Fisher information estimates for Boltzmann’s collision operator

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FISHER INFORMATION ESTIMATES FOR BOLTZMANN’S COLLISION OPERATOR

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Abstract. We derive several estimates for Boltzmann’s collision operator in terms of Fisher’s information. In particular, we prove that Fisher’s information is decreasing along solutions of the Boltzmann equation with Maxwellian cross-section, in any dimension of velocity space, thus generalizing results by G. Toscani, E. Carlen and M. Carvalho.

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1. Introduction

Let $f$ be a probability density on $\mathbb{R}^N$, $N \geq 1$. Fisher’s quantity of information associated to $f$ is defined as the (possibly infinite) nonnegative number

$$I(f) = \int_{\mathbb{R}^N} \frac{|\nabla f|^2}{f} = 4 \int_{\mathbb{R}^N} \left| \nabla \sqrt{f} \right|^2. \quad (1)$$

This formula defines a convex, isotropic functional $I$, which was first used by Fisher [11] for statistical purposes, and plays a fundamental role in information theory.

In 1959, Linnik [12] used this functional (therefore also called Linnik’s functional) to give an information-theoretic proof of the central limit theorem (see [1, 10] for recent improvements of Linnik’s methods).

Some years later, McKean [14], drawing an analogy between the central limit theorem and the trend to equilibrium in kinetic theory, adapted the work of Linnik to the kinetic theory of gases. In this way he
obtained the first explicit bound from below for the speed of approach to equilibrium in Kac’s model, which is a one-dimensional caricature of the Boltzmann equation (we note that the optimal bound, conjectured by McKean, was recently derived by Carlen, Gabetta and Toscani [9], using a completely different technique).

The key observation by McKean was that \( I \), like the classical Boltzmann \( H \)-functional, is nonincreasing with time along solutions of Kac’s model. This monotonicity property was extended by Toscani [18] to the two-dimensional Boltzmann equation for Maxwellian molecules. It is our purpose here to generalize this result to higher dimensions of velocity space, and to give related estimates in a larger setting.

The fact that \( I \) is a Lyapunov functional for the Boltzmann equation with Maxwellian molecules has many applications. For instance, Toscani [19] used it to derive strengthened limit theorems. Moreover, it entails also a propagation of smoothness for the solution to the Boltzmann equation, some applications of which are given in [9].

Bobylev and Toscani on one hand, Carlen and Carvalho on the other, noticed that the decreasing property of \( I \) can be seen as a consequence of an inequality which is reminiscent of well-known inequalities in information theory. To understand this, let us go a little bit into the details of the Boltzmann equation.

In the (spatially homogeneous) Boltzmann equation, the unknown is a nonnegative integrable function \( f(t,v) \), standing for the probability distribution at time \( t \) of the velocity \( v \) of the molecules in a gas. The equation governing the evolution of \( f \) is

\[
\partial_t f = Q(f,f),
\]

where Boltzmann’s collision operator \( Q(f,f) \) is defined by

\[
Q(f,f) = \int dv_* d\sigma \ B(v - v_*,\sigma) \ (f' f'_* - f f_*) \equiv Q^+(f,f) - Q^-(f,f),
\]

with the usual conventions \( f_* = f(v_*) \), \( f' = f(v') \), \( f'_* = f(v'_*) \), and

\[
\begin{align*}
 v' &= \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma \\
 v'_* &= \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma.
\end{align*}
\]

The weight-function \( B : \mathbb{R}^N \times S^{N-1} \to \mathbb{R}^+ \) is the so-called “cross-section”, depending on the interaction between particles. On physical grounds it is always assumed that \( B(z,\sigma) \) depends only on \( |z| \) and
In the case of the so-called Maxwellian molecules, i.e., repelling each other with an inverse-power force law of exponent $2N - 1$, $B$ depends only upon $z/|z| \cdot \sigma$. More generally, we shall define a Maxwellian cross-section as a weight-function $B(z/|z| \cdot \sigma)$.

Under very little assumptions on the initial datum $f(t = 0) = f_0 \in L^1(\mathbb{R}^N)$ and on the cross-section $B$, one can show that the equation (2) admits a unique nonnegative solution, whose total mass is preserved with time. Therefore, we shall always assume that $f$ is a probability density.

Let $B$ be a Maxwellian cross-section. Then, by rotational invariance, $\int B(k \cdot \sigma) \, d\sigma$ is independent of the unit vector $k$. Assuming its value to be 1, $Q^-$ is simply

$$Q^-(f, f) = f \left( \int f \right) = f. \tag{5}$$

Therefore, due to the convexity of $I$, to prove that Fisher’s information is decreasing with time along solutions of the Boltzmann equation with Maxwellian cross-section, it is sufficient to prove that

$$I(Q^+(f, f)) \leq I(f). \tag{6}$$

Since $Q^+$ acts more or less like a (rescaled) convolution operator, this inequality is strongly reminiscent of the well-known Blachman-Stam inequality [2, 5, 16]. If $f$ is a probability density, let us define $f_\alpha(v) = \alpha^{-N/2} f(\alpha^{-1/2} v)$; then, if $f$ and $g$ are any two probability densities, the Blachman-Stam inequality reads

$$I(f_\alpha * g_{1-\alpha}) \leq \alpha I(f) + (1 - \alpha) I(g). \tag{7}$$

It was in fact proven by Bobylev and Toscani [4], using the Fourier-transform representation of Boltzmann’s equation, that if the inequality (7) holds for a arbitrary functional $I$, then the inequality (6) also holds in dimension 2 of velocity space (the general case is still open).

Carlen and Carvalho [7] also proved the inequality (6) for arbitrary dimension $N$, in the case when $B$ is constant. Here we shall prove this inequality in full generality, by a direct and entirely elementary computation, relying on apparently new representations for $\nabla Q^+(f, f)$, that have interest on their own. In fact, one can draw a slightly better parallel with the Blachman-Stam inequality: let $f$ and $g$ be two probability densities, and define

$$Q^+(f, g) = \frac{1}{4} \left( Q^+(f + g, f + g) - Q^+(f - g, f - g) \right).$$

$$= \frac{1}{2} \int_{\mathbb{R}^N \times S^{N-1}} dv_\sigma d\sigma B(v - v_\sigma, \sigma)(f' g' + g' f' - f g' - g f').$$
Then, we shall prove that
\begin{equation}
I(Q^+(f, g)) \leq \frac{1}{2} \left[ I(f) + I(g) \right].
\end{equation}

We also investigate briefly the case of arbitrary potentials, and show
precisely why the Maxwellian case seems to depart from the others.
Some of our methods are directly inspired from [10] and [7].
To conclude this introduction, we want to emphasize the fact that
even though the decreasing property of $I$ can apparently be shown only
for Maxwellian cross-sections, the consequences that can be drawn from
this property go far beyond this setting. This can be seen for instance
in the work by Carlen and Carvalho [7, 8]. In a forthcoming joint work
with G. Toscani, we shall show that the decreasing property of $I$ can
be related to the properties of the Ornstein-Uhlenbeck regularization
of the entropy dissipation functional in the Boltzmann equation, and
give new estimates for the speed of trend towards equilibrium in a very
general setting.

