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On the symmetry of solutions of the Ginzburg-Landau equations for small domains

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Abstract

In this paper, we study the Ginzburg-Landau equations for a 2 dimensional domain which has small size. We prove that if the domain is small, then the solution has no zero, that is no vortex. Additionally, we obtain that if the domain is a disc of small radius, then the solution is symmetric. Then, in the case of a slab, that is a one dimensional domain, we use the same method to derive that solutions are symmetric. The proof uses a priori estimates and the Poincaré inequality.

1 Introduction and main results

In this paper, we study the properties of a superconducting cylinder submitted to an exterior magnetic field \( \mathbf{H}_0 \) parallel to the axis of the cylinder. According to the Ginzburg-Landau theory of superconductivity, the sample is in a state that minimizes the following energy:

\[
E_\kappa(\Psi, \mathbf{A}) = \int_\Omega \left| \left( \frac{1}{\kappa} \nabla - i \mathbf{A} \right) \Psi \right|^2 + \frac{1}{2} \left| \left( |\Psi|^2 - 1 \right) \right|^2 + (\text{curl } \mathbf{A} - \mathbf{H}_0)^2 \, d\Omega. \tag{1.1}
\]

Here \( \Omega \) is a simply connected domain in \( \mathbb{R}^2 \) with characteristic length \( d \). As usual, \( \kappa \) is the Ginzburg-Landau parameter that characterizes the type

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of superconductor, \( A(x, y) \) is the vector potential, so that \( \text{curl} \ A \) is the magnetic field and \( \Psi(x, y) \) is the order parameter. Because \( H_0 \) is along the \( z \) axis, we can assume without loss of generality that \( A = (A_1, A_2, 0) \). For a detailed description of the model, one may refer to [4] or [11].

The minimization process yields the classical Ginzburg-Landau equations, where we have chosen the special gauge where \( \text{div} \ A = 0 \) and \( A \cdot n = 0 \) on \( \partial \Omega \) (see [4] for more details):

\[
\begin{aligned}
\left\{ \begin{array}{l}
\left( \frac{1}{\kappa} \nabla - iA \right)^2 \Psi = \Psi(|\Psi|^2 - 1) \quad \text{in} \quad \Omega, \\
-\text{curl curl} \ A = \frac{1}{\kappa^2} (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) + A |\Psi|^2 \quad \text{in} \quad \Omega, \\
\frac{\partial \Psi}{\partial n} = 0 \quad \text{on} \quad \partial \Omega, \\
\text{curl} \ A \times n = H_0 \times n \quad \text{on} \quad \partial \Omega.
\end{array} \right.
\]  

(GL)

In this paper, we are interested in the case where \( d \), the characteristic length of \( \Omega \) is small, in particular compared to \( 1/\kappa \). In this setting, it is expected that there is no zero of \( \Psi \), that is no vortices, since a vortex is of size \( 1/\kappa \). This is what we prove:

**Theorem 1** Assume that \( D \) is a fixed simply connected bounded domain and let \( \Omega = dD \). For any \( d_0 > 0 \), there exists \( d_1 > 0 \), such that if \( d < \min(d_0, d_1/\kappa) \), then any solution of (GL) is such that \( \Psi \) has no zero.

Note that in the case where \( d \) is fixed and \( \kappa \) is large, then it is proved [10] that the solution can have a lot of vortices for certain magnetic field.

Additionally, we prove a symmetry result for solutions in small discs:

**Theorem 2** Let \( \Omega \) be a disc of radius \( d \). There exist constants \( d_0 \) and \( d_1 \), such that if \( d < \min(d_0, d_1/\kappa) \), then any solution of (GL) is radially symmetric, that is \( \Psi(x, y) = \Psi(r) \) and \( A(x, y) = A(r) e_\theta \), where \( r = \sqrt{x^2 + y^2} \).

The radial symmetry of local minimizers has already been studied in [9] for a different system with no magnetic field. Here the proof holds for any solution.

