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A PLANCHEREL FORMULA ON $\text{Sp}_4$

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Abstract. In [HC], Harish-Chandra derived the Plancherel formula on $p$-adic groups. However, to have an explicit formula, one will have to compute the measures appearing in the formula. Here, we compute Plancherel measures on $\text{Sp}_4$ over $p$-adic fields explicitly.

Introduction

Let $k$ be a $p$-adic field with odd residue characteristic and let $G$ be $\text{Sp}_4$, the symplectic group of four variables. In this paper, we compute Plancherel measures on $G$ based on types constructed in [Ki1, Ro]. In [HC] (see also [Wald]), Harish-Chandra derived the Plancherel formula on $p$-adic groups. However, to have an explicit formula, we need to to compute the measures appearing in the formula. For real groups, the Plancherel measures are completely understood via good knowledge of discrete series, characters and orbital integrals. However, analogous tools for $p$-adic cases are not well understood. Complete Plancherel formulas have been found for relatively simple cases such as some of groups of type $A_n$ [Mat, Sh, JKM, SS, AKM].

There have been two major approaches in finding Plancherel measures: F. Shahidi found an expression of Plancherel measures as a product of local root numbers in his work on Langlands conjecture. In particular, he could find all Plancherel measures on $\text{GL}_n$ in this way. It has been also known that by the work of Howe-Moy and Bushnell-Kutzko, Plancherel measures can be computed via Hecke algebra isomorphisms. This method is especially working effectively for computing formal degrees of discrete series.

Plancherel measures have also been computed for some representations: spherical representations [Mac], anti-spherical representations [HO], representations induced from rank one parabolic [Sh, AKM] and some of unipotent representations [Re2]. In case of spherical representations, the problem can be reduced to computing Plancherel measures on polynomial algebras which are abelian. F. Shahidi found Plancherel measures for representations induced from supercuspidal representations of rank one parabolic subgroups with his method. For those representations, their associated Hecke algebras are of rank one whose Plancherel measures are computed [Mat, AKM]. Combining above methods, M. Reeder [Re2] computed Plancherel measures for unipotent representations coming from irreducible principal series representations of associated Hecke algebras.

However, collecting all these is not enough to have complete Plancherel formula for $\text{Sp}_4$. Since complete Plancherel formula on Hecke algebras of higher rank is not available,
some of Iwahori-spherical representations in the support Plancherel measures cannot be
treated via Hecke algebra isomorphisms. Those are parabolically induced from non
supercuspidal discrete series of proper maximal Levi subgroups. For those cases, we will use
intertwining operators (see [Sh, Jan]) to compute Plancherel measures. As a corollary,
we get Plancherel formula on Iwahori Hecke algebras of $\text{Sp}_4$ of rank 2. There is one more
case that we run into Hecke algebras of rank 2. Again, we use intertwining operators for
those.

For the rest of cases, we will compute Plancherel measures via Hecke algebra isomor-
phisms using Plancherel formula on rank one Hecke algebras [AKM, Mat]. Implicit in
[My] and [J], we know the admissible dual of $\text{Sp}_4$. They are explicitly constructed in
[Kil] and Hecke algebra isomorphisms were established in [Ki2, My]. Other than Iwahori
Hecke algebras, the Hecke algebras occurring for $\text{Sp}_4$ are at most rank one. Since Hecke
algebra isomorphisms preserve Plancherel measures up to normalization and we know
Plancherel formula for these cases [AKM, Mat], we will transfer Plancherel measures on
Hecke algebras to our cases. Computation can be treated case by case based on the
classification of the unitary dual of $\text{Sp}_4$ given by P. Sally and M. Tadic [ST].

Throughout this paper, we assume that the residue characteristic of $k$ is odd. In Sec-
tion I, we give preliminaries on Harish-Chandra’s Plancherel theorem. We also realize
$\text{Sp}_4(k)$ to fit for our purpose and compute basic constants $c$- and $\gamma$-factors which appears
in Plancherel formula (see 1.2 for definition). In §I3, we describe two basic methods that
we use to compute $\mu$-factors.

In Section II, we compute Plancherel measures for principal series. We use the types
and Hecke algebras constructed by A. Roche [Ro]. Based on the classification of unitary
dual of $\text{Sp}_4(k)$ in [ST], we deal each case via Hecke algebra isomorphisms except when the
associated Hecke algebras are of rank 2 (for example, Iwahori spherical representations).
In those cases, we use intertwining operators to compute $\mu$-factors. As a corollary, we
get Plancherel formula on the Iwahori Hecke algebra of type $C_2$.

In the beginning of Section III, we summarize the construction of types in [Kil] and
specialize it to $G = \text{Sp}_4$. In the rest of Section III, we also compute formal degrees
of supercuspidal representations. Lastly, in Section IV, we find Plancherel measures of
generalized principal series induced from maximal parabolic subgroup. Since the con-
struction in [Kil] is valid only when $\frac{1}{4} > \frac{2\text{ord}_p \mu}{p-1} + \frac{1}{p-1}$, the results in III and IV are
valid under same assumption.

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I. Preliminaries

§1. Plancherel Formula [HC, Si, Walds]

1.1. Let $k$ be a $p$-adic field and let $G$ be the group of the rational $k$-points of a
connected reductive algebraic group $G$ defined over $k$. Let $M_0$ be a fixed $k$-rational
Levi component of some minimal parabolic subgroup $P_0$ defined over $k$. Any parabolic
subgroup $P$ defined over $k$ of $G$ which contains $M_0$ has a unique Levi component $M =$
\( M \) which contains \( M_0 \). Both \( M \) and the unipotent radical \( N = N_\mathcal{P} \) of \( P \) are defined over \( k \). We will denote respectively by \( M, N \) and \( P \) the \( k \)-points of \( M, N \) and \( P \). For such \( M \), we denote by \( A_M \) the split component of its center. Hence \( M = Z_G(A_M) \). Let \( A_G \) be the group of \( k \)-points of \( A_G \) and let \( W \) be the Weyl group of \( G \) with respect to \( A_G \), that is, \( W = N_G(A_G)/Z_G(A_G) \), where \( N_G(A_G) \) and \( Z_G(A_G) \) denote the normalizer and the centralizer of \( A_G \) in \( G \) respectively.

1.2. \( \gamma \) and \( c \)-factors appearing in Harish-Chandra’s Plancherel formula are defined here. Let \( K_0 \) be a maximal special parahoric subgroup of \( G \) such that \( K_0 \) and \( M_0 \) are in good relative position. We set

\[
\gamma(G/P) := \gamma(G, K_0) = \int_{N} \delta_p(\overline{n})^{-1} d\overline{n}
\]

where \( N \) denotes the negative unipotent radical of \( P \) (i.e., containing the negatives of the roots in \( N \)), where \( \delta_p \) is the extension of the character \( \delta \) of \( P \) defined by \( \delta(p) := |\det \text{Ad}(p)||\text{Lie}(N)| \) to \( G = K_0 P \) by left \( K_0 \)-invariance and where \( d\overline{n} \) is the Haar measure on \( N \) assigning volume 1 to \( N \cap K_0 \). Note that

\[
\gamma(G/P) = \int_{N} \delta_{\mathcal{P}}(\overline{n}) d\overline{n}
\]

where \( d_{\mathcal{P}} \) is the right \( K_0 \)-invariant extension of \( \delta \).

To define \( c(G/P) \), we first note that \( A_0 := A_{M_0} \) is a maximal \( k \)-split torus in \( G \). Let \( \Delta \) be the set of roots of \( A_0 \) in \( G \) and \( \Pi \) be the subset of the simple roots in \( \Delta \). We denote by \( \Pi^+ \) the positive roots in \( \Pi \). Let \( \alpha \) be a reduced root of \( A_M \) in \( G \). Let \( A_\alpha \) denote the maximal torus contained in the kernel of \( \alpha \) and let \( M_\alpha \) be the centralizer of \( A_\alpha \) in \( G \). We set \( \gamma_\alpha := \gamma(M_\alpha, K_0 \cap M_\alpha) \). It is clear that \( \gamma_{-\alpha} = \gamma_\alpha \). Then

\[
c(G/P) := c(G, K_0, A_0) = \gamma(G/P)^{-1} \prod_{\alpha \in \Delta_\mathcal{P}} \gamma_\alpha,
\]

where \( \Delta_\mathcal{P} \) is the set of all reduced roots of \( (P, A_0) \).

1.3. Harish-Chandra’s Plancherel formula. Let \( \mathcal{E}_2(M) \) be the set of equivalence classes of irreducible square integrable mod center representations of \( M \). For \( f \) in the Schwartz space on \( G \), define

\[
f_M(g) := c(G/P)^{-2} \gamma(G/P)^{-1} W_M^{-1} \int_{\mathcal{E}_2(M)} \mu(\omega) d(\omega)(\Theta_\omega, r(g)f) d\omega, \quad g \in G.
\]

Here, \( r(g) \) denotes the right translation by \( g \), \( d(\omega) \) is the formal degree of \( \omega \in \mathcal{E}_2(M) \), \( \mu(\omega) \) is Plancherel measure on the series of unitary induced representations \( \text{Ind}_G^P(\omega) \) of \( G \) and \( \Theta_\omega \) is the character of \( \text{Ind}_G^P(\omega) \). \( W_M \) is \( N_G(A_M)/M \), the Weyl group associated to \( A_M \). Then Harish-Chandra Plancherel Theorem states that

\[
f = \sum_{M=3}^3 f_M
\]
where $M$ runs over the set of Levi subgroups in $G$ up to conjugacy. The set $\mathcal{E}(M)$ consists of a countable union of components where each component is a compact torus and the measure restricts to a normalized Haar measure on each component. More precisely, we recall the complex structure on the set $\mathcal{E}(M)$ of smooth irreducible representations of $M$. Let $X(M)$ and $X(A_M)$ be the groups of $k$-rational characters of $M$ and $A_M$ respectively. Then we have an injective homomorphism $X(A_M) \to X(M)$. Let $a = \text{Hom}(X(A_M), \mathbb{R}) = \text{Hom}(X(M), \mathbb{R})$ be the real Lie algebra of $A_M$ and let $H_P : M \to \text{Hom}(X(M), \mathbb{R})$ be the mapping determined by $|\chi(m)| = q^{\langle \chi, H_P(m) \rangle}$ for $m \in M$. Let $a^* = X(A_M) \otimes \mathbb{R}$ and $a^*_C = X(A_M) \otimes \mathbb{C}$. Then $\nu \in a^*_C$ defines a quasiclassical character $\chi_{\nu}$ of $M$ by $\chi_{\nu}(m) = q^{\langle \nu, H_P(m) \rangle}$ and $a^*_C$ acts on $\mathcal{E}(M)$ in an obvious way. Now, $\mathcal{E}_2(M)$ splits into the orbit under the action of unitary characters and each orbit is homeomorphic to a product of $S^1$'s.

§2. $\text{Sp}_4(k)$

In this section, we realize $\text{Sp}_4(k)$ in a convenient form for our computation. We also compute $\gamma(G/P)$ and $c(G/P)$-factors for each case. Similar description can be found in [My].

2.1. Let $k$ be a $p$-adic field as before and let $\mathbb{F}_q$ be the residue field of $k$. Let $\mathcal{O}_k$ be the ring of integers and let $\mathfrak{p}_k$ the prime ideal in $\mathcal{O}_k$ with the prime element $\varpi_k$. From now on, let $G$ be the algebraic group $\text{Sp}_4$ and let $G = \text{Sp}_4(k)$ be the group of $k$-rational points of $G$. We will realize $G$ as a subgroup of $\text{GL}_4(k)$ which consists of elements preserving the symplectic form on $k^4$ defined by

$$J = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix},$$

that is, $\langle x, y \rangle = xJ^t y$. Note that $G$ acts on $V$ from the left. Then we have the maximal $k$-split torus $A_0$ consisting of diagonal elements in $G$ and the Borel subgroup consisting of upper triangular matrices in $G$. Now, we adopt the notation defined in (1.2) (e.g. $\Pi, \Pi^+$) to our case. Then the Weyl group $W = N_G(A_0)/A_0$ is generated by the images in $W$ of

$$s_1 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad s_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$

Let $\alpha_i$ be the positive simple root corresponding to $s_i$, $i = 1, 2$ and let $\Pi = \{\alpha_1, \alpha_2\}$ be the set of positive simple roots and $\Delta^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$. For any subset $I$ of $\{s_1, s_2\}$, let $P_I$ be the standard parabolic subgroup generated by $B$ and $I$. Then we
can write
\[ P_\emptyset = B, \quad P_{\{s_1, s_2\}} = G; \]
\[ P_s := P_{\{s_1\}} = \left\{ \begin{pmatrix} A & \ast \\ w_0^t A^{-1} w_0^{-1} \end{pmatrix} : A \in \text{GL}_2(k) \right\}, \quad w_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]
\[ P_h := P_{\{s_2\}} = \left\{ \begin{pmatrix} a & \ast \\ \ast & a^{-1} \end{pmatrix} : A \in \text{Sp}_2(k) \right\}. \]

For \( \Phi \subset \Pi \), let \( P_\Phi \) be the standard parabolic subgroup associated to \( \Phi \). Then we have
\[ G = P_{\{s_1, s_2\}} = P_{\{\alpha_1, \alpha_2\}}, \quad P_s = P_{\{s_1\}} = P_{\{\alpha_1\}}, \quad P_h = P_{\{s_2\}} = P_{\{\alpha_2\}}. \]

