p)} \leq C \left( \|q_0\|_{B^s_p} + \|m_0\|_{B^{s-1}_p} + \|F\|_{\tilde{L}^p_T (B^{s-2+1/\rho_1}_p)} + \|\mathbf{G}\|_{\tilde{L}^p_T (B^{s-3+2/\rho_1}_p)} \right).
\]

**Proof:** Apply operator $\Delta$ to the first equation and operators div and curl (with $\text{curl} g := \partial_2 g^j - \partial_1 g^j$) to the second one. Denoting $\bar{\nu} = \bar{\lambda} + 2\bar{\mu}$, we obtain

\[
\begin{align*}
\begin{cases}
\partial_t \Delta q + \Delta \text{div} m &= \Delta F, \\
\partial_t \text{div} m - \bar{\nu} \Delta \text{div} m - \bar{\kappa} \Delta^2 q &= \text{div} \mathbf{G}, \\
\partial_t \text{curl} m - \bar{\mu} \Delta \text{curl} m &= \text{curl} \mathbf{G}.
\end{cases}
\end{align*}
\]

(63)

Proposition 2 gives the following estimates for the third equation, which decouples from the first two equations

\[
\|\text{curl} m\|_{\tilde{L}^p_T (B^{s+2/\rho_1}_p)} \leq C \left( \|\text{curl} m_0\|_{B^{s-2}_p} + \|\text{curl} \mathbf{G}\|_{\tilde{L}^p_T (B^{s+4+2/\rho_1}_p)} \right).
\]

(64)

The following lemma points out a smoothing effect for the first two equations:
Lemma 3 Let \( s \in \mathbb{R}, (p, \rho_1) \in [1, +\infty]^2 \) and \( T \in [0, +\infty] \). Suppose that \((c_0, v_0) \in (B^{q}_p)^2 \) and \((h, k) \in (L_{-\infty}^{\rho_1}(B^{s+22/\rho_1}_p))^{2} \). Then the system

\[
\begin{align*}
\partial_t c + \Delta v &= h, \\
\partial_t v - \bar{\nu} \Delta v - \bar{\kappa} \Delta c &= k, \\
(c, v)|_{t=0} &= (c_0, v_0),
\end{align*}
\]

has a unique solution \((c, v)\) in \((\bar{C}_T(B^{q}_p) \cap \bar{L}_{-\infty}^{\rho_1}(B^{s+22/\rho_1}_p))^{2}\). Moreover, for all \( \rho \in [\rho_1, +\infty] \), there exists \( C > 0 \) depending only on \( \bar{\nu}, \bar{\kappa}, \rho, \rho_1 \) such that

\[
\| (c, v) \|_{\bar{C}_T^{(q, \rho, \rho_1)}(B^{q}_p)} \leq C \left( \| (c_0, v_0) \|_{B^{q}_p} + \| (h, k) \|_{\bar{L}_{-\infty}^{\rho_1}(B^{s+22/\rho_1}_p)} \right).
\]

Using (64), lemma 3 with \( c_0 = \Delta q_0, v_0 = \text{div} m_0, h = \Delta f, k = \text{div} g \) and noticing that \( \Delta m = \nabla \text{div} m - \text{curl} \text{curl} m \), proposition 6 is now obvious. \( \square \)

**Proof of lemma 3:** Denoting by \( U(t) \) the semi-group associated to (65), we deduce from Duhamel’s formula that

\[
\begin{align*}
\begin{pmatrix} c(t) \\ v(t) \end{pmatrix} &= U(t) \begin{pmatrix} c_0 \\ v_0 \end{pmatrix} + \int_0^t U(t - s) \begin{pmatrix} h(s) \\ k(s) \end{pmatrix} ds,
\end{align*}
\]

with \( U(t) = e^{-tA(D)} \) and

\[
A(\xi) = \begin{pmatrix} 0 & -|\xi|^2 \\ \bar{\kappa} |\xi|^2 & \bar{\nu} |\xi|^2 \end{pmatrix}
\]

Straightforward computations show that

\[
e^{-tA(\xi)} = e^{-\frac{\bar{\nu} |\xi|^2 t}{2}} \begin{pmatrix} h_1(t, \xi) + \bar{\nu} h_2(t, \xi) & h_2(t, \xi) \\ -\bar{\kappa} \bar{\nu} h_2(t, \xi) & h_1(t, \xi) - \bar{\nu} h_2(t, \xi) \end{pmatrix}
\]

with

\[
h_1(t, \xi) = \cos(\nu |\xi|^2 t), \quad h_2(t, \xi) = \frac{\sin(\nu |\xi|^2 t)}{\nu'}, \quad \text{if} \quad \bar{\nu}^2 < 4\bar{\kappa},
\]

\[
h_1(t, \xi) = 1, \quad h_2(t, \xi) = t |\xi|^2, \quad \text{if} \quad \bar{\nu}^2 = 4\bar{\kappa},
\]

\[
h_1(t, \xi) = \cosh(\nu |\xi|^2 t), \quad h_2(t, \xi) = \frac{\sinh(\nu |\xi|^2 t)}{\nu'}, \quad \text{if} \quad \nu^2 > 4\bar{\kappa}
\]
and $\nu' = \sqrt{\kappa - \beta^2/4}$. Let $\tilde{\varphi}$ be a smooth function supported in $C(0,1/2,2)$, such that $\tilde{\varphi} \equiv 1$ on $\text{Supp} \varphi$, denote $a_{ij}(t, \xi)$ the coefficients of the matrix $e^{-tA(\xi)}$ and

$$H_{ij}^q(t, \mathbf{x}) = (2\pi)^{-d} \int e^{ix\cdot\xi}a_{ij}(t, \xi)\tilde{\varphi}(2^{-q}\xi)\,d\xi.$$ 

