Semi-infinite hodge structures and mirror symmetry for projective spaces

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Abstract. We express total set of rational Gromov-Witten invariants of $\mathbb{C}P^n$ via periods of semi-infinite Hodge structure associated with their mirror partners.

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1. INTRODUCTION

Motivated by higher-dimensional mirror symmetry we introduced in [B2] periods of non-commutative deformations of complex manifolds. When it is just a deformation of complex structure then the corresponding quantum periods specialize to usual periods. In another simplifying situation quantum periods specialize to oscillating integrals. In this paper we look closely at this case and associated variations of semi-infinite Hodge structures. It turns out that these variations give mirror partners to projective spaces. Our principal result in this paper is the formula (5.7) expressing the generating function for total collection of rational Gromov-Witten invariants of $\mathbb{C}P^n$ via periods of variations of semi-infinite Hodge structure of its mirror partner.

Let us consider a pair $(X, f)$ where

$$X = \{x_0 : x_1 \cdot \ldots \cdot x_n = 1\} \subset \mathbb{C}P^{n+1}, \ f : X \to \mathbb{A}^1, \ f = x_0 + \ldots + x_n$$

Such data was conjectured in ([G], [EHX]) to be mirror partner of projective space $\mathbb{C}P^n$. Generating function (potential) for genus $= 0$ Gromov-Witten invariants of $\mathbb{C}P^n$ reads as (see [KM] eq.(5.15))

$$F_{\mathbb{C}P^n} = \frac{1}{6} \sum_{0 \leq i,j,k \leq n} y^i y^j y^k \delta_{i+j+k} + \sum_{d,m_2, \ldots , m_n} N(d; m_2, \ldots , m_n) \frac{(y^2)^{m_2} \ldots (y^n)^{m_n}}{m_2! \ldots m_n!}$$

(1.1)

here $y \in H^*(\mathbb{C}P^n, \mathbb{C})$ and the Taylor coefficients $N(d; m_2, \ldots , m_n)$ are the virtual numbers of rational curves of degree $d$ in $\mathbb{C}P^n$ intersecting $m_a$ hyperplanes of codimension $a$, $a = 2, \ldots , n$. 

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Let us put

\begin{equation}
\hat{F}(x; t) = f + \sum_{m=0}^{n} t^{m} \left( \sum_{i=0}^{n} x_{i} \right)^{m}
\end{equation}

Any oscillating integral \( \psi_{k} = \int_{\Delta} \exp \left( \frac{\hat{F}(x(t))}{\hbar} \right)(a_{0} + h a_{1} + \ldots) \frac{dx_{0} \ldots dx_{n}}{d(x_{0} \ldots x_{n})} \) can be expressed via integrals

\[ \varphi_{k}^{m}(t, \hbar) = \int_{\Delta} \exp \left( \frac{\hat{F}(x(t))}{\hbar} \right) (x_{i})^{m} \frac{dx_{0} \ldots dx_{n}}{d(x_{0} \ldots x_{n})}, \quad \varphi_{k}^{m} = \hbar \frac{\partial \psi_{k}^{0}(t, \hbar)}{\partial t^{m}} \]

as linear combination \( \psi_{k} = \sum_{m=0}^{n} u_{m} \varphi_{k}^{m} \) with coefficients \( u_{m} = u_{m}^{(0)} + h u_{m}^{(1)} + \ldots \). Let us consider an element \( \psi(t, \hbar) \) satisfying the following normalization condition:

\[ \psi_{k}(t, \hbar) = \sum_{m=0}^{n} u_{m} \varphi_{k}^{m}(0, \hbar) \]

for some functions \( u_{m}(t, \hbar) = \delta_{m,0} + h^{-1} u_{m}^{(-1)}(t) + \ldots, \) \( m = 0, \ldots, n \). We will show that if one makes a change of parameters \( y^{m} = u_{m}^{(-1)}(t) \) then Picard-Fuchs equations for oscillating integrals \( \psi_{k} \) take the form

\begin{equation}
\frac{\partial^{2} \psi_{k}}{\partial y^{i} \partial y^{j}} = h^{-1} \sum_{m} A_{ij}^{m}(y) \frac{\partial \psi_{k}}{\partial y^{m}}
\end{equation}

Our principal formula can be written then as

\begin{equation}
\forall i, j, m, \quad A_{ij}^{m-n}(y) = \frac{\partial^{3}}{\partial y^{i} \partial y^{j} \partial y^{m}} \mathcal{F}^{n_{-m}}(y)
\end{equation}