In either case, we have \( P_\emptyset = B \). For each case, we have the Levi decomposition \( M_I N_I \) of \( P_I \) for \( I \subset \{s_1, s_2\} \) or \( \Pi \) such that its Levi subgroup \( M_I \) consisting of diagonal block matrices. That is,
\[ M_\emptyset = A_0 \simeq \text{GL}_1(k) \times \text{GL}_1(k), \]
\[ M_s = \left\{ \begin{pmatrix} A & 0 \\ 0 & w_0^t A^{-1} w_0^{-1} \end{pmatrix} : A \in \text{GL}_2(k) \right\} \simeq \text{GL}_2(k) \times \text{Sp}_0(k) \]
\[ M_h = \left\{ \begin{pmatrix} a & \ast \\ \ast & a^{-1} \end{pmatrix} : A \in \text{Sp}_2(k) \right\} \simeq \text{GL}_1(k) \times \text{Sp}_2(k). \]

Note that any Levi subgroup \( M \) can be written as \( \left( \prod_{i=1}^{\ell} \text{GL}_{n_i}(k) \right) \times \text{Sp}_{2m}(k) \) with \( \sum_{i=1}^{\ell} n_i + m = 2 \). For \( \tau_i \in (\text{GL}_{n_i}(k))^\circ \) and \( \sigma \in (\text{Sp}_{2m}(k))^\circ \), let \((\otimes_i \tau_i) \times \sigma\) be the (normalized) parabolically induced representation from \((\otimes_i \tau_i) \otimes \sigma\) as in [ST]. That is,
\[ (\tau_1 \times \cdots \times \tau_\ell) \times \sigma = \text{Ind}_P^G((\otimes_i \tau_i) \otimes \sigma). \]

Let \( K_0 = G(O_k) \) be the special maximal compact group and \( B_0 \) be the Iwahori subgroup consisting of elements in \( K_0 \) which are upper triangular mod 1. The affine Weyl group
\[ W^{\text{aff}} = N_G(A_0)/A_0(O_k) \]
is generated by the images in \( W^{\text{aff}} \) of \( s_1, s_2 \) and
\[ s_0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \]

Then we can define a length function \( l : W^{\text{aff}} \rightarrow \mathbb{Z}^+ \) such that \([B_0 w B_0 : B_0] = q^{l(w)}\). For \( I \subset \{s_0, s_1, s_2\} \), let \( P_I \) be the parahoric subgroup generated by \( B_0 \) and \( I \). Then parahoric subgroups containing \( B_0 \) in \( G \) are as follows:
\[ B_0 := P_\emptyset, \quad P_0 := P_{\{s_0\}}, \quad P_s := P_{\{s_1\}}, \quad P_h := P_{\{s_2\}} \]
\[ K_0 := P_{\{s_1, s_2\}}, \quad K_1 := P_{\{s_0, s_2\}}, \quad K_2 := P_{\{s_0, s_1\}}. \]
Corresponding to each affine root \((\alpha, m)\) with \(\alpha \in \Delta\) and \(m \in \mathbb{Z}\), we can find a root subgroup as follows:

\[
U_{\alpha_1,m} := \left\{ x_{\alpha_1}(u) = \begin{pmatrix} 1 & u & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -u \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad u \in \omega^m \mathfrak{O}_k \right\},
\]

\[
U_{\alpha_2,m} := \left\{ x_{\alpha_2}(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & u & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad u \in \omega^m \mathfrak{O}_k \right\},
\]

\[
U_{\alpha_1+\alpha_2,m} := \left\{ x_{\alpha_1+\alpha_2}(u) = \begin{pmatrix} 1 & 0 & u & 0 \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad u \in \omega^m \mathfrak{O}_k \right\},
\]

\[
U_{2\alpha_1+\alpha_2,m} := \left\{ x_{2\alpha_1+\alpha_2}(u) = \begin{pmatrix} 1 & 0 & 0 & u \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad u \in \omega^m \mathfrak{O}_k \right\},
\]

and \(U_{\alpha,m} := U_{-\alpha,m}\) if \(\alpha \in \Delta^-\), where \(\Delta^-\) denotes the set of negative roots in \(\Delta\). From now on, we normalize the Haar measure \(dg\) on \(G\) such that the volume of \(K_0\) is 1. Then we can compute the volumes of parahoric subgroups as follows:

\[
v(B_0) = \frac{1}{(1+q^2)(1+q)^2},
\]

\[
v(P_0) = v(P_s) = v(P_h) = \frac{1}{(1+q^2)(1+q)},
\]

\[
v(K_0) = v(K_1) = 1, \quad v(K_2) = \frac{1}{1+q^2}.
\]

2.2. \(\gamma(G/P)\) and \(c(G/P)\). Since all parabolics in \(G\) are self opposed and parameters occurring in its Iwahori Hecke algebra are equal, to find \(\gamma(G/P)\), we may either use the formula in Prop 11.1 [Re2] or can find it by evaluating Poincaré polynomial of the flag manifold \(LG/LP\) at \(q^{-1}\) where \(L\) denotes Langlands dual. Then we have

\[
\gamma(G/P_s) = \frac{1-q^{-2} 1-q^{-3} 1-q^{-4}}{1-q^{-1} 1-q^{-2} 1-q^{-3}} = (1+q^{-1})(1+q^{-2}),
\]

\[
\gamma(G/P_h) = \frac{1-q^{-2} 1-q^{-3} 1-q^{-4}}{1-q^{-1} 1-q^{-2} 1-q^{-3}} = (1+q^{-1})(1+q^{-2}),
\]

\[
\gamma(G/B) = \frac{1-q^{-2} 1-q^{-2} 1-q^{-3} 1-q^{-4}}{1-q^{-1} 1-q^{-2} 1-q^{-3}} = (1+q^{-1})^2(1+q^{-2}),
\]

\[
c(G/P_s) = 1, \quad c(G/P_h) = 1, \quad c(G/B) = \gamma(G/B)^{-1}(1+q^{-1})^4.
\]
§3. Main Methods

§3.1. Plancherel measures via Hecke algebra isomorphisms [Ku]

We will use results from [Ku] to pass from the group side to the Hecke algebra side or vice versa.

3.1.1. Let \( J \) be an arbitrary open compact subgroup of \( G \) and let \( (\rho, V_\rho) \) be an irreducible representation of \( J \). The Hecke algebra (or spherical function algebra) \( \mathcal{H}(G//J, \rho) \) of \( G \) with respect to \( (J, \rho) \) consists of the compactly supported functions \( f : G \rightarrow \text{End}(V_\rho) \) such that

\[
f(jgj') = \rho(j)f(g)\rho(j'), \quad \text{for } j, j' \in J.
\]

Then \( \mathcal{H}(G//J, \rho) \) is an algebra under the convolution with respect to the Haar measure \( dg \) on \( G \):

\[
f_1 * f_2(g) := \int_G f_1(x)f_2(x^{-1}g)dx \quad \text{for } f_1, f_2 \in \mathcal{H}(G//J, \rho)
\]

. If \( \rho \) is unitary, then \( \mathcal{H}(G//J, \rho) \) possesses a natural involutive \(*\)-operation defined by

\[
f^*(g) := f(g^{-1})^*, \quad \text{for } f_1, f_2 \in \mathcal{H}(G//J, \rho)
\]

where \( * \) on the right side is the adjoint operation on \( \text{End}(V_\rho) \). We may then make the algebra \( \mathcal{H}(G//J, \rho) \) into a standard Hilbert algebra by setting [Ku 2.3(2)]

\[
\langle f_1, f_2 \rangle := \frac{v(J)}{\dim \rho} \text{tr}(f_1 * f_2^*(1)), \quad f_1, f_2 \in \mathcal{H}(G//J, \rho)
\]

where \( v(J) \) is the volume of \( J \). We recall that a Hilbert algebra (see [D, A 54]) is an involutory algebra which is at the same time a pre-Hilbert space. We call it standard if it has an identity \( 1_A \), the scalar product \( \langle , \rangle \) is definite and \( \langle 1_A, 1_A \rangle = 1 \). The contragredient representation \( (\hat{\rho}, V_{\hat{\rho}}) \) of \( (\rho, V_\rho) \) is unitary, since \( (\rho, V_\rho) \) is and so \( \text{End}_\mathbb{C}(V_\rho) \) is naturally a standard Hilbert algebra with the involution given by

\[
\langle u^* v_1, v_2 \rangle = \langle v_1, u v_2 \rangle, \quad u \in \text{End}_\mathbb{C}(V_\rho), \quad v_1, v_2 \in V_\rho,
\]

and the scalar product given by

\[
\langle u_1, u_2 \rangle := \frac{1}{\dim \rho} \text{tr}(u_1 u_2^*), \quad u_1, u_2 \in \text{End}_\mathbb{C}(V_\rho).
\]

We then may make the algebra \( \mathcal{H}(G//J, \rho) \otimes_\mathbb{C} \text{End}(V_\rho) \) into a standard Hilbert algebra by setting

\[
(f \otimes u)^* := f^* \otimes u^*, \quad f \in \mathcal{H}(G//J, \rho), \quad u \in \text{End}_\mathbb{C}(V_\rho).
\]

We write \( \mathcal{H}(G) \) for the convolution algebra consisting of smooth, compactly supported functions \( G \rightarrow \mathbb{C} \). Let \( \hat{\nu} \) denote the Plancherel measure on the reduced dual \( \hat{G}_r \) of
G, that is, the unique positive Borel measure on $\hat{G}_r$ (see [D, 18.8.1]) such that for all functions $f \in \mathcal{H}(G)$, we have

$$f(1_G) = \int_{\hat{G}_r} \text{tr}_H(f) d\tilde{\nu}(H).$$

We define an idempotent $e_\rho \in \mathcal{H}(G)$ [BK 4.2.1] by

$$e_\rho(g) := \begin{cases} \dim_{\mathbb{C}} \text{tr}(\rho(g^{-1})) & \text{if } \rho \in J, \\ 0 & \text{otherwise.} \end{cases}$$

Then the space $e_\rho \ast \mathcal{H}(G) \ast e_\rho$ is therefore a subalgebra of $\mathcal{H}(G)$ and it has a unit element namely $e_\rho$. Note that $\mathcal{H}(G)$ is a dense subalgebra of the full $C^*$-algebra of $G$, $C^*(G)$ (with the same involution). The algebra $e_\rho \ast \mathcal{H}(G) \ast e_\rho$ is a standard Hilbert algebra with respect to the scalar product given by

$$\langle f_1, f_2 \rangle := \frac{1}{e_\rho(1)} f_1 \ast f_2^* (1_G), \quad f_1, f_2 \in e_\rho \ast \mathcal{H}(G) \ast e_\rho.$$

### 3.1.2. Let $A_\rho$ be the closure of $\mathcal{H}(G) \ast e_\rho \ast \mathcal{H}(G)$ in the reduced $C^*$-algebra $C^*_r(G)$ of $G$. Then $A_\rho$ is a closed two-sided ideal in $C^*_r(G)$ and it is also a liminal $C^*$-algebra [P11, P12]. We set

$$\hat{G}_r^\rho := \{H \in \hat{G}_r \mid A_{\rho}H \neq 0\}.$$

Then [D, 3.2.1], $\hat{G}_r^\rho$ is an open subset of $\hat{G}_r$ and restriction induces a homeomorphism of $\hat{G}_r^\rho$ with $\hat{A}_\rho$, the set of equivalence classes of irreducible non-degenerate left $A_\rho$-modules. Identifying $\hat{G}_r^\rho$ with $\hat{A}_\rho$ via this homeomorphism, let $\hat{\nu}^\rho$ be the restriction of $\hat{\nu}$ to $\hat{A}_\rho$.

We note that $e_\rho \ast A_\rho \ast e_\rho$ is a sub-$C^*$-algebra of $A_\rho$ which is, in addition, unital. Let $H$ be an irreducible left $A_\rho$-module. Then the closed subspace $e_\rho H$ of $H$ is stable under $e_\rho \ast A_\rho \ast e_\rho$ and hence has a structure of $(e_\rho \ast A_\rho \ast e_\rho)$-module, and the map $H \mapsto He_\rho$ induces an homeomorphism of $\hat{A}_\rho$ with $(e_\rho \ast A_\rho \ast e_\rho)^\sim$. For any subset $S$ of $\hat{A}_\rho$, set

$$S_\rho := \{He_\rho : H \in S\},$$

and define a Borel measure $\tilde{\nu}_{e_\rho}$ on $(e_\rho \ast A_\rho \ast e_\rho)^\sim$ by $\tilde{\nu}_{e_\rho}(S_\rho) := \tilde{\nu}^\rho(S)$ where $S$ is any Borel set in $\hat{A}_\rho$.

Let $C^*_r(e_\rho \ast \mathcal{H}(G) \ast e_\rho)$ denote the closure of the image of $e_\rho \ast \mathcal{H}(G) \ast e_\rho$ in the algebra $\mathcal{L}(e_\rho \ast L^2(G) \ast e_\rho)$ of bounded operators on the completion of $e_\rho \ast L^2(G) \ast e_\rho$. It is isomorphic with $e_\rho \ast A_\rho \ast e_\rho$ [Ku, Corollary 1.9]. Let $H$ be an irreducible left $A_\rho$-module. Then the left $(e_\rho \ast \mathcal{H}(G) \ast e_\rho)$-module structure on $He_\rho$ may be extended in a unique way to make $He_\rho$ into a left $C^*_r(e_\rho \ast \mathcal{H}(G) \ast e_\rho)$-module and the map $H \mapsto He_\rho$ induces a homeomorphism of $\hat{A}_\rho$ with $(C^*_r(e_\rho \ast \mathcal{H}(G) \ast e_\rho))^\sim$. We then may define a Borel measure, still denoted by $\tilde{\nu}_{e_\rho}$, on $(C^*_r(e_\rho \ast \mathcal{H}(G) \ast e_\rho))^\sim$ in the analogous way as for $(e_\rho \ast A_\rho \ast e_\rho)^\sim$.

For any standard Hilbert algebra $A$, let $H(A)$ denote its completion with respect to scalar product. The left regular $A$-module structure on $A$ extends to make $H(A)$ into a
faithful left $A$-module. Identifying $A$ with its image in $\mathcal{L}(H(A))$ and taking its closure, we obtain a $C^*$-algebra which we denote by $C^*_r(A)$.