We assume that the following inequality holds:

$$\left\| H_{ij}^q \right\|_{L^1} \leq C e^{-c\min(\eta_1, \eta_2)2^q\tau t}$$  \hspace{1cm} (66)

where $C$ depends only on $\tilde{\nu}$, $d$ and $\kappa$, and $c$ is a universal constant. We clearly have

$$U(t)\left( \frac{\Delta_q c}{\Delta_q v} \right)(\mathbf{x}) = \left( (H_{11}^q(t, \cdot) * \Delta_q c)(\mathbf{x}) + (H_{12}^q(t, \cdot) * \Delta_q v)(\mathbf{x}) \right)$$  

\hspace{1cm} \left( (H_{21}^q(t, \cdot) * \Delta_q c)(\mathbf{x}) + (H_{22}^q(t, \cdot) * \Delta_q v)(\mathbf{x}) \right)$$

so estimate (66) yields

$$\left\| U(t)(\Delta_q c, \Delta_q v) \right\|_{L^p} \leq C e^{-c\min(\eta_1, \eta_2)2^q\tau t} \left( \left\| \Delta_q c \right\|_{L^p} + \left\| \Delta_q v \right\|_{L^p} \right).$$

Now, we complete the proof of lemma 3 coming back to the definition of Besov spaces, and using convolution inequalities.

In order to prove (66), we first remark that $\left\| H_{ij}^q \right\|_{L^1} = \left\| h_{ij}^q \right\|_{L^1}$ with

$$h_{ij}^q(t, \mathbf{y}) = (2\pi)^{-d} \int e^{ix\cdot\eta}a_{ij}(t, 2^q\eta)\tilde{\varphi}(\eta)\,d\eta,$$

so all the functions $h_{ij}^q$ are of the type

$$h^q(t, \mathbf{x}) = \int e^{ix\cdot\xi}f(2^{2q}\xi^2t)\tilde{\varphi}(\xi)\,d\xi,$$ \hspace{1cm} (67)

for a function $f \in C^\infty(\mathbb{R}^+)$. Using integrations by parts and Leibniz’ formula, we get for all $\alpha \in \mathbb{N}_d$,

$$(-ix)^\alpha h^q(\mathbf{x}) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \int e^{ix\cdot\xi}\partial^\beta f(2^{2q}\xi^2t)\partial^{\alpha-\beta}\tilde{\varphi}(\xi)\,d\xi.$$ \hspace{1cm} (68)

Next, from Faà-di-Bruno’s formula, we deduce that

$$\partial^\beta f(2^{2q}|\xi|^2t) = \sum_{\gamma_1 + \cdots + \gamma_m = \beta} f^{(m)}(2^{2q}|\xi|^2t)(2^{2q}t)^m \left( \prod_{j=1}^m \partial^{\gamma_j}(|\xi|^2) \right).$$ \hspace{1cm} (69)
Let us suppose first that $\tilde{\nu}^2 < 4\tilde{\kappa}$. Then, we just have to prove that
\[ \|h^0\|_{L^1} \leq C e^{-c_2 q t} \] (70)
for $f(u) = e^{i\nu' u} e^{-\tilde{\nu} u/2}$ and $\nu' = \sqrt{\tilde{\kappa} - \tilde{\nu}^2/4}$. We have
\[ f^{(m)}(u) = \left(i\nu' - \frac{\tilde{\nu}}{2}\right)^m e^{i\nu' u} e^{-\tilde{\nu} u/2}, \]
so that $|f^{(m)}(u)| \leq (\nu' + \tilde{\nu}/2)^m e^{-\tilde{\nu} u/2}$. Using (68), (69) and the fact that Supp $\tilde{\varphi} \subset C(0, 1/2, 2)$, we prove the existence of constants $C_{\alpha, \beta, m}$ such that
\[ |x^\alpha h^0(x)| \leq \sum_{\beta \leq \alpha} \sum_{m=1}^{|\beta|} C_{\alpha, \beta, m} (2^q t)^m e^{-c_2 q t}. \]

For any constant $c < 1$ and $m \in \mathbb{N}$, there exists $C_m$ such that $u^m e^{-u} \leq C_m e^{-cu}$. This clearly yields (70).

When $\tilde{\nu}^2 = 4\tilde{\kappa}$, we must verify (70) for $f(u) = u e^{-\tilde{\nu} u/2}$ and $f(u) = e^{-\tilde{\nu} u/2}$. This is obvious in view of (69) and Leibniz’ formula. When $\tilde{\nu}^2 > 4\tilde{\kappa}$, we must verify (70) for
\[ f(u) = \exp\left(-\frac{\tilde{\nu}}{2} \left(1 \pm \sqrt{1 - \frac{4\tilde{\kappa}}{\tilde{\nu}^2}}\right)\right). \]
Using again (69) we thus get
\[ |x^\alpha h^0(x)| \leq C e^{-c_2 q t} \left(1 \pm \sqrt{1 - \frac{4\tilde{\kappa}}{\tilde{\nu}^2}}\right) \leq C e^{-c_2 q t} \]
and we conclude to (66). \hfill \square

### 6.2 Local well-posedness for an initial density close to a constant

In this section, we agree that $B^\alpha$ stands for $B^\alpha$. Let us introduce the functional space needed in the local existence theorem. We will prove existence in
\[ E^p_T = \tilde{C}_T(B^{d/p} \times (B^{d/p-1})^d) \cap L^1_T(B^{d/p+2} \times (B^{d/(p+1)})^d) \]
and uniqueness in
\[ \tilde{E}^p_T = \tilde{C}_T(B^{d/p} \times (B^{d/p-1})^d) \cap \tilde{L}^2_T(B^{d/p+1} \times (B^{d/p})^d). \]

We have the following result:
Theorem 5. Let $p \in [1, +\infty]$. Then there exists $\eta > 0$ such that if $q_0 \in B^{\frac{1}{p}}$, $m_0 \in (B^{\frac{1}{p}-1})^d$ and
\[
\|q_0\|_{B^{\frac{1}{p}}} \leq \eta,
\]
then there exists $T > 0$ such that system (60)(61)(62) has a unique solution $(q, m)$ in $E_T^{p}$. In addition, $(q, m)$ belongs to $E_T^{p}$.

