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1. Introduction.— Let $X$ be an algebraic curve over the field $\mathbb{C}$ of complex numbers which is assumed to be smooth, connected and projective. For simplicity, we assume that the genus of $X$ is $> 2$. Let $G$ be a simple simply connected group and $M_G(X)$ the coarse moduli scheme of semistable $G$-bundles on $X$. Any linear representation determines a line bundle $\Theta$ on $M$ and some non-negative integer $l$ (the Dynkin index of the representation, cf. [KNR], [LS]). Its is known that the choice of a (closed) point $x \in X(\mathbb{C})$ (and, a priori, of a formal coordinate near $x$) of $X$ determines an isomorphism (see (5.4)) between the projective space of conformal blocks $P_{\mathcal{B}_l}(X)$ (for $G$) of level $l$ and the space $PH^0(M_G(X)_x, \Theta)$ of generalized theta functions (see [BL], [F1], [KNR], [LS]). In fact, it is observed in [T] that there is a coordinate free description of $\mathcal{B}_l(X)$.

When the pointed curve $(X, x)$ runs over the moduli stack $M_{g, 1}$ of genus $g$ pointed curve, these 2 projective spaces organize in 2 projective bundles $P\Theta$ and $P\mathcal{B}_l$. We first explain (see (5.7)) how to identify these 2 projective bundles (this a global version of the identification above. The projective bundle $P\Theta$ has a canonical $\Omega\text{at}$ connection: the Hitchin connection [H] and $P\mathcal{B}_l$ has a $\Omega\text{at}$ connection, which we call the WZW connection coming from conformal field theory (see [TUY] or [S]). In the rest of the paper, we prove that this canonical identification (5.7)

$$\kappa : P\Theta \cong P\mathcal{B}_l$$

is $\Omega\text{at}$ (theorem (9.3)).

1.1. Let me roughly explain how to prove the $\Omega\text{at}$-ness. Let $M$ be the smooth open subvariety of $M_G(X)$ parameterizing regularly stable bundles $E$ (such that $\operatorname{Aut}_G(E) = Z(G)$, the center of $G$). The cup-product

$$\operatorname{H}^1(X, T_X) \otimes \operatorname{H}^0(X, \text{ad}(E) \otimes \omega_X) \to \operatorname{H}^1(X, \omega_X) = \mathbb{C}.$$ 

defines a morphism $T_{[X]} : \mathcal{M}_g \to \mathbb{S}^2 T_{[X]} M$ which globalizes in

$$(*) \quad T_{[X]} : \mathcal{M}_g \to \mathbb{H}^0(M, \mathbb{S}^2 TM).$$

Let $s$ be a generalized theta function and $d_i s$ the length 1 complex

$$d_i s : \mathcal{D}'(\Theta) \xrightarrow{i} \mathcal{D}_{s} \xrightarrow{\Theta} \Theta$$

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which evaluates the differential operator $D$ of order $\leq i$ on $s$. The symbol exact sequence

$$0 \rightarrow d_1 s \rightarrow d_2 s \rightarrow S^2 TM \rightarrow 0$$

defines a Bockstein operator $\delta : H^i(S^2 TM) \rightarrow H^{i+1}(d_1 s)$. Let $w_s$ be the composite morphism

$$w_s = H^1(X, T_X) \rightarrow H^i(S^2 TM) \rightarrow H^{i+1}(d_1 s)$$

Let $\tilde{\mathcal{M}}$ be the image of a tangent vector on $\mathcal{M}_G$ by $(\ast)$. The main ingredient in the computation of Hitchin's connection is the computation of $w_s(\tilde{t})$. If $(U_\alpha)$ is an affine cover of $U$, the class $w_s(\tilde{t})$ can be represented by a pair $(D_\alpha - D_{\beta_1} - D_{\beta_2})$ where $s$ is some second order differential operator defined on $U_\alpha$. It is well known that $G$-bundles trivialized on punctured curve $X^* = X \backslash \mathfrak{a}$ are parameterized by an infinite dimensional homogeneous ind-scheme $Q = G(\text{Frac}(\mathcal{O}_x))/G(\mathcal{O}_x)$ (see [LS]). Let $Q^0$ be the open sublocus of $Q$ parameterizing regularly stable $G$-bundles. The crucial point (cf. [DS]) is that $Q^0 \rightarrow M$ is a locally trivial torsor (for the étale topology). The idea of the paper is to use the cover $Q^0 \rightarrow M$ to compute some representative of $w_s(\tilde{t})$, even though the author does not control all second order differential operators on $Q^0$. Let $t$ be a meromorphic tangent vector on $D^*$ projecting on $\tilde{t}$ (7.3). To avoid too much abstract nonsense on differential operators on ind-schemes, we use an étale quasi-section (8)

$$\sigma$$

of $Q^0 \rightarrow M$ to construct a second order differential operator $\theta(t) \in H^0(N, D^2(\sigma^* \Theta))$ computing $w_s(\tilde{t})$. In a certain sense, $\theta(t)$ is the "pull-back" of the Sugawara tensor $T(t)$ (see definition (8.11)). The theorem follows easily, because the only non trivial term in the formula defining the WZW connection is the Sugawara tensor $(9.1)$.

1.2. Under the hypothesis $\text{codim}_{\mathcal{M}_G}(\mathcal{M}_G \backslash \mathcal{M}^0_G) > 2$, Hitchin constructs the connection not only for the bundle $\mathbf{P} \Theta$ of theta functions coming from determinantal line bundles on $\mathcal{M}_G$, but also for the bundle $\mathbf{P} p_* \mathcal{L}$ where $\mathcal{L}$ is any line bundle on $\mathcal{M}^0_G$ and $p : \mathcal{M}^0_G \rightarrow \mathcal{M}_G$ is the universal family of coarse moduli spaces of regularly stable bundles. The codimension assumption is used to identify $H^i(\mathcal{M}_G, F)$ with $H^i(\mathcal{M}^0_G, F)$, $i = 0, 1$ for any vector bundle $F$ on $\mathcal{M}_G$. This identification shows that the formation of the direct image $p_* \mathcal{L}$ commutes with base change. The flatness result is written in this context.

1.3. For completeness, we compute the Picard group of the universal moduli stack of $G$-bundles over $\mathcal{M}_{g,1}$. This allows to compare a determinantal line bundle and the line bundle $\mathcal{L}$ (5.6).

**Notations.-** We work over the field $\mathbb{C}$ of complex numbers. We fix a simple Lie algebra $\mathfrak{g}$ with a Borel subalgebra $\mathfrak{b}$. Let $\theta$ be the longest root (relative to $\mathfrak{b}$) and $\mathfrak{s}_2(\theta) = (X_\theta, X_{-\theta}, H_\theta)$.
a corresponding $\mathfrak{sl}_2$-triple. Finally, $(\omega, \theta) = 2$. If $\rho$ is half of the sum of the positive roots, the dual Coxeter number is $h^\vee = 1 + \frac{\rho}{\theta^\vee}$. Let $G$ be the simply connected algebraic group of Lie algebra $\mathfrak{g}$. The symbol $X$ (resp. $x$) will always define a smooth, connected and projective complex curve of genus $g > 2$ (resp. a point of $X(\mathbb{C})$). If $\mathcal{X} \to S$ is family of genus $g$ pointed curve, we'll denote by $\mathcal{X}$ the formal neighborhood of the marked section $S \to \mathcal{X}$.

**Conformal blocks and theta functions over $\mathcal{M}_{g,1}$**

We want to identify over $\mathcal{M}_{g,1}$ the projective bundle of conformal blocks $P\Theta$ and the the projective bundle generalized theta function $P\Phi_l$ as done in [BL] in the absolute case. The precise statement is in (5.7)

2. **Residues**  
We denote by $K$ the field of fractions of $\mathcal{O} = \mathcal{O}_{X,x}$. The dualizing sheaf $\omega$ of $X$ is the biggest quotient of $\Omega_{X/\mathbb{C}}$ which is separated for the $x$-adic topology. Let me denote by $d : \mathcal{O} \to \omega$ the projection of the universal derivation $\mathcal{O} \to \Omega_{X/\mathbb{C}}$ on $\omega$. If $z$ is a formal coordinate at $x$, the $\mathcal{O} = \mathbb{C}[[z]]$-module $\omega$ is the free module $\mathbb{C}[[z]].dz$ and $\omega = K \otimes \mathcal{O} \omega = C((z)).dz$. Recall that there exists a residue map $\text{res} : \omega \to C$ which is given in coordinates by $\text{res}(\sum_{n \geq N} a_n z^n dz) = a_{-1}$.

2.1. Let $\pi : (X, x) \to S$ be a pointed curve over an affine $\mathbb{C}$-scheme $S = \text{Spec}(R)$ and $\omega^\pi$ (resp. $\omega^\pi$) be the relative dualizing sheaf of $X \to S$ (resp. $\mathcal{X}^* \to S$). Because formal coordinates along $x$ exists Zariski locally in $\mathcal{X}$, the residue is defined as a (functorial) $R$-morphism $\text{res} : \omega^\pi \to R$. Let $A_X$ be the algebra $\Gamma(S, \pi_* \mathcal{O}_{X\setminus x})$ which is embedded in $K = \Gamma(S, \pi_* \mathcal{O}_{X\setminus x})$ by Taylor expansion.

**Lemma 2.**  
Let $f \in A_X$. Then, $\text{res}(f) = 0$.

**Proof:** because $\mathcal{M}_{g,1}$ is a smooth $\mathbb{C}$-stack, one can assume that $S$ is a least reduced of finite type over $\mathbb{C}$. The residue theorem says that $\text{res}(f)(r) = 0$ for all $r \in S(\mathbb{C})$ which implies that $\text{res}(f) = 0$.