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2. Main results

Our first estimate concerns arbitrary cross-sections. It relies on the
following elementary identity.

**Proposition 1.** Let $f$ and $g$ be two smooth probability densities with
fast decay at infinity, and let $Q$ be a Boltzmann collision operator asso-
ciated with a cross-section $B \in L^1_{in}(\mathbb{R}^N \times S^{N-1})$ (at most polynomially
increasing at infinity). Then
\begin{equation}
\nabla Q^\pm(f, g) = Q^\pm(f, \nabla g) + Q^\pm(\nabla f, g),
\end{equation}
where, of course, $[Q^\pm(f, \nabla g)]_i = [Q^\pm(\nabla g, f)]_i = Q^\pm(f, \partial_i g)$.

**Remark.** Under suitable assumptions on $B$, formula (9) can be given
a distributional sense even if $f$ is not as smooth as required above. We
shall however not try to do so because this proposition suffices to our
purposes.

**Theorem 2.** Let $B(z, \sigma) \in L^\infty(\mathbb{R}^N; L^1(S^{N-1}))$ be an arbitrary cross-
section, and define
\begin{equation}
A(z) = \int_{S^{N-1}} d\sigma \ B(z, \sigma).
\end{equation}
Let \( a = \|A\|_{L^\infty(\mathbb{R}^N)} \). Then, for any two probability densities \( f \) and \( g \),
\[
I(Q^+(f, g)) \leq a[I(f) + I(g)].
\]

The noticeable point in this inequality is that it depends on \( A \) only through \( L^b \) bounds, and does not require any smoothness for the cross-section. As a corollary, we give a simple (certainly not optimal) theorem of (local in time) propagation of Fisher information bounds.

**Corollary.** Let \( B(z, \sigma) \) be an arbitrary cross-section, and let \( A \) be defined by the formula (10). Assume that \( a \equiv \|A\|_{L^\infty} < \infty \), and \( b \equiv \|\nabla A\|_{L^\infty} < \infty \). Let \( f_0 \) be a probability density with finite variance such that \( I(f_0) < \infty \), and let \( f(t, v) \) be the solution of the Boltzmann equation with cross-section \( B \) and initial datum \( f_0 \). Then, for all time \( t \), \( I(f(t)) < \infty \) and there exists a constant \( C < \infty \) depending only on \( a \) and \( b \) such that
\[
I(f(t)) \leq e^{4st} (2I(f_0) + C(1 + t^3)).
\]

For a Maxwellian cross-section, we can impose that \( A(z) = 1 \). Then Theorem 2 yields
\[
I(Q^+(f, g)) \leq I(f) + I(g).
\]

But, due to the particular structure of the Maxwellian case, this inequality can be improved by a factor 2. To this purpose, we establish the following representation.

**Proposition 3.** Let \( B(k \cdot \sigma) \) be a Maxwellian cross-section such that for any unit vector \( k \), \( \int d\sigma B(k \cdot \sigma) = 1 \), and let \( f \) be a smooth probability density with fast decay at infinity. Then
\[
\nabla Q^+(f, f) = \frac{1}{2} \int dv_\sigma d\sigma B(k \cdot \sigma) \left(f_\sigma'(I + P_{\sigma k})(\nabla f)' + f'(I - P_{\sigma k})(\nabla f)'\right)
\]
where \( k = (v - v_\sigma)/|v - v_\sigma| \), \( I : \mathbb{R}^N \rightarrow \mathbb{R}^N \) is the identity, and \( P_{\sigma k} : \mathbb{R}^N \rightarrow \mathbb{R}^N \) is the linear transformation defined by
\[
P_{\sigma k}(x) = (k \cdot \sigma)x + (\sigma \cdot x)k - (k \cdot x)\sigma.
\]

A formula similar to (12) is obtained for \( Q^+(f, g) \) by the usual doubling procedure.

**Theorem 4.** Let \( f \) and \( g \) be two probability densities, and let \( B(k \cdot \sigma) \) be a Maxwellian cross-section such that for any unit vector \( k \), \( \int d\sigma B(k \cdot \sigma) = 1 \). Then
\[
I(Q^+(f, g)) \leq \frac{1}{2}[I(f) + I(g)].
\]
From this theorem follows the decrease of Fisher’s information with time along solutions of the Boltzmann equation. Moreover, thanks to the uniqueness theorem given in [20] by G. Toscani and the author, we obtain the

**Corollary 4.1.** Let $B$ be a Maxwellian cross-section such that for any unit vector $k$,

$$
\int_{S^{N-1}} d\sigma \, B(k \cdot \sigma) (1 - k \cdot \sigma) < \infty.
$$

Let $f_0$ be a probability density with finite variance and such that $I(f_0) < \infty$, and let $t \mapsto f(t)$ be the unique solution to the Boltzmann equation with cross-section $B$ and initial datum $f_0$. Then $I(f(t))$ is nonincreasing with time.