We make a change of variables \( x' = x/d, \quad y' = y/d \) so that the new variable lies in a domain of unit size \( D \). We also define \( B = \kappa dA \) and \( h_0 = \kappa d^2 H_0 \). Recall that \( \text{div} \ B = 0 \). The equations then become

\[
\begin{aligned}
\left\{ \begin{array}{l}
(\nabla - iB)^2 \psi = \kappa^2 d^2 \psi(|\psi|^2 - 1) \quad \text{in} \quad D, \\
\Delta B = d^2 \left( \frac{1}{\kappa^2} (\psi^* \nabla \psi - \psi \nabla \psi^*) + B |\psi|^2 \right) \quad \text{in} \quad D, \\
\frac{\partial \varphi}{\partial n} = 0 \quad \text{on} \quad \partial D, \\
\text{curl} \ B \times n = h_0 \times n \quad \text{on} \quad \partial D.
\end{array} \right.
\]  

(GL)
Note that another way of writing the equation for $B$ is
\[
\Delta B = -d^2 (\bar{\psi}, \nabla \psi - iB\psi),
\] (1.2)
where $(\ldots)$ is the scalar product in $C$.

The proof consists in obtaining a priori estimates for the solutions $(\psi, B)$. This is done in Section 2. In Section 3, we use these a priori estimates to derive that $\psi$ is nearly constant, hence has no zero. This will prove Theorem 1. Then, in Section 4, we define the functions
\[
\tilde{\psi}(x, y) = \psi(-x, y), \quad \tilde{B}(x, y) = \left( \begin{array}{c} -B_1(-x, y) \\ B_2(-x, y) \end{array} \right),
\]
\[
w(x, y) = \frac{1}{|\psi|_\infty} (\psi(x, y) - \tilde{\psi}(x, y)), \quad z(x) = B(x, y) - \tilde{B}(x, y),
\] (1.3)
which satisfy elliptic PDE’s with small right hand side terms. Then we use that $\psi$ is nearly constant and hence we get that $w$ and $z$ are identically zero. This will prove Theorem 2.

**Remark:** If $\Omega$ is simply connected and symmetric in the $y$ direction, then our proof also gives that $\psi$ and $B$ are symmetric, in the sense that $\tilde{\psi} = \psi$ and $\tilde{B} = B$.

Finally in Section 5, we show how our proof can be adapted to the one dimensional case to provide a symmetry result in this setting.

## 2 A priori estimates

**Proposition 3** Fix $p > 1$. For all constants $d_0$ and $d_1$, if $d < \min(d_0, d_1/\kappa)$, then $\psi$ and $B$ are bounded in $W^{2,p}(D)$ by constants independent of $d$ and $\kappa$. Moreover, for fixed $\kappa$,
\[
\lim_{d \to 0} \|B\|_{W^{2,p}(D)} = 0.
\] (2.1)

The goal of this section consists in proving Proposition 3.

### 2.1 First estimates for $\psi$

First recall from Du-Gunzburger-Peterson [4] that
\[
|\psi| \leq 1 \quad \text{a.e.}
\] (2.2)
Next we define
\[ u(x, y) = |\psi(x, y)|^2 \quad \text{and} \quad \Pi = \nabla - i\mathbf{B}. \] (2.3)

One can compute easily that
\[ \frac{1}{2} \Delta u + \kappa^2 d^2 u(1 - u) = |\Pi \psi|^2. \] (2.4)

Integrate (2.4) in \( D \), use the boundary condition \( \partial \psi / \partial n = 0 \) and get the key estimate for \( \psi \):
\[ \|\Pi \psi\|_{L^2(D)} \leq \kappa d \|\psi\|_{L^2(D)}. \] (2.5)

### 2.2 Estimates for \( \mathbf{B} \)

We decompose \( \mathbf{B} = h_0 \mathbf{b}_0 + \mathbf{b}_r \) where \( \mathbf{b}_0 \) is chosen such that

curl curl \( \mathbf{b}_0 \) = 0 and \( \text{div} \mathbf{b}_0 = 0 \) in \( D \), curl \( \mathbf{b}_0 = \mathbf{e}_z \) and \( \mathbf{b}_0 \cdot \mathbf{n} = 0 \) on \( \partial D \),
and \( \mathbf{b}_r = \mathbf{B} - h_0 \mathbf{b}_0 \).