Proof: For $T > 0$ and $p \in [1, +\infty]$, we denote
\[
\| (q, m) \|_{F_T^p} = \| q \|_{L_T^p(B^{\frac{1}{p}})} + \| q \|_{L_T^{1}(B^{\frac{1}{p}+2})} + \| m \|_{L_T^2(B^{\frac{1}{p}})} + \| m \|_{L_T^1(B^{\frac{1}{p}+1})}.
\]

Let $(q_0, m_0)$ be as in theorem 5 and denote by $(q_L, m_L)$ the solution of the linearized pressure-less system on the interval $[0, T]$:
\[
\begin{cases}
\partial_t q + \text{div } m = 0, \\
\partial_t m - \bar{\mu} \Delta m - (\bar{\lambda} + \bar{\mu}) \text{div } m - \bar{\kappa} \nabla \Delta q = 0, \\
(q, m)|_{t=0} = (q_0, m_0).
\end{cases}
\]

Denoting by $V(t)$ the semi-group generated by the above system, we have
\[
(q_L, m_L)(t) = V(t)(q_0, m_0).
\]

Let us define
\[
\Phi_{q_L, m_L}(q, m) \overset{\text{def}}{=} \int_0^t V(t-s)(0, G(q_L + \bar{q}, m_L + \bar{m})(s)) \, ds.
\]

In order to prove the existence part of the theorem, we just have to show that $\Phi_{q_L, m_L}$ has a fixed point in $E_T^{p}$. Since $E_T^{p}$ is a Banach space, we are going to prove that $\Phi_{q_L, m_L}$ satisfies the hypotheses of Picard's theorem in a ball $B(0, R)$ of $E_T^{p}$ for sufficiently small $R$.

1st step: Stability of $B(0, R)$

Denote $q = q_L + \bar{q}$ and $m = m_L + \bar{m}$. According to proposition 6, we have
\[
\| \Phi_{q_L, m_L}(q, m) \|_{E_T^{p}} \leq C \| G(q, m) \|_{L_T^1(B^{\frac{1}{p}+1})},
\]

Under the assumption
\[
\| q \|_{L^\infty([0,T] \times \mathbb{R}^d)} \leq 1/2,
\]

(\mathcal{H})
and using proposition 3 and lemma 1, we state the following estimates:

\[
\begin{align*}
\|G_1(q, m)\|_{L^1_T(B^{d/p-1})} &\lesssim T \|q\|_{L^\infty_T(B^{d/p})}, \\
\|G_2(q, m)\|_{L^1_T(B^{d/p-1})} &\lesssim (1 + \|q\|_{L^\infty_T(B^{d/p})}) \|m\|_{L^1_T(B^{d/p})}^2, \\
\|G_3(q, m)\|_{L^1_T(B^{d/p-1})} &\lesssim \|\nabla q\|_{L^2_T(B^{d/p})}^2 + \|q\|_{L^\infty_T(B^{d/p})} \|\nabla^2 q\|_{L^1_T(B^{d/p})}, \\
\|G_4(q, m)\|_{L^1_T(B^{d/p-1})} &\lesssim \|q\|_{L^\infty_T(B^{d/p})} \|
abla m\|_{L^1_T(B^{d/p})} \\
&\quad + (1 + \|q\|_{L^\infty_T(B^{d/p})}) \|\nabla q\|_{L^2_T(B^{d/p})} \|m\|_{L^2_T(B^{d/p})}. 
\end{align*}
\]

This leads to the following inequality:

\[
\|\Phi_{q_L,m_L}(\bar{q}, \bar{m})\|_{E^p_T} \leq C (1 + \|q_L\|_{L^\infty_T(B^{d/p})}) \left( \|(q_L, m_L)\|_{F^p_T} + \|(\bar{q}, \bar{m})\|_{F^p_T} \right) \\
\times \left( T + \|(q_L, m_L)\|_{F^p_T} + \|(\bar{q}, \bar{m})\|_{F^p_T} \right). 
\]

Let \(c\) be a constant such that \(\|\cdot\|_{B^{d/2}} \leq c\) implies \(\|\cdot\|_{L^\infty} \leq 1/5\). We choose

\[
R = \min((10C)^{-1}c, 1),
\]

and suppose that \(\|q_0\|_{B^{d/p}} \leq R/2\). Then, since \(q_L \in C([0, T]; B^{d/p})\), we have \(\|(q_L, m_L)\|_{F^p_T} \leq R\) for \(T\) small enough so that \((\mathcal{H})\) is fulfilled. We also suppose that \(T \leq R/4\) and we get

\[
\Phi_{q_L,m_L}(B(0, R)) \subset B(0, 9R/10).
\]