The next two corollaries are conveniently proved by the use of “Bobylev’s lemma” [3] : if $f$ and $g$ are probability densities and $M$ is a Maxwellian distribution, then

$$Q^+(f \ast M, g \ast M) = Q^+(f, g) \ast M.$$

**Corollary 4.2.** Let $f$ be a smooth positive function, rapidly decreasing at infinity, and $B$ a Maxwellian cross-section satisfying the assumption (13). Then

$$-\frac{1}{8} I'(f) \cdot Q(f, f) \equiv \int_{\mathbb{R}^N} Q(f, f) \frac{\Delta \sqrt{f}}{\sqrt{f}} \geq 0.$$

Finally, let us recall the definition of Boltzmann’s $H$-functional, or entropy :

$$H(f) = \int f(v) \log f(v) dv.$$ 

By adapting the arguments given in [10], we shall obtain easily the

**Corollary 4.3.** Let $f$ and $g$ be two probability densities with the same mean and variance, and let $B(k \cdot \sigma)$ be a Maxwellian cross-section such that for any unit vector $k$, $\int d\sigma \, B(k \cdot \sigma) = 1$. Then

$$H(Q^+(f, g)) \leq \frac{1}{2} \left[ H(f) + H(g) \right].$$

This last inequality is clearly reminiscent of the well-known Shannon-Stam inequality [16, 10],

$$H(f_\alpha * g_{1-\alpha}) \leq \alpha H(f) + (1 - \alpha) H(g).$$
We give an application to numerical simulations. Consider the explicit Euler scheme for the Boltzmann equation with Maxwellian cross-section

\[
\begin{align*}
    f^{n+1} - f^n &= \varepsilon Q(f^n, f^n), \\
    f^0 &= f_0
\end{align*}
\]

Then

\[
H(f^{n+1}) = H((1 - \varepsilon)f^n + \varepsilon Q^+(f^n, f^n)) \leq (1 - \varepsilon)H(f^n) + \varepsilon H(Q^+(f^n, f^n)) \leq H(f^n).
\]

Thus, the entropy of the solution \( (f^n) \) to this scheme is nonincreasing with the time step. This phenomenon can be clearly seen in numerical simulations [15]. We note that the decrease of the entropy always holds for an implicit Euler scheme of the Boltzmann equation.

The plan of the paper is as follows. Arbitrary cross-sections are considered in section 3; then we turn to the Maxwellian case in section 4. The use of Bobylev’s regularization and its consequences are left to section 5.

3. ARBITRARY CROSS-SECTIONS

We first establish Proposition 1. Let \( f \) and \( g \) be smooth probability densities with fast decay at infinity, and let \( B \) be a smooth cross-section. Then, by standard theorems of differentiation of integrals depending on a parameter,

\[
\nabla Q^+(f, g) = \int dv_\ast d\sigma \nabla_v [B(v - v_\ast, \sigma)] \left( \frac{f'g'_* + g'f'_*}{2} \right) + \int dv_\ast d\sigma B(v - v_\ast, \sigma) \nabla_v \left[ \frac{f'g'_* + g'f'_*}{2} \right] = -\int dv_\ast d\sigma \nabla_v [B(v - v_\ast, \sigma)] \left( \frac{f'g'_* + g'f'_*}{2} \right) + \int dv_\ast d\sigma B(v - v_\ast, \sigma) \nabla_v \left[ \frac{f'g'_* + g'f'_*}{2} \right].
\]

Integrating the first term by parts, we find

\[
\nabla Q^+(f, g) = \int dv_\ast d\sigma B(v - v_\ast, \sigma)(\nabla + \nabla_\ast) \left[ \frac{f'g'_* + g'f'_*}{2} \right].
\]
Now, from the formulas (4), one easily deduces that if $g$ is an arbitrary $(L^1_{\text{loc}})$ function, then, in distributional sense (or in classical sense if $g$ is smooth),

\[
\begin{align*}
\nabla (g') &= \frac{1}{2} (\nabla g)' + \frac{1}{2} \frac{v - v_*}{|v - v_*|} \sigma \cdot (\nabla g)', \\
\nabla_*(g') &= \frac{1}{2} (\nabla g)' - \frac{1}{2} \frac{v - v_*}{|v - v_*|} \sigma \cdot (\nabla g)'
\end{align*}
\]

(18)

\[
\begin{align*}
\nabla (g'_*) &= \frac{1}{2} (\nabla g)' - \frac{1}{2} \frac{v - v_*}{|v - v_*|} \sigma \cdot (\nabla g)'_*, \\
\nabla_*(g'_*) &= \frac{1}{2} (\nabla g)' + \frac{1}{2} \frac{v - v_*}{|v - v_*|} \sigma \cdot (\nabla g)'_*. \\
\end{align*}
\]

In particular,

\[
\begin{align*}
(\nabla + \nabla_*)(g') &= (\nabla g)', \\
(\nabla + \nabla_*)(g'_*) &= (\nabla g)'_.
\end{align*}
\]

(19)

As a consequence,

\[
\nabla Q^+ (f, g) = \int dv_\sigma \, d\sigma \, B(v - v_*, \sigma) \left[ (\nabla f)' g'_* + f'(\nabla g)'_* + (\nabla g)' f'_* + g'(\nabla f)'_* \right] = \frac{1}{2} Q^+(f, \nabla g) + Q^+(g, \nabla f).
\]

The computation for $Q^-$ is exactly the same (in fact simpler). By density, Proposition 1 extends to nonsmooth cross-sections.

From this we derive Theorem 2. Let $f$ and $g$ be smooth functions with fast decay at infinity. Using the identity $(\nabla f)/(2\sqrt{f}) = \nabla \sqrt{f}$, we can write

\[
\begin{align*}
\nabla Q^+ (f, g) &= \int dv_\sigma \, d\sigma \, B(v - v_*, \sigma) \sqrt{f} g'_* \left[ (\nabla \sqrt{f})' \sqrt{g'_*} + (\nabla \sqrt{g})' \sqrt{f'_*} \right] \\
&\quad + \int dv_\sigma \, d\sigma \, B(v - v_*, \sigma) \sqrt{g f'_*} \left[ (\nabla \sqrt{g})' \sqrt{f'_*} + (\nabla \sqrt{f})' \sqrt{g'_*} \right].
\end{align*}
\]
From now on, we omit the argument of $B$ for simplicity. Using first the convexity of the square norm, then the Cauchy-Schwarz inequality,

$$|
abla Q^+(f, g)|^2 \leq 2 \left( \int dv_* d\sigma B \sqrt{f'g_*} \left[ \left( \nabla \sqrt{f} \right)' \sqrt{g_*} + \left( \nabla \sqrt{g} \right)' \sqrt{f'} \right]^2 \right)$$

$$+ 2 \left( \int dv_* d\sigma B \sqrt{g'f_*} \left[ \left( \nabla \sqrt{g} \right)' \sqrt{f_*} + \left( \nabla \sqrt{f} \right)' \sqrt{g'} \right]^2 \right)$$

$$\leq 2 \left( \int dv_* d\sigma B f'g_* \right) \left( \int dv_* d\sigma B \left( \nabla \sqrt{f} \right)' \sqrt{g_*} + \left( \nabla \sqrt{g} \right)' \sqrt{f'} \right)^2$$

$$+ 2 \left( \int dv_* d\sigma B g'f_* \right) \left( \int dv_* d\sigma B \left( \nabla \sqrt{g} \right)' \sqrt{f_*} + \left( \nabla \sqrt{f} \right)' \sqrt{g'} \right)^2.$$ 