The functions \( \psi_{k}(t, \hbar) \) can be viewed as periods of variation of semi-infinite Hodge structures attached to \((X, f)\) in the following way. The semi-infinite analog of Hodge filtration associated with a point \( t \) of moduli space of deformations of \( f \) is the subspace \( L(t) \) generated by cohomology classes of oscillating integrals in the space of sections of local system over \( \mathbb{C} \setminus \{0\} \) whose fiber over \( \hbar \) is the relative cohomology group \( H^{n}(X, \text{Re} \frac{\hat{F}(x(t))}{\hbar}) \ll \mathbb{C} \). The normalization condition for elements \( \psi(t) \in L(t) \) says that \( \psi(t) \) should lie also in covariantly constant affine subspace.

Another aim of this paper is to consider in details in explicit case the general relation, which we described in [B1], between variations of semi-infinite Hodge structures and Frobenius manifolds. We construct in particular in §4 a family of solutions to WDVV equations parameterized by open part in isotropic Sato Grassmanian. Same result holds true for any pair \((X, f)\) satisfying a homological condition, see the remark after prop.4.9.

Concerning the previously known results of similar type we should mention the paper [M] dealing with the case of \( \mathbb{CP}^{2} \) in which after some change of variables the associativity equation \( \mathcal{F}^{y_{1}y_{2}y_{3}} = \mathcal{F}^{y_{1}y_{2}y_{3}} - \mathcal{F}^{y_{1}y_{2}y_{3}} \mathcal{F}^{y_{1}y_{2}y_{3}} \) for non-trivial piece of the potential \( \mathcal{F}^{\mathbb{CP}^{2}} \) was identified with differential equation on (multi)sections of pencil of elliptic curves \( z^{2} = x(x - 1)(x - t) \) written in terms of \( \int_{\infty}^{t} dt \frac{dz}{z} \).

To simplify the notations we replace sometimes \( \frac{\partial}{\partial z_{1}} \) by \( \partial_{a} \).
2. Moduli space.

The deformations of the pair \((X, f)\) represented by \((X, \tilde{F}(t))\) where \(\tilde{F}(t_0, \ldots, t_n) \in \Gamma_{Zar}(X, \mathcal{O}_X) \otimes \mathcal{O}_{\mathbb{C}^{n+1}}\) is the family of functions defined in (1.2) are described in general with help of differential graded Lie algebra of polyvector fields

\[ g(X, f) = \oplus_i g^i(X, f)[-i], \quad g^i(X, f) := \Gamma_{Zar}(X, \Lambda^i \text{T}), \quad d := [f, \cdot]\]

equipped with the Schouten bracket on polyvector fields. Explicitly, the elements \(\gamma \in g(X, f)\) can be uniquely written in the form

\[ \gamma = \sum_{i \in \{1, \ldots, n\}} \gamma_i(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}) \partial_{x_i} \wedge \ldots \wedge \partial_{x_n}\]

where \(\gamma_i \in \mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]\). The space of first order deformations \(T_{f(X, f)} \mathcal{M}\) at the base point is equal to the total cohomology group of the complex \((g(X, f), [f, \cdot])\).

Simple calculation shows that the cohomology of the complex \((g_{\mathbb{A}^n}(X, f), [f, \cdot])\) are zero everywhere except in degree zero where they are \((n+1)\)-dimensional. Notice that the partial derivatives \(\frac{\partial \tilde{F}^{(i)}}{\partial t^i} \) \(i = 0, \ldots, n\) form a basis in the cohomology of the complex \((g(X, f), [f, \cdot])\). It follows that \(g(X, f)\) is formal as differential graded Lie algebra and that the set \(\mathcal{M}^{(X, f)}(R)\) of equivalence classes of solutions to Maurer-Cartan equation in \(g \otimes R\) for an Artin algebra \(R\) can be identified with

\[ \mathcal{M}^{(X, f)}(R) = \{ F = f + \tilde{f} \in \Gamma_{Zar}(X, \mathcal{O}_X) \otimes \mathfrak{M}_R \}/\{ \varphi(0) = Id \varphi \in Aut_{Zar}(X) \otimes \mathfrak{M}_R \}\]

In other words \(\mathcal{M}^{(X, f)}\) is the moduli space of deformations of the function \(f\). The family \(\tilde{F}(t_0, \ldots, t_n)\) is a mini-versal “solution” depending analytically on the parameters of deformation. In the sequel it will be convenient for us to denote by \(\mathcal{U}\) the analytic germ of \((n+1)\)-dimensional smooth space of the base of mini-versal deformation and by \(\tilde{X} = X \times \mathcal{U}\) the total space on which the family \(\tilde{F}(t_0, \ldots, t_n)\) is defined.