2nd step: Contraction properties

Suppose that \((\bar{q}_1, \bar{m}_1)\) and \((\bar{q}_2, \bar{m}_2)\) belong to the ball \(B(0, R)\) of \(E^p_T\), and denote \(q_1 = q_L + \bar{q}_1, m_1 = m_L + \bar{m}_1, q_2 = q_L + \bar{q}_2\) and \(m_2 = m_L + \bar{m}_2\). Using again proposition 6, we get

\[
\|\Phi_{q_L,m_L}(\bar{q}_2, \bar{m}_2) - \Phi_{q_L,m_L}(\bar{q}_1, \bar{m}_1)\|_{E^p_T} \leq C \|G(q_2, m_2) - G(q_1, m_1)\|_{L^1_T(B^{d/p-1})}. 
\]
Under assumption \((\mathcal{H})\) for \(q_1\) and \(q_2\), we can derive estimates for \(G_i(q_2, m_2) - G_i(q_1, m_1)\). We apply proposition 3 and lemma 1 to the following identities:

\[
G_1(q_2, m_2) - G_1(q_1, m_1) = -\nabla \left( P(\tilde{p}(1 + q_2)) - (P(\tilde{p}(1 + q_1)) \right),
\]

\[
G_2(q_2, m_2) - G_2(q_1, m_1) = \text{div} \left( (m_1 \otimes m_1)(\tilde{q}_2 - \tilde{q}_1) - \tilde{q}_2 (m_2 \otimes (m_2 - m_1) + (m_2 - m_1) \otimes m_1) \right),
\]

\[
G_3(q_2, m_2) - G_3(q_1, m_1) = -\frac{\tilde{r}}{2} \nabla (q_2 - q_1) \cdot \nabla (q_2 + q_1) - k \text{div} ((q_2 - q_1) \nabla^2 q_2 + q_1 \nabla^2 (q_2 - q_1)),
\]

\[
G_4(q_2, m_2) - G_4(q_1, m_1) = -\tilde{p} \text{div} \left( \nabla \tilde{q}_2 \otimes (m_2 - m_1) + \tilde{q}_1 \nabla (m_2 - m_1) + \nabla (\tilde{q}_2 - \tilde{q}_1) \otimes m_2 + (\tilde{q}_2 - \tilde{q}_1) \nabla m_2 \right) - (\tilde{r} + \tilde{p}) \nabla (\cdots),
\]

where \(\tilde{q}_i \triangleq q_i/(1 + q_i)\). We finally get a constant \(C\) such that

\[
\| \Phi_{q_1, m_2}(\tilde{q}_2, \tilde{m}_2) - \Phi_{q_1, m_2}(\tilde{q}_1, \tilde{m}_1) \|_{E_T^p} \leq CT\|\tilde{q}_2 - \tilde{q}_1\|_{L_{T}^\infty(B_{R/2})} + C\|\tilde{q}_2 - \tilde{q}_1\|_{F_T^p} \left( \|q_1, m_1\|_{F_T^p} + \|q_2, m_2\|_{F_T^p} \right). \quad (78)
\]

We make the same assumption on \(R\) and \(T\) as in the first step (replacing \(C\) with a larger constant if necessary) and get

\[
\| \Phi_{q_1, m_2}(\tilde{q}_2, \tilde{m}_2) - \Phi_{q_1, m_2}(\tilde{q}_1, \tilde{m}_1) \|_{E_T^p} \leq \frac{9}{10}\|q_2 - q_1, m_2 - m_1\|_{F_T^p}.
\]

This completes the proof of the existence of a solution \((q, m)\) for (60)(61)(62) in \(E_T^p\), which in addition satisfies

\[
\|q\|_{L^\infty([0, T] \times \mathbb{R}^d)} \leq \frac{2}{5} \quad \text{and} \quad \|q\|_{L_T^\infty(B_{R/2})} \leq \frac{1}{5C}. \quad (79)
\]

3rd step: Uniqueness

Uniqueness is a straightforward application of the following lemma, which yields also the global uniqueness result of section 4.

**Lemma 4** There exists a constant \(C\) depending only on \(d\) and \(p\) such that if \(T \in [0, +\infty)\) and \((q_i, m_i) \ (i \in \{1, 2\})\) are two solutions of (60)(61)(62) belonging to \(E_T^p\) and \((q_1, m_1)\) satisfies (79) with the constant \(C\), then \((q_2, m_2) \equiv (q_1, m_1)\) on \([0, T]\).
Proof: let $T^*$ be the largest time such that $(\mathcal{H})$ is satisfied by $q_2$. As \(\|q_2(0)\|_{L^\infty} \leq 1/5\) and $q_2 \in C([0, T] \times \mathbb{R}^d)$, we have $T^* > 0$. Let $[0, T_m] \subset [0, T^*]$ be the biggest interval such that the two solutions coincide on $[0, T_m]$. Suppose that $T_m < T$. Let

\[
(q_i(t), \bar{m}_i(t)) \overset{\text{def}}{=} q_i((t - T_m), m_i(t - T_m)).
\]

Continuity in time for $q_i$ implies that $(\mathcal{H})$ is satisfied by $\tilde{q}_1$ and $\tilde{q}_2$ on an interval $[0, \epsilon]$ for $\epsilon > 0$ small enough. Moreover, $(\tilde{q}_i, \bar{m}_i))_{1 \leq i \leq 2}$ belongs to $\tilde{E}^p \times \tilde{E}^p$.