Dividing by $Q^+(f, g)$ and integrating with respect to $v$, then using the inequality $\lambda x + (1 - \lambda)y \leq \max(x, y)$ if $0 \leq \lambda \leq 1$, we get

$$I(Q^+(f, g)) = \int dv \frac{|
abla Q^+(f, g)|^2}{Q^+(f, g)}$$

$$\leq 4 \left( \int dv_* d\sigma B f'g_* \right) \left( \int dv_* d\sigma B \left( \nabla \sqrt{f} \right)' \sqrt{g_*} + \left( \nabla \sqrt{g} \right)' \sqrt{f'} \right)^2$$

$$+ 4 \left( \int dv_* d\sigma B g'f_* \right) \left( \int dv_* d\sigma B \left( \nabla \sqrt{g} \right)' \sqrt{f_*} + \left( \nabla \sqrt{f} \right)' \sqrt{g'} \right)^2$$

$$\leq 4 \max \left( \int dv_* d\sigma B \left( \nabla \sqrt{f} \right)' \sqrt{g_*} + \left( \nabla \sqrt{g} \right)' \sqrt{f'} \right)^2,$$

$$\int dv_* d\sigma B \left( \nabla \sqrt{g} \right)' \sqrt{f_*} + \left( \nabla \sqrt{f} \right)' \sqrt{g'} \right)^2 \right).$$

By the involutive change of variables with unit Jacobian $(v, v_*) \leftrightarrow (v', v_*')$, the right-hand side of this last expression is

$$4 \max \left( \int dv_* d\sigma B \left( \nabla \sqrt{f} \right)' \sqrt{g_*} + \left( \nabla \sqrt{g} \right)' \sqrt{f'} \right)^2,$$

$$\int dv_* d\sigma B \left( \nabla \sqrt{g} \right)' \sqrt{f_*} + \left( \nabla \sqrt{f} \right)' \sqrt{g'} \right)^2 \right).$$

By the exchange of variables $v$ and $v_*$, both integrals above are equal. Since by assumption, $\int d\sigma B(v - v_*, \sigma) \leq a$, we obtain, expanding the
square norms,
\[
I(Q^+(f,g)) \leq 4a \int dv \, dv_\ast \left( |\nabla \sqrt{f}|^2 g_\ast + (|\nabla \sqrt{g}|^2)_\ast f \right) \\
+ a \int dv \, dv_\ast \left( \nabla f \cdot (\nabla g)_\ast + \nabla g \cdot (\nabla f)_\ast \right).
\]

Since
\[
\int dv \, \nabla f = \int dv \, \nabla g = 0,
\]
the last integral in the previous expression vanishes. Moreover, since
\[
\int dv \, f = \int dv \, g = 1,
\]
we obtain by Fubini’s theorem
\begin{equation}
I(Q^+(f,g)) \leq a \left( I(f) + I(g) \right),
\end{equation}
and this is the desired result in the case when \( f \) and \( g \) are smooth and rapidly decreasing.

In the general case, one can find sequences of smooth, rapidly decreasing densities \((f^n)\) and \((g^n)\) such that
\[
f^n \overset{L^1}{\to} f, \quad g^n \overset{L^1}{\to} g, \quad I(f^n) \to I(f), \quad I(g^n) \to I(g).
\]
Then, it is easy to check that \(Q^+(f^n, g^n)\) converges towards \(Q^+(f, g)\), at least weakly in \(L^1\), and by the weak lower semicontinuity of \(I\) the conclusion follows.

Remarks.

1. The use of the Cauchy-Schwarz inequality above can be seen as a variant of the proof given by Carlen and Carvalho in the case of a constant kernel, using another representation for \(v^c\) and \(v^s\) (Cf. [7], lemma 3.3).

2. In establishing Proposition 1, we have used the structure of \(Q^+\) to report the derivatives of \(B\) onto \(f\) and \(g\). One could wonder whether the inverse manipulation is possible, thus obtaining an estimate of \(I(Q^+(f,g))\) depending only on the smoothness of \(B\), like \(I(Q^+(f,g)) \leq C\|f\|^{2}_{L^1}\). But such a result is clearly false: if \(\delta\) denotes any Dirac measure, then for a Maxwellian cross-section, \(Q^+(\delta, \delta) = \delta\).

3. We recall that the main result in this respect is the estimate that was first obtained by Lions [13]:
\[
\|Q^+(f,f)\|_{\mathcal{H}^{\frac{1}{2}}_{\frac{3}{2}}} \leq C\|f\|_{L^2} \|f\|_{L^1}.
\]
if $B(z, \sigma)$ is smooth, compactly supported and subject to certain additional technical assumptions.

Now, we deduce from Theorem 2 a result of propagation of Fisher information bounds. Let $B$ be a cross-section such that $A$, defined by (10), belongs to $W^{1, \infty}(\mathbb{R}^N)$. If we set

$$Lf = A \ast f,$$

the solution of the Boltzmann equation with initial datum $f_0$ can be written as

\begin{equation}
(21) \quad f(t, v) = \int_0^t e^{-\int_s^t Lf(\tau, v) d\tau} Q^+(f(s, v)) \, ds + f_0(v) e^{-\int_0^t Lf(\tau, v) d\tau}.
\end{equation}

Therefore, by convexity and homogeneity of $I$,

\begin{equation}
(22) \quad I(f(t)) \leq I \left( \int_0^t e^{-\int_s^t Lf(\tau, v) d\tau} Q^+(f(s, f(s))) \, ds \right) + I \left( f_0 e^{-\int_0^t Lf(\tau, v) d\tau} \right)
\end{equation}

\begin{equation}
\leq \int_0^t ds \, I \left( e^{-\int_s^t Lf(\tau, v) d\tau} Q^+(f(s, f(s))) \right) + I \left( f_0 e^{-\int_0^t Lf(\tau, v) d\tau} \right).
\end{equation}