There is an isomorphism of Hilbert algebras from $\mathcal{H}(G//J, \rho) \otimes \mathcal{C} \operatorname{End}(V_\rho)$ to $e_\rho * \mathcal{H}(G) * e_\rho$ (see [BK, 4.3.3]), which induces a natural isomorphism of $C^*$-algebras (see [Ku, Corollary 2.5]) from $C^*_r(\mathcal{H}(G//J, \rho) \otimes \mathcal{C} \operatorname{End}(V_\rho))$ to $C^*_r(\mathcal{H}(G//J, \rho) \otimes \mathcal{C} \operatorname{End}(V_\rho))$. Hence we may transfer the measure $\tilde{\nu}_{e_\rho}$, and we get a (positive) Borel measure, still denoted by $\tilde{\nu}_{e_\rho}$, on $(C^*_r(\mathcal{H}(G//J, \rho) \otimes \mathcal{C} \operatorname{End}(V_\rho)))^\sim$. Note that the map $H \mapsto H \otimes V_\rho$ induces a homeomorphism of $(C^*_r(\mathcal{H}(G//J, \rho)))^\sim$ with $(C^*_r(\mathcal{H}(G//J, \rho)) \otimes \mathcal{C} \operatorname{End}(V_\rho)))^\sim$. For $S \subset (C^*_r(\mathcal{H}(G//J, \rho)))^\sim$, set

$$S^\rho := \{S \otimes V_\rho : H \in S\} \subset (C^*_r(\mathcal{H}(G//J, \rho) \otimes \mathcal{C} \operatorname{End}(V_\rho)))^\sim,$$

and, for any Borel subset $S$ of $(C^*_r(\mathcal{H}(G//J, \rho)))^\sim$, set

$$\tilde{\nu}_\rho(S) := \langle \dim(\rho) \tilde{\nu}_{e_\rho}(S^\rho).$$

Then $\tilde{\nu}_\rho$ is Plancherel measure on $(C^*_r(\mathcal{H}(G//J, \rho)))^\sim$, that is, the unique positive Borel measure $\tilde{\nu}_\rho$ with the property that for all $f \in \mathcal{H}(G//J, \rho)$,

$$\langle f, \mathcal{H}(G//J, \rho) \rangle = \int_{(C^*_r(\mathcal{H}(G//J, \rho)))^\sim} \operatorname{tr}_H(f) d\tilde{\nu}_\rho(H).$$

For $H \in \hat{A}_\rho$, let $H_\rho := \operatorname{Hom}_J(V_\rho, H)$. Then the map $H \mapsto H_\rho$ from $\hat{A}_\rho$ to $(C^*_r(\mathcal{H}(G) * e_\rho * \mathcal{H}(G)))^\sim$ is a composition of the various homeomorphisms given above. It is therefore a homeomorphism.

**Theorem.** [Ku] For any subset $S \subset \hat{A}_\rho$ let $S_\rho = \{H_\rho : H \in S\}$. For any Haar measure $\nu$ on $G$, let $\hat{\nu}$ be Plancherel measure on $\hat{G}_\tau$. Then for all Borel subsets $S \subset \hat{A}_\rho$, we have

$$\hat{\nu}_\rho(S_\rho) = \frac{\nu(J)}{\dim(\rho)} \hat{\nu}(S)$$

where $\hat{\nu}_\rho$ is Plancherel measure on $(C^*_r(\mathcal{H}(G//J, \rho)))^\sim$.

**Corollary.** [Ku] Suppose that $G_i$, $i = 1, 2$, are reductive groups over $k$, that $J_i$ is a compact open subgroup of $G_i$ and the $(\rho_i, V_i)$ is an irreducible representation of $J_i$. Fix Haar measure $\nu_i$ on $G_i$ and suppose that there is an isomorphism of Hilbert algebras

$$j : \mathcal{H}(G_1//J_1, \rho_1) \cong \mathcal{H}(G_2//J_2, \rho_2).$$

Let $\hat{G}_i(\rho_i)$ be the subset of $\hat{G}_i$ consisting of representations $(\pi, H)$ with $\pi(\rho_i)H \neq 0$. Then $\hat{G}_i(\rho_i)$ is open in $\hat{G}$ and $j$ induces a homeomorphism $\hat{j} : \hat{G}_2(\rho_2) \to \hat{G}_1(\rho_1)$. Let $\hat{\nu}_i$ be the restriction to $\hat{G}_i(\rho_i)$ of Plancherel measure on $\hat{G}_i$. Then for all Borel sets $S \subset \hat{G}_2(\rho_2)$, we have

$$\frac{\nu_1(J_1)}{\dim(\rho_1)} \hat{\nu}_1(j^*(S)) = \frac{\nu_2(J_2)}{\dim(\rho_2)} \hat{\nu}_2(S).$$

Analogous formula holds for the formal degrees of discrete series, see [BK 7.7.11] and [Re, Prop. 9.1]. Formal degrees for some supercuspidal representations of symplectic groups have been obtained by Kariyama [Ka, Theorem 4.15].
§3.2. Plancherel measures via Intertwining operators

It is known [HC, Sh] that Plancherel $\mu$-factors can be computed by analyzing standard intertwining operators. Since we do not always have a good information of Plancherel measures of higher rank Hecke algebras other than rank one cases, for those cases rather than transferring Plancherel measures from Hecke algebra side, we need to compute Plancherel measures from intertwining operators.

For $\Phi \subseteq \Pi$, let $P_\Phi = M_\Phi N_\Phi$ be a standard parabolic associated to $\Phi$ as before. Let $W(\Phi) = \{ w \in W \mid w \cdot \Phi = \Phi \}$. For a representation $\sigma$ of $M = M_\Phi$, let $I_M(\sigma) = \text{Ind}_P^G \sigma$ be the normalized parabolically induced representation. Then, the standard intertwining operator $A_w$ for $w \in W$ is formally defined as

$$A_w(\sigma) f(g) = \int_{N_w} f(w^{-1}ng) \, dn$$

for $f \in I_M(\sigma)$ and $N_w = N_\Phi \cap w^{-1}\overline{N}_\Phi w$ where $\overline{N}_\Phi$ is the opposite unipotent radical of $N_\Phi$. The intertwining operator $A_w(\sigma)$ converges under suitable conditions on the exponent associated to $\sigma$ and has meromorphic continuation. It intertwines $I_M(\sigma)$ and $I_M(w\sigma)$. Moreover, they have the following properties [Jan, Sh]:

1. If $w_1, w_2 \in W_\Phi$ with $l(w_1 w_2) = l(w_1) + l(w_2)$,
   $$A_{w_1 w_2}(\sigma) = A_{w_1}(w_2 \sigma) A_{w_2}(\sigma).$$

2. If $\sigma \hookrightarrow \text{Ind}_{A_0}^M(\chi)$, then $I_M(\sigma) \hookrightarrow I_{A_0}(\chi)$ and for $w \in W(\Phi)$,
   $$A_w(\sigma) = A_w(\chi)|_{I_M(\sigma)}.$$

3. If $w_\Phi$ is the longest element in $W(\Phi)$, $\mu(\sigma)$ satisfies
   $$A_{w_\Phi}(\sigma) A_{w_\Phi}^{-1}(w_\Phi \cdot \sigma) = \mu(\sigma)^{-1} r(G/P)^2.$$

II. Plancherel measures on Principal series

These cases correspond to $\Sigma = (\Gamma, B_0, \varrho)$ where $\Gamma$ is a semisimple element in a split torus $\mathfrak{a}_0 := \text{Lie}(A_0)$ in the Lie algebra $\mathfrak{g}$ of $G$ and $\varrho$ is a tamely ramified character (i.e., a character which factors through the maximal pro-$p$ subgroup of $B_0$) in terms of the data in [Ki1] (see also Chapter III). However, to avoid the restriction on the residue characteristic, we will use principal types constructed by A. Roche [Ro]. His constructions (and hence the results in this section) are valid if the residue characteristic of $k$ is odd.

0.1. Recalling his construction, let

$$(*) \quad \chi = \chi_1 \otimes \chi_2$$
be a character of $A_0 = A_0(\mathcal{O}_k)$. For a character $\xi$ of $\mathcal{O}_k^\times$, denote its conductor by $c(\xi)$. In particular, for each $\alpha \in \Delta$, $c(\chi \circ \alpha^\vee)$ is the conductor of the character $\chi \circ \alpha^\vee$ of $\mathcal{O}_k^\times$. Define ([Ro, Definition 3.3]) also

$$f_\chi(\alpha) = \begin{cases} \frac{c(\chi \circ \alpha^\vee)}{2} & \text{for } \alpha \in \Delta^+ \\ \frac{c(\chi \circ \alpha^\vee)}{2} & \text{for } \alpha \in \Delta^- . \end{cases}$$

Let $J_\chi$ be the open compact subgroup generated by $A_0$ and $U_{\alpha, f_\chi(\alpha)}$, $\alpha \in \Delta$. Here, $[x]$ (resp. $[x]$) denotes the greatest integer not greater than $x$ (resp. the least integer not less than $x$).

0.2. By analyzing intertwining operators, unitary tempered representations supported on the minimal parabolic subgroups are classified in [ST]. We list those (other than discrete series) from [ST, Theorem 5.2] below. We also refer to [ST] for notations:

**Theorem** [ST]. Let $\eta, \eta_1, \eta_2 \in (k^\times)^-$ be unitary characters and let $\xi_1, \xi_2, \xi_3$ be characters of $k^\times$ of order two.

1. $(\eta_1 \times \eta_2) \rtimes 1$ is irreducible if neither $\eta_1$ nor $\eta_2$ is of order two.
2. Write $\xi \rtimes 1 = T_1^1 + T_2^2$ as a sum of irreducible representations of $\text{Sp}_2(k)$. Suppose either $\xi = \eta$ or $\eta$ is not of order two. Then $\eta \rtimes T_1^1$ and $\eta \rtimes T_2^2$ are irreducible representations which are not equivalent.
3. If $\xi_1 \neq \xi_2$, then the representation $\xi_1 \rtimes \xi_2 \rtimes 1$ is a multiplicity one representation of length four.
4. Suppose that $\eta$ is not of order two and $\eta \neq 1$. Then $\eta \rtimes St_{\text{Sp}_2}$ is irreducible.
5. Suppose $\xi^2 = 1$. Then $\xi \rtimes St_{\text{Sp}_2}$ is a sum of two representations which are not equivalent.
6. Representations $\eta St_{\text{GL}_2} \rtimes 1$ are irreducible.

The representations in (1)-(6) are disjoint. They consist of non-square integrable reducible tempered representations of $\text{Sp}_4(k)$ which are supported in the minimal parabolic subgroups.

The Steinberg representation $St_{\text{Sp}_4}$ and the unique irreducible subrepresentation of $\xi \nu_1 \rtimes T_1^1$ for $i = 1, 2$, where $\xi$ runs over all the characters of $k^\times$ of order two, are square integrable representations. They are inequivalent and they exhaust all the square integrable representations of $\text{Sp}_4(k)$ which are supported on the minimal parabolic subgroup [ST, Theorem 5.1].

### §1. Plancherel measures of Principal series: Rank 2 Hecke algebras

As is mentioned in §I.3.2, since we do not have a good information of Plancherel measures of rank two Hecke algebras, we will compute Plancherel measures from intertwining operators on group side. As a result, we deduce a Plancherel formula on Iwahori Hecke algebras of type $C_2$. We have two cases when we get rank two Hecke algebras: One case is when $\chi_1 = \chi_2 = 1$ and the other case is when $\chi_1 = \chi_2 = sgn$. In §1.1 and §1.2, we will treat each case.
§1.1. Iwahori spherical representations

The Iwahori Hecke algebra $\mathcal{H}(G//B_0) = \mathcal{H}(G//B_0, 1)$ is the Hecke algebra associated to the trivial representation of Iwahori subgroup $B_0$. That is, this case occurs when $\chi_1 = \chi_2 = 1$. We remark that $(J_\chi, \chi) = (B_0, 1)$ corresponds to the case $\Sigma = (0, B_0, 1)$ in [Kil].

The Iwahori Hecke algebra $\mathcal{H}(G//B_0)$ is linearly spanned by $\{f_w \mid w \in W^{\text{aff}}\}$ where $f_w$ is the characteristic function of the double coset $B_0 w B_0$. Algebraically it is generated by $f_{s_i}$, $i = 0, 1, 2$ with relations

1. $f_w * f_{w'} = f_{w w'}$ if $l(w w') = l(w) + l(w')$
2. $f_s * f_s = q f_1 + (q - 1) f_s$

1.1.1. Discrete Series

Since all discrete series with Iwahori fixed vectors (there are three [ST]) are coming from 1-dimensional characters of $\mathcal{H}(G//B_0)$, following Borel’s method [Bo], we can find the formal degrees from special values of Poincaré series [Mac]. The following 3 characters $\phi_{St}, \phi_1, \phi_2$ of $\mathcal{H}(G//B_0)$ give rise to discrete series of $G$:

$$\phi_{St}(f_{s_i}) = -1$$
$$\phi_1(f_{s_0}) = q, \quad \phi_1(f_{s_i}) = -1, \quad i = 1, 2$$
$$\phi_2(f_{s_0}) = q, \quad \phi_2(f_{s_i}) = -1, \quad i = 0, 1$$

Denote $\sigma_i$ the discrete series of $G$ corresponding to the character $\phi_i$ with $i \in \{St, 1, 2\}$ of the Iwahori Hecke algebra. Poincaré series associated to $C_2$ (or $B_2$) are given as follows:

$$\widetilde{W}_{C_2}(t_0, t_1, t_2) = \frac{(1 - t_1)(1 + t_0)(1 + t_2)}{(1 - t_1)(1 - t_1t_0t_2)}$$

Then we can find formal degrees

$$d(\sigma_{St}) = \left(\frac{1}{q} \frac{1}{q^3} \frac{1}{q^2}\right)^{-1} v(B_0)^{-1} = \frac{(q^3 - 1)(q - 1)}{(q^2 + 1)(q + 1)^2 \cdot v(B_0)} = (q^3 - 1)(q - 1)$$

$$d(\sigma_1) = \left(\frac{1}{q} \frac{1}{q^3} \frac{1}{q^2}\right)^{-1} v(B_0)^{-1} = \frac{q(q - 1)^2}{2}$$

$$d(\sigma_2) = \left(\frac{1}{q} \frac{1}{q^3} \frac{1}{q^2}\right)^{-1} v(B_0)^{-1} = \frac{q(q - 1)^2}{2}$$

We remark that these can be also deduced from [Re1] and [My].