We now use the decomposition

\[
G(q_i, m_i) = G_1(q_i, m_i) + G_2(q_i, m_i) + G_3'(q_i, m_i) + G_4'(q_i, m_i) + G_5'(q_i, m_i)
\]

with

\[
\begin{align*}
G_3'(q_i, m_i) &= -\frac{\tilde{\kappa}}{2} \nabla(|\nabla q_i|^2) + \tilde{\kappa} \text{div} (\nabla q_i \otimes \nabla q_i), \\
G_4'(q_i, m_i) &= -\tilde{\mu} \Delta \left( \frac{q_i m_i}{1 + q_i} \right) - (\tilde{\lambda} + \tilde{\mu}) \nabla \text{div} \left( \frac{q_i m_i}{1 + q_i} \right), \\
G_5'(q_i, m_i) &= -\tilde{\kappa} \Delta (q_i \nabla q_i).
\end{align*}
\]

Proposition 3 provides us with estimates for $G_1(q_i, m_i)$, $G_2(q_i, m_i)$ and $G_3'(q_i, m_i)$ in $L^1_T(B^{d/p-1})$, and for $G_4'(q_i, m_i)$ and $G_5'(q_i, m_i)$ in $L^2_T(B^{d/p-2})$. Thanks to proposition 6, we get

\[
\|(\tilde{q}_2 - \tilde{q}_1, \bar{m}_2 - \bar{m}_1)\|_{\tilde{E}^p} \leq Z(\epsilon)\|(\tilde{q}_2 - \tilde{q}_1, \bar{m}_2 - \bar{m}_1)\|_{\tilde{E}^p},
\]

with

\[
Z(\epsilon) = C\epsilon + C \sup_{i \in \{1, 2\}} \left( \|q_i\|_{L^\infty_T(B^{d/p})} + \|q_i\|_{L^2_T(B^{d/p+1})} + \|m_i\|_{L^2_T(B^{d/p})} \right).
\]

From (79) for $q_1$ and the definition of the space $\tilde{C}_T^d(B^{d/p})$, we infer that

\[
\lim_{\epsilon \to 0} Z(\epsilon) = 2C\|q_1(T_m)\|_{B^{d/p}} \leq 4/5,
\]

thus $Z(\epsilon) < 1$ for $\epsilon$ small enough. We therefore get $(\tilde{q}_2, \bar{m}_2) = (\tilde{q}_1, \bar{m}_1)$ on $[0, \epsilon]$. This achieves the proof of lemma 4. \(\square\)

**Remark 3** In theorem 5, we supposed that the external forcing term $f$ vanishes. We can easily show that the local existence and uniqueness still holds if $f$ belongs to $L^1_T(B^{d/p-1})$ where $p < 2d$. Indeed, the usual product maps $B^{d/p} \times B^{d/p-1}$ into $B^{d/p-1}$, provided $p < 2d$. 35
7 Further remarks

7.1 Weak solutions in dimension 2

We now focus on the 2-dimensional problem and study existence results of weak solutions. A weak solution \((\rho, \mathbf{u})\) of (NSK) in \(\mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^2)\) is required to satisfy the finite energy a priori bounds (16) (17) for initial data verifying (15). More precisely, we require that for all test functions \((\psi, \phi) \in C_0^\infty([0, \infty) \times \mathbb{R}^2) \times C_0^\infty([0, \infty) \times \mathbb{R}^2)^2\),

\[
\int_{\mathbb{R}^2} \rho \psi(t) \, dx = \int_{\mathbb{R}^2} \rho_0 \psi(0) \, dx - \int_0^t \int_{\mathbb{R}^2} \rho \mathbf{u} \cdot \nabla \psi \, dx,
\]

\[
\int_{\mathbb{R}^2} \rho \mathbf{u}(t) \cdot \phi(t) \, dx = \int_{\mathbb{R}^2} \rho_0 \mathbf{u}_0 \cdot \phi(0) \, dx + \int_0^t \int_{\mathbb{R}^2} \left( \rho \mathbf{u} \otimes \mathbf{u} : \mathbf{D}(\phi) - \mu \mathbf{D}(\mathbf{u}) : \mathbf{D}(\phi) - (\lambda + \mu) \text{div} \mathbf{u} \text{div} \phi + \rho(\rho) \text{div} \phi + \kappa \left( \frac{1}{2} \text{div} \phi + \nabla \rho \otimes \nabla \rho : \mathbf{D}(\phi) - \frac{\rho^2}{2} \Delta \text{div} \phi \right) \right) \, dx.
\]

As pointed out in the introduction, we are not able to prove a global existence theorem like in [17] when \(\kappa \neq 0\) because of the quadratic terms \(\nabla \rho \otimes \nabla \rho\), even though \(\nabla \rho\) is priori bounded in \(L^\infty((0, T); L^2(\mathbb{R}^d))\). We focus now on weak solutions in dimension \(d = 2\) near a stable equilibrium, i.e. solutions \((\rho, \mathbf{u})\) close to \((\bar{\rho}, 0)\), where \(\bar{\rho} > 0\) satisfies \(P'(\bar{\rho}) > 0\). Let us introduce

\[
\delta(t) = \left\| \frac{\rho(t) - \bar{\rho}}{\bar{\rho}} \right\|_{L^\infty},
\]

which measures the density fluctuation in \(L^\infty\). Unfortunately, such an a priori bound does not seem to be available for finite energy initial data. However, in view of the results of the preceding sections, it is valid for suitably smooth initial data and small enough time, which motivates the following result

**Proposition 7** There exists \(\eta > 0\) such that as long as

\[
\mathcal{E}_0 + \sup_{t \in [0, T]} \delta(t) \leq \eta,
\]

there exists a weak solution \((\rho, \mathbf{u})\) on \((0, T)\) of (NSK) such that \(\rho - \bar{\rho} \in L^2(0, T; H^1(\mathbb{R}^2) \cap H^2(\mathbb{R}^2)), \ u \in L^2(0, T; H^1(\mathbb{R}^2))^2 \cap L^\infty(0, T; L^2(\mathbb{R}^2))^2\).