3. Variation of semi-infinite Hodge structures associated with \((X, \tilde{F})\).

In this section we define semi-infinite analog of variation of Hodge structures associated with the moduli space \(\mathcal{M}^{(X, f)}\). The analogy with usual variations of Hodge structures associated with families of Kahler manifolds is explained in [B2] §3.

Let \(\mathcal{R}_f\) denotes the sheaf of relative cohomology whose fiber over \(h \in \mathbb{C} \setminus \{0\}\) equals to

\[ \mathcal{R}_{f, h} := H^n(X, \text{Re} \frac{f}{h} \ll 0; \mathbb{C}) \]

Since there is a natural integral structure on this sheaf: \(\mathcal{R}_{f, h} = H^n(X, \text{Re} \frac{f}{h} \ll 0; \mathbb{Z}) \otimes \mathbb{C}\), therefore it is equipped with Gauss-Manin connection which we will denote by \(\mathcal{D}\). Denote also via \((\mathcal{R}_{\tilde{F}(t)}, \mathcal{D}^{\mathcal{R}_{\tilde{F}(t)}})\) the analogous sheaf with connection associated with the function \(\tilde{F}(t), t \in \mathcal{U}\). There is a flat connection \(\nabla^{\mathcal{M}}, [\nabla^{\mathcal{M}}, \mathcal{D}^{\mathcal{R}_{\tilde{F}(t)}}] = 0\) on the total family of sheaves equipped with covariant derivatives \((\mathcal{R}_{\tilde{F}(t)}, \mathcal{D}^{\mathcal{R}_{\tilde{F}(t)}})\).

Let us consider sections of sheaf \(\mathcal{R}_{\tilde{F}(t)}\) which are represented by oscillating integrals. In other words we consider the subspace \(L_{\text{an}}(t)\) which consists of sections given by integrals of elements of the form \([\exp \left( \frac{1}{h} \tilde{F}(t) \right) \sum_{i \geq 0} h^i \varphi_i]\), where
\[ \sum_{i \geq 0} \frac{1}{\hbar^{i+1}} \in \Gamma(X \times \mathbb{A}^1_{\text{can}}, \Omega^n_{X \times \mathbb{A}^1_{\text{can}}/\mathbb{A}^1_{\text{can}}}) \] (here \( \mathbb{A}^1_{\text{can}} \) denotes the spectrum of the algebra of analytic functions on \( \mathbb{C} \)). For example at \( t = 0 \) the subspace of oscillating integrals can be described explicitly as follows. It is convenient to introduce an auxiliary variable \( P \). Let us denote via \( \xi_k(h) \) the coefficient in front of \( P^k \) in the expression

\[
(3.2) \quad \xi(P, h) := \exp(-P(n+1) \log h)^{\infty} \sum_{d=0}^{\infty} \frac{1}{h^{(n+1)d}(P + 1) \ldots (P + d)^{n+1}}
\]

where \( \frac{1}{P+d} = \frac{1}{d} (1 - \frac{P}{d} + (\frac{P}{d})^2 - \ldots) \)

**Proposition.** ([G])

\[
(3.3) \quad \int_{\Delta_k \subset X} \exp \left( \sum_{i=0}^{n} \frac{x_i}{h} \right) \frac{dx_0 \ldots dx_n}{dx_0 \ldots dx_n} = \xi_k(h), \quad k = 0, \ldots, n
\]

for some locally constant basis \( \{\Delta_k(h)\} \) in \( H_n(X, \Re \frac{t}{h} \ll 0; \mathbb{C}) \).

Consequently the subspace of sections of sheaf \( \mathcal{R}_f \) represented by oscillating integrals can be described explicitly as subspace generated by elements which in the locally constant frame dual to \( \{\Delta_k\} \) are written as

\[ h^{2l+m} \frac{\partial^{k} \xi_k(h)}{\partial h^l} \delta_{k=0, \ldots, n}, \quad l \in \{0, \ldots, n\}, \quad m \in \mathbb{N} \cup \{0\} \]

It is convenient to introduce the algebra \( \mathbb{C}[P]/P^{n+1} \) and to identify \( \{P^k\} \) with locally constant frame dual to \( \{\Delta_k\} \). Then monodromy transformation around \( h = 0 \) acting on cohomology \( H^n(X, \Re \frac{t}{h} \ll 0; \mathbb