1.1.2. Unitary tempered Iwahori spherical principal series

In this section, we start discussing standard intertwining operators after [Jan]. For more details, you may refer to [HC, Sh2].

Let $B = A_0 N_0$ be the Borel subgroup as before. Let $\nu_1$ be the character defined on $k^\times$ as $\nu_1(x) = |x|$ and for $s \in \mathbb{C}$, define a character $\nu_s$ of $k^\times$ as $\nu_s(x) = |x|^s$. For a character $\nu = \nu_{x_1} \otimes \nu_{x_2}$ of $A_0$, denote $\text{Ind}_B^G(\nu)$ by $I(x_1, x_2)$ and $\mathcal{A}_w(\nu)$ by $\mathcal{A}_w(x_1, x_2)$. 

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As is in [Jan], we can just look at the behavior of intertwining operators on Iwahori invariants $I(x_1, x_2)^{B_0}$ of $I(x_1, x_2)$. Then $\dim I(x_1, x_2)^{B_0} = |W| = 8$ for our cases and we can choose the basis $\{f_w \mid w \in W\}$ (see [Jan]), where

$$f_w(g) = \begin{cases} \delta^\kappa \nu(p) & \text{if } g = pwB_0 \\ 0 & \text{otherwise.} \end{cases}$$

Fixing the order of basis in column vectors as $f_1, f_{s_1}, f_{s_2}, f_{s_1s_2}, f_{s_1s_2}, f_{s_1s_2s_1}, f_{s_1s_2s_2}, f_{s_1s_2s_1s_2}$, from [Jan], we can find

$$\mathcal{A}_{s_1}(x_1, x_2) = \begin{bmatrix} \alpha & 1 \\ \frac{1}{q} & \beta \\ & \alpha & 1 \\ & \frac{1}{q} & \beta \\ & & \alpha & 1 \\ & & \frac{1}{q} & \beta \end{bmatrix}$$

with $\alpha = \left(1 - \frac{1}{q}\right)\frac{q^{x_1-x_2}}{1-q^{x_1-x_2}}$, $\beta = \left(1 - \frac{1}{q}\right)\frac{1}{1-q^{x_1-x_2}}$.

$$\mathcal{A}_{s_2}(x_1, x_2) = \begin{bmatrix} \alpha & 1 \\ \frac{1}{q} & \beta \\ & \alpha & 1 \\ & \frac{1}{q} & \beta \\ & & \alpha & 1 \\ & & \frac{1}{q} & \beta \end{bmatrix}$$

with $\alpha = \left(1 - \frac{1}{q}\right)\frac{q^{x_1-x_2}}{1-q^{x_1-x_2}}$, $\beta = \left(1 - \frac{1}{q}\right)\frac{1}{1-q^{x_1-x_2}}$.

Then note that the Steinberg representation over the residue field $\mathbb{F}_q$ is generated by

$$v_{\text{St}}(\nu) = v_{\text{St}}(x_1, x_2) = \sum_{w \in W} (-1)^{l(w)}q^{-l(w)}f_w.$$  

Moreover, under $\mathcal{A}_w$, $v_{\text{St}}(\nu)$ is mapped to constant multiple of $v_{\text{St}}(w \cdot \nu)$. Denote the constant by $\lambda_w(x_1, x_2)$, that is,

$$\mathcal{A}_w(x_1, x_2)v_{\text{St}}(x_1, x_2) = \lambda_w(x_1, x_2)v_{\text{St}}(w \cdot (x_1, x_2)).$$

Then direct computation shows

$$\lambda_{s_1}(x_1, x_2) = \frac{q - q^{x_1-x_2}}{q(1 - q^{x_1-x_2})} = -c_{\alpha_1}(\nu_{x_1} \otimes \nu_{x_2})$$

$$\lambda_{s_2}(x_1, x_2) = \frac{q - q^{x_2}}{q(1 - q^{x_2})} = -c_{\alpha_2}(\nu_{x_1} \otimes \nu_{x_2})$$

where $c_\alpha$ for $\alpha \in \Delta$ denote the usual c-function [Ca] associated to $\alpha$. The following cases indicate the cases in Theorem 0.2. We note that Case (3) in the theorem does not yield any Iwahori spherical representations.
Case (1)-(2): $\nu_{x_1} \times \nu_{x_2} \times 1$ is irreducible with $x_i \in i\mathbb{R} \setminus \{ \frac{\pi n}{h} \mid n \in \mathbb{Z} \}$. In this case, the associated discrete series $\nu_{x_1} \otimes \nu_{x_2}$ are also supported on $A_0 \subset B = P_0$. Obviously, we have

$$d(\nu_{x_1} \otimes \nu_{x_2}) = 1.$$ 

The longest element $w_0$ in $W(\theta)$ is $s_1s_2s_1s_2 = s_2s_1s_2s_1$. Hence

$$\lambda_{w_0}(x_1, x_2) = \lambda_{s_2}(-x_1, x_2)\lambda_{s_1}(x_2, -x_1)\lambda_{s_2}(x_2, x_1)\lambda_{s_1}(x_1, x_2)$$

$$= \frac{q - q^{x_2}}{q(1 - q^{x_2})} \frac{q - q^{x_3 + x_2}}{q(1 - q^{x_3 + x_2})} \frac{q - q^{x_1}}{q(1 - q^{x_1})} \frac{q - q^{-x_1 - x_2}}{q(1 - q^{-x_1 - x_2})}$$

$$= \prod_{\alpha \in \Delta^+} c_\alpha(\nu_{x_1} \otimes \nu_{x_2})$$

and

$$\mu(\nu_{x_1} \otimes \nu_{x_2}) = \gamma(G/B)^2 (\lambda_{w_0}(-x_1, -x_2)\lambda_{w_0}(x_1, x_2))^{-1}$$

$$= \gamma(G/B)^2 \left( \prod_{\alpha \in \Delta} c_\alpha(\nu_{x_1} \otimes \nu_{x_2}) \right)^{-1}.$$ 

Case (4)-(5): Let $\eta = \nu_x$ with $x \in i\mathbb{R}$. In these cases, the associated discrete series $\nu_x \otimes \text{St}_{\text{Sp}_2}$ are in $E_2(M_\eta)$. 

$$d(\nu_x \otimes \text{St}_{\text{Sp}_2}) = d(\text{St}_{\text{Sp}_2}) = \frac{q - 1}{(q + 1)v(B_{\text{Sp}_2})} = q - 1$$

and $w_{\{\alpha_2\}} = s_1s_2s_1$. Since $\text{Ind}_{P_\eta}^G (\nu_x \otimes \text{St}_{\text{Sp}_2}) \hookrightarrow \text{Ind}(\nu_x \otimes \nu_1)$, from the property I.3.2-(2) of intertwining operators, we can compute $\mu$ factor from $\lambda_{s_1}$ and $\lambda_{s_2}$:

$$\lambda_{w_{\{\alpha_2\}}}(x_1, x_2) = \lambda_{s_1}(1, -x)\lambda_{s_2}(1, x)\lambda_{s_1}(x_1, 1)$$

$$= \frac{q - q^{x_2}}{q(1 - q^{x_2})} \frac{q - q^x}{q(1 - q^x)} \frac{q - q^{-x}}{q(1 - q^{-x})}$$

$$= \prod_{\alpha \in \Delta^+ \setminus \{\alpha_2\}} (-c_\alpha(\nu_x \otimes \nu_1))$$

$$\mu(\nu_x \otimes \text{St}_{\text{Sp}_2}) = \gamma(G/P_\eta)^2 (\lambda_{w_{\{\alpha_2\}}}(-x_1, -1)\lambda_{w_{\{\alpha_2\}}}(x_1, 1))^{-1}$$

$$= \gamma(G/P_\eta)^2 \left( \prod_{\alpha \in \Delta \setminus \{\pm \alpha_2\}} c_\alpha(\nu_x \otimes \nu_1) \right)^{-1}.$$ 

Case (6): Let $\eta = \nu_x$ with $x \in i\mathbb{R}$. Then $\nu_x \text{St}_{\text{GL}_2} \in E_2(M_\eta)$. 

$$d(\nu_x \text{St}_{\text{GL}_2}) = d(\text{St}_{\text{GL}_2}) = \frac{q - 1}{2(q + 1)v(B_{\text{GL}_2})} = \frac{q - 1}{2}$$

and $w_{\{\alpha_1\}} = s_2s_1s_2$. Since $\text{Ind}_{P_\eta}^G (\nu_x \otimes \text{St}_{\text{GL}_2}) \hookrightarrow \text{Ind}(\nu_{x + \frac{1}{4}} \otimes \nu_{x - \frac{1}{4}})$, we can compute $\mu$ factor as follows:

$$\lambda_{w_{\{\alpha_1\}}}(x + \frac{1}{2}, x - \frac{1}{2}) = \lambda_{s_2}(-x + \frac{1}{2}, x + \frac{1}{2})\lambda_{s_1}(x + \frac{1}{2}, -x + \frac{1}{2})\lambda_{s_2}(x + \frac{3}{2}, x - \frac{1}{2})$$

$$= \frac{q - q^{x - \frac{3}{4}}}{q(1 - q^{x - \frac{3}{4}})} \frac{q - q^{2x}}{q(1 - q^{2x})} \frac{q - q^{x + \frac{1}{4}}}{q(1 - q^{x + \frac{1}{4}})}$$

$$= \prod_{\alpha \in \Delta^+ \setminus \{\alpha_1\}} c_\alpha(\nu_{x + \frac{1}{4}} \otimes \nu_{x - \frac{1}{4}})$$
\[ \mu(\nu_x \otimes \text{St}_{\text{GL}_2}) = \gamma(G/P_s)^2 (\lambda_{w_{(\alpha_1)}}(-x + \frac{1}{2}, -x - \frac{1}{2})\lambda_{w_{(\alpha_1)}}(x + \frac{1}{2}, x - \frac{1}{2}))^{-1} \]

\[ = \gamma(G/P_s)^2 \prod_{\alpha \in \Delta \setminus \{\pm \alpha_1\}} (-c_\alpha(\nu_{x+\frac{1}{2}} \otimes \nu_{x-\frac{1}{2}})) \]

1.1.3. Plancherel formula on the Iwahori Hecke algebra

Applying §1.3 to the Iwahori Hecke algebra \( \mathcal{H}(G/B_0, 1_{B_0}) \), that is, putting \((J, \rho) = (B_0, 1_{B_0})\), we can deduce Plancherel formula on the Iwahori Hecke algebra from the formula in Corollary I.3.1.3. Let \( \check{G}^1_{r_{B_0}} \) be the set of irreducible Iwahori spherical representations and let \( \check{\mathcal{H}} \) be the set of finite simple \( \check{\mathcal{H}}(G/B_0, 1_{B_0}) \)-modules.

**Corollary.** For \( S \subset \check{G}^1_{r_{B_0}} \), let \( S_{1_{B_0}} \) be the corresponding subset in \( \check{\mathcal{H}} \). That is, \( S_{1_{B_0}} = \{ V_{1_{B_0}} = \text{Hom}_{B_0}(V, 1_{B_0}) \mid V \in S \} \). Let \( \check{\nu}_{1_{B_0}} \) be the Plancherel measure on \( \check{\mathcal{H}} \) and let \( \check{\nu} \) be the Plancherel measure on \( \check{G}_r \). Then we have

\[ \check{\nu}_{1_{B_0}}(S_{1_{B_0}}) = \frac{\nu(B_0)}{\text{dim}(1_{B_0})} \frac{1}{\check{\nu}(S)} \frac{(q + 1)^2(q - 1)^2}{(1 + q^2)(1 + q)^2} \check{\nu}(S). \]

§1.2. The case \( \chi_1 = \chi_2 = \text{sgn} \)

In this case, \( J_\chi = B_0 \) and the Hecke algebra \( \mathcal{H}(G/B_0, \chi) \) is \( L^2 \)-isomorphic to the Iwahori Hecke algebra of \( \mathcal{H}(O_4/B_{O_4}) \). Here, \( O_4 \) is the split orthogonal group of 4 variables [Ro].

1.2.1. Discrete series.

There are four discrete series containing \((J_\chi, \chi)\). Since \( \mathcal{H}(G/B_0, \chi) \simeq \mathcal{H}(O_4/B_{O_4}) \) and the affine Weyl group \( W^\text{aff}_{O_4} \) of \( O_4 \) is isomorphic to \((Z_2 \times Z_2) \rtimes (W_{Sp_2} \times W_{Sp_2})\), we find all four discrete series have the formal degree

\[ \frac{(q - 1)^2}{4(q + 1)^2 \nu(B_0)} = \frac{(q - 1)^2(1 + q^2)}{4}. \]

**Notation.** For a character \( \nu = \nu_{x_1} \otimes \nu_{x_2} \) of \( A_0 \), denote \( \text{Ind}^G_B(\nu \otimes \chi) \) by \( I(x_1, x_2) \) and \( A_w(\nu \otimes \chi) \) by \( A_w'(x_1, x_2) \).