Now, for any two nonnegative functions $g$ and $h$,

\begin{equation}
(23) \quad I(gh) = \int \frac{d}{gh} \left| \nabla (gh) \right|^2 = \int \frac{d}{gh} \left| g \nabla h + h \nabla g \right|^2
\end{equation}

\begin{equation}
\leq 2 \left( \int \frac{g^2 \left| \nabla h \right|^2}{gh} + \int \frac{h^2 \left| \nabla g \right|^2}{gh} \right) = 2 \left( \int \frac{\left| \nabla h \right|^2}{h} + \int \frac{\left| \nabla g \right|^2}{g} \right)
\end{equation}

\begin{equation}
\leq 2 \left\| g \right\|_{L^\infty} I(h) + 2 \left\| \frac{\left| \nabla g \right|^2}{g} \right\|_{L^\infty} \| h \|_{L^1}.
\end{equation}

We estimate the different terms appearing in the right-hand side of (21). First,

\begin{equation}
(24) \quad e^{-\int_s^t Lf(\tau, v) d\tau} \leq 1, \quad I \left( Q^+(f, f) \right) \leq 2a I(f).
\end{equation}

Next, since

$$\nabla \left( e^{-\int_s^t Lf(\tau, v) d\tau} \right) = -e^{-\int_s^t Lf(\tau, v) d\tau} \int_s^t \nabla (Lf)(v, \tau) \, d\tau,$$
we obtain the bound
\[
\left( \frac{\nabla \left( e^{-\int_{\tau}^{t} L f(\tau) d\tau} \right)}{e^{-\int_{\tau}^{t} L f(\tau) d\tau}} \right)^2 = e^{-\int_{\tau}^{t} L f(\tau) d\tau} \left( \int_{s}^{t} d\tau \nabla A * f(\tau) \right)^2 \leq \left( \int_{s}^{t} d\tau \| \nabla A \|_{L^\infty} \| f \|_{L^1} \right)^2 \leq (t-s)^2 b^2.
\]
Finally, we note that
\[
\int dv \, Q^+(f, f) = \int dv \, Q^-(f, f) \leq \alpha.
\]
Putting together (22), (23), (24), (25) and (26), it follows that
\[
I(f(t)) \leq 4a \int_{0}^{t} I(f(s)) \, ds + 2a \int_{0}^{t} (t-s)^2 b^2 + 2I(f_0) + 2t^2 b^2.
\]
Our estimate of $I(f(t))$ follows by Gronwall’s lemma.

4. MAXWELLIAN CROSS-SECTIONS

Let $B(k \cdot \sigma)$ be a smooth Maxwellian cross-section, such that for any unit vector $k$, $\int d\sigma \, B(k \cdot \sigma) = 1$. We shall refine the computation done in the proof of Theorem 2, taking advantage of the structure of $B$; instead of writing
\[
\nabla_v (B(v - v_*, \sigma)) = -\nabla_\sigma (B(v - v_*, \sigma)),
\]
we shall report the derivatives on the variable $\sigma$. We begin with an elementary lemma in differential calculus, whose proof we omit.

**Lemma 1.** \( \nabla_v \left[ B \left( \frac{v - v_*}{|v - v_*|} \cdot \sigma \right) \right] = \frac{1}{|v - v_*|} B' \left( \frac{v - v_*}{|v - v_*|} \cdot \sigma \right) \Pi_{k^\perp} \sigma, \)
where $\Pi_{k^\perp}$ is the orthogonal projection upon $k^\perp$, that is (if $\|k\| = 1$),
\[
\Pi_{k^\perp} \sigma = \sigma - (\sigma \cdot k)k.
\]

Our second lemma can be considered as a particular integration by parts on the unit sphere.

**Lemma 2.** Let $k$ be a fixed unit vector, and $F$ a smooth function on $\mathbb{R}^N$. Then
\[
\int_{\mathbb{S}^{N-1}} d\sigma \, B'(k \cdot \sigma) F(\sigma) \Pi_{k^\perp} \sigma = \int_{\mathbb{S}^{N-1}} d\sigma \, B(k \cdot \sigma) M_{\sigma k} \nabla F(\sigma),
\]
where the linear operator $M_{\sigma k} : \mathbb{R}^N \to \mathbb{R}^N$ is defined by
\[
M_{\sigma k}(x) = (k \cdot \sigma) x - (k \cdot x) \sigma.
\]
Proof of the lemma. Let $u$ be a smooth function defined on $\mathbb{R}^N$; we define its spherical gradient at point $\sigma \in S^{N-1}$ by

$$\nabla_\sigma u(\sigma) = \Pi_{\sigma \perp} \nabla u(\sigma).$$

We note first that

$$\int_{S^{N-1}} d\sigma \nabla_\sigma u(\sigma) = (N - 1) \int_{S^{N-1}} d\sigma u(\sigma)\sigma.$$

To prove (30), we introduce a smooth function $q$ on $\mathbb{R}_+$, identically vanishing near 0 and $\infty$. Then we set

$$w(x) = u \left( \frac{x}{|x|} \right) q(|x|),$$

so that

$$\nabla w(x) = \frac{1}{|x|} \nabla_\sigma u \left( \frac{x}{|x|} \right) q(|x|) + u \left( \frac{x}{|x|} \right) q'(|x|) \frac{x}{|x|}.$$ 

Hence, by integration and spherical change of variables,

$$0 = \int_{\mathbb{R}^N} \frac{1}{|x|} \nabla_\sigma u \left( \frac{x}{|x|} \right) q(|x|) + \int_{\mathbb{R}^N} u \left( \frac{x}{|x|} \right) q'(|x|) \frac{x}{|x|}$$

$$= \left( \int_{S^{N-1}} \nabla_\sigma u(\sigma) d\sigma \right) \left( \int_0^\infty q(r) r^{N-2} dr \right) + \left( \int_{S^{N-1}} u(\sigma)\sigma d\sigma \right) \left( \int_0^\infty q'(r) r^{N-1} dr \right).$$

Since $\int_0^\infty q'(r) r^{N-1} dr = -(N - 1) \int q(r) r^{N-2} dr$, and since $q$ is arbitrary, the formula (30) follows.