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Proof: First, multiplying the equation of momentum conservation by $\nabla \rho^2$ and integrating by part, we easily infer
\[
\frac{d}{dt} \int_{\mathbb{R}^2} m \cdot \nabla \rho^2 \, dx + \frac{k}{2} \int_{\mathbb{R}^2} (\Delta \rho^2)^2 \, dx = \int_{\mathbb{R}^2} \left( 2\rho |\text{div} \, m|^2 + 2\kappa |\nabla \rho|^4 + \nabla^2 \rho^2 : (\rho \otimes u) \right)
\] 

\[-(\lambda + 2\mu) \Delta \rho^2 \text{div} \, u - 2\rho P'(\rho)|\nabla \rho|^2 \bigg) \, dx, \quad (81)
\]
hence, denoting $Z(s) = 2s P'(s)$, we obtain
\[
\frac{d}{dt} \int_{\mathbb{R}^2} m \cdot \nabla \rho^2 \, dx + \int_{\mathbb{R}^2} \left( \tilde{Z}(\bar{\rho}) |\nabla \rho|^2 + \frac{k}{4} |\Delta \rho|^2 \right) \, dx \leq C(1 + \|\rho\|^3_{L^\infty}) \|\nabla u\|^2_{L^2}
\] 
\[+ C\|\rho\|_{L^\infty} \|u \cdot \nabla \rho\|^2_{L^2} + C \|\nabla \rho\|^4_{L^4} + C \|\rho\|^2_{L^\infty} \|u\|^4_{L^4} + C \|Z(\rho) - Z(\bar{\rho})\|_{L^\infty} \|\nabla \rho\|^2_{L^2}. \]

Let us now recall Gagliardo Nirenberg’s inequality
\[
\|f\|^2_{L^4} \leq C \|f\|_{L^2} \|\nabla f\|_{L^2}. \quad (82)
\]
Thus, we obtain assuming that $\delta(t) < 1/2$
\[
\|u\|^4_{L^4} \leq C \|u\|^2_{L^2} \|\nabla u\|^2_{L^2},
\] 
\[
\|\nabla\rho\|^2_{L^4} \leq C \|\rho^{-1}\|^2_{L^\infty} \|\nabla \rho^2\|^2_{L^4} \leq C(1 + \delta(t)^2) \|\nabla \rho^2\|_{L^2} \|\Delta \rho^2\|_{L^2}
\] 
\[\leq C(1 + \delta(t)^3) \|\nabla \rho\|_{L^2} \|\Delta \rho^2\|_{L^2},
\] 
\[
\|u \cdot \nabla \rho\|^2_{L^2} \leq \|u\|^2_{L^4} \|\nabla \rho\|^2_{L^4} \leq C(1 + \delta(t)^3) \|u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla \rho\|_{L^2} \|\Delta \rho^2\|_{L^2},
\] 
\[
\|u\|_{L^2} \leq C(1 - \delta(t))^{-\frac{3}{2}} \|\sqrt{\rho} u\|_{L^2}.
\]
Hence, for $\delta(t) < \eta$ (where $\eta > 0$ depends on $Z'$), we have for some $C_0 > 0$
\[
\frac{d}{dt} \int_{\mathbb{R}^2} m \cdot \nabla \rho^2 \, dx + C_0 \left( \|\nabla \rho\|^2_{L^2} + \|\Delta \rho^2\|^2_{L^2} \right) \leq C \|\nabla u\|^2_{L^2}
\] 
\[+ C \|u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla \rho\|_{L^2} \|\Delta \rho^2\|^2_{L^2} + C \|\nabla \rho\|_{L^2} \|\Delta \rho^2\|^2_{L^2} + C \|u\|^2_{L^2} \|\nabla u\|^2_{L^2},
\] 
\[
\leq C \|\nabla u\|^2_{L^2} \left( 1 + \|u\|^2_{L^2} \right) + C \|\nabla \rho\|^2_{L^2} \|\Delta \rho^2\|^2_{L^2}
\] 
\[\leq C \|\nabla u\|^2_{L^2} \left( 1 + \|\sqrt{\rho} u\|^2_{L^2} \right) + \|\nabla \rho\|^2_{L^2} \|\Delta \rho^2\|^2_{L^2},
\] 
\[
\leq C_{E_0} \left( \|\nabla u\|^2_{L^2} + \|\Delta \rho^2\|^2_{L^2} \right) + C \|\nabla u\|^2_{L^2}.
\]
Therefore, as soon as $E_0$ is small enough, we have for some constant $C_1 > 0$
\[
\frac{d}{dt} \int_{\mathbb{R}^2} m \cdot \nabla \rho^2 \, dx + C_1 \left( \|\nabla \rho\|^2_{L^2} + \|\Delta \rho^2\|^2_{L^2} \right) \leq C \|\nabla u\|^2_{L^2}. \quad (83)
\]
Let $\alpha > 0$ and define $w_\alpha$ by

$$w_\alpha(t) = \frac{1}{2} \| \sqrt{\rho_0} \|_{L^2}^2 + \| \pi(\rho) - \pi(\bar{\rho}) \|_{L^1} + \frac{\kappa}{2} \| \nabla \rho \|_{L^2}^2 + \alpha \int_{\mathbb{R}^d} \mathbf{m} \cdot \nabla \rho^2 \, dx.$$ 