1.2.2. As in Iwahori spherical cases, we can just look at the behavior of intertwining operators on \( \chi \)-isotypic components \((I(x_1, x_2) \otimes \chi^{-1})B_0 \) of \( I(x_1, x_2) \). Then \( \dim(I(x_1, x_2) \otimes \chi^{-1})B_0 = |W| = 8 \) for our cases and we can choose the basis \( \{ f_w \mid w \in W \} \), where

\[ f_w(g) = \begin{cases} \delta^2 \chi(p) \otimes \chi(p) & \text{if } g = pwb \in PwB_0 \\ 0 & \text{otherwise.} \end{cases} \]

Fixing the order of basis in column vectors as \( f_1, f_{s_1}, f_{s_2}, f_{s_1s_2}, f_{s_2s_1}, f_{s_1s_2s_1}, f_{s_1s_2s_2}, f_{s_2s_1s_2}, f_{s_1s_2s_1s_2} \), by similar computation as in [Jan], we have
explained in Chapter I. In at most one. We will transfer Plancherel measures via Hecke algebra isomorphisms as on rank at most one groups, which are necessary for operators directly. On the other hand, again, since $Sp\,\text{ associated to these principal types are isomorphic to Hecke algebras on groups of rank in each case, we will find Plancherel measures of representations containing $(J, \chi)$. Then direct computation shows that

$$
\mu(\nu(x_1, x_2) \otimes \chi) = \gamma(G/B)2^{q} \frac{(1 - q^{x_1+x_2})(1 - q^{x_1-x_2})(1 - q^{x_1+x_2})(1 - q^{x_1-x_2})(1 - q^{x_1+2x_2})}{(q - q^{x_1+x_2})(q - q^{x_1-x_2})(q - q^{x_1+x_2})(q - q^{x_1-x_2})(q - q^{x_1+2x_2})}
$$

$$
\mu(\nu_x \otimes \text{St}_{GL_2}) = \gamma(G/P_{\text{ss}})2^{q} \frac{(q - 1/2)(q + 1)}{2(q + 1)} \cdot (q + 1) = \frac{q - 1}{2}
$$

§2. Plancherel measures of Principal series: Rank one Hecke algebras

For the rest of principal series, we can divide into following cases:

<table>
<thead>
<tr>
<th>Case</th>
<th>$\chi_1$</th>
<th>$\chi_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\neq$</td>
<td>$\neq$</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>$\neq$</td>
</tr>
<tr>
<td>3</td>
<td>$\neq$</td>
<td>$\neq$</td>
</tr>
<tr>
<td>4</td>
<td>$\neq$</td>
<td>$\neq$</td>
</tr>
<tr>
<td>5</td>
<td>$\neq$</td>
<td>$\neq$</td>
</tr>
</tbody>
</table>

In each case, we will find Plancherel measures of representations containing $(J, \chi)$. Since $Sp_4(k)$ is small enough, it is possible to compute Plancherel measures using intertwining operators directly. On the other hand, again, since $Sp_4(k)$ is small, Hecke algebras associated to these principal types are isomorphic to Hecke algebras on groups of rank at most one. We will transfer Plancherel measures via Hecke algebra isomorphisms as explained in Chapter I. In §2.1, we first formulate Plancherel formulas of principal series on rank at most one groups, which are necessary for §2.2 and Chapter III.
§2.1. Plancherel measures: Rank one groups

In this section, let $G$ be a group of semisimple rank at most one. If $G$ is a torus, that is, $G = k^\times$, we can easily see that $\mu = 1$ and also the other constants are 1. In the following, assume that $G$ is a group of semisimple rank 1 defined over $k$. In all cases, let $A$ be a maximal torus, $B$ be the Borel containing $A$ in $G$ and let $B_0$ be the corresponding Iwahori subgroup. Then we have

$$\gamma = \gamma(G/B) = 1 + \frac{1}{q}, \quad c(G/B) = 1, \quad d(\chi) = 1, \quad |W| = 2$$

where $\chi$ is a character of $A$.

(1) Iwahori spherical principal series on $GL_2(k)$:

$$\mu(\nu_x \otimes \nu_{x_2}) = \gamma^2 q^2 \frac{(1 - q^{x_1 - x_2})(1 - q^{x_2 - x_1})}{(q - q^{x_1 - x_2})(q - q^{x_2 - x_1})}$$

$$d(\nu_x \cdot \text{St}) = \frac{1}{v(B_0)} \frac{q - 1}{2(q + 1)} = \frac{q - 1}{2}$$

(2) Iwahori spherical principal series on $Sp_2(k)$:

$$\mu(\nu_x) = \gamma^2 q^2 \frac{(1 - q^x)(1 - q^{-x})}{(q - q^x)(q - q^{-x})}$$

$$d(\text{St}) = \frac{1}{v(B_0)} \frac{q - 1}{q + 1} = q - 1$$

(3) Iwahori spherical principal series on $U(1, 1)$: Let $E/k$ be the quadratic extension with respect to which $U(1, 1)$ is defined.

$$\mu(\nu_x) = \begin{cases} 
\gamma^2 q^2 \frac{(1 - q^{2x})(1 - q^{-2x})}{(q - q^{2x})(q - q^{-2x})} & \text{if } E/k \text{ is unramified,} \\
\gamma^2 q^2 \frac{(1 - q^x)(1 - q^{-x})}{(q - q^x)(q - q^{-x})} & \text{if } E/k \text{ is ramified.}
\end{cases}$$

$$d(\text{St}) = \frac{1}{v(B_0)} \frac{q - 1}{q + 1} = q - 1$$

(4) $Sp_2(k)$ or $U(1, 1)(E/k)$ with $(A_0, sgn)$ where $E/k$ is ramified and $sgn$ is a ramified quadratic character with $sgn(\varpi) = 1$ and where $A_0$ is the maximal compact subgroup of $A$.

$$\mu(\nu_x \cdot sgn) = \gamma^2 q.$$ 

§2.2. Plancherel measures of Principal series

In the following, let $Q$ denote the constant $q^{(c(\chi_1) + c(\chi_2) + c(\chi_1 \chi_2) + c(\chi_1 \chi_2^{-1}))}$. We also let $\text{St}_G$ be the Steinberg representation of $G$ and let $B_G$ be the Iwahori subgroup of $G$. 17
Case 1: $\chi_1 \neq \chi_2$, $\chi_1^2 = \chi_2^2 = 1$. We may assume that $\chi_1 = \text{sgn}$, $\chi_2 = 1$. In this case, $\mathcal{I}(G/J, \chi) \simeq \mathcal{I}(\text{Sp}_2(k)/B_{\text{Sp}_2}) \otimes \mathcal{I}(\text{Sp}_2(k)/B_{\text{Sp}_2}, \text{sgn}) \circ \mathcal{I}(\text{Sp}_2(k)/B_{\text{Sp}_2}, 1)$. We use the formula in I.1.3-(1), Corollary I.3.1.3 and (2), (4) in §2.1. Then

$$\frac{v'(B_{\text{Sp}_2})}{\dim(\chi_1)} c(\text{Sp}_2/B_{\text{Sp}_2})^{-2} \gamma(\text{Sp}_2/B_{\text{Sp}_2})^{-1}|W_A|^{-1} \mu_1(\nu_{x_1} \chi_1)$$

$$- \frac{v'(B_{\text{Sp}_2})}{\dim(\chi_2)} c(\text{Sp}_2/B_{\text{Sp}_2})^{-2} \gamma(\text{Sp}_2/B_{\text{Sp}_2})^{-1}|W_A|^{-1} \mu_2(\nu_{x_2} \chi_2).$$

$$= \frac{v'(B_{\text{Sp}_2})}{\dim(\chi_1) \oplus \chi_2} \gamma(\text{Sp}_4/B_0)^{-1}|W_A|^{-1} \mu(\nu_{x_1} \chi_1 \otimes \nu_{x_2} \chi_2)$$

Here $v'$ is the Haar measure on $\text{Sp}_2$ with $v'(B_{\text{Sp}_2}) = \frac{1}{q+1}$, $A'$ is the diagonal maximal split torus in $\text{Sp}_2$ and $\mu_1$, $\mu_2$ are the $\mu$-factors from (4) and (2) respectively. Plugging in known values for $c$, $\mu_i$- and $\gamma$-factors, we can compute

$$\mu(\nu_{x_1} \chi_1 \otimes \nu_{x_2}) = \gamma(G/B)^2 Q \frac{q^2(1 - q^{x_2})(1 - q^{-x_2})}{(q - q^{x_2})(q - q^{-x_2})}.$$  

Similarly,

$$\mu(\nu_{x_1} \chi_1 \otimes \text{St}_{\text{Sp}_2}) = \gamma(G/P_h)^2 q^3, \quad d(\nu_{x_1} \chi_1 \otimes \text{St}_{\text{Sp}_2}) = q - 1.$$

Case 2: $\chi_2 = 1$, $\chi_1^2 \neq 1$. $\mathcal{I}(G/J, \chi) \simeq \mathcal{I}(\text{Sp}_2(k)/B_{\text{Sp}_2}) \otimes \mathcal{I}(k^x/\text{O}_h^x)$ and

$$\mu(\nu_{x_1} \chi_1 \otimes \nu_{x_2}) = \gamma(G/B)^2 Q \frac{q^2(1 - q^{x_2})(1 - q^{-x_2})}{(q - q^{x_2})(q - q^{-x_2})}$$

$$\mu(\nu_{x_1} \chi_1 \otimes \text{St}_{\text{GL}_2}) = \gamma(G/P_h)^2 q^{3c(\chi)} \quad d(\nu_{x_1} \chi_1 \otimes \text{St}_{\text{Sp}_2}) = q - 1$$

Case 3: $\chi_1 = \text{sgn}$, $\chi_2^2 \neq 1$. $\mathcal{I}(G/J, \chi) \simeq \mathcal{I}(\text{Sp}_2(k)/B_{\text{Sp}_2}, \text{sgn}) \circ \mathcal{I}(k^x/\text{O}_h^x)$ and

$$\mu(\nu_{x_1} \chi_1 \otimes \nu_{x_2} \chi_2) = \gamma(G/B)^2 Q$$

Case 4: $\chi_1 = \chi_2 \neq 1$. $\mathcal{I}(G/J, \chi) \simeq \mathcal{I}(\text{GL}_2(k)/B_{\text{GL}_2})$ and

$$\mu(\nu_{x_1} \chi_1 \otimes \nu_{x_2} \chi_2) = \gamma(G/B)^2 Q \frac{q^2(1 - q^{x_2-x_1})(1 - q^{x_2-x_1})}{(q - q^{x_2-x_1})(q - q^{x_1-x_2})}$$

$$\mu(\nu_{x_1} \chi_1 \otimes \text{St}_{\text{GL}_2(k)}) = \gamma(G/P_h)^2 q^{3c(\chi)} \quad d(\nu_{x_1} \chi_1 \otimes \text{St}_{\text{GL}_2(k)}) = q - 1$$

Case 5: $\chi_1 \neq \chi_2$, $\chi_1^2 \neq 1$. Then $\mathcal{I}(G/J, \chi) \simeq \mathcal{I}(A/A_0)$ where $A_0$ is the maximal compact subgroup of $A$ and

$$\mu(\nu_{x_1} \chi_1 \otimes \nu_{x_2} \chi_2) = \gamma(G/B)^2 Q.$$
III. Formal degrees of supercuspidal representations

§1. Summary of Construction of Types

We briefly summarize the construction of \((J_\Sigma, \rho_c)\) from [Kil] for our case. Three digits labellings in this section are referring to those in [Kil]. The construction is based on the data \(\Sigma = (\Gamma, P'_0, \varrho)\) where \(\Gamma\) is a semisimple element of certain “truncated form” as in (1.3.2) in [Kil], \(P'_0\) is a parahoric subgroup of the centralizer \(C_G(\Gamma)\) of \(\Gamma\) in \(G\) and \(\varrho\) is a cuspidal representation of finite reductive quotient of \(P'_0\). This constructions is valid when the residue characteristic is large. In our case when \(G = \text{Sp}_4(k)\), we assume

\[
\frac{1}{4} > \frac{2\text{ord}_k p}{p - 1} - \frac{p}{p - 1} + \frac{1}{p - 1}.
\]

S1.1. \(\Sigma = (\Gamma, P'_0, \varrho)\)

S1.1.1. Semisimple element \(\Gamma\) and Tamely Ramified Tori. Let \(\mathfrak{g}\) be the Lie algebra of \(G\) and let \(\Gamma \subset \mathfrak{g}\) be a semisimple element and let \(t\) be a maximal torus in \(\mathfrak{g}\) which is maximally \(k\)-split among tori in \(\mathfrak{g}\) containing \(\Gamma\). Let \(T\) be the torus in \(G\) with its Lie algebra \(t\). Let \(A[t]\) and \(A[T]\) be the subalgebra of \(\text{End}_k(V)\) generated by \(t\) and \(T\) respectively. Then \(A[t] = A[T]\) and it can be written as a direct sum of tamely ramifed extensions over \(k\). On the other hand, as \(t\)-, \(T\)-module, \(V \simeq A[t] = A[T]\). Now \(V\) can be decomposed as follows;

\[
V = \sum_{i=1}^{m} V_i \simeq A[t] = A[T]
\]

where \(V_i = F_i \oplus \cdots \oplus F_i\) for some tamely ramified extension \(F_i\) over \(k\) with involution \(\sigma_i\) and where each \(V_i\) is equipped with a sesquilinear form \(f_{V_i}\) such that \(\langle , \rangle = \sum \text{Tr}_{F_i/k} \sigma_i f_{V_i}\). We can write \(V\) with respect to Witt basis (with respect to a fixed ordering) as follows;

\[
V_i = V_i^+ \oplus V_i^- \oplus V_i^\delta \oplus V_i^{\delta'}
\]

where

\[
V_i^+ \oplus V_i^- = F_i^{d_i} \oplus \cdots \oplus F_i^1 \oplus F_i^{-1} \oplus \cdots \oplus F_i^{-d_i}
\]

with \(V_i^+\) a maximal isotropic subspace in \(V\) and \(V_i^-\) its dual with respect to \(f_{V_i}\) and where

\[
V_i^\delta = 0, \quad F_i^\delta \text{ or } F_i^{\delta_1} \oplus F_i^{\delta_2}, \quad \text{and} \quad V_i^{\delta'} = 0, \quad F_i^{\delta'} \text{ or } F_i^{\delta_1'} \oplus F_i^{\delta_2'}.
\]

We refer to [Kil, §1.4] for details and notation. Then under the above identifications (1) and (2), \(\Gamma \subset t\) can be written as \(\Gamma = (\cdots, \gamma_i, \cdots, \gamma_i, \cdots, -\gamma_i, \cdots)\) with \(\gamma_i \in F_i\). Moreover, \(G' = C_G(\Gamma)\) is \(\prod_{i=1}^{m} G'_i\) where \(G'_i\) is either isomorphic to \(\text{GL}(V_i^+\) or to the group of isometries on \((V_i, f_{V_i})\). That is,

\[
G = \prod_{i=1}^{m} G'_i \quad \text{where} \quad G_i = \text{GL}(V_i^+) \text{ or } G(V_i, f_{V_i}).
\]
From now on, we assume \((\Gamma, t)\) satisfies \((P)\) (recall it is defined in (1.3.2)).