As a consequence, if $k$ denotes a fixed unit vector and $u, v$ are smooth functions on $\mathbb{R}^N$, the following formulas of integration by parts hold.

$$\int_{S^{N-1}} d\sigma u(\sigma) \nabla_\sigma v(\sigma) = - \int_{S^{N-1}} d\sigma \nabla_\sigma u(\sigma)v(\sigma) + (N - 1) \int d\sigma u(\sigma)v(\sigma)\sigma;$$

$$\int_{S^{N-1}} d\sigma u(\sigma) \left[ k \cdot \nabla_\sigma v(\sigma) \right] = - \int_{S^{N-1}} d\sigma \left[ k \cdot \nabla_\sigma u(\sigma) \right] v(\sigma)$$

$$+ (N - 1) \int d\sigma (\sigma \cdot k) u(\sigma)v(\sigma).$$

Now, let us be interested in the right-hand side of (28). Writing $\nabla F = \nabla_\sigma F + (\sigma \cdot \nabla F)\sigma$, the terms involving $\sigma \cdot \nabla F$ cancel out, and this expression is

$$\int_{S^{N-1}} d\sigma B(k \cdot \sigma) \left[ (k \cdot \sigma)\nabla_\sigma F(\sigma) - (k \cdot \nabla_\sigma F(\sigma)) \sigma \right] \equiv (a) - (b).$$


which depends only on the values of $F$ on $S^{N-1}$. Integrating by parts,

$$(33) \quad (a) = \int_{S^{N-1}} d\sigma \, B(k \cdot \sigma)(k \cdot \sigma) \nabla_\sigma F(\sigma) = - \int_{S^{N-1}} d\sigma \, \nabla_\sigma \left[ B(k \cdot \sigma) k \cdot \sigma \right] F(\sigma)$$

$$+ (N-1) \int_{S^{N-1}} d\sigma \, B(k \cdot \sigma) (k \cdot \sigma) \sigma F(\sigma);$$

On the other hand, choosing an orthonormal basis $(e_i)$, the $i$-th component of $(b)$ is

$$\int_{S^{N-1}} d\sigma \, B(k \cdot \sigma) (k \cdot \nabla_\sigma F(\sigma)) \sigma_i = - \int_{S^{N-1}} d\sigma \, \nabla_\sigma (\sigma_i B(k \cdot \sigma) ) \cdot k F(\sigma)$$

$$+ (N-1) \int_{S^{N-1}} d\sigma \, B(k \cdot \sigma) (k \cdot \sigma) F(\sigma) \sigma_i.$$

Therefore,

$$(34) \quad (b) = - \int_{S^{N-1}} d\sigma \sum_i \nabla_\sigma (\sigma_i B(k \cdot \sigma) ) \cdot k e_i F(\sigma)$$

$$+ (N-1) \int_{S^{N-1}} d\sigma \, B(k \cdot \sigma) (k \cdot \sigma) F(\sigma) \sigma_i.$$ 

In view of (33) and (34), our lemma is proven provided that the following identity holds:

$$(35) \quad \sum_i \nabla_\sigma \left[ (\sigma \cdot e_i) B(k \cdot \sigma) \right] \cdot k e_i = \nabla_\sigma \left[ B(k \cdot \sigma) k \cdot \sigma \right] = B'(k \cdot \sigma) \Pi_{\sigma \perp} \sigma.$$

Let us compute the left-hand side of (35), this is

$$\sum_i (\Pi_{\sigma \perp} e_i \cdot k) B(k \cdot \sigma) e_i + \sum_i (\sigma \cdot e_i) B'(k \cdot \sigma)(\Pi_{\sigma \perp} k \cdot k) e_i$$

$$- B'(k \cdot \sigma) \Pi_{\sigma \perp} k (k \cdot \sigma) - B(k \cdot \sigma) \Pi_{\sigma \perp} k.$$

$$= \sum_i (e_i \cdot \Pi_{\sigma \perp} k) e_i B(k \cdot \sigma) + \sum_i (\sigma \cdot e_i) e_i B'(k \cdot \sigma)(\Pi_{\sigma \perp} k \cdot k)$$

$$- B'(k \cdot \sigma) \Pi_{\sigma \perp} k (k \cdot \sigma) - B(k \cdot \sigma) \Pi_{\sigma \perp} k.$$

$$= \Pi_{\sigma \perp} k B(k \cdot \sigma) + \sigma B'(k \cdot \sigma)(\Pi_{\sigma \perp} k \cdot k) - B'(k \cdot \sigma) \Pi_{\sigma \perp} k (k \cdot \sigma) - B(k \cdot \sigma) \Pi_{\sigma \perp} k.$$
Since the terms involving $B(k \cdot \sigma)$ cancel out, it only remains to compute
\[
B'(k \cdot \sigma) \left( (\Pi_{k \perp} k \cdot \sigma) - \Pi_{k \perp} k (k \cdot \sigma) \right)
\]
\[
= B'(k \cdot \sigma) (\sigma - (\sigma \cdot k)^2 \sigma - k(k \cdot \sigma) + (\sigma \cdot k)^2 \sigma)
\]
\[
= B'(k \cdot \sigma) (\sigma - k(k \cdot \sigma))
\]
\[
= B'(k \cdot \sigma) \Pi_{k \perp} \sigma.
\]

\[\square\]

**Remark.** If $N = 2$, let us denote by $\sigma^\perp$ the unit vector obtained from $\sigma$ by a counterclockwise rotation of angle $\pi/2$. Then
\[
M_{\sigma k}(x) = (\sigma^\perp \cdot x) k^\perp.
\]

By the lemmas 1 and 2 above, if we set
\[
k = \frac{v - v_*}{|v - v_*|},
\]
\[
F(x) = f \left( \frac{v + v_*}{2} + \frac{|v - v_*|}{2} x \right) f \left( \frac{v + v_*}{2} - \frac{|v - v_*|}{2} x \right),
\]
then we obtain
\[
(36) \quad \int dv_* d\sigma \nabla_v \left[ B(k \cdot \sigma) \right] f' f_*'
\]
\[
= \frac{1}{2} \int dv_* d\sigma B(k \cdot \sigma) \left[ f'_* M_{\sigma k} (\nabla f)'_i - f' M_{\sigma k} (\nabla f)'_i \right].
\]

Proposition 3 follows thanks to (18), with
\[
P_{\sigma k}(x) = (\sigma \cdot x) k + M_{\sigma k}(x).
\]

We note that $P_{\sigma k}$ is an odd function of both $k$ and $\sigma$.

Let us be interested in the norm of the linear operator $P_{\sigma k}$.