3. **Loop algebras**  
We start with our pointed curve $(X, x)$ and the simple algebra $\mathfrak{g}$. Let $l$ be a positive integer. We would like to give an explicit coordinate free description of the vector spaces $B_l(X)$ of conformal blocks of level $l$ on $(X, x)$ which coincide with the usual one once a coordinate has been chosen and which globalizes when the pointed curve moves.

3.1. The loop algebra $\hat{L}\mathfrak{g} = L\mathfrak{g} \oplus \mathbb{C} c$ of $\mathfrak{g}$ is the universal central extension of $L\mathfrak{g} = \mathfrak{g} \oplus K$ by $C = \mathbb{C}.c$ with bracket

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + (X | Y) \text{res}(gdf).$$

Let me denote by $\hat{L}\mathfrak{g}$ the Lie subalgebra $L^+\mathfrak{g} \oplus \mathbb{C}.c$ of $L\mathfrak{g}$ (where $L^+\mathfrak{g} = \mathfrak{g} \oplus \mathcal{O}$).
Let $\lambda$ be a dominant weight of level $l$ (i.e., $(\lambda, \theta) \leq l$) and $M$ be the simple $\mathfrak{g}$-module with highest weight $\lambda$ and highest weight vector $v_\lambda$. Let $M_l$ be the $L^+\mathfrak{g}$-module structure on $M$ where the action of $L^+\mathfrak{g}$ (resp. $\mathfrak{c}$) is induced by $L^+\mathfrak{g} \to \mathfrak{g}$ (resp. is the multiplication by $l$).

We denote by $V_{\lambda,l}$ the Verma module of weight $(\lambda, l)$

$$V_{\lambda,l} = U(\widehat{\mathfrak{g}}) \otimes_{U(L^+\mathfrak{g})} M_l$$

and by $v_{\lambda,l}$ the highest weight vector $1 \otimes v_\lambda$.

**Lemma 3.2.** Let $z$ be a formal coordinate of $X$ at $x$. The line $C.(X_{\theta} \otimes z^{-1})^{l+1-(\lambda, \theta)}v_l$ of $V_{\lambda,l}$ does not depend on the choice of $z$.

**Proof:** Let $u(z)$ (with $u(0) = 0$ and $u'(0) \neq 0$) another coordinate (set $a = \frac{1}{u'(0)}$). Then,

$$X_{\theta} \otimes u(z)^{-1} = aX_{\theta} \otimes z^{-1} \mod CX_{\theta} \oplus L^0\mathfrak{g}$$

where $L^0\mathfrak{g}$ is the kernel of $\mathfrak{g} \otimes \mathcal{O} \to \mathfrak{g}$. Then,

$$(X_{\theta} \otimes u(z)^{-1})^{l+1-(\lambda, \theta)} = a^{l+1-(\lambda, \theta)}(X_{\theta} \otimes z^{-1})^{l+1-(\lambda, \theta)} \mod U(\widehat{\mathfrak{g}})(CX_{\theta} \oplus L^0\mathfrak{g})$$

(because $l+1-(\lambda, \theta)$ is positive) and the lemma follows because $X_{\theta}$ kills $v_\lambda$ and $L^0\mathfrak{g}$ kills even the whole $M$.

In the most interesting case for us, namely when $\lambda = 0$ (i.e. $M = C$), we denote $V_{(\lambda, l)}$ simply by $V_l$.

**Definition 3.3.** We denote by $Z_l$ the $U(\widehat{\mathfrak{g}})$-submodule generated by $C.(X_{\theta} \otimes z^{-1})^{l+1}$ (a formal coordinate at $x$) and by $H_l$ the quotient $V_l/Z_l$.

The usual theory of representation of affine algebras says that $H_l$ is the fundamental representation of level $l$ of $\widehat{\mathfrak{g}}$ (see [B]). In particular, the canonical embedding of $\mathfrak{g}$-modules $C \hookrightarrow H_l$ has image the annihilator of $L^+\mathfrak{g}$.

By the residue theorem, the embedding $L_{X\mathfrak{g}} = \mathfrak{g} \otimes A_{X} \hookrightarrow \mathfrak{g}$ lifts canonically to an embedding $L_{X\mathfrak{g}} \hookrightarrow \widehat{\mathfrak{g}}$.

**Definition [TUY] 3.4.** The (finite dimensional) vector space

$$B_l(X) = \text{Hom}_{U(\mathfrak{g})}(H_l, C) = (H_l/L_{X\mathfrak{g}}H_l)^*$$

is the space of vacua (or conformal blocks) of level $l$.

**3.5.** Let $\pi : (\mathcal{X}, x) \to S = \text{Spec}(\mathcal{R})$ be a family of genus $g$ pointed curve. One the relative version

$$(\widehat{\mathfrak{g}}, L^+\mathfrak{g}, L_{X\mathfrak{g}}, V_l(\pi))$$

of $(\widehat{\mathfrak{g}}, L^+\mathfrak{g}, L_{X\mathfrak{g}}, V_l)$ exactly as before. Now, because formal coordinates along $x$ exists Zariski locally in $S$, one defines as in definition (3.3) the submodule $Z_l(\pi)$ of $V_l(\pi)$ and correspondingly the $L^+\mathfrak{g}$-modules

$$H_l(\pi) = V_l(\pi)/Z_l(\pi).$$
The Lie algebra $L_X \mathfrak{g}$ embeds canonically in $\widehat{L_\mathfrak{g}}$ (2.2). One defines the module (which is in fact a projective $R$-modules by [TUY]) of covacua by the equality

$$B_\mathfrak{g}(\pi) = \mathcal{H}_\mathfrak{g}(\pi)/L_X \mathfrak{g} \cdot \mathcal{H}_\mathfrak{g}(\pi)$$

and the module of vacua by

$$B_\mathfrak{g}(\pi) = \text{Hom}_R(B_\mathfrak{g}(\pi), R).$$

The construction $\pi \mapsto B_\mathfrak{g}(\pi)$ (resp. $\pi \mapsto B_\mathfrak{g}(\pi)$) is functorial in $\pi$: this defines two vector bundles $B_\mathfrak{g}$ and $B_\mathfrak{g}$ on $\mathcal{M}_{g,1}$ which are dual to each other. If $\pi$ is the fixed curve $(X, x) \to \text{Spec}(C)$, the fiber $B_\mathfrak{g}(\pi)$ is $B_\mathfrak{g}(X)$ [TUY].

4. Loop groups.— Let us first recall the construction of the Kac-Moody group $\widehat{L_\mathfrak{g}}$ (of Lie algebra $\widehat{L_\mathfrak{g}}$) in the absolute case and of the corresponding generator $L$ of the Picard group of $Q = LG/L^\times G$ (see [LS]).

4.1. The adjoint action of $L_\mathfrak{g}$ on $\widehat{L_\mathfrak{g}}$ can be integrated explicitly as follows. Let $LG$ be the loop group of $G$ (whose $R$-points are $G(X_\mathfrak{g})$ or simply $G(R((z)))$ once a formal coordinate $z$ at $x$ has been chosen). Let $\Omega$ be a point of $LG(R)$: the cotangent morphism of the morphism

$$\Omega : X^*_R \to G$$

defines a morphism

$$\mathfrak{g}^* \otimes H^1(X_\mathfrak{g}^*, \mathcal{O}_{X_\mathfrak{g}^*}) \to \Omega_{X_\mathfrak{g}^*/R} \otimes \omega^*.$$

Let me denote by $\Omega^{-1} d\Omega$ the corresponding element of $\mathfrak{g} \otimes \omega^*$. 

**Remark 4.2.**— Suppose that $G$ is embedded in some $GL_N$ and that a coordinate $z$ has been chosen. Then, $\Omega$ is some invertible matrix $\Omega(z)$ of rank $N$ with coefficients in $R((z))$ and $\Omega^{-1} d\Omega$ is the matrix product $\Omega(z)^{-1} \Omega'(z) dz \in \omega^* = \mathfrak{g} \otimes C R((z)) dz$

Let $\alpha \in L_\mathfrak{g}(R)$ and $r \in R$. Then, $\Omega$ acts on $\alpha + r.c \in \widehat{L_\mathfrak{g}}(R)$ by

$$(4.1) \quad \text{Ad}(\Omega)(\alpha + r.c) = \text{Ad}(\Omega).\alpha + (s + \text{res}(\Omega^{-1} d\Omega \mid \alpha)).c. $$

4.3. Let me recall the following integrability property (result which is due to Faltings, see [BL], lemma A.3) of the basic integrable representation $\rho : \widehat{L_\mathfrak{g}} \to \text{End}(H_1)$:

**Proposition (Faltings) 4.4.**— Let $R$ be a $C$-algebra and $\Omega \in LG(R)$. Locally over $\text{Spec}(R)$, there exists an automorphism $u$ of $H_1 \otimes R$, uniquely determined up to $R^*$, satisfying

$$u_{R}(\alpha)u_{R}^{-1} = \rho(R(\text{Ad}(\Omega)).\alpha)$$

for any $\alpha \in \widehat{L_\mathfrak{g}}(R)$. This proves that the representation $\widehat{L_\mathfrak{g}} \to \text{End}(H_1)/C.\text{Id}$ is the derivative of an algebraic (i.e morphism of $C$-groups) representation $\rho : LG \to \text{PGL}(H_1)$. 