**S1.1.2. Special Open Compact subgroups.** To have appropriate parahoric subgroups \(P'_0\) in \(\Sigma\), we fixed an Iwahori subgroup of \(C_G(\Gamma)\) in [K11, §1.5.A]. To find lattices whose structure fit with \(t\) or \(T\) and which are normalized by the parahoric subgroup \(P'_0\), we had to fix a maximal open compact subgroup \(K'_0\) of \(C_G(\Gamma)\) containing \(P'_0\). This is done in [K11, §1.5.B]. Roughly speaking, the maximal compact subgroup is chosen such that its associated lattice in \(V\) is the “most self-dual” with respect to \(f\) among those containing \(P'_0\).

**S1.2. Root Decomposition and Lattices in \(\mathfrak{g}\)**

Recall that we have a list of notation and definitions in (2.1.1). We will use them throughout this paper.

Decomposing \(\mathfrak{g}\) as a sum of irreducible \(t\)-modules (see (2.2.9) for details and notation),

\[
\mathfrak{g} = \bigoplus_{(\nu, \tau) \in (\nu, \tau)} \overline{M}_\nu^\tau
\]

On each \(t\)-root space \(\overline{M}_\nu^\tau\), we have a lattice structure induced from fractional ideals in \(\overline{F}_\nu^\tau\). However, to produce a lattice in \(\mathfrak{g}\), we need to work with “shifted” (by \(\frac{1}{2}a_\nu\)) lattices as in (2.3.3) due to non self-duality of lattices associated to the parahoric subgroup \(P'_0\). That is, for any \(s \in \mathbb{Q}_p\), the lattice \(\overline{M}_\nu^\tau(s)\) (it is also denoted by \(\overline{M}_\nu^\tau(s + \frac{1}{2}a_\nu)\)) corresponds to \(p^{n_s}_{\overline{F}_\nu^\tau}\) where \(n_s = \left[e(\overline{F}_\nu^\tau/k_0) \cdot (s + \frac{1}{2}a_\nu)\right]\) with \(a_\nu\) defined in (2.1.1). Then the following lattices defined in (2.3.9) are normalized by \(I'_0\):

\[
\mathcal{A}_c(s) = \bigoplus_{\nu \in \mathbb{Y}, \tau \in \text{Gal}^\nu} \overline{M}_\nu^\tau(s) = \bigoplus_{\nu \in \mathbb{Y}, \tau \in \text{Gal}^\nu} \overline{M}_\nu^\tau(s + \frac{1}{2}a_\nu),
\]

\[
\mathcal{A}_c(s^+) = \bigoplus_{\nu \in \mathbb{Y}, \tau \in \text{Gal}^\nu} \overline{M}_\nu^\tau(s^+) = \bigoplus_{\nu \in \mathbb{Y}, \tau \in \text{Gal}^\nu} \overline{M}_\nu^\tau(s + \frac{1}{2}a_\nu)^+.
\]

Let \(\mathcal{Y}_c\) and \(\mathcal{Y}_c^\dagger\) be defined as in (3.3.3). More explicitly, we have

\[
\mathcal{Y}_c = \mathcal{K}_c^1 + \sum_{\overline{M}_\nu^\tau \notin \mathfrak{g}} \overline{M}_\nu^\tau \left(\frac{1}{2}(-1 - \text{ord}(\gamma_i^\tau - \gamma_j^\tau))\right)^+
\]

\[
\mathcal{Y}_c^\dagger = \mathcal{K}_c^1 + \sum_{\overline{M}_\nu^\tau \notin \mathfrak{g}} \overline{M}_\nu^\tau \left(\frac{1}{2}(-1 - \text{ord}(\gamma_i^\tau - \gamma_j^\tau))\right)
\]

where \(\mathcal{K}_c^1 = \mathfrak{g} \cap \mathcal{A}_c(0^+)\). Then in (3.3.3), two open compact subgroups \(J'_\Sigma, J_\Sigma\) are defined as follows:

\[
J'_\Sigma = P'_0 \cdot Y'_\tau, \quad J_\Sigma = P'_0 \cdot Y_\tau
\]
where
\[ Y'_t = \exp(Y'_r), \quad Y_r = \exp(Y_r'). \]

Let \( \Sigma \) be \((\Gamma, P'_0, \varrho) \) as before. Since \( Y_r \) is normalized by \( P'_0 \), we can extend \( \varrho \) to \( J_x \) by letting \( Y_r \) act trivially on \( V_\varrho \), i.e., \( \varrho(t \cdot b) = \varrho(t) \) for \( t \cdot b \in P'_0 \cdot Y'_r \). On \( Y_r' \), \( \Gamma \) defines a character as \( \chi_r(y) = \theta(\text{Tr}(\Gamma \log(y))) \) for \( y \in Y_r' \) where \( \theta \) is the additive character with the conductor \( \mathcal{O}_k \) fixed in \((2.4.1)\). Now we extend this to the whole \( J'_x \) and then to \( J_x \). Up to \( J'_x \), we can extend it as a character as follows: For a given \( \Gamma \), we fix a character \( \chi_r^o \) of the maximal compact subgroup \( T_0 \) of \( T \), which coincides with \( \chi_r \) on \( T_0 \cap Y'_r \) and which induces a character of \( P'_0 \) via the determinant of \( G' \). We still denote this extended character by \( \chi_r^o \). Define the extended character \( \tilde{\chi}_r \) of \( \chi_r \) to \( J'_x \) as follows;

\[
(1) \quad \tilde{\chi}_r(t \cdot b) = \chi_r^o(t)\chi_r(b) \quad \text{for} \quad t \cdot b \in P'_0 \cdot Y'_r.
\]

If \( J_x = J'_x \), define \( \rho_x \) as \( \chi_r \otimes \varrho \):

\[
\rho_x = \tilde{\chi}_r \otimes \varrho.
\]

If \( J_x \neq J'_x \), we extend \( \chi_r \) to a representation \( \rho_x^o \) of \( J_x \) via theory of oscillator representations. We refer to \((3.4.3)\) for details.

\section*{§2. Computing Formal degrees}

Formal degrees of supercuspidal representations of the form \( \pi = \text{Ind}_{J_x}^G \rho_x \) can be found as

\[
d(\pi) = \frac{\dim(\rho_x)}{v(J_x)}.
\]

According to the algebra \( A[\mathfrak{t}] \) generated by \( \mathfrak{t} \) in \((1.5.2)\) in \( \text{GL}_4(k) \), we can divide the cases as follows (here, \( U(.) (E/F) \) denotes a unitary group defined with respect to a quadratic extension \( E \) over \( F \)):

<table>
<thead>
<tr>
<th>Case</th>
<th>( A[\mathfrak{t}] )</th>
<th>( C_G(\Gamma) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( E_1 \oplus E_2 ), ( (E : k) = 4 )</td>
<td>( U(1) ) (generators)</td>
</tr>
<tr>
<td>2</td>
<td>( E_1 \oplus E_2 \oplus k^2 ), ( (E : k) = 4 )</td>
<td>( U(2) ) or ( U(1, 1)(E/k) )</td>
</tr>
<tr>
<td>3</td>
<td>( E \oplus E_1 ), ( (E : k) = 2 )</td>
<td>( U(2) )</td>
</tr>
<tr>
<td>4</td>
<td>( E \oplus k^2 ), ( (E : k) = 2 )</td>
<td>( \text{Sp}_2(k) \times U(1)(E_1/k) )</td>
</tr>
<tr>
<td>5</td>
<td>( k^4 )</td>
<td>( G )</td>
</tr>
</tbody>
</table>

To find a volume of \( J_x \), we use the fact that \( J_x \) is \( P'_0 \ltimes Y_r \). Since \( Y_r \) is exponentiated from a lattice on \( \mathfrak{g}_r \), finding \( v(J_x) \) is almost reduced to counting indices on the Lie algebra level.

**Case 1** The case when \( A[\mathfrak{t}] = E \) with \( (E : k) = 4 \). Then \( P'_0 = C_G(\Gamma) \) and \( \varrho \) is a character of \( C_G(\Gamma) \).

(i) \( E/k \) is totally unramified.
Then $\Gamma = \tilde{c}_{-n}(\Gamma)$ with $n \in \{2, 3, \cdots\}$. Let Gal be generated by $\tau$ with $\tau^2 = \sigma$, the involution on $E$. Then we can write $\mathfrak{g}$ more explicitly:
\[
\mathfrak{g} = \tilde{M}_{\nu_0}^1 \oplus \tilde{M}_{\nu_0}^3 \oplus \tilde{M}_{\nu_0}^7
\]
where $\nu_0 = (1, 1, \delta, \delta)$. We have $\tilde{M}_{\nu_0}^1 \simeq \ker(\text{Tr}_{E/E^\sigma})$ and $\tilde{M}_{\nu_0}^3 \simeq E \simeq \tilde{M}_{\nu_0}^7$.
\[
\mathcal{Y}_r = \tilde{M}_{\nu_0}^3 (\lfloor \frac{d}{2} \rfloor) \oplus \tilde{M}_{\nu_0}^5 (\lfloor \frac{d}{2} \rfloor) \oplus \tilde{M}_{\nu_0}^7 (\lfloor \frac{d}{2} \rfloor).
\]
We may assume that $Y_r$ is contained in $(K_0)_p$ if $f_Y(x, y) = xy^\sigma$ or $(K_2)_p$ if $f_Y(x, y) = \omega_kxy^\sigma$ by choosing symplectic basis in $\mathcal{O}_E^\times$ over $k$. Here $(K_0)_p$ (resp. $(K_2)_p$) is maximal pro-$p$ normal subgroup of $K_0$ (resp. $K_2$). If $n$ is odd, $\rho_{\sigma}$ is a Heisenberg representation of dimension $\# \{\mathcal{Y}_r/\mathcal{Y}_r^{\#$} = q^{4((\frac{d}{2})-1)}\}$.

Putting them together,
\[
v(J_\Sigma) = \frac{v(K_0)}{[K_0 : J_\Sigma]} = \frac{1}{q^{8(\frac{d}{2})-1}(q + 1)^2(q - 1)^2 q^4} \quad \text{with} \quad \alpha = 0, 2.
\]

Putting them together,
\[
d(\pi) = \frac{\dim(\rho_\Sigma)}{v(J_\Sigma)} = q^{8(\frac{d}{2})-1}(q + 1)^2(q - 1)^2 q^4 = q^{4n-4}(q^2 - 1)^2.
\]

(ii) $E/k$ is totally ramified.

$\Gamma = \tilde{c}_{-n}(\Gamma)$ where $n = -n_1 + \frac{1}{4}$ or $-n_1 + \frac{3}{4}$ with $n \in \{2, 3, \cdots\}$. The computation can be done similarly as above. In this case, symplectic basis can be chosen such that $Y_r \subset (B_0)_p$ and $\dim(\rho_\Sigma) = 1$.

\[
v(J_\Sigma) = \frac{v(B_0)}{[B_0 : J_\Sigma]} = \frac{2}{(1 + q^2)(1 + q)^2 q^{-4n-5} \cdot (q - 1)^2}
\]

Choosing coordinates in $V$ such that $Y_r \subset P_s$, we can compute
\[
[v(B_0) : v(J_\Sigma)] = (q + 1)(q^2 + 1)q^{3(n_1 - 1)}q^{(n_2 - 1)} \frac{q(q - 1)(q^2 - q)}{2} = d(\pi)
\]

(iii) $e(E/k) = f(E/k) = 2$.

(1) $E/E^\sigma$ is ramified. Then $\Gamma = a\sqrt{\omega_k} + b\sqrt{\varepsilon \omega_k}$ where $\varepsilon \in \mathcal{O}_E^\times$ is non-square. We may assume $v(a\sqrt{\omega_k}) = -\frac{2n_1 + 1}{2} < v(b\sqrt{\omega_k}) = -\frac{2n_2 + 1}{2}$ without loss of generality.

\[
\mathcal{Y}_r = \mathcal{Y}_r^{\prime} = \tilde{M}_\sigma(\frac{n_1}{2}) \oplus \tilde{M}_\sigma(\frac{n_2}{2}) \oplus \tilde{M}_\sigma(\frac{n_2}{2})
\]

Choosing coordinates in $V$ such that $Y_r \subset P_s$, we can compute
\[
[K_0 : P_s][P_s : J_\Sigma] = (q + 1)(q^2 + 1)q^{3(n_1 - 1)}q^{(n_2 - 1)} \frac{q(q - 1)(q^2 - q)}{2} = d(\pi)
\]

(2) $E/E^\sigma$ is unramified. We can write $\Gamma = a\sqrt{\varepsilon} + b\sqrt{\varepsilon \omega_k}$. The computation is similar to the case (1) and we will just make a list. Let $v(a\sqrt{\varepsilon}) = -n_1$ and $v(b\sqrt{\varepsilon \omega_k}) = -n_2 - \frac{1}{2}$.