**Lemma 3.** Let $k, \sigma, x$ be vectors of $\mathbb{R}^2$, such that $||k|| = ||\sigma|| = 1$. Then
\[
||P_{\sigma k} x|| = ||x||.
\]

**Proof of the lemma.** By the remark following the proof of lemma 2,
\[
P_{\sigma k} x = (\sigma \cdot x) k + M_{\sigma k} x = (\sigma \cdot x) k + (\sigma^\perp \cdot x) k^\perp.
\]

Hence,
\[
||P_{\sigma k} x||^2 = (\sigma \cdot x)^2 + (\sigma^\perp \cdot x)^2 = ||x||^2.
\]

\[\square\]
Lemma 4. For all $\sigma, k, x \in \mathbb{R}^N$ such that $\|k\| = \|\sigma\| = 1$, one has

$$\|P_{\sigma k} x\| \leq \|x\|,$$

with equality only if $\sigma, k$ and $x$ belong to the same plane.

Proof of the lemma. First note that if $\sigma$ and $k$ are colinear, then $P_{\sigma k} = \pm I$, and equality holds. Therefore, let us assume that $k$ and $\sigma$ are free, and let $\Pi$ be the orthogonal projector onto the plane directed by $\sigma$ and $k$. We write

$$x = \Pi x + \Pi_\perp x,$$

with $\Pi_\perp x \cdot \sigma = 0$, $\Pi_\perp x \cdot k = 0$. Then,

$$P_{\sigma k} x = (k \cdot \sigma) \Pi_\perp x + [(k \cdot \sigma) \Pi x + (\sigma \cdot \Pi x) k - (k \cdot \Pi x) \sigma]$$

$$= (k \cdot \sigma) \Pi_\perp x + P_{\sigma k}(\Pi x).$$

It is obvious that $P_{\sigma k}(\Pi x)$ lies in the plane directed by $\sigma$ and $k$. In view of lemma 3, $\|P_{\sigma k}(\Pi x)\| = \|\Pi x\|$, and therefore

$$\|P_{\sigma k} x\|^2 = (k \cdot \sigma)^2 \|\Pi_\perp x\|^2 + \|\Pi x\|^2 \leq \|\Pi_\perp x\|^2 + \|\Pi x\|^2 = \|x\|^2,$$

with equality only if $\Pi_\perp x = 0$, since $(k \cdot \sigma)^2 < 1$. \qed

Now, we are ready to adapt the computation in section 3 to our new representation (12) of $Q^+$. In the sequel, we only deal with $Q^+(f, f)$ for simplicity, but the proof goes through for $Q^+(f, g)$ as well.

By Cauchy-Schwarz inequality, reasoning as in section 3,

\begin{equation}
I(Q^+(f, f)) = \int dv \frac{[\nabla Q^+(f, f)]^2}{Q^+(f, f)}
\leq \int dv \, dv_\ast \, d\sigma \, B \left| \sqrt{f'}(I + P_{\sigma k})(\nabla \sqrt{f})' + \sqrt{f'}(I - P_{\sigma k})(\nabla \sqrt{f})' \right|^2
\end{equation}

with $k = (v - v_\ast)/[v - v_\ast]$.

By the change of variables $(v, v_\ast) \longleftrightarrow (v', v'_\ast)$, which exchanges $k$ and $\sigma$,

\begin{equation}
I(Q^+(f, f)) \leq \int dv \, dv_\ast \, d\sigma \, B \left| \sqrt{f'}(I + P_{k\sigma})(\nabla \sqrt{f}) + \sqrt{f'}(I - P_{k\sigma})(\nabla \sqrt{f})' \right|^2
\end{equation}

$$= \int dv \, dv_\ast \, d\sigma \, B \left| (\sqrt{f'} \nabla \sqrt{f} + \sqrt{f'} (\nabla \sqrt{f})') + P_{k\sigma} (\sqrt{f'} \nabla \sqrt{f} - \sqrt{f'} (\nabla \sqrt{f})') \right|^2.$$

We expand the square norm, and notice that the cross product vanishes,

$$\int dv \, dv_\ast \, d\sigma \, B \left( \sqrt{f'} \nabla \sqrt{f} + \sqrt{f'} (\nabla \sqrt{f})' \right) \cdot P_{k\sigma} \left( \sqrt{f'} \nabla \sqrt{f} - \sqrt{f'} (\nabla \sqrt{f})' \right) = 0,$$
because $P_{\sigma k}$ is an odd function of $\sigma$. Therefore,

$$I(Q^+(f, f)) \leq \int dv dv_* d\sigma B \left( \left| \sqrt{f_*} \nabla \sqrt{f} + \sqrt{f} \nabla \sqrt{f}_* \right|^2 + \left| P_{\sigma k} \left( \sqrt{f_*} \nabla \sqrt{f} - \sqrt{f} \nabla \sqrt{f}_* \right) \right|^2 \right).$$  \hspace{1cm} (39)

$$\leq \int dv dv_* d\sigma B \left( \left| \sqrt{f_*} \nabla \sqrt{f} + \sqrt{f} \nabla \sqrt{f}_* \right|^2 + \left| \sqrt{f_*} \nabla \sqrt{f} - \sqrt{f} \nabla \sqrt{f}_* \right|^2 \right).$$  \hspace{1cm} (40)

Using $\int d\sigma B(k \cdot \sigma) = 1$ and the symmetry in $(v, v_*)$, we obtain

$$I(Q^+(f, f)) \leq 4 \int dv f_* \left| \nabla \sqrt{f} \right|^2 = I(f).$$

Finally, this inequality extends to arbitrary probability distributions by density.

**Remark.** By the "Cauchy-Schwarz equality",

$$\left( \int u^2 \right) \left( \int v^2 \right) - \left( \int uv \right)^2 = \int dx dy (u(x)v(y) - u(y)v(x))^2,$$

we can obtain an explicit lower bound for the dissipation of Fisher information (see [8] for related arguments): if we set

$$T f(v, v_*, \sigma) = \sqrt{f_0'(I + P_{\sigma k})(\nabla \sqrt{f})'} + \sqrt{f'(I - P_{\sigma k})(\nabla \sqrt{f})'_*},$$

with $k = (v - v_*)/|v - v_*|$ and

$$g' = g \left( \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \right) \equiv g' [\sigma],$$

$$g'_* = g \left( \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \right) \equiv g'_* [\sigma],$$

we obtain

$$I(f) - I(Q^+(f, f)) \geq \int dv dv_* d\sigma dw dw_* d\tau \left\{ B \left( \frac{v - v_*}{|v - v_*|} \cdot \sigma \right) \sqrt{f_0' f_*'[\tau]} T f(w, w_*, \tau) - B \left( \frac{w - w_*}{|w - w_*|} \cdot \tau \right) \sqrt{f_0' f_*'[\tau]} T f(v, v_*, \sigma) \right\}^2,$$  \hspace{1cm} (41)

with equality in dimension 2.
5. Related inequalities and analogy between $Q^+$ and the rescaled convolution

In this section, we fix a collision operator $Q$ with Maxwellian cross-section $B$, such that for any unit vector $k$, $\int d\sigma B(k \cdot \sigma) = 1$. We begin with a very useful proposition, which we shall call “Bobylev’s lemma”.