A particular choice for \( \mathbf{b}_0 \) when \( \Omega \) is a disc is \( \mathbf{b}_0 = \frac{1}{2}(-y, x, 0) \). Note that in any case \( \mathbf{b}_0 \) and \( \mathbf{b}_r \) are vectors in the \( x - y \) plane and their curl are in the \( z \) direction. Making the difference between the equations for \( \mathbf{B} \) and \( \mathbf{b}_0 \), we obtain the equation for \( \mathbf{b}_r \)
\begin{align*}
\text{curl curl } \mathbf{b}_r &= d^2 (i\psi, \Pi \psi) \quad \text{and} \quad \text{div } \mathbf{b}_r = 0 \quad \text{in } D, \quad (2.6) \\
\text{curl } \mathbf{b}_r &= 0 \quad \text{and} \quad \mathbf{b}_r \cdot \mathbf{n} = 0 \quad \text{on } \partial D. \quad (2.7)
\end{align*}

Since \( \text{div } \mathbf{b}_r = 0 \) and \( \mathbf{b}_r \cdot \mathbf{n} = 0 \) on \( \partial D \), then \( \|\text{curl } \mathbf{b}_r\|_{L^p(D)} \) is a norm in \( W^{1,p}(D) \), that is
\[ \|\mathbf{b}_r\|_{W^{1,p}(D)} \leq C \|\text{curl } \mathbf{b}_r\|_{L^p(D)}. \] (2.8)

Note that the boundary condition (2.7) also gives that curl \( \mathbf{b}_r \) is orthogonal to constants. Now Cauchy-Schwarz inequality and (2.6) imply that
\[ |\text{curl curl } \mathbf{b}_r|_2 \leq d^2 |\Pi \psi|_2. \] (2.9)

Note that \( |\cdot|_2 \) is the 2-norm for vectors in \( \mathbb{R}^2 \). From (2.9), we get the \( W^{2,2} \) bound for \( \mathbf{b}_r \) and then we bootstrap to get the \( W^{2,p} \) bound (see [7]). Thus, because of (2.5), (2.9) and the boundary condition (2.7), it follows from the Poincaré inequality that
\[ \|\text{curl } \mathbf{b}_r\|_{W^{1,p}(D)} \leq C \kappa d^3. \] (2.10)
We gather (2.8) and (2.10) to obtain the key estimate for $b_r$:

$$\|b_r\|_{W^{2,p}(D)} \leq C \kappa d^3, \quad (2.11)$$

where $C$ is independent of $d$ and $\kappa$. In order to get the bounds for $B$, we only need to prove that $\lim_{d \to 0} h_0 = 0$. Let us assume that $h_0$ is bounded below by some constant $m$. Then we use an estimate by Giorgi and Phillips [8] (Lemma 2.8 p.349): there exists $C_2$ (which depends on $m$) such that

$$C_2 h_0 \int_D |\psi|^2 \leq \int_D \left| \nabla \psi - i h_0 b_0 \psi \right|^2. \quad (2.12)$$

In order to bound the right hand side of (2.12), we write $h_0 b_0 = B - b_r$ and use (2.5) and (2.11) to get

$$(C_2 h_0)^{1/2} (\int_D |\psi|^2)^{1/2} \leq \kappa d (\int_D |\psi|^2)^{1/2} (1 + C_1 d^2). \quad (2.13)$$

So we obtain a contradiction with the lower bound for $h_0$ when $d$ is small enough. Hence $B$ is bounded in $W^{2,p}$ independently of $d$ and $\kappa$ and (2.1) is true for $B$.