Then, choosing $\alpha$ small enough, we deduce that

$$\frac{1}{2} w_\alpha(t) \leq w_\alpha(t) \leq \frac{3}{2} w_\alpha(t),$$

and from the energy bounds and (83) that

$$\frac{d}{dt} w(t) + C_2 \left( \| \nabla u \|_{L^2}^2 + \| \nabla \rho \|_{L^2}^2 + \| \Delta \rho \|_{L^2}^2 \right) \leq 0,$$

so that we have the claimed a priori bounds. Now considering a suitable approximate problem, we can easily pass to the limit in quadratic terms like $\nabla \rho \otimes \nabla \rho$, since $\nabla \rho$ is bounded in $L^2(0, T; H^1(\mathbb{R}^d))$, and also in $C([0, T]; H^{-1}(\mathbb{R}^d))$, as can be seen by writing a linear transport equation on $\nabla \rho$.

\begin{equation}
\tag{84}
\end{equation}

### 7.2 Blow-up of solutions with compactly supported density

Let us finally make a few remarks based upon the work of Z. Xin [22] in the noncapillary case. We consider the full Navier-Stokes Korteweg system (1)(2)(3) (with energy equation), for which Hattori and Li [14] proved global existence of $H^s$ solutions (for $s > 0$ large) close enough to constant states $\|(\rho_0, \mathbf{m}_0, \theta_0) - (\bar{\rho}, 0, \bar{\theta})\| < 1$ when the heat conduction parameter $\alpha$ is positive.

We expect that the preceding results, namely global well-posedness of the Korteweg system, still hold in scale invariant Besov spaces for solutions close to constant states.

In contrast to the preceding approach where $\rho$ is close to a constant, we assume now that the initial data satisfy

$$A(0) \overset{\text{def}}{=} \int_{\mathbb{R}^d} \left( \rho_0 \frac{\| \mathbf{u}_0 - \mathbf{x} \|_2^2}{2} + \rho_0 c_0 \right) \, dx < +\infty,$$

which is the case for instance if $\rho_0$ has compact support. For the sake of simplicity, we consider a perfect gas law $p = \rho R \theta$, $e = c_v \theta$, and set $\gamma = 1 + R/c_v$. 

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When $\gamma \in (1, 1+2/d]$, we define $\sigma(t) = (1+t)$, whereas when $\gamma \in (1+2/d, \infty)$, we take $\sigma(t) = t$. Then, we deduce from easy computations (see [22]) that

$$\frac{d}{dt} A(t) = \sigma(t) \int_{\mathbb{R}^d} (2\rho e - dp) \, dx,$$

where

$$A(t) = \int_{\mathbb{R}^d} \left( p \frac{|u\sigma(t) - x|^2}{2} + \sigma(t)^2 \rho e \right)(t) \, dx + \kappa \int_{\mathbb{R}^d} \sigma(s) \int_{\mathbb{R}^d} |\nabla \rho(s, x)|^2 ds \, dx.$$

Using the fact that $p = (\gamma - 1)\rho e$ and denoting $\delta = 2 - d(\gamma - 1)$, we obtain

$$\dot{A}(t) \leq \delta \frac{A(t)}{\sigma(t)}; \quad \text{hence} \quad A(t) \leq A(0)\sigma(t)^{\max(\delta, 0)}.$$

Therefore, we have

$$\sigma(t)^{\min(2, \frac{d(\gamma-1)}{\delta})} \int_{\mathbb{R}^d} p \, dx \leq (\gamma - 1)A(0). \quad (85)$$

As a consequence, we obtain in the isothermal case $\theta = \bar{\theta}$

$$RM\bar{\theta}\sigma(t)^{\min(2, \frac{d(\gamma-1)}{\delta})} \leq (\gamma - 1)A(0),$$

denoting by $M$ the total mass, which proves that solutions blow up after some critical time $T_0$ as soon as $A(0)$ is finite.

Let us remark that similar observations can be done in the isentropic case, as well as in the nonisentropic case when the thermal diffusion is neglected (i.e. $\alpha = 0$). In order to get blow up estimates, we have to consider initial densities compactly supported in $\mathbb{R}^d$ (see [22]), and observe that the support of the density does not grow when the time evolves. Then, blow up estimates stem from estimate (85) and Hölder’s inequality.

**Appendix**

This section is devoted to a commutation lemma that we used to prove proposition 5.
Lemma 5 Suppose $A \in \tilde{L}_T^2(B^{d/2+1})$ and $B \in \tilde{L}_T^2(B^{d/2})$. Then the following estimate holds on $[0, T]$:

$$
\| \partial_h [A, \Delta_x \partial_j] B \|_{L_T^2(L^2)} \leq C \sum_{l=0}^{d} 2^{-l(\frac{d}{2} - 1)} \| A \|_{\tilde{L}_T^2(B^{d/2+1})} \| B \|_{\tilde{L}_T^2(B^{d/2})},
$$

where $C$ depends only on $d$, and $\sum_{l} c_l \leq 1$.