4.5. Let
\[ 1 \rightarrow \mathbb{G}_m \rightarrow \widehat{L\mathbb{G}} \rightarrow L\mathbb{G} \rightarrow 1 \]
the pull back of the extension
\[ 1 \rightarrow \mathbb{G}_m \rightarrow GL(H_1) \rightarrow PGL(H_1) \rightarrow 1. \]
The corresponding central extension of Lie algebras
\[
(4.2) \quad 0 \rightarrow C \rightarrow \text{Lie}(\widehat{L\mathbb{G}}) \rightarrow L\mathfrak{g} \rightarrow 0
\]
is the pull-back pull-back of
\[ 0 \rightarrow C \rightarrow \text{End}(H_1) \rightarrow \text{End}(H_1)/C.\text{Id} \rightarrow 0 \]
by \( d\tilde{\rho} \).

**Lemma 4.6.**— The central extension \((4.2)\) is the universal central extension
\[ 0 \rightarrow C \rightarrow \widehat{L\mathfrak{g}} \rightarrow L\mathfrak{g} \rightarrow 0 \]
of \((3.1)\).

**Proof:** as a vector space, \( \widehat{L\mathfrak{g}} = L\mathfrak{g} \oplus C.\mathfrak{c} \). Let \( \Phi \) be the morphism \( \Phi : L\mathfrak{g} \rightarrow \text{Lie}(\widehat{L\mathfrak{g}}) \) defined by \( \Phi(a, b, c) = [a, d\tilde{\rho}(a) + b, c] \) for \( a \in \widehat{L\mathfrak{g}} \) and \( b \in C \). By construction, \( \Phi \) is a Lie algebra isomorphism. \( \blacksquare \)

With the identification of the above lemma, the derivative of
\[ \tilde{\rho} : \widehat{L\mathbb{G}} \rightarrow GL(H_1) \]
is \( \rho \).

4.7. Let \( L^+\mathbb{G} \hookrightarrow L\mathbb{G} \) be the \( C \)-space whose \( R \)-points are \( G(X_R) \). Notice that \( L^+\mathbb{G} \) is an (infinite dimensional) affine \( C \)-scheme.

**Lemma 4.8.**— There exists a unique splitting \( \chi : \widehat{L^+\mathbb{G}} \rightarrow \mathbb{G}_m \) of
\[ 1 \rightarrow \mathbb{G}_m \rightarrow \widehat{L\mathbb{G}} \rightarrow L\mathbb{G} \rightarrow 1 \]
over \( L^+\mathbb{G} \).

**Proof:** by construction, the line \( C.x_1 \) of \( H_1 \) is stable by \( \widehat{L^+\mathbb{G}} \) and therefore defines the character \( \chi \) which is a splitting. Because every character of \( L\mathbb{G} \) is trivial, this splitting is unique. \( \blacksquare \)

4.9. If now we allow the pointed curve \((X, x)\) to move, ie if we consider our family \( \pi \) of pointed curve over a finite type basis \( S = \text{Spec}(R) \) (which is possible because \( \mathcal{M}_{g,1} \) is locally of finite type), one can construct the relative version \( L^\pi\mathbb{G} \) of \( \widehat{L\mathbb{G}} \) by integration of the representation \( \mathcal{H}_\pi(\pi) \) as in lemma (4.4). First of all, by unicity of the representation \( \tilde{\rho} \), the problem is local in \( S \). One can therefore assume that a formal coordinate \( z \in \Gamma(X_R \mathcal{O}) \) identifies \( X \) with \( X_R \) and \( \mathcal{H}_\pi \) with \( H_1 \otimes_C R \), reducing the problem to the absolute case. The details are left to the reader.
5. The universal Verlinde’s isomorphism.— Let us first recall in the absolute case how loop groups allows to uniformize the moduli stack \( \mathcal{M}_G \) of \( G \)-bundle over \( X \) and accordingly to describe generalized theta functions in terms of conformal blocks (see [LS]).

5.1. Let \( Q = LG/I^+G \) be the grassmannian parameterizing families of pairs \( (E, \rho) \) where \( E \) is a \( G \)-bundle over \( X \) and \( \rho \) is a trivialization of \( E \) over \( X^* \). Let \( L_XG \hookrightarrow LG \) be the ind-group parameterizing automorphisms of the trivial \( G \)-bundle \( X^* \times G \). Then, the forgetful morphism

\[
\begin{array}{ccc}
Q & \to & \mathcal{M}_G \\
(E, \rho) & \mapsto & E
\end{array}
\]

is a \( L_XG \)-torsor. The character \( \chi : L^+G \to \mathbb{G}_m \) of lemma (4.8) defines a \( \widehat{LG} \)-linearized line bundle \( \mathcal{L} \) on \( Q = \widehat{LG}/L^+G \) which is a generator of \( \text{Pic}(Q) \) (see [LS]).

The line bundle \( \mathcal{L} \) is associated to \( \chi^{-1} \) (cf. example 3.9 of [BL]). Sections of \( \mathcal{L} \) are functions \( f \) on \( \widehat{LG} \) such that

\[
f(gh) = \chi(h)f(g), \quad g \in \widehat{LG}(R), h \in L^+G(R).
\]

With this section, \( \mathcal{L} \) is the positive generator of \( Q \).

5.2. Let us recall the argument of [So] proving that \( L_XG \) is a subgroup of \( \widehat{LG} \). The fibred product

\[
\widehat{L_XG} = \widehat{LG} \times_{\widehat{L}G} L_XG
\]

certainly acts on the finite dimensional vector space of level 1 conformal blocks

\[
B_1(X) = \left( H_1/L_Xg\mathfrak{h}_1 \right)^*.
\]

The differential at the origin of the projective action

\[
L_XG \to \text{PGL}(B_1(X))
\]

is the natural morphism

\[
L_Xg \to \text{End}(B_1(X))/C.\text{Id}
\]

and is therefore trivial. Because \( L_XG \) is integral (see [LS]), \( \widehat{L_XG} \) acts by a character on \( B_1(X) \) defining the embedding \( L_XG \hookrightarrow \widehat{LG} \).

5.3. In particular, \( \mathcal{L} \) is \( L_XG \)-linearized and defines a line bundle still denoted by \( \mathcal{L} \) on \( \mathcal{M}_G = L_XG \setminus Q \) which generates \( \text{Pic}(\mathcal{M}_G) \). Let \( \mathcal{M}^0_G \) be the open substack of \( \mathcal{M}_G \) parameterizing regularly stable bundles (bundles \( E \) such that \( \text{Aut}_G(E) = Z(G) \), the center of \( G \)). Because \( Z(G) \) acts trivially on \( V_1 \), the center \( Z(G) \) acts trivially on the restriction of \( \mathcal{L} \) to \( Q^0 \) and \( \mathcal{L} \) is therefore \( L_XG/Z(G) \)-linearized. Therefore, \( \mathcal{L} \) comes from a line bundle (still denoted by \( \mathcal{L} \)) on the (smooth and quasi-projective) coarse moduli space \( M = M^0_G \) of regularly stable bundles (because \( Q^0 \to M \) is an isotrivial \( L_XG/Z(G) \)-torsor).

5.4. The space of generalized theta functions of level \( l \) is by definition

\[
H^0(M_G, \mathcal{L}^l) = H^0(Q, \mathcal{L}^l)^{L_XG}.
\]
By a codimension argument, it is also \( \mathbb{H}^0(M_G^0, \mathcal{L}^i) \) which is in turn \( \mathbb{H}^0(M_G^0, \mathcal{L}^i) \) (see [BL], [LS]). By [Ku], [M], the \( L^0 - \)module \( \mathbb{H}^0(Q_i, \mathcal{L}^i) \) is the (algebraic) dual \( H_i^* \) of \( H_i \), the isomorphism being unique up to nonzero scalar by Schur's lemma. Let us explicit the associated Verlinde isomorphism (see [BL], [Fa], [KNR], [LS])

\[
\kappa : \mathbb{P} \mathbb{B}_i(X) \cong \mathbb{P} \mathbb{H}^0(M_G^0, \mathcal{L}^i) = \mathbb{P} \mathbb{H}^0(M_G^0, \mathcal{L}^i).
\]

Let \( u \in B_i(X) \) be a \( L_XG \)-invariant form on \( H_i \). After an eventual étale base change, any smooth morphism \( S \to M_G^0 \) can be defined by a family of bundles. Therefore, let us consider \( S \to M_G \) a smooth morphism where \( S \) is a \( \mathbb{C} \)-scheme of finite type defined by a family of \( G \)-bundles \( E \). Étale locally in \( S \), let us choose a formal cocycle \( \Omega \in L^0G(S) \) defining \( E \). The multivalued function \( u_E \)

\[
(s) \mapsto u(\Omega(s).v_1)
\]

defines a divisor on the smooth scheme \( S \) (\( E \) is generic by assumption and therefore \( u_E \) is generically nonzero). The gluing of these divisors defines \( \kappa(u) \).

5.5. If now the curve \( \pi : (X, x) \to S = \text{Spec}(\mathbb{R}) \) is non constant, the family of ind-groups \( (L_XG)_s \) glue to give an ind-group \( L_XG \) over \( S \) which is a subgroup of \( L_xG \). As in (5.2), the action of \( L_XG \) on the vector bundle of level 1 vacua \( B_1 \) defines a character \( L_XG \to \mathbb{G}_m(S) \) and therefore an embedding (over \( S \))

\[
L_XG \hookrightarrow L_\pi G
\]

Recall that the action of \( L_\pi G \) on the trivial line bundle \( O_S. v_1 \hookrightarrow H_1(\pi) \) defines a character

\[
\chi : \begin{cases} 
L_\pi G & \to \mathbb{G}_m(S) \\
S & \to \mathbb{G}_m(S)
\end{cases}
\]

Of course, this construction is functorial in \( \pi \) and all the above construction are universal over \( M_{g,1} \).