Once we have bounded $B$, the equation for $\psi$ can be thought of as a linear elliptic equation with bounded coefficients, so that the classical elliptic estimates imply bounds for $\psi$.

3 $\psi$ is nearly constant

**Proposition 4** For all $d_0 > 0$, there exists $d_1 > 0$, such that if $d < \min(d_0, d_1/\kappa)$, then $\|\nabla \psi/|\psi|_{\infty}\|_{\infty}$ and $|\psi/|\psi|_{\infty} - 1|$ are small, thus $\psi$ has no zero.

We define $\phi = \psi/|\psi|_{\infty}$, then the equation for $\phi$ can be written as

$$\Delta \phi - 2iB \cdot \nabla \phi = \kappa^2 d^2 \phi (|\psi|_{\infty}^2 |\phi|^2 - 1) + |B|^2 \phi$$

and

$$|B \cdot \nabla \phi|_{L^p(D)} \leq |B \cdot \Pi \phi|_{L^p(D)} + |B^2 \phi|_{L^p(D)} \leq \frac{1}{|\psi|_{\infty} \Pi \phi \Pi \phi} |B|_{L^p} \|B\|_{L^{2,p}} + |B^2|_p. \quad (3.1)$$

Since $|\phi| \leq 1$, and because of (2.5) and the previous estimates for $B$, we see that the right hand side of (3.1) is small when $d$ is small. Hence we go back to the equation for $\phi$ and the Agmon Douglis Nirenberg estimates.
imply a bound for $\phi$ in $W^{2,p}$ for any $p$, hence in $C^1$. In particular, $\nabla \phi$ is equicontinuous. Now, when $d$ tends to 0, we know that $\|B\|_\infty$ tends to 0, so if we multiply the equation for $\phi$ by $\phi^*$ and integrate, we find that $\|\nabla \phi\|_{L^2(D)}$ is small. Thus, since $\nabla \phi$ is equicontinuous, we see that $\|\nabla \phi\|_\infty$ tends to 0 with $d$, which proves Proposition 4.

In fact we can get a precise estimate for the smallness of $|\psi/|\psi|_\infty - 1|$. Take the equation (2.4), multiply by $u$ and integrate to get

$$\int_D |\nabla u|^2 \leq \kappa^2 d^2 \int_D u^2 (1 - u).$$

(3.2)

Next we define $\phi = \psi/|\psi|_\infty$ and $v = u/|\psi|_\infty^2$. Then (3.2) implies that

$$\int_D |\nabla v|^2 \leq \kappa^2 d^2.$$

We call $v_{\text{mean}}$ the mean value of $v$ in $D$. The previous estimate and the Poincaré inequality yield

$$\|v - v_{\text{mean}}\|_{L^2(D)} \leq C\kappa d.$$

(3.3)

Let us call $r$ the upper bound for $\nabla v$. Note that $r$ is independent of $d$ and $\kappa$ as we have seen in the first part of the proof.

Let us now prove that

$$|v(x) - v_{\text{mean}}| \leq \alpha \quad \forall x \in D, \text{ where } \alpha^2 = 16r\kappa d.$$

(3.4)

Assume that (3.4) is not true. Then there exists a point $x_0$ where this does not hold. Set $\eta = \alpha/4r$. Note that when $\kappa d$ is small, then $\eta$ is small. If $\text{dist}(x_0, \partial D) \leq 2\eta$, then for $\kappa d$ small enough, there always exists a point $x_1$ in $D$ with $\text{dist}(x_1, \partial D) \geq 2\eta$ and $\text{dist}(x_1, x_0) \leq 2\eta$. If $\text{dist}(x_0, \partial D) \geq 2\eta$ then we set $x_1 = x_0$. Now we have $|v(x) - v_{\text{mean}}| > \alpha/4$ for $|x - x_1| \leq \eta$. Define $D_1 = D(x_1, \eta)$, then

$$\|v - v_{\text{mean}}\|_{L^2(D_1)} > \eta \sqrt{\pi} \alpha/4.$$

This provides a contradiction with the definition of $\alpha$ and (2.5).