**Lemma (Bobylev).** Let $f$ and $g$ be two probability densities, and let $M$ be a Maxwellian density. Then

$$Q^+(f \ast M, g \ast M) = Q^+(f, g) \ast M.$$  

The proof can be found for instance in [3].

From this lemma we easily deduce Corollary 4.2. Let $f_0$ be a smooth probability density, nonvanishing and rapidly decreasing at infinity. Then there exists a unique solution $f(t, v)$ to the Boltzmann equation $\partial_t f = Q(f, f)$. Let $M_\delta(v) = \exp(-v^2/(2\delta))/(2\pi\delta)^N$. By Bobylev’s lemma, $f \ast M_\delta$ is also a solution of the Boltzmann equation. Moreover, since $\partial_t(f \ast M_\delta) = Q(f, f) \ast M_\delta$, $f \ast M_\delta$ lies in $C^\infty([0, \infty) \times \mathbb{R}^N)$ and does not vanish. This enables to justify the formula

$$\frac{d}{dt} I(f(t) \ast M_\delta) = -8 \int \frac{\Delta \sqrt{f \ast M_\delta}}{\sqrt{f \ast M_\delta}} Q(f \ast M_\delta, f \ast M_\delta).$$

Passing $\delta$ to 0, we obtain in particular

$$\int \frac{\Delta \sqrt{f_0}}{\sqrt{f_0}} Q(f_0, f_0) \geq 0,$$

which proves Corollary 4.2.

We note that we are not aware of a direct proof for this last inequality.

Now, to prove Corollary 4.3, we recall another property related to the rescaled convolution.

**Lemma 5.** Let $f$ and $g$ be two probability distributions, and let $Q$ be a Boltzmann operator with Maxwellian cross-section. Then, for any $\lambda > 0$,

$$Q^+(f_\lambda, g_\lambda) = [Q^+(f, g)]_\lambda.$$
Proof of the lemma. This is a straightforward computation:

\[
2Q^+(f, g)_\lambda = \frac{1}{\sqrt{\lambda}} \int dv_* d\sigma B \left( \frac{v/\sqrt{\lambda} - v_*}{\left| v/\sqrt{\lambda} - v_* \right|} \right)
\]

\[
\left[ f \left( \frac{v/\sqrt{\lambda} + v_*}{2} + \frac{|v/\sqrt{\lambda} - v_*|}{2} \right) g \left( \frac{v/\sqrt{\lambda} + v_*}{2} - \frac{|v/\sqrt{\lambda} - v_*|}{2} \right) + g \left( \frac{v/\sqrt{\lambda} + v_*}{2} + \frac{|v/\sqrt{\lambda} - v_*|}{2} \right) f \left( \frac{v/\sqrt{\lambda} + v_*}{2} - \frac{|v/\sqrt{\lambda} - v_*|}{2} \right) \right].
\]

By the change of variables \( w_* = \sqrt{\lambda} v_* \), the previous expression is equal to:

\[
\frac{1}{\sqrt{\lambda}} \int dw_* d\sigma B \left( \frac{v - w_*}{\left| v - w_* \right|} \right) \left[ f \left( \frac{v + w_*}{2\sqrt{\lambda}} + \frac{|v - w_*|}{2\sqrt{\lambda}} \right) g \left( \frac{v + w_*}{2\sqrt{\lambda}} - \frac{|v - w_*|}{2\sqrt{\lambda}} \right)
\]

\[
\left. g \left( \frac{v + w_*}{2\sqrt{\lambda}} + \frac{|v - w_*|}{2\sqrt{\lambda}} \right) f \left( \frac{v + w_*}{2\sqrt{\lambda}} - \frac{|v - w_*|}{2\sqrt{\lambda}} \right) \right] = 2Q^+(f, g)_{\lambda}.
\]

Corollary. The semigroup associated with the Boltzmann equation with Maxwellian cross-section commutes with the adjoint Ornstein-Uhlenbeck semigroup.

This corollary (due to Bobylev and Carlen [6]) is a direct consequence of the last two lemmas, since the adjoint Ornstein-Uhlenbeck semigroup is obtained by rescaling and convolution with a Maxwellian distribution. It was first used by Carlen and Carvalho in the case of a constant cross-section.

Now, corollary 4.3 follows easily. Let \( f \) and \( g \) be any two fixed probability densities with the same mean and variance. Let us denote by \( M \) the Maxwellian distribution with the same mean and variance as \( f, g \) and \( Q^+(f, g) \), and by \( S_t h \) the solution of the Fokker-Planck (or Ornstein-Uhlenbeck) equation with initial datum \( h \), i.e.

\[
\partial_t (S_t h) = \nabla \cdot (\nabla (S_t h) + (S_t h)v).
\]

As \( t \) goes to infinity, \( S_t h \rightarrow M \) in all Sobolev norms (Cf. [17]). By an easy computation, and the same arguments as in [17],

\[
-\frac{d}{dt} H(S_t[Q^+(f, g)]) = I(S_t[Q^+(f, g)]) - I(M).
\]
Integrating this inequality in time from 0 to $\infty$ and using the last Corollary, we get

$$H(Q^+(f, g)) - H(M) = \int_0^\infty dt \left( I \left( Q^+(S_t f, S_t g) \right) - I(M) \right).$$

By inequality (8),

$$H(Q^+(f, g)) - H(M) \leq \frac{1}{2} \int_0^\infty dt \left( I(S_t f) - I(M) \right) + \frac{1}{2} \int_0^\infty dt \left( I(S_t g) - I(M) \right)$$

$$= \frac{1}{2} [H(f) + H(g)] - H(M),$$

and the proof is complete. Rigorous justification for such manipulations can be found in full detail in [10, 17].

REFERENCES


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