Proof: The proof of the above lemma requires some paradifferential calculus. We have to recall here that paradifferential calculus enables to define a generalized product between distributions, which is continuous in many functional spaces where the usual product does not make sense (see the pioneering work of J.-M. Bony in [4]). The paraproduct between $u$ and $v$ is defined by

$$
T_u v \overset{\text{def}}{=} \sum_{\ell \in \mathbb{Z}} S_{\ell - 1} u \Delta_{\ell} v.
$$

We thus have the following formal decomposition (modulo a polynomial):

$$
uv = T_u v + T'_u u, \quad \text{with} \quad T'_u u \overset{\text{def}}{=} \sum_{\ell \in \mathbb{Z}} S_{\ell - 2} v \Delta_{\ell} u.
$$

Coming back to the proof of lemma 5, we split $\partial_h [A, \Delta_x \partial_j] B$ into

$$
\partial_h [A, \Delta_x \partial_j] B = T'_{\Delta_x \partial_j \partial_k} A - \Delta_x \partial_j T'_{\partial_k} A + 2^{\ell} [T_A, \Delta'_x] \partial_k B + \partial_h A \Delta_x \partial_j B - \Delta_x \partial_j (B \partial_k A)
$$

with $\Delta'_x = \varphi^j(2^{-\ell} D)$ and $\varphi^j(x) = -i \xi^j \varphi(\xi)$. From now on, we agree that $(c_\ell)_{\ell \in \mathbb{Z}}$ denotes a positive sequence such that $\sum_{\ell} c_\ell \leq 1$. According to (20) and to the definition of $T'$, we have

$$
T'_{\Delta_x \partial_j \partial_k} A = \sum_{m \geq \ell - 2} S_{m + 2} \Delta_x \partial_j \partial_k B \Delta_m A.
$$

Thus, in view of Bernstein’s lemma and the definition of $\tilde{L}_T^2(B^s)$ spaces,

$$
\| T'_{\Delta_x \partial_j \partial_k} A \|_{L_T^2(L^2)} \lesssim \sum_{m \geq \ell - 2} 2^{\frac{d}{2} m} \| \Delta_x \partial_j \partial_k B \|_{L_T^2(L^2)} \| \Delta_m A \|_{L_T^2(L^2)}
$$

$$
\lesssim 2^{\frac{d}{2} \ell} \| \Delta_x B \|_{L_T^2(L^2)} \sum_{m \geq \ell - 2} 2^{m(\frac{d}{2} + 1)} \| \Delta_m A \|_{L_T^2(L^2)},
$$

$$
\lesssim 2^{-\ell(\frac{d}{2} - 1)} \left( 2^{\frac{d}{2}} \| \Delta_x B \|_{L_T^2(L^2)} \right) \sum_{m \geq \ell - 2} 2^{m(\frac{d}{2} + 1)} \| \Delta_m A \|_{L_T^2(L^2)},
$$

$$
\lesssim c_\ell 2^{-\ell(\frac{d}{2} - 1)} \| B \|_{\tilde{L}_T^2(B^{d/2})} \| A \|_{\tilde{L}_T^2(B^{d/2+1})}.
$$
We use classical estimates for the paraproduct to bound the second term of the right-hand side (see [7] and [8]). We get

\[ \| T' \delta_k A \|_{L^1_T(B^{d/2})} \lesssim \| \partial_k B \|_{L^2_T(B^{d/2-1})} \| A \|_{L^2_T(B^{d/2+1})}. \]

Using spectral localization of \( \Delta_\ell \) and definition of \( \tilde{L}^s_T(B^s) \) spaces, we get

\[ \| \partial_j \Delta_\ell T' \delta_k A \|_{L^1_T(L^2)} \lesssim e\ell 2^{-\ell(\frac{d}{2}-1)} \| \partial_k B \|_{\tilde{L}^2_T(B^{d/2-1})} \| A \|_{\tilde{L}^2_T(B^{d/2+1})}. \]

To estimate the third term, we observe that the definition of paraproduct and (20) yields

\[ [T_A, \Delta_\ell] \partial_k B = \sum_{m=4}^{4} [S_{m-1} A, \Delta_\ell] \Delta_m \partial_k B. \]

Denoting \( h^j = \mathcal{F}^{-1} \varphi^j \) and applying first order Taylor’s formula, we get for \( x \in \mathbb{R}^d \),

\[ [S_{m-1} A, \Delta_\ell] \Delta_m \partial_k B(x) = 2^{-\ell} \int_{\mathbb{R}^d} h^j(y) \left( \sum_{k=1}^{d} \nabla A(x - 2^{-\ell} y) \right) \Delta_m \partial_k B(x - 2^{-\ell} y) d\tau dy. \]

Convolution inequality thus yields

\[ \left\| [S_{m-1} A, \Delta_\ell] \Delta_m \partial_k B \right\|_{L^2} \lesssim 2^{-\ell} \| \nabla A \|_{L^\infty} \| \Delta_m \partial_k B \|_{L^2}, \]

hence

\[ \| 2^\ell [T_A, \Delta_\ell] \partial_k B \|_{L^1_T(L^2)} \lesssim 2^{-\ell(\frac{d}{2}-1)} \| \nabla A \|_{L^\infty_T(L^\infty)} \left( 2\ell(\frac{d}{2}-1) \| \Delta \nabla B \|_{L^\infty_T(L^\infty)} \right), \]

\[ \lesssim e\ell 2^{-\ell(\frac{d}{2}-1)} \| \nabla A \|_{L^\infty_T(L^\infty)} \| \nabla B \|_{\tilde{L}^2_T(B^{d/2-1})}. \]

Finally, we have

\[ \| \partial_k A \Delta_\ell \partial_j B \|_{L^1_T(L^2)} \leq \| \partial_k A \|_{L^1_T(L^\infty)} \| \Delta_\ell \partial_j B \|_{L^2_T(L^2)} \]

and, according to proposition 3,

\[ \| B \partial_k A \|_{L^1_T(B^{d/2})} \lesssim \| B \|_{\tilde{L}^2_T(B^{d/2})} \| \nabla A \|_{\tilde{L}^2_T(B^{d/2})}, \]

so the proof of lemma 5 is achieved.
References


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