Now recall that $v$ is necessarily 1 for at least one point in $D$. So we use (3.4) to get

$$|\psi(x)/|\psi|_\infty - 1| \leq C\kappa d \quad \forall x \in D.$$

This implies in particular that for small $\kappa d$, $\psi$ is never equal to zero.
4 Radial symmetry of the solutions in the case of a disc

In this Section, we assume that $\Omega$ is a disc of radius $d$. We are going to use the previous a priori estimates to obtain radial symmetry. We define the functions $w$ and $z$ as in (1.3). They satisfy the following equations

$$
\Delta w - 2i B \cdot \nabla w - \frac{2i}{|\psi|_\infty} z \cdot \nabla \tilde{\psi} = \kappa^2 d^2 w(|\psi|^2 + \tilde{\psi}(\psi + \tilde{\psi}) - 1) + w|B|^2 + \frac{\tilde{\psi}}{|\psi|_\infty} z \cdot (B + \tilde{B}) \quad \text{in } D, \quad (4.1)
$$

$$
\frac{\partial w}{\partial n} = 0 \quad \text{on } \partial D, \quad (4.2)
$$

$$
curl \, curl \, z = d^2[(i|\psi|_\infty w, \Pi \psi) + (i\tilde{\psi}, |\psi|_\infty \nabla w - iB|\psi|_\infty w - i\tilde{\psi}z)] \quad \text{and div } z = 0 \quad \text{in } D, \quad (4.3)
$$

$$
z \cdot \mathbf{n} = 0 \quad \text{and} \quad \text{curl } z \times \mathbf{n} = 0 \quad \text{on } \partial D. \quad (4.4)
$$

**Proposition 5** There are constants $d_0$ and $d_1$ such that, if $d < \min(d_0, d_1/\kappa)$, then $w \equiv z \equiv 0$.

This will end the proof of Theorem 2. Indeed, once we know that $w$ and $z$ are identically zero, then it implies as in [6] that $\psi$ and $|B|$ are radial, where $|\cdot|$ is the modulus. Thus, we see that either $B$ vanishes everywhere on the circle $|(x, y)| = r$ and there is nothing to prove, or nowhere on this circle. In this case, we can argue that $B(r, \theta)$ is in the direction of $e_\theta = (-\sin \theta, \cos \theta)$. Indeed, since $z = 0$, it implies in particular that $B_1(r, 0) = 0$, hence $B(r, 0)$ is in the direction of $(0, 1)$. Then we use a reflection in the axis $T$ in the direction of $(\cos \theta, \sin \theta)$ and derive that the corresponding $w$ and $z$ are zero. This implies that $B(r, \theta) \cdot e_r = 0$. Since $B_1(r, 0) = 0$, a continuity argument implies that $B(r, \theta) = B(r)e_\theta$, as we had claimed.

Now let us prove Proposition 5. Let us multiply (4.1) by $w$ and integrate. The first term only gives an integral of the gradient of $w$ because of the boundary condition (4.2). We recall that $|\psi| \leq 1$ and $|w| \leq 2$ to obtain

$$
\int_D |\nabla w|^2 \leq 4\|B\|_\infty \int_D |\nabla w| + (4\kappa^2 d^2 + \|B\|_\infty^2) \int_D w^2 + 4(\|\nabla \frac{\psi}{|\psi|_\infty}\|_\infty + \|B\|_\infty) \int_D |z|. \quad (7)
$$
Then we use our previous estimates for $\psi$ and $\mathbf{B}$ in Proposition 3 and 4. In particular we recall that $\|\mathbf{B}\|_\infty$ and $\|\sqrt{d}\nabla \psi\|_\infty$ are small when $d$ is small to get: for all $\varepsilon > 0$, if $d$ is small enough, then

$$\|\nabla w\|_2 \leq \varepsilon \|w\|_2 + \varepsilon \|\mathbf{z}\|_2.$$ 

Because of the definition of $w$, one can see easily that $w$ has mean value zero. We can use the Poincaré inequality to get