Independent of unitary forms defining $f_Y$,
\[
d(\pi) = \begin{cases} q^{4(n_2 - 1)}q^{n_2 - 1}q^{2(n_1 - 1)}(q - 1)^2(q + 1)q^2(q^2 + 1) & \text{if} \ -n_1 < -n_2 - \frac{1}{2} \\ q^{2(n_2 - 1)}q^{n_2 - 1}q^{n_1 - 1}(q - 1)^2(q + 1)q^2(q^2 + 1) & \text{if} \ -n_1 > -n_2 - \frac{1}{2} \end{cases}
\]
**Case 2** The case when \( A[t] = E_1 \oplus E_2 \) with \( (E_i : k) = 2, i = 1, 2 \) and \( C_G(\Gamma) = U(1)(E_1/k) \times U(1)(E_2/k) \). Then \( V = E_1 \oplus E_2 = V^0 \oplus V^\beta \) with \( \alpha, \beta \in \{ \delta, \delta' \} \).

\[
\mathfrak{g} = \tilde{M}_{\nu_{11}} \oplus \tilde{M}_{\nu_{11}}^* \oplus \tilde{M}_{\nu_{22}} \oplus \tilde{M}_{\nu_{22}}^* \oplus \tilde{M}_{\nu_{12}}
\]

where \( \nu_{11} = (1, 1, \alpha, \alpha), \nu_{22} = (2, 2, \beta, \beta) \) and \( \nu_{12} = (1, 2, \alpha, \beta) \).

(i) \( E_1 \oplus E_2, \quad E_1 = E_2 \) and they are unramified over \( k \).

Write \( \Gamma = \sqrt{\varepsilon}(a, b) \) with \( a \neq b \). Then we may assume \( v(a \sqrt{\varepsilon}) \leq v(b \sqrt{\varepsilon}) \). Let \( v(a \sqrt{\varepsilon}) = -n_1, \quad v(b \sqrt{\varepsilon}) = -n_2 \). Then

\[
\mathfrak{g}^+ \cap g^+ = \tilde{M}_{\nu_{11}}(\frac{2n_1}{2}) \oplus \tilde{M}_{\nu_{22}}(\frac{2n_2}{2}) \oplus \tilde{M}_{\nu_{12}}(\frac{2n_1}{2})
\]

Independent of the anti-Hermitian forms defining \( U(1)(E_1/k) \times U(1)(E_2/k) \), we have

\[
d(\pi) = q^{2(n_1-1)}q^{2n_1}(\frac{2n_1}{2}-1) \cdot q^{4} \cdot (q^2 + 1)^2(q - 1)^2q^4
\]

(ii) \( E_1 \oplus E_2 \) with \( E_1 \) ramified and \( E_2 \) unramified.

Write \( \Gamma = (a\sqrt{\varepsilon}, b\sqrt{\varepsilon}) \) with \( v(a\sqrt{\varepsilon}) = -\frac{2n_1+1}{2}, v(b\sqrt{\varepsilon}) = -n_2 \).

Again, independent of the anti-Hermitian forms defining \( U(1)(E_1/k) \times U(1)(E_2/k) \), we have

\[
d(\pi) = \begin{cases} 
q^{(n_1-1)}q^{2n_1}(\frac{2n_1}{2}-1) \cdot \frac{2n_1}{2} \cdot (q^2 - 1)(q^2 + 1) & \text{if } v(a\sqrt{\varepsilon}) < v(b\sqrt{\varepsilon}) \\
q^{(n_1-1)}q^{2n_2}(\frac{2n_2}{2}-1) \cdot \frac{2n_2}{2} \cdot (q^2 - 1)(q^2 + 1) & \text{if } v(a\sqrt{\varepsilon}) < v(b\sqrt{\varepsilon})
\end{cases}
\]

(iii) \( E_1 \oplus E_2, \quad E_1, E_2 \) are ramified over \( k \).

Write \( \Gamma = (a, b) \) with \( a \neq b \). Then we may assume \( v(a) \leq v(b) \) and let \( v(a) = -\frac{2n_1+1}{2}, v(b) = -\frac{2n_2+1}{2} \).

\[
d(\pi) = q^{3(n_1-1)}q^{n_2-1}(q^2 - 1)(q^2 - q)(q^2 + 1)(q + 1).
\]

**Case 3** The case when \( A[t] = E \oplus E \) with \( (E : k) = 2, \) and \( C_G(\Gamma) = U(2)(E/k) \) or \( U(1, 1)(E/k) \).

(i) \( E/k \) is unramified. We have \( \Gamma = \tilde{\epsilon}_n(\Gamma) \).

If \( C_G(\Gamma) = U(1, 1)(E/k) \), in \( \Sigma = (\Gamma, P_\alpha, \varrho), P_\alpha \) should be maximal and \( \dim(\varrho) = q - 1 \).

Computing \( v(J_\varrho) = (q^{3(n-2)}q^3(q^2 + 1)(q - 1)^2)^{-1} \) as before, we have

\[
d(\pi) = q^{3(n-2)}q^3(q^2 + 1)(q - 1)^2.
\]
If $C_G(\Gamma) = U(2)(E/k)$, a compact unitary group,
\[ d(\pi) = q^{3n-3}(q^2 + 1)(q - 1)^2 \]

(iii) $E/k$ is ramified. We have $\Gamma = \tilde{c}_{-n-\frac{1}{2}}(\Gamma)$.

For $\text{Ind}_{\mathbb{Z}_p}^{\mathbb{Q}_p} \rho$, to be supercuspidal, $C_G(\Gamma)$ should be compact and $\dim(\varrho) = 1$ or $2$.
\[ d(\pi) = q^2\left(\frac{q^2 - 1}{q^2 + 1}\right) \frac{(q^2 - 1)(q^2 - q)}{2} \frac{(q^2 + 1)(q + 1)}{2} \dim(\varrho) \]

**Case 4** $A[t] = E \otimes k^2$, $(E : k) = 2$ and $G = \text{Sp}_2(k) \times U(1)(E_1/k)$. Then $P'_0$ is maximal and $\dim(\varrho) = q - 1$ or $\frac{q - 1}{2}$.

(i) $E/k$ is unramified. We have $\Gamma = \tilde{c}_{-n}(\Gamma)$.

Independent of maximal $P'_0$ or the unitary form defining $U(1)$, we have
\[ d(\pi) = q^{3(n-1)}(q^2 + 1)(q - 1) \dim \varrho. \]

(ii) $E/k$ is ramified. We have $\Gamma = \tilde{c}_{-n-\frac{1}{2}}(\Gamma)$.
\[ d(\pi) = q^{3n-1} \frac{q - 1}{2}(q + 1)(q^2 + 1) \dim \varrho. \]

**Case 5** $\Gamma = 0$ and $P'_0 = K_0$ or $K_2$. If $P'_0 = K_0$, $\varrho$ is a cuspidal representation of $\text{Sp}_4(\mathbb{F}_q)$. Then $\dim(\varrho) = (q - 1)^2(q^2 + 1)$, $(q^2 - 1)^2$ or $\frac{d(q-1)}{2}$ and
\[ d(\pi) = \dim(\varrho). \]

If $P'_0 = K_2$, $\varrho$ is a cuspidal representation of $\text{Sp}_2(\mathbb{F}_q) \times \text{Sp}_2(\mathbb{F}_q)$. Then $\dim(\varrho) = (q - 1)^2$, $\frac{(q - 1)^2}{2}$ or $\frac{(q - 1)^2}{4}$ and
\[ d(\pi) = \dim(\varrho)(q^2 + 1). \]

**IV. Plancherel measures on Generalized Principal series**

We now compute Plancherel measures of generalized principal series induced from $P_s$ or $P_h$.

§1. Generalized Principal series on $P_s$

We can consider related types case by case as follows:

Case 1: $\Sigma = (0, P_s, \varrho)$. This gives a type of depth zero [MP, Mo].

Case 2: $\Sigma = (\Gamma, B_{U(1,1)}, \varrho)$ where $C_G(\Gamma) = U(1,1)(E/k)$ is a unitary group of two variables and $B_{U(1,1)}$ is its Iwahori subgroup or $\Sigma = (\Gamma, O_{E}^1, \varrho)$ where $C_G(\Gamma) \simeq E^X$, a quadratic extension.

We will treat Case 1 and Case 2 in §1.1 and §1.2 respectively.
§1.1. Depth zero types

In this case, \((J_x, \rho_x) = (P, \vartheta)\) and \(\vartheta\) induces a representation of \(\text{GL}_2(\mathcal{O}_k)\) and \(\text{GL}_2(\mathbb{F}_q)\). We use the same notation for the corresponding representation of \(\text{GL}_2(\mathbb{F}_q)\). Let \(\tilde{\rho}_x\) be a unitary extension to the center of \(\text{GL}_2(k)\).

1.1.1. Case: \(\vartheta\) is self contragredient, that is, it is isomorphic to its contragredient. Then \(\mathcal{H}(G/P, \vartheta) \simeq \mathcal{H}(\text{Sp}_2(k)/B_{\text{Sp}_2(k)}, 1)\) and we have a discrete series \(\pi\) containing \((P, \vartheta)\) of formal degree

\[
d(\pi) = \frac{(q-1) \dim(\vartheta)}{(q+1) v(P)} = (q-1)^2(q^2 + 1).
\]

Again, this can be computed by transferring the formal degree of the unique discrete series of \(\text{Sp}_2(k)\). Otherwise, we have

\[
c(G/P)^2 \gamma(G/P)^{-1} d(\nu_x \otimes \tilde{\rho}_x) \mu(\nu_x \otimes \tilde{\rho}_x)
= \frac{\dim(\vartheta)}{v(P)} \frac{q(1 - q^{-2x})(1 - q^{2x})}{(q - q^{-2x})(q - q^{2x})}
= (q^4 - 1) \frac{q(1 - q^{-2x})(1 - q^{2x})}{(q - q^{-2x})(q - q^{2x})}
\]

and hence

\[
d(\nu_x \otimes \tilde{\rho}_x) = (q-1)
\]

\[
\mu(\nu_x \otimes \tilde{\rho}_x) = c(G/P)^{-2} \gamma(G/P) \frac{(q^4 - 1) q(1 - q^{-2x})(1 - q^{2x})}{(q - q^{-2x})(q - q^{2x})}
\]

1.1.2. Case: \(\vartheta\) is not self contragredient. Then there is no discrete series containing \((J_x, \rho_x)\) and

\[
d(\nu_x \otimes \tilde{\rho}_x) = (q-1)
\]

\[
\mu(\nu_x \otimes \tilde{\rho}_x) = c(G/P)^{-2} \gamma(G/P) d(\nu_x \otimes \tilde{\rho}_x)^{-1} \frac{\dim(\vartheta)}{v(P)}
= c(G/P)^{-2} \gamma(G/P) \frac{(q^4 - 1)}{(q-1)}
\]

§1.2. Ramified types

These cases occur when \(\Gamma \in \Sigma\) is of the form \(\Gamma = (\gamma, -\overline{\gamma})\) where \(\gamma\) is in a quadratic extension \(E\) of \(k\).

1.2.1. Case: \(C_G(\Gamma) \simeq E^\times\).

If \(E/k\) is unramified, let \(\gamma = a + b\sqrt{\varepsilon}\) with \(v(a) = -m\) and \(v(b) = -n\). Then

\[
c(G/P)^2 \gamma(G/P)^{-1} d(\nu_x \otimes \tilde{\rho}_x) \mu(\nu_x \otimes \tilde{\rho}_x)
= \frac{\dim(\vartheta)}{v(J_x)} q^2(q^2 - 1) q^{2(q^2 - 1)} q^4(q - 1)^2(q + 1)^2(q^2 + 1)
= \frac{q^2 - 1}{q^2 - 1}
\]
where \( f_{\Gamma} = \max(m, n) \).

If \( E/k \) is ramified, let \( \gamma = a + b\sqrt{\omega} \) with \( v(a) = -m \) and \( v(b\sqrt{\omega}) = -(n + \frac{1}{2}) \).

\[
c(G/P)^2 \gamma (G/P)^{-1} d(\nu_x \otimes \tilde{\rho}_x) \mu(\nu_x \otimes \tilde{\rho}_x) = \frac{\dim(\rho)}{v(J_{\Sigma})}
\]

\[
= q^{m+n-1}(q+1)(q^4-1) \left\{ \begin{array}{ll}
q^{3(n+\frac{1}{2})} & \text{if } m < -n - \frac{1}{2} \\
q^{3(n+\frac{1}{2})} & \text{if } m > -n - \frac{1}{2}
\end{array} \right.
\]

1.2.2. Case: \( C_{\Gamma}(\Gamma) \simeq U(1,1)(E/k) \) with \( E/k \) unramified and \( \rho = 1 \) or \( \rho \) satisfies \( \rho(N(E/k)(x)) = 1 \) for \( x \in \mathcal{O}_E^\times \). Let \( \gamma = b\sqrt{\omega} \) with \( v(b) = -n \). Then there is a unique discrete series \( \pi \) containing \( (J_{\Sigma}, \rho_x) \) and its formal degree is

\[
d(\pi) = \frac{q-1}{q+1} \dim(\rho_x) = \frac{q-1}{q+1} q^{6(n-\frac{1}{2})-1} q^{3(n^2+1)}(q^2-1).
\]

\[
c(G/P)^2 \gamma (G/P)^{-1} d(\nu_x \otimes \tilde{\rho}_x) \mu(\nu_x \otimes \tilde{\rho}_x) = \frac{\dim(\rho)}{v(J_{\Sigma})} \frac{q(1-q^{-2x})(1-q^{2x})}{(q-q^{-2x})(q-q^{2x})}
\]

1.2.3. Case: \( C_{\Gamma}(\Gamma) \simeq U(1,1)(E/k) \) with \( E/k \) ramified and \( \rho = 1 \). Let \( \gamma = b\sqrt{\omega} \) with \( v(b) = -n - \frac{1}{2} \). Then there is a unique discrete series \( \pi \) containing \( (J_{\Sigma}, \rho_x) \) and its formal degree is

\[
d(\pi) = \frac{q-1}{q+1} \cdot (q^2+1)(q+1)^2 q^{6(n-\frac{1}{2})} q(q-1).
\]

\[
c(G/P)^2 \gamma (G/P)^{-1} d(\nu_x \otimes \tilde{\rho}_x) \mu(\nu_x \otimes \tilde{\rho}_x) = \frac{\dim(\rho)}{v(J_{\Sigma})} \frac{q(1-q^{-2x})(1-q^{2x})}{(q-q^{-2x})(q-q^{2x})}
\]

1.2.4. Case: \( C_{\Gamma}(\Gamma) \simeq U(1,1)(E/k) \) with \( \rho \) nontrivial character. In all cases,

\[
c(G/P_s)^2 \gamma (G/P_s)^{-1} d(\nu_x \otimes \tilde{\rho}_x) \mu(\nu_x \otimes \tilde{\rho}_x) = \frac{\dim(\rho)}{v(J_{\Sigma})} = q^{3n-2}(q+1)(q^4-1)
\]

Remark. In [Sh2], F. Shahidi has also computed Plancherel measures for generalized principal series supported on \( P_s \) using local root numbers. One can show they match with the above results.