5.6. The relative version of (5.4) goes as follows. Consider the relative grassmannian \( Q_\pi = L_XG/L_\pi G \) over \( S \) and the line bundle \( L \) on \( Q_\pi \) defined by \( \chi^{-1} \). Because \( L_XG \) embeds in \( L_\pi G \), the line bundle \( L \) is \( L_XG \)-linearized and therefore defines a line bundle \( L \) on the universal moduli stack \( L_XG\backslash Q \). The projection

\[
q_\pi : Q_\pi \to S
\]

is locally trivial for the Zariski topology (the choice of a formal coordinate along \( x \) defines such a trivialization). The formula (5.4) defines a morphism

\[
\iota_\pi : H_i(\pi)^* \to q_\pi^* \mathcal{L}^i.
\]
Because $q$ is locally trivial, showing that $\iota_\pi$ is an isomorphism remains to show that $\iota_\pi \odot C(s)$ is so for every $s \in S(C)$ which is the above theorem of [Ku], [M]. As in (5.1), let me consider the $L_XG$-torsor

$$r_\pi : Q_\pi \to L_XG \setminus Q = \mathcal{M}_{G,\pi}$$

If $p_\pi$ denotes the projection $\mathcal{M}_{G,\pi} \to S$, the sheaf $p_\pi_* \mathcal{L}^I$ of global sections of $\mathcal{L}^I$ is the invariant sheaf

$$(q_\pi_* \mathcal{L}^I)^{L_XG} = (\mathcal{H}(\pi)^*)^{L_XG}.$$ 

5.7. These constructions are functorial in $\pi$. Let $\mathcal{M}^0_G$ (resp. $\mathcal{M}^0_{G,\pi}$) be the universal coarse moduli space (resp. moduli stack) of regularly stable bundles. Let $p : \mathcal{M}^0_G \to \mathcal{M}_{G,\pi}$ be the projection, $X$ be the universal curve and $\mathcal{H}_l$ the universal family of basic representations of level $l$. As in the absolute case, the restriction of $\mathcal{L}$ to $\mathcal{M}^0_G$ defines a line bundle $\mathcal{L}$ on $\mathcal{M}^0_G$. By the above discussions, the global Verlinde’s isomorphism is the isomorphism

$$\kappa : PB^1 = \mathbb{P}(\mathcal{H}_l^*)^{L_XG} \cong \mathbb{P}_p \mathcal{L}^I$$

which is explicitly described by formula (5.4). 

Computation of the connections

We chose a positive integer $l$. We denote by $M$ the regularly stable locus of $M_G(X)$ and by $\Theta$ the line bundle $\mathcal{L}^I$ on $M$ (5.3). As explained above, the line bundle $\Theta$ exists over $\mathcal{M}_{G,\pi}$. 

6. Deformations of global sections and connections. — Let $U_i$, $i \in I$ be an affine open cover of any smooth variety $V$. Let $s$ be a global section of the line bundle $L$ on $V$. For the convenience of the reader, let me first recall some deformation theory of the triple $(V, I, s)$ (see [W]). We denote by $(V_\epsilon, I_\epsilon, s_\epsilon)$ a deformation of $(V, I, s)$ over $D_\epsilon = \text{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)$. 

6.1. The restriction $U_\epsilon$ of $V_\epsilon$ to $U_i$ is trivial (because $U_i$ is smooth and affine). Let us chose an isomorphism

$$\iota_i : \mathcal{O}_{U_i}[\epsilon] = \mathcal{O}_{U_i} \otimes \mathbb{C}[\epsilon] \cong \mathcal{O}_{U_i,\epsilon}$$

which restricts to $\text{Id}$ when $\epsilon = 0$. The matrix of $\iota_i^{-1} \circ \iota_i$ is of the form

$$
\begin{pmatrix}
\text{Id} & 0 \\
\xi_{i,j} & \text{Id}
\end{pmatrix}
$$

where $\xi_{i,j}$ is a derivation of $\mathcal{O}_{U_i,\epsilon}$. The image of the cocycle $(\xi_{i,j})$ in $\mathbf{H}^{1}(V, T_V)$ is the Kodaira-Spencer class of the deformation $V_\epsilon$. One checks that this procedure identifies isomorphism classes of infinitesimal deformations of $V$ and $\mathbf{H}^{1}(V, T_V)$.

6.2. As above, the restriction $L_{U_i,\epsilon}$ of $L_{\epsilon}$ to $U_i$ is trivial. Let us therefore chose a morphism

$$\phi_i : L_{U_i}[\epsilon] = L_{U_i} \otimes \mathbb{C}[\epsilon] \to L_{U_i,\epsilon}$$

which restricts to $\text{Id}$ when $\epsilon = 0$. The morphism $\phi_i$ is an isomorphism and the matrix of $\phi_i^{-1} \circ \phi_i$ is of the form

$$
\begin{pmatrix}
\text{Id} & 0 \\
\eta_{i,j} & \text{Id}
\end{pmatrix}
$$
where $\eta_{i,j}$ is a first order differential operator of symbol $\eta_{i,j}$ of $L_{U_i} \otimes U_j$. Let $\mathcal{D}^i(L), i \in \mathbb{N}$ be the sheaf of differential operators of order $\leq i$ on $L$. The image of the cocycle $(\eta_{i,j})$ in $H^1(V, \mathcal{D}^i(L))$ is the Kodaira-Spencer class of the deformation $(V, L)$. One checks that this procedure identifies isomorphism classes of infinitesimal deformations of $(V, L)$ and $H^1(V, \mathcal{D}^1(L))$.

6.3. There exists a (uniquely defined) section $\sigma_i$ of $L_{U_i}$ such that the restriction $s_{U_i \epsilon}$ of $s_\epsilon$ to $U_i$ can be written

$$s_{U_i \epsilon} = \phi_i(s_{U_i} + \epsilon \sigma_i).$$

One has the tautological relation $s_{U_i} = s_{U_j}$ on $U_i \cap U_j$ and, by definition of $\eta$, one has the equality

$$(*) \quad \sigma_j - \sigma_i = \eta_{i,j}(s)$$

Let $d, s, i \in \mathbb{N}$ be the complex

$$d_i s = \left\{ \begin{array}{ll} \mathcal{D}^i(L) & \epsilon s_i \\ \deg(0) & \deg(1) \end{array} \right.$$ 

The equality $(*)$ means that

$$(\eta_{i,j}, \sigma_i) \in C^1(\{U_i\}, d_1 s) = C^1(\{U_i\}, \mathcal{D}^1(L)) \oplus C^0(\{U_i\}, L)$$

is a cocycle and therefore defines a class in $H^1(d_1 s)$. One checks that this procedure identifies isomorphism classes of infinitesimal deformations of $(V, L, s)$ and $H^1(d_1 s)$.

7. **How to compute Hitchin’s connection.**— Let us first explain why it is enough to compute the covariant derivative.

7.1. Let $E$ be a vector bundle on a (smooth) variety $V$ and $\nabla$ be a connection on the projective bundle $PE$ of lines of $E$. Let $\tau$ be a vector field defined on some open subset $U$ of $V$ and let $s$ be a section of $E$ on $U$. Let $u$ be a point of $U(C)$ and $v$ be the tangent vector $\tau(u)$. Let us denote by $(u, \tau(u))^\nabla$ the tangent vector of $PE$ at $s(u)$ which is the horizontal lifting of $v$. Then, the difference

$$(\text{7.1}) \quad \nabla_\tau(s)[u] = ds(v) - (u, \tau(u))^\nabla \in T_{s(u)}PE$$

is tangent to the fiber $PE_u$ and therefore lives in $T_{s(u)}PE_u = E \otimes C(u)/C.s(u)$. Because the space of connection is an affine space under $H^0(V, \Omega_V \otimes \text{End}(E)/\Omega_V \cdot 1)$ and because $V$ is reduced, the collection $\nabla_\tau(s)[u], u \in U(C)$ determine the connection $\nabla$.

7.2. Let $\tilde{\iota} \in H^1(X, T_X)$ and $\tilde{\iota}_t : D_t \to \mathcal{M}_g$ the corresponding morphism. Let us denote the pull-back $\tilde{\iota}_*(\mathcal{M}_g^0, X, \Theta)$ of the universal data simply by $(X_t, M_t, \Theta_t)$ and by $(X, M, \Theta)$ its restriction to $(\epsilon = 0)$.

**Remark 7.3.**— Recall that for any vector bundle $F$ on $X$, the Čech complex $\mathcal{C}_F$

$$H^0(D, F) \oplus H^0(X^*, F) \xrightarrow{\delta_F} H^0(X^*, F)$$
associated to the Ω at cover \( D \sqcup X^* \rightarrow X \) of \( X \) calculates the cohomology \( H^*(X, F) \). In particular, the complex \( C_{TX} \) defines a projection from the vector space of meromorphic vector fields \( T_D \) on \( D^* \) onto \( H^1(X, T_X) \). If \( t \) is a meromorphic vector field on \( D \) which projects to \( \mathfrak{t} \), then the infinitesimal deformation \( X_\varepsilon \) of \( X \) over \( D \) can be described in the following manner: one glue the 2 trivial deformations \( X[\varepsilon] \) and \( D[\varepsilon] \) of \( X^* \) and \( D \) respectively along \( D[1] \) thanks to the automorphism of \( D^*[\varepsilon] \) defined by

\[
\begin{cases}
\mathcal{O}_D[\varepsilon] & \rightarrow \mathcal{O}_D^*[\varepsilon] \\
f & \mapsto f + \varepsilon < t, df >
\end{cases}
\]

In particular, a formal coordinate \( z \) on \( X \) lifts canonically to a formal coordinate on \( X_\varepsilon \).

7.4. Let \( s \) be a global section of \( \Theta \). To construct Hitchin’s connection, one has to lift \( s \) to a global section \( s^\nabla \) of \( \Theta_\varepsilon \). The basic observation of Hitchin’s construction is that the cup-product pairing

\[
H^1(X, T_X) \otimes H^0(X, \text{Ad}(E) \otimes \omega_X) \rightarrow H^1(X, \text{Ad}(E))
\]

where \( E \) is a regularly stable bundle on \( X \) induces by Serre duality a morphism

\[
\tau : H^1(X, T_X) \rightarrow S^2 H^1(\text{ad}(E)) = (S^2 T_M)[E]
\]

which globalizes when \( [E] \) runs over \( M(\mathbb{C}) \) to give the quadratic differential

\[
(7.3) \\
\tau : H^1(X, T_X) \rightarrow H^0(M, S^2 T_M).
\]

The short exact sequence of complexes

\[
(7.4) \\
0 \rightarrow d_1 s \rightarrow d_2 s \rightarrow S^2 T_M[0] \rightarrow 0
\]

gives a morphism

\[
\delta : H^0(M, S^2 T_M) \rightarrow H^1(d_1 s).
\]

Let

\[
w_\varepsilon(\mathfrak{t}) : H^1(X, T_X) \rightarrow H^1(d_1 s)
\]

be the composition \( \delta \circ \tau \).

Lemma 7.5. [H] The deformation of \( (M, \Theta) \) defined by the projection of \(-w_\varepsilon(\mathfrak{t})/(2l + 2h^\vee)\) in \( H^4(D^4(\Theta)) \) is isomorphic to \( (M_\varepsilon, \Theta_\varepsilon) \).

Proof: let \( \Lambda \) be the integer defined by \( \omega_{M_0} = \mathcal{O}(\Lambda) \) where \( \mathcal{O}(1) \) is the determinant bundle. One has the equality \( \Lambda = 2h^\vee \) (see [KNR] for instance). By theorem 3.6 of [H], the projection \(-w_\varepsilon(\mathfrak{t})/(2l + \Lambda)\) in \( H^4(M, T_M) \) is the Kodaira-Spencer class of \( M_\varepsilon \). Because the codimension the non regularly stable locus is at least 2 (see the appendix), \( H^1(M, \mathcal{O}_M) \) is zero and the symbol map \( H^1(M, D^4(\Theta)) \rightarrow H^1(M, T_M) \) is injective. Because the image of \( (M_\varepsilon, \Theta_\varepsilon) \) in \( H^1(M, T_M) \) is (tautologically) the Kodaira-Spencer class of \( M_\varepsilon \), the lemma follows. ■
Remark 7.6. — Strictly speaking, only the case where $G = \text{SL}_r$ is treated in [H]. But the proof in [H] can be straightforward adapted to the general case if $\Lambda$ is defined by the equality $\omega_{\text{M}_0} = \mathcal{O}(-\Lambda)$ as above.

7.7. By the lemma, $-w_\epsilon(\tilde{t})/(2l+2h^\vee)$ defines a section over $\tilde{t}$ of $\Theta_\epsilon$, denoted by $(s, \tilde{t})^\vee$ well-defined up to $\text{Ker}(\text{Aut}(\Theta_\epsilon) \longrightarrow \text{Aut}(\Theta)) = 1 + \epsilon C$ which is the horizontal lifting (for Hitchin’s connection) of $\tilde{t}$ through $s$. If $s_\epsilon$ is a section of $\Theta_\epsilon$ restricting on $s$ when $\epsilon = 0$, the difference $s_\epsilon - (s, \tilde{t})^\vee$ lives in $\epsilon \mathcal{H}^0(M, \Theta)/\mathcal{C}$.s and one has the equality (cf. (7.4))

\begin{equation}
\epsilon(\nabla s_\epsilon)(0) = s_\epsilon - (s, \tilde{t})^\vee.
\end{equation}

7.8. The explicit calculation of $w_\epsilon(\tilde{t})$ goes as follows. Choose second order differential operators $D_i$ on $\Theta_{U_i}$ whose symbols are $\tau(\tilde{t})/(2l+2h^\vee)$ on $U_i$. Then one has the equality in $\mathcal{H}^1(d_1s)$

\begin{equation}
w_\epsilon(\tilde{t}) = [D_i - D_j, -D_1s]
\end{equation}

(compare with (3.17) of [H]). With the notations above, one has

\[\eta_{i,j} = \text{symbol}(D_i - D_j)\text{ and } \sigma_i = -D_1s.\]

7.9. Suppose that the diagram

\[N \times_M N = \sqcup_{i,j} U_i \cap U_j \Longrightarrow N = \sqcup U_i \longrightarrow M\]

is replaced by

\[N_1 = N \times_M N \xrightarrow{\delta} N \longrightarrow M\]

where $N \longrightarrow M$ is any étale epimorphism such that $r^*(M, \Theta_\epsilon)$ is trivial. We suppose also that the pull-back of the quadratic differential $\tau(\tilde{t})$ is the image of a second order differential operator $\theta(t) \in \mathcal{H}^0(N, \mathcal{D}^2(r^*\Theta))$ by the composite

\[\mathcal{H}^0(N, \mathcal{D}^2(r^*\Theta)) \longrightarrow \mathcal{H}^0(N, \mathcal{S}^2\mathcal{T}_N) \xrightarrow{r^*} \mathcal{H}^0(N, r^*\mathcal{S}^2\mathcal{T}_M).\]

The degree one piece $\mathcal{C}^1(r, d_1s)$ of the Čech complex of $r$ is

\[\mathcal{C}^1(r, d_1s) = \mathcal{H}^0(N_1, \rho^*\mathcal{D}^1(\Theta)) \oplus \mathcal{H}^0(N, r^*\Theta)\]

where $\rho = r \circ p = r \circ q$. Because coherent cohomology can be calculated using the étale topology, one has a canonical morphism

\[\mathcal{C}^1(r, d_1s) \rightarrow \mathcal{H}^1(d_1s).\]

Then, as in (7.6), one has the equality in $\mathcal{H}^1(d_1s)$

\begin{equation}
w_\epsilon(t) = [q^*r_\epsilon \theta(t) - p^*r_\epsilon \theta(t), -\theta(t).r^*s]
\end{equation}
and the infinitesimal lifting \((s, \tilde{t})^\nabla\) defined by the class \(-w_s(\tilde{t})/(2l + 2h^\vee)\) is given on \(N\) by
\[
(s, \tilde{t})^\nabla = r^*s + \frac{\epsilon}{(2l + 2h^\vee)} \theta(\tilde{t})r^*s.
\]

Suppose that the global section \(s_\epsilon\) of \(\Theta_\epsilon\) is given on \(N\) by
\[
s_\epsilon = u + \epsilon v, \ u, v \in H^0(N, r^*\Theta).
\]

Then, the formula (7.5) gives
\[
(7.8) \quad \nabla r s_\epsilon(0) = v - \frac{\theta(\tilde{t}).u}{2l + 2h^\vee} \in H^0(N, r^*\Theta)/C. u.
\]

**8. Sugawara tensors and differential operators.**— Recall that \(r^*\Theta\) is the homogeneous line bundle \(L_\lambda\) where \(\lambda\) is the character \(\chi^{-l}\) of \(\mathbb{L}^\vee\). If \(\mathbb{L}\) were finite dimensional, one would have a morphism
\[
U(\mathbb{L}_\Theta) \xrightarrow{\text{opp}} H^0(\mathcal{Q}, \mathcal{D}(L_\lambda))
\]
and the Sugawara tensor \(T(t)\) would define a second order differential operator on \(L_\lambda\), a natural candidate for \(\theta(\tilde{t})\) (see (7.9)). Let \(\mathcal{L}_\Theta^0\) (resp. \(\mathcal{Q}^0\)) be the regularly stable locus of \(\mathbb{L}_\Theta\) (resp. \(\mathbb{Q}\)). To avoid too much abstract nonsense about differential operators on ind-schemes, one use an étale quasi-section
\[
\begin{array}{ccc}
\mathcal{Q}^0 & \xrightarrow{\sigma} & M \\
\pi \downarrow & & \downarrow \pi \\
N & \xrightarrow{r} & M
\end{array}
\]
(cf. [DS]) of \(\pi : \mathcal{Q}^0 \to M\) to construct the differential operator \(\theta(t)\) using \(T(t)\) (formally, one just pull-back \(T(t)\) by \(\sigma\)). By convention, all cohomology groups of any coherent sheaf on \(N\) are endowed with the discrete topology.

**8.1.** Let us first define the ”differential”
\[
\sigma^*d\pi : \mathbb{L}_\Theta \to H^0(N, r^*TM) \xrightarrow{\sim} H^0(N, TN).
\]
Let \(n \in N(R)\) and \(x\) be an element of \(\mathbb{L}_\Theta\). The image of
\[
\exp(\epsilon x) \cdot \sigma(n(\epsilon)) \in \mathcal{Q}^0(R[\epsilon])
\]
by \(\pi\) is a point \(m(\epsilon)\) of \(M[\epsilon]\) which restricts to \(r(n)\) when \(\epsilon = 0\) (recall that \(\mathcal{Q}^0\) is open in \(\mathcal{Q}\)). Because \(r\) is étale, there exists a unique point \(\nu(\epsilon)\) of \(N[\epsilon]\) such that \(\nu(0) = n\) and \(r(\nu(\epsilon)) = m(\epsilon)\). If \(f\) is a regular function defined near \(n\), the expansion
\[
f(\nu(\epsilon)) = f(n) + \epsilon x \cdot f(n)
\]
defines a regular function near \(n\). The corresponding vector field is denoted by \(\sigma^*d\pi(x)\). One checks that
\[
\sigma^*d\pi : \mathbb{L}_\Theta^\text{opp} \to H^0(N, T_N)
\]
is a morphism of Lie algebras and therefore induces a morphism of filtered algebras

\[ U(\hat{\mathfrak{g}})^{\text{opp}} \to H^0(N, \mathcal{D}(\mathcal{O}_N)). \]

8.2. We want to extend (8.1) to a completion of \( \hat{\mathcal{U}}(\hat{\mathfrak{g}}) \) in which leaves the Sugawara tensors. Let \( U \) be the enveloping algebra of \( \mathfrak{g} \otimes \hat{K} \). For \( n \geq 0 \), let \( U^n \) be the subspace of \( u \in U \) which is of order \( \leq n \). We define a filtration \( F^iU^n, i > 0 \) by

\[ F^iU^n = U_{\mathfrak{g}_i} \cap U^n \]

where \( \mathfrak{g}_i \) is the kernel of the projection \( \mathfrak{g} \otimes \mathcal{O} \to \mathfrak{g} \otimes \mathcal{O}_{ix} \). The family \( F^iU_n, i > 0 \) defines a topology on \( U_n \); let \( \hat{U}_n \) be the corresponding completion and \( \hat{U} = \cup_{n \in \mathbb{N}} \hat{U}_n \) be our completion of \( U \). It is a complete associative algebra which is by definition filtered and which acts on every integrable representation. Let us chose a formal coordinate \( z \) at \( x \). For \( x \in \mathfrak{g} \), and \( i \in \mathbb{Z} \), let me denote by \( x(i) \) the vector \( X \otimes z^i \).

**Lemma 8.3.** There exists an integer \( i \) such that

\[ \sigma^*d\pi(x(j)) = 0 \]

for all \( x \in \mathfrak{g} \) and \( j \geq i \).

**Proof:** because \( N \) is of finite type, there exists \( i \) such that

\[ \text{Ad}(\Omega) \cdot \exp(cx(j)) \in L^*G(R[t]/(t^2)) \]

for all \( j \geq i \) and \( \Omega \in \sigma(N(R)) \).

The lemma follows because \( \pi \) is right \( L^*G \)-invariant. \( \blacksquare \)

In particular, we get continuous morphisms (see (8.2) for the definition of the completion \( \hat{U}_n, \text{opp}(\hat{\mathfrak{g}}) \))

\[ \hat{U}_n, \text{opp}(\hat{\mathfrak{g}}) \to H^0(N, \mathcal{D}^n(\mathcal{O}_N)). \]

8.4. Let \( n \) be a point of \( N \). Let us consider \( \sigma(n) \) as a pair \((E, \rho)\) where \( \rho \) is a trivialization of \( E|_{X^\times} \). The geometric interpretation of

\[ \sigma^*d\pi_n : \hat{\mathfrak{g}} \to T_n N = H^1(X, \text{Ad}(E)) \]

goes as follows. Let \( x \in \hat{\mathfrak{g}} \) and let \( E_x \) be the underlying \( G \)-bundle on \( X[t] \) of \( \exp(c)\sigma(n) \).

The family \( E_v \) defines a Kodaira-Spencer map

\[ T_v \mathcal{D} \to H^1(X, \text{Ad}(E)). \]

Then, the image of \( d/c \in T_v \mathcal{D} \) by the Kodaira-Spencer map is \( \sigma^*d\pi_n(x) \).

8.5. One can of course explicitly calculate this map. The trivialization \( \rho \) defines an isomorphisms between \( \mathfrak{g}_{\text{Ad}(E)} \) (cf. (7.3)) and

\[ H^0(\mathcal{D}, \text{Ad}(E)) \oplus \mathfrak{g} \otimes \Lambda X \to \mathfrak{g} \otimes K. \]
The corresponding surjection

\[ \tilde{L}_\varphi \longrightarrow \mathfrak{g} \otimes K \longrightarrow H^1(X, \text{Ad}(E)) \]

is the differential \( \sigma^* d\pi_n \).

8.6. Let \( t \in T_{D^*} \) which projects to \( \tilde{t} \in H^1(X, T_X) \) (7.3) and \( \tau(\tilde{t}) \in H^0(M, S^2 T_M) \) the corresponding quadratic tensor (7.3). One can compute the value

\[ r^* \tau(\tilde{t})|_n \in S^2 T_n N = S^2 H^1(X, \text{Ad}(E)) \]

of \( r^* \tau(\tilde{t}) \in H^0(S^2 T N) \) at \( n \) as follows. The Killing form of \( \mathfrak{g} \) defines an isomorphism between \( \text{Ad}(E) \) and its dual. The residue theorem says that the residue \( \text{res} : \Omega_{D^*} \to \mathbb{C} \) factors through

\[ \Omega_{D^*}/(\Omega_X \oplus \Omega_D) \simeq H^1(X, \omega_X) \]

to give the canonical isomorphism \( H^1(X, \omega_X) \simeq \mathbb{C} \) defined by the meromorphic form \( dt/t \).

By Serre duality, \( r^* \tau(\tilde{t})|_n \) is therefore a quadratic form on \( H^0(X, \text{Ad}(E) \otimes \omega_X) \). By (7.2), \( r^* \tau(\tilde{t})|_n \) is induced by the cup-product

\[ H^1(X, T_X) \otimes H^0(X, \text{ad}(E) \otimes \omega_X) \longrightarrow H^1(X, \text{Ad}(E)). \]

The trivialization \( \rho \) defines an injection

\[ H^0(X, \text{Ad}(E) \otimes \omega_X) \hookrightarrow \mathfrak{g} \otimes \Omega_{X^*}. \]

The Killing form defines a pairing

\[ \text{tr} : [\mathfrak{g} \otimes \Omega_{X^*}] \otimes [\mathfrak{g} \otimes K] \longrightarrow K \otimes \Omega_{X^*} \simeq \Omega_{D^*}. \]

The tensor \( \tau(\tilde{t})|_n \in S^2 H^1(X, \text{Ad}(E)) \) of (7.2) is characterized by the formula

\[ \tau(\tilde{t})(\phi \otimes \phi) = \text{res} \, \text{tr}(\tilde{\phi} \otimes t \tilde{\phi}) \]

for every \( \phi \in H^0(X, \text{Ad}(E) \otimes \omega_X) \) mapping to \( \tilde{\phi} \in \mathfrak{g} \otimes \Omega_{X^*} \) and \( t \in T_{D^*} \) (the contraction \( t \tilde{\phi} \) is thought as an element of \( \mathfrak{g} \otimes K \)).

8.7. The twisted version is analogous. Consider the commutative diagram with cartesian square

\[
\begin{array}{ccc}
\hat{N} & \longrightarrow & \hat{L}_\mathfrak{g}^0 \\
\downarrow & & \downarrow \\
N & \longrightarrow & \hat{\Omega}^0 \\
\downarrow \sigma & & \downarrow \pi \\
\hat{M} & \longrightarrow & \hat{M}
\end{array}
\]
The morphism of $C$-space $N \to N$ is a $L^+G$-torsor and sections of $r^*\Theta = \sigma^*L_\lambda$ are functions on $N$ which are $\lambda$-equivariant. Let $f$ be such a function and let $\hat{n} = (n, \Omega)$ be a point of $\hat{N}(R)$. With the notation above,

$$exp(\epsilon)n := (\nu(\epsilon), exp(\epsilon)\Omega)$$

is a point of $\hat{N}(R[\epsilon])$ restricting to $\hat{n}$ when $\epsilon = 0$. The expansion

$$f(exp(\epsilon)\hat{n}) = f(\hat{n}) + \epsilon f(n)$$

defines a morphism of Lie algebras

$$\left\{ \begin{array}{c}
L^{\text{opp}} \to H^0(N, r^*\Theta) \\
x \mapsto (f \mapsto x.f)
\end{array} \right.$$ 

As above, the lemma (8.3) allows to define continuous morphisms

$$\tilde{U}^{n, \text{opp}}(\hat{L}_G) \to H^0(N, D^n(r^*\Theta)).$$

The arrows (8.6) and (8.2) are compatible, meaning that the symbol diagram

(8.7)

$$\tilde{U}^{n, \text{opp}}(\hat{L}_G) \xrightarrow{\text{symbol}} H^0(N, D^n(r^*\Theta)) \quad \xleftarrow{\text{symbol}} H^0(N, S^n T_N)$$

is commutative.

8.8.

Let me recall the definition of $T_n \in \tilde{U}$ (see [Ka], (12.8.4)). Let $x_i$ be an orthonormal basis of $\mathfrak{g}$ (for the Killing form). The sequence of operators

$$T_n = \sum_{m \in \mathbb{Z}} x_i(-m)x_i(m + n)$$

is well defined and does not depend on the choice of the $x_i$'s.

Remark 8.9.--- The notation is not standard. Usually, $(1/2l + 2\lambda^+)T_n$ is denoted by $L_n$ and the formal power serie $-\sum L_n u^{-n-2}$ is denoted by $T(u)$.

8.10. Suppose that $n$ is positive. Then, because $x_i(-m)$ and $x_i(n + m)$ commute in $U(L_G)$, one has $x_i(-n)x_i(n + m) \in F[n/2]U^2(\hat{L}_G)$ for every integer $m$. Therefore,

$$T_n \in F[n/2]U^2(\hat{L}_G) \text{ and } \lim_{n \in \mathbb{N}} T_n = 0.$$ 

Let $d_z$ be the meromorphic tangent vector $\frac{dz}{dz}$ (and not $-\frac{dz}{dz}$ as usual).
Definition 8.11.— Let $t = \sum_{n \geq -N} t_n d_n$ be a meromorphic vector field on $D^*$. The Sugawara tensor $T(t) \in \hat{U}^2(\hat{\mathfrak{g}})$ is defined by the equality

$$T(t) = \sum_{n \geq -N} t_n T_n.$$  

The second order differential operator $\theta(t) \in H^0(N, D^2(r^*\Theta))$ is the image of $T(t)$ by the morphism

$$\hat{U}^2(\hat{\mathfrak{g}}) \to H^0(N, D^2(r^*\Theta))$$

of (8.6).

8.12. Let $\phi \in H^0(X, \text{Ad}(E) \otimes \omega_X)$ mapping to $\tilde{\phi} \in \mathfrak{g} \otimes \Omega_{X^*}$ and $t \in T_{D^*}$. The series

$$\sum_{m \in \mathbb{Z}} \sum_i < \tilde{\phi}, x_i(-m) > < \tilde{\phi}, x_i(m+n) >$$

has finite support which allows to define

$$(8.9) \quad < \tilde{\phi} \otimes \tilde{\phi}, T_n > = \sum_{m \in \mathbb{Z}} \sum_i < \tilde{\phi}, x_i(-m) > < \tilde{\phi}, x_i(m+n) >$$

One defines $< \tilde{\phi} \otimes \tilde{\phi}, T_0 >$ by the analogous formula. By (8.3) and (8.7), the symbol of $\theta(t)$ evaluated at

$$\phi \otimes \phi \in S^2 T_{n^*} N = S^2 H^0(X, \text{Ad}(E) \otimes \omega_X)$$

is equal to the finite sum

$$\sum_{n \in \mathbb{Z}} t_n < \tilde{\phi} \otimes \tilde{\phi}, T_n > = \sum_{n \leq \text{val} \tilde{\phi}} t_n < \tilde{\phi} \otimes \tilde{\phi}, T_n >.$$  

Proposition 8.13.— The symbol of $\theta(t)$ is the quadratic differential $\tau(t)$ of $\tilde{\phi}$.

Proof: by (8.5) (keeping the notations above), one has to prove the equality

$$\text{res } \text{tr} (\tilde{\phi} \otimes t \tilde{\phi}) = < \tilde{\phi} \otimes \tilde{\phi}, T_{\text{symb}}(t) > .$$

Observe the preceding expression still makes sense if $\tilde{\phi}$ lives in $\mathfrak{g} \otimes \Omega_{D^*}$. Now, if the valuation $\text{val}(\tilde{\phi})$ is big enough, both the scalars $< \tilde{\phi} \otimes \tilde{\phi}, T_{\text{symb}}(t) >$ and $\text{res } \text{tr}(\tilde{\phi} \otimes t \tilde{\phi})$ are zero. One can therefore assume that $\tilde{\phi} = x_j(l)dz$ for some $l \in \mathbb{Z}$. One can also assume that $t = d_n, n \in \mathbb{Z}$. Now, one computes

$$< x_j(l)dz \otimes x_j(l)dz, T_n > = \delta_{n+2l, -2} = \text{res}(z^{n+1+2l}dz)$$

(even in the case where $n = 0$). One has

$$\text{res } \text{tr}(x_j(l)dz \circ d_n, x_j(l)dz) = \text{res } \text{tr}(x_j \otimes z^l dz \circ z^{n+1} d/dz, x_j \otimes z^l dz) = \text{res}(z^{n+1+2l}dz).$$
The computation of the Hitchin’s covariant derivative $\nabla_{\tau_s}(0), s \in H^0(D, \Xi)$ is now easy. Let us chose a local coordinate on $X$ which lifts to a local coordinate on $X_\tau$ along $x$ (see remark (7.3)), identifying the universal pair $(Q^0, \Theta)$ over $D$ to the trivial deformation $(Q^0[\epsilon], \Theta[\epsilon])$. We pick an étale quasi-section

$$\sigma$$

$$\begin{array}{c}
\sigma \\
\downarrow \\
N \\
\rightarrow \\
M
\end{array}$$

of $\tau : Q^0 \rightarrow M$. We define $\theta(t)$ as in (8.11). One is under the hypothesis of (7.9). By (5.7), there exists 2 linear forms $u, v$ on $H_1$ such that

$$\kappa(u + v) = s_\epsilon.$$

With the notation of (8.7), recall that $r^* \kappa(u)$ can be thought of as a $\lambda$-equivariant function on $N$ defined by (5.1)

$$\hat{n} = (n, \Omega) \mapsto r^* \kappa(u)(\hat{n}) = u(\Omega v_\epsilon)$$

where $v_\epsilon$ is the highest weight vector of $H_1$. The action (8.7) of $x \in \hat{L}_0^\infty$ on $r^* \kappa(u)$ is defined by the $\epsilon$-derivative of

$$r^* \kappa(u)(\exp(\epsilon x).\hat{n}) = u(\Omega v_\epsilon) - \epsilon x.u(\Omega v_\epsilon).$$

Therefore, one has the equality

$$x.r^* \kappa(u) = -\kappa(x.u).$$

The formula (7.9) becomes therefore

$$\nabla_{\tau_s}(u + v)(0) = \kappa(v - T(t)/(2l + 2h^\vee)).u \mod u. \quad (8.10)$$

9. WZW connection. Let me recall how the WZW connection on $V_t$ can be explicitly computed (see [S], spec. definition 2.7.4).

9.1. We start with a versal deformation $X \rightarrow S$ of the pointed curve $X_0$. Let $t$ be a meromorphic vector field on $\mathcal{D}$ which projects to the image by the Kodaira-Spencer map of some tangent vector $\tau \in T_0 S$. If $f$ is a function on $S$ and $u$ a linear form on $H_1$, the WZW-connection $\Delta$ on $V_t^*$ is defined by the formula

$$\Delta_x(u \otimes f) = u \otimes t.f - T(t)/(2l + 2h^\vee)u \otimes f \mod (u \otimes f) \quad (9.1)$$

(see [S], definition 2.7.4).

9.2. The tangent vector $\tau$ defines a morphism $D_t \rightarrow (S, 0)$ such that $\partial/\partial t$ maps to $\tau$. Let us pull-back the situation by this morphism. The first order expansion of (9.1) gives then

$$\Delta_{\partial/\partial t}(u + v) = v - T(t)/(2l + 2h^\vee).u \mod u \quad (9.2)$$

which is precisely $\nabla_{\partial/\partial t} \kappa(u + v)$ (see (8.10)). We endow $P(V_{\partial_\epsilon})^*$ with the WZW connection and $P_{\partial_\epsilon} \Theta_\epsilon$ with the Hitchin’s connection. Comparing (8.10) and (9.2), we have proved
Theorem 9.3. — With the notation of (5.7), the morphism $\kappa$

$$PB_l \sim p_s L^l$$

is a $\Omega_1$-isomorphism of $\Omega_1$ projective bundles over $\mathcal{M}_{g,1}$.

10. The Picard group of $\mathcal{Q}$ — We know that the Picard group of each fiber of $q^{-1}(s)$ ($s$ a complex point of $\mathcal{M}_{g,1}$) is $Z\mathcal{L}_s$ (see the appendix): this defines an integer $\deg(L)$ of every line bundle on $\mathcal{Q}$ which is the exponent $\epsilon$ such that $L_s = \mathcal{L}_s^{\otimes \epsilon}$ (recall that $\mathcal{M}_{g,1}$ is connected).

Proposition 10.1. — The sequence

$$0 \to \text{Pic}(\mathcal{M}_{g,1}) \xrightarrow{\kappa^*} \text{Pic}(\mathcal{Q}) \xrightarrow{\deg} Z \to 0$$

is exact and the morphism

$$\begin{cases}
Z & \to \text{Pic}(\mathcal{Q}) \\
\epsilon & \mapsto \mathcal{L}_s^{\otimes \epsilon}
\end{cases}$$

is a splitting.

Proof: the Grassmannian $\mathcal{Q}$ is the direct limit $\lim\mathcal{Q}_w$ where $w \in W_{\text{aff}}/W = \mathcal{Q}(R^\vee)$ and $\mathcal{Q}_w$ is the relative Schubert variety of index $w$ which can be geometrically described as follows. Let $L^G$ be the inverse image of $1$ by the evaluation $L^G \to \mathbb{T}$ and let $\mathcal{Q}_w$ be the direct image of the $G_m$-torsor $\mathcal{V}(\mathcal{O}(-x)) \setminus 0$ by $w: G_m \to G$: because $\mathcal{O}(-x)$ is canonically trivial on $X^*$, the $G$-bundle $\mathcal{Q}_w$ is trivialized on $X^*$ and defines therefore a point of $\mathcal{Q}$. The Schubert variety $\mathcal{Q}_w$ is as usual the union $\mathcal{Q}_w = \cup_{w' \leq w} L^{G_w}$. The Schubert variety $\mathcal{Q}_w$ of $\mathcal{Q} = \mathcal{Q}(s,w)$ is as usual the union $\mathcal{Q}_w = \cup_{w' \leq w} L^{G_w}$. By construction, $\mathcal{Q}_w = \mathcal{Q}(s,w)$ is projective and integral. Moreover, the natural morphism $\text{Pic}(\mathcal{Q}) \to \text{Pic}(\mathcal{Q}_w)$ is an isomorphism. By construction, the restriction of $\mathcal{Q} = \mathcal{Q}(s,w)$ to $\mathcal{Q}_w$ is trivial. Because $\mathcal{M}_{g,1}$ is reduced, the base change theorem implies that the direct image $q_w \mathcal{M}_w$ of $\mathcal{M}_w$ to $\mathcal{Q}_w$ is a line bundle $\overline{\mathcal{M}}_w$ on $\mathcal{M}_{g,1}$ and that the morphism $q_w^* q_w: \mathcal{M}_w \to \overline{\mathcal{M}}_w$ is surjective and therefore an isomorphism. The isomorphisms $(\mathcal{M}_w)_{q_w^*} \mathcal{M}_w$ for $w' \leq w$ induce the isomorphisms $\overline{\mathcal{M}}_{w'} \sim \overline{\mathcal{M}}_w$: let $\overline{\mathcal{M}}_w$ be the direct limit $\lim\overline{\mathcal{M}}_w$ (which is isomorphic to each of the $\overline{\mathcal{M}}_w$). By construction, $L = L^{G(w)} \otimes q^* \overline{\mathcal{M}}_w$.

Remark 10.2. — In particular, the Picard group of $\mathcal{Q}$ is $Z^3$.

Lemma 10.3. — Let $H$ be a $C$-group. Let $H_1, H_2$ be 2 $C$-subgroups of $H$ and $\psi_2: H_2 \to G_m$ a character defining a line bundle $\mathcal{L}_2$ on $H/H_2$. The pull-back $\mathcal{L}_{1,2}$ on $H_1/H_1,2$ (where $H_{1,2} = H_1 \cap H_2$) of $\mathcal{L}_2$ is the line bundle associated to the restriction $\psi_{1,2}$ of $\psi_2$ to $H_{1,2}$.

Proof: by definition, $\mathcal{L}_2$ is defined by the morphism $H/H_2 \to BH_2 \to BG_m$. 

$$H/H_2 \to BH_2 \to BG_m$$
where $H/H_2 \rightarrow BH$ is defined by the ($H_2$-equivariant) morphism $H \times H/H_2$ ($H$ being seen as an $H_2$-torsor over $H/H_2$ and $BH_2 \rightarrow BG_m$ being $B\psi_2$). The pull-back on $H_1/H_1,2$ is defined by the composite

$$H_1/H_1,2 \rightarrow H/H_2 \rightarrow BH_2 \rightarrow BG_m.$$ 

The diagram

$$\begin{array}{ccc}
H_1/H_1,2 & \rightarrow & H/H_2 \\
& \searrow & \nearrow \\
& & BH_1,2
\end{array}$$

is 2-commutative ($BH_1,2 \rightarrow BH_2$ being the natural morphism deduced from $H_1,2 \hookrightarrow H_2$).

The proposition follows because the composite $BH_1,2 \rightarrow BH_2 \rightarrow BG_m$ is $B\psi_{1,2}$. 

10.4. Let $\sigma$ the section of $Q$ defined by the trivial $G$-bundle (with its canonical trivialization on the punctured curve) over $X \times M_{g,1}$. It corresponds to the unit section of $LG \rightarrow M_{g,1}$. The above lemma proves that $\sigma^*L$ is trivial. We can therefore rewrite the proposition (10.1) in the following form: for every $L \in \text{Pic}(Q)$, one has the formula

$$(10.1) \quad L = L^{\deg(L)} \otimes q^*(\sigma^*L).$$

10.5. Let $\rho : G \rightarrow SL_N$ be a linear representation of $G$, which can be assumed to be non-trivial. Let $\mathcal{E}$ be the universal $G$-bundle on $Q \times M_{g,1}$, $X$ and $L_\rho$ the line bundle on $Q$

$$(10.2) \quad L_\rho = \text{det}(R\Gamma\mathcal{E}(C^N))^{-1}.$$ 

The degree $\deg(L_\rho)$ is the Dynkin index $d_\rho$ of the representation $\rho$ (see [LS]). The formula (10.1) gives therefore an isomorphism of $L_\rho G$-linearized bundles

$$(10.3) \quad L_\rho \otimes q^* \text{det} R\Gamma\mathcal{O}_X \cong L^{d_\rho},$$

well defined up to $H^0(M_{g,1},\mathcal{O}^*)$.

Remark 10.6. – Both sides of (10.3) descends to the universal moduli space. The corresponding projective bundles of global sections

$$\text{PR}L_\rho$$

and

$$\text{PR}L^{d_\rho}$$

have therefore a Hitchin’s connection and are isomorphic (as projective bundles). The construction of Hitchin’s connection is certainly functorial and the preceding isomorphism is $\Omega at$.

11. Appendix. – For completeness, let me prove a codimension estimate (see [F2], theorem II.6 for similar statements) which is certainly well known to the experts.

\[\text{One can show that this group is in fact } \mathbf{C}^*, \text{ proving that (10.3) is well-defined up to a non-zero scalar.}\]
Lemma 11.1.— Let $\pi : E \to S$ be a (right) $G$-bundle over a connected $C$-scheme $S$ with $G$ reductive. Assume that $E$ has a non central automorphism of finite order. Then $E$ has an $L$-structure $F$ where $L$ is a proper Levi subgroup of $G$.

Proof: let $e$ be a point of $E(C)$ and $g \in G(C)$ the (unique) point such that $\phi(e) = e.g$. Then, $g$ is of finite order as $\phi$. Let $L$ be the centralizer of $g$ : it is a proper Levi subgroup of $G$ (see [□]). Let $T \to S$ be an $S$-scheme and $F(T)$ be the set

$$F(T) = \{ \epsilon \in \text{Hom}_S(T, E) \text{ such that } \phi(\epsilon) = \epsilon \}.$$ 

The obvious functor

$$F : \left( \text{Schemes}^{\text{opp}} \to \text{Ens} \right)$$

is a formally principal homogeneous space under $L$. Let $T$ be a maximal torus of $G$ and $W = N(G, T)/T$ the Weyl group. Because $g$ is of finite order, $g$ is semi-simple and therefore belongs to a maximal torus $h^{-1}Th$ for some $h$ in $G(C)$ well defined up to $N(G, T)$. Then, $hgh^{-1}$ is a well defined element of $T/W$ which is of torsion. By construction, this class depend only on $s = \pi(\epsilon)$ and we have defined a map

$$S(C) \to [T/W]_{\text{tors}}$$

from $S(C)$ to the finite group $[T/W]_{\text{tors}}$ which is continuous (for the étale topology of $S$). By connectedness of $S$, this map is constant and all the $F(s)_{s \in S(C)}$ are nonempty and $F$ is therefore (representable by) an $F$-torsor which is an $L$-structure of $E$. ■

Remark 11.2.— Notice that $L$ is defined up to inner-automorphism and that the choice of $L$ determines $F$ up to isomorphism. One can therefore (slightly incorrectly) that $F$ is the $L$-structure of $E$ determined by $L$.

11.3. Let $X$ be a smooth complete and projective complex curve and $G$ a reductive algebraic group.

Definition 11.4.— A regularly stable $G$-bundle on $X$ is a stable bundle with $\text{Aut}(E) = Z(G)$.

The locus of regularly stable bundles in the moduli space of semi-stable $G$-bundles $M_G(X)$ on $X$ is certainly open.

Proposition 11.5.— The closed subset $B$ of $M_G(X)$ parameterizing semi-stable bundles $E$ which are not regularly stable is of codimension $\geq 3$ for $g \geq 3$.

Proof: one can assume that $G$ is semi-simple (divide by the neutral component of $Z(G)$). Let $E$ be a semi-stable bundle which is not regularly stable. If $E$ is not stable, there exists a unique standard parabolic subgroup $P$ and a $P$-structure $\tilde{F}$ of $F$ such that $F = \tilde{F}/\text{rad}_P P$ is stable (as $P/\text{rad}_P P$-bundle). If $L$ is a Levi subgroup of $P$, this shows that $[E] = [\tilde{F}(G)]$ is in the image of $M_L(X) \to M_G(X)$ in this case. If now $E$ is assumed stable with $\text{Aut}(E) \neq Z(G)$, let
us choose a non central automorphism of $\phi$. Let $F$ be the $L$-structure of $\text{gr}(E)$ determined by $\phi$ (see remark (11.2)). Then, $F$ is stable as $E$ and $E$ is in the image of the natural morphism $M_L(X) \to M_G(X)$. We have therefore to compare

$$\dim M_L(X) = (g - 1) \dim(L) + \dim(Z(L))$$

and

$$\dim M_G(X) = (g - 1) \dim(G) = \dim M_G(X).$$

Using that the center of a reductive group is the intersection of the kernel of the roots, one checks by induction that $\dim(G) - \dim(L) \geq 2 \dim(Z(L))$ and therefore

$$\dim M_G(X) - \dim M_L(X) \geq (2g - 3) \dim(Z(L)) \geq 3$$

because $g \geq 3$ and $\dim(Z(L)) > 0$. $lacksquare$

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