$$\mu \|w\|_2 \leq \varepsilon \|w\|_2 + \varepsilon \|\mathbf{z}\|_2,$$  

where $\mu > 0$ is independent of $d$ and $\kappa$. Thus

$$\|w\|_2 \leq \frac{\varepsilon}{\mu - \varepsilon} \|\mathbf{z}\|_2 \leq \|\mathbf{z}\|_2,$$  

provided $\varepsilon/(\mu - \varepsilon)$ is less then 1.

Similarly, we multiply equation (4.3) by $\mathbf{z}$, integrate and use that the boundary condition implies $\text{curl} \mathbf{z} = 0$ on $\partial D$ to get

$$\int_D \text{curl} \mathbf{z}_2 \leq d^2 \int_D |\mathbf{z}_2| \|\mathbf{w} + \nabla \psi + \mathbf{B} + \mathbf{z}_2|.$$ 

We can apply the previous estimates which give bounds on $\psi$ and $\mathbf{B}$ to derive

$$\|\nabla \mathbf{z}\|_2 \leq \varepsilon \|\mathbf{z}\|_2.$$  

Then we write $\mathbf{z} = \mathbf{z}_m + \mathbf{z}_t$ where $\mathbf{z}_m$ is the mean value of $\mathbf{z}$ (so $\mathbf{z}_m$ is a vector) and $\mathbf{z}_t$ has mean value zero. Again we use the Poincaré inequality for $\mathbf{z}_t$ and equation (4.7) to get

$$\|\mathbf{z}_t\|_2 \leq \frac{\varepsilon}{\mu - \varepsilon} \|\mathbf{z}_m\|_2 \quad \text{and} \quad \|w\|_2 \leq \frac{\varepsilon \mu}{(\mu - \varepsilon)^2} \|\mathbf{z}_m\|_2.$$  

Now we integrate (4.3) over $D$, use the boundary conditions which imply that the $\Delta$ term contribution vanishes and we get

$$\int_D \frac{|\psi|^2}{|\psi|_\infty^2} \mathbf{z} = \int_D (i\mathbf{w}, \mathbf{\Pi} \psi) + (i\bar{\psi}, \nabla \mathbf{w} - i\mathbf{B} \mathbf{w}).$$  

This yields

$$|\mathbf{z}_m| \int_D \frac{|\psi|^2}{|\psi|_\infty^2} \leq \|\mathbf{w}\|_2 \|\mathbf{\Pi} \psi\|_2 + \|\psi\|_2 \|\nabla \mathbf{w} - i\mathbf{B} \mathbf{w}\|_2 + \int_D |\mathbf{z}_t| \leq C \varepsilon |\mathbf{z}_m|.$$
The last inequality comes from (4.8) and the bounds for $\psi$ and $B$ in Section 3. If $z_m$ is not zero, the boundedness of $\Pi \psi$, $B$ and (4.8) imply that

$$\int_D |\psi|^2 / |\psi|_\infty^2$$

is small. This provides a contradiction with Proposition 4, where we have proved that $|\psi| / |\psi|_\infty$ is nearly 1 for small $d$. This finishes the proof.

5 The 1-dimensional case

When the superconducting material is an infinite slab of thickness 2$d$ between the planes $x = -d$ and $x = d$, it is usual to assume that both $\psi$ and $A$ are uniform in the $y$ and $z$ directions, and that the exterior magnetic field is tangential to the slab, that is $h_0 = (0, 0, h_0)$. A suitable gauge can then be chosen so that $\psi = f(x)$ is a real function, and $A = q(x)e_y$, where $e_y$ is the unit vector along the $y$ direction (see [5] for more details). The model can then be simplified to a system of 2 coupled ODE’s $(f, q)$ satisfy

$$\begin{cases} \frac{1}{\kappa} f'' = f(f^2 + q^2 - 1) \quad \text{in} \quad (-d, d), \\ f'(\pm d) = 0, \\ q'' = qf^2 \quad \text{in} \quad (-d, d), \\ q'(\pm d) = h_0. \end{cases} \quad (gl_d)$$

In this setting, there are two types of solutions: symmetric solutions where $f$ is even and $q$ is odd, and asymmetric solutions. A complete numerical study of the number and symmetry of these solutions has been done in [1].