\section{Generalized Principal series on \( P_h \)}

We again consider related types dividing into following cases:

Case 1: \( \Sigma = (0, P_s, \rho) \). This gives a type of depth zero [MP].

Case 2: \( \Sigma = (\Gamma, \mathcal{O}_E^\times \times U(1), \rho) \) where \( \Gamma \in \mathfrak{g} \cap (k \oplus E \oplus k) \) and \( E/k \) is a quadratic extension.
§2.1. Depth zero types

As in the previous case, \((J_{\Sigma}, \rho_{\Sigma}) = (P_h, \varrho)\) and \(\varrho = \chi \otimes \varrho_0\) induces a representation of \(\mathcal{O}_k^X \times \text{Sp}_2(\mathcal{O}_k)\) and \(\mathbb{F}_q^X \times \text{Sp}_2(\mathbb{F}_q)\). We use the same notation for the corresponding representation of \(\mathbb{F}_q^X \times \text{Sp}_2(\mathbb{F}_q)\). Note that \(\chi\) can be extended to \(k\) with \(\chi(\pi_k) = 1\). Let \(\rho_0 = c\text{-Ind}_{\text{Sp}_2(\mathcal{O}_k)}^{\text{Sp}_2(k)}(\varrho_0)\) and \(\tilde{\rho}_x = \chi \otimes \rho_0\).

2.1.1. Case 1: \(\Sigma = (0, P_h, \chi \otimes \varrho_0)\) with \(\chi\) trivial. The Hecke algebra \(\mathcal{H}(G//P_h, \rho_\Sigma)\) is isomorphic to the Iwahori Hecke algebra of \(\text{Sp}_2(k)\) and the unique discrete series containing \((P_h, \rho_\Sigma)\) has formal degree

\[
\frac{q - 1}{q + 1} \frac{\dim(\varrho_0)}{v(P_h)} = (q - 1)^2(q^2 + 1) \quad \text{or} \quad q - 1 \quad \frac{1}{2} \frac{q - 1}{q + 1} \frac{1}{v(P_h)} = \frac{(q - 1)^2(q^2 + 1)}{2}
\]

depending on whether \(\dim(\varrho_0) = q - 1\) or \(\frac{q - 3}{2}\).

\[
c(G/P_h)^2 \gamma(G/P_h)^{-1} d(\nu_x \otimes \tilde{\rho}_x) \mu(\nu_x \otimes \tilde{\rho}_x)
= \frac{\dim(\varrho)}{v(J_\Sigma)} \frac{q(q^x + 1)(q^{-x} + 1)}{(q^x + q)(q^{-x} + q)}.
\]

where \(\frac{\dim(\varrho)}{v(J_\Sigma)} = (q^4 - 1)\) or \(\frac{q^4 - 1}{2}\).

2.1.2. Case 2: \((0, P_h, \chi \otimes \varrho)\) with \(\chi = sgn\), a nontrivial character of order two. Let \(r_0 = s_0\) and \(r_1 = s_1 s_2 s_1\). Then \(\mathcal{H}(G//J_\Sigma, \rho_\Sigma)\) is generated by \(f_{r_0}\) and \(f_{r_1}\) with following relations (see [Myi])

\[
f_{r_0} \ast f_{r_0} = q\chi(-1)f_1,
\]

\[
f_{r_1} \ast f_{r_1} = \begin{cases} 
q^3\chi(-1)f_1 & \text{if } \dim(\varrho_0) = q - 1 \\
q^3\chi(-1)f_1 + (q^2 - 1)(\gamma_+ - \gamma_-)\kappa f_{r_1} & \text{if } \dim(\varrho_0) = \frac{q - 1}{2}
\end{cases}
\]

where \(\kappa = \sigma(-1)\) and \(\gamma_\pm = \frac{-1 \pm \sqrt{1 - 4q^2}}{2q^2}\). Hence if \(\dim(\varrho_0) = \frac{q - 1}{2}\), we have two discrete series containing \((P_h, \rho_\Sigma)\) coming from two characters \(f_{r_1} \to -(\gamma_+ - \gamma_-)\kappa, f_{r_0} \to \pm(\gamma_+ - \gamma_-)\). Their formal degrees are

\[
\frac{(q^2 - 1)}{2(q^2 + 1)} \frac{\dim(\varrho)}{v(P_h)} = \frac{q^4 - 1}{4}.
\]

\[
c(G/P_h)^2 \gamma(G/P_h)^{-1} d(\nu_x \otimes \tilde{\rho}_x) \mu(\nu_x \otimes \tilde{\rho}_x)
= \frac{\dim(\varrho)}{v(J_\Sigma)} \frac{q^2(1 - q^2x)(1 - q^{-2x})}{(q^x - q^{-x})(q - q^{-2x})}.
\]

If \(\dim(\varrho_0) = q - 1\), there is no discrete series containing \((P_h, \rho_\Sigma)\) and

\[
c(G/P_h)^2 \gamma(G/P_h)^{-1} d(\nu_x \otimes \tilde{\rho}_x) \mu(\nu_x \otimes \tilde{\rho}_x) = \frac{\dim(\varrho)}{v(J_\Sigma)} \cdot q
\]
2.1.3. Case 3: \((0, P_h, \chi \otimes \rho)\) with \(\chi \neq 1, sgn\). Then \(\mathcal{H}(G//J_\chi, \rho_\chi)\) is abelian and there is no discrete series containing \((P_h, \rho_\chi)\).

\[
c(G/P_h)^2 \gamma(G/P_h)^{-1} d(\nu_x \otimes \tilde{\rho}_x) \mu(\nu_x \otimes \tilde{\rho}_x) = \frac{\dim(q)}{v(J_\chi)} \cdot q
\]

§2.2. Ramified types: Rank one Hecke algebras

Let \(\Sigma = (\Gamma, \mathcal{O}^\times \times U(1), \varrho)\) where \(\Gamma \in \mathfrak{g} \cap (k \oplus E \oplus k)\) and \(E/k\) is a quadratic extension. Then \(\rho_\chi\) induces a supercuspidal representation on \(M_h\) of the form \(\tilde{\rho}_\chi = \chi \otimes \rho_0\) where \(\chi\) is a character of \(k^\times\) and \(\rho_0\) is a supercuspidal representation of \(\text{Sp}_2(k)\). To have rank one Hecke algebra, \(\chi\) is either trivial or \(sgn\).

2.2.1. \(E/k\) is unramified. Then let \(\Gamma = (0, \gamma, 0)\) with \(v(\gamma) = -n\).

If \(\chi = 1\), then \(\mathcal{H}(G//J_\chi, \rho_\chi)\) is isomorphic to the Iwahori Hecke algebra of \(\text{Sp}_2(k)\) and there is a unique discrete series containing \((J_\chi, \rho_\chi)\) of formal degree

\[
q - 1 \dim(\rho_\chi) \qquad \frac{q - 1}{q + 1} \cdot \frac{q - 1(q^4 - 1)q^{3 + 3n}}{v(J_\chi)}
\]

and

\[
c(G/P_h)^2 \gamma(G/P_h)^{-1} d(\nu_x \otimes \tilde{\rho}_\chi) \mu(\nu_x \otimes \tilde{\rho}_\chi) = \frac{\dim(q)}{v(J_\chi)} \cdot q = (q^4 - 1)q^{4 + 3n}
\]

If \(\chi = sgn\), \(\mathcal{H}(G//J_\chi, \rho_\chi)\) is isomorphic to \(\mathcal{H}(\text{Sp}_2(k)//B_{\text{Sp}_2}, sgn)\). There is no discrete series containing \((J_\chi, \rho_\chi)\) and

\[
c(G/P_h)^2 \gamma(G/P_h)^{-1} d(\nu_x \otimes \tilde{\rho}_\chi) \mu(\nu_x \otimes \tilde{\rho}_\chi) = \frac{\dim(q)}{v(J_\chi)} \cdot q = (q^4 - 1)q^{4 + 3n}
\]

2.2.2. \(E/k\) is ramified. Let \(\Gamma = (0, \gamma, 0)\) with \(v(\gamma) = -n - \frac{1}{2}\).

If \(\chi = 1\), \(\mathcal{H}(G//J_\chi, \rho_\chi) \simeq \mathcal{H}(\text{Sp}_2(k)//B_{\text{Sp}_2}, sgn)\). If \(\chi = sgn\), \(\mathcal{H}(G//J_\chi, \rho_\chi) \simeq \mathcal{H}(\text{Sp}_2(k)//B_{\text{Sp}_2}, 1)\) and we have a discrete series of formal degree

\[
q - 1 \dim(\rho_\chi) \qquad \frac{1}{q + 1} \cdot \frac{q^{2(n - 1)}2^{-1} \cdot q^4(q - 1)^2}{v(J_\chi)}
\]

and

\[
c(G/P_h)^2 \gamma(G/P_h)^{-1} d(\nu_x \otimes \tilde{\rho}_\chi) \mu(\nu_x \otimes \tilde{\rho}_\chi) = \left\{
\begin{array}{ll}
\frac{\dim(q)}{v(J_\chi)} \cdot q & \text{if } \chi = 1 \\
\frac{\dim(q)}{v(J_\chi)} \cdot q(1 - q^x)(1 - q^{-x}) & \text{if } \chi = sgn
\end{array}
\right.
\]

where \(\frac{\dim(q)}{v(J_\chi)} = q \cdot q^{2(n - 1)}2^{-1} \cdot q^4(q - 1)^2\).
2.3. Ramified types: Abelian Hecke algebras

Let $\Sigma = (\Gamma, O_k^\times \times U(1), \varrho)$ where $\Gamma = (a, \gamma, -a) \in \mathfrak{g} \cap (k \oplus E \oplus k)$ with $a \neq 0$ and $E/k$ is a quadratic extension. Then $\rho_\Sigma$ induces a supercuspidal representation on $M_h$ of the form $\tilde{\rho}_\Sigma = \chi \otimes \rho_0$ as in previous section. However, in this case, $\chi$ is a character of conductor $-v(a)$.

2.3.1. $E/k$ is unramified. Then let $\Gamma = (a, \gamma, -a)$ with $v(a) = -m$ and $v(\gamma) = -n$.

\[
c(G/P_h)^2 \gamma(G/P_h)^{-1} d(\nu_x \otimes \tilde{\rho}_\Sigma) \mu(\nu_x \otimes \tilde{\rho}_\Sigma) = \dim(\tilde{\rho}_\Sigma) = \frac{\dim(\tilde{\rho}_\Sigma)}{v(J_\Sigma)}
\]

\[
= q^{2(\frac{1}{2} - \frac{1}{2})} q^{2(\frac{1}{2} - \frac{1}{2})} q^{2(\frac{1}{2} - 1)} q^{\frac{1}{2}} (q + 1)^2 (q^2 + 1) - q^{4(\frac{1}{2} - \frac{1}{2})} q^{2(\frac{1}{2} - 1)} q^{\frac{1}{2}} (q + 1)^2 (q^2 + 1)
\]

2.3.2. $E/k$ is ramified. Then let $\Gamma = (a, \gamma, -a)$ with $v(a) = -m$ and $v(\gamma) = -(n + \frac{1}{2})$.

\[
c(G/P_h)^2 \gamma(G/P_h)^{-1} d(\nu_x \otimes \tilde{\rho}_\Sigma) \mu(\nu_x \otimes \tilde{\rho}_\Sigma) = \dim(\tilde{\rho}_\Sigma) = \frac{\dim(\tilde{\rho}_\Sigma)}{v(J_\Sigma)}
\]

\[
= \begin{cases} 
q^{2(\frac{1}{2} - \frac{1}{2})} q^{4(\frac{1}{2} - \frac{1}{2})} q^{2(\frac{1}{2} - 1)} q^{\frac{1}{2}} (q + 1)^2 (q^2 + 1) & \text{if } -m < -(n + \frac{1}{2}) \\
q^{4(\frac{1}{2} - \frac{1}{2})} q^{4(\frac{1}{2} - \frac{1}{2})} q^{2(\frac{1}{2} - 1)} q^{\frac{1}{2}} (q + 1)^2 (q^2 + 1) & \text{if } -m > -(n + \frac{1}{2})
\end{cases}
\]

Remark. It can be checked that $\mu(\nu_x \otimes \tilde{\rho}_\Sigma) = \gamma(G/P_h)^2 q^{c(\chi) + c(\rho_0)}$ where $c(\chi)$ and $c(\rho_0)$ are the conductors of $\chi$ and $\rho_0$ respectively.

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