Using the same techniques as in the previous Sections, we can prove

**Theorem 6** There exist constants $d_0$ and $d_1$, such that if $d < \min(d_0, d_1/\kappa)$, then any solution of $(gl_d)$ is symmetric, that is $f$ is even and $q$ is odd.

This theorem together with the result of uniqueness for symmetric solutions proved in [2] give a global uniqueness result for the solutions of $(gl_d)$ with small $d$.

The proof, as in the previous sections consists first in deriving a priori estimates for the functions $f$ and $q$. We non dimensionalize the distance by $d$. We recall from [2] that a solution is such that $f$ has a unique maximum, which we call $x_0$, with $\beta = f(x_0) = ||f||_\infty$, and $q$ is increasing with a unique zero $x_1$. A similar proof to what we did in Section 2 and 3 yields:
Proposition 7 There exist constants $d_0$ and $d_1$, such that if $d < \min(d_0, d_1/\kappa)$, then
\[
\frac{d}{\beta}(x - 1) \leq Ckd, \text{ and } |q(x) - h_0x| \leq Ckd^3.
\] (5.1)

Using similar techniques as in [2], one can get more precise estimates, which in particular give a relation between $\beta$ and $h_0$.

Proposition 8 Let $0 < d < \min(1, 1/2\kappa)$, then
\[
\begin{align*}
\beta(1 - \frac{\kappa^2d^2}{2}(x - x_0)^2) &\leq f(x) \leq \beta \quad \forall x \in (-1, 1), \\
\alpha(x - x_1) \leq q(x) \leq \alpha(x - x_1)(1 + 2\beta d) &\quad \forall x \in (-1, x_1), \\
\alpha(x - x_1)(1 - 2\beta d) \leq q(x) \leq \alpha(x - x_1) &\quad \forall x \in (x_1, 1),
\end{align*}
\] (5.2) (5.3) (5.4)
\[
|x_1| \leq 8\kappa^2d^2 + 8d, \quad \sqrt{3(1 - \beta^2)} \leq \alpha \leq \sqrt{3(1 - \beta^2) + 2d^2\kappa^2} \quad \sqrt{1 - d^2\kappa^2/2}
\] (5.5) (5.6)

These estimates mean that $f$ is nearly constant and that $q$ is nearly equal to $\alpha x$, because its zero $x_1$ is very small and recall from Proposition 7 that $\alpha$ and $h_0$ are very close. We won’t give the proof of Proposition 8 since Proposition 7 is enough for our purposes.

The second step of the proof consists in defining the functions $w$ and $z$ as before:
\[w(x) = \frac{1}{\|f\|_\infty}(f(x) - f(-x)) \quad \text{and} \quad z(x) = q(x) + q(-x).\] (5.7)

We derive the equations satisfied by $w$ and $z$, multiply them respectively by $w$ and $z$ and use the estimates of Proposition 7 to obtain
\[
|w''| \leq \varepsilon|w| + \varepsilon|z| \quad \text{and} \quad |z''| \leq \varepsilon|w| + \varepsilon|z|.
\]

Then we use the Poincaré inequality to get the equivalent of (4.6) and (4.7).

Finally, we see that since $\int_{-1}^{1} f^2q = 0$, then
\[
\int_{-1}^{1} \frac{z^2}{\beta^2} = \int_{-1}^{1} qw\frac{(\tilde{f} + f)}{\beta}.
\] (5.8)

As previously, we derive that if $z_m$ is different from zero, (5.8) provides a contradiction with (5.1), which means that $f(x)/\beta$ is nearly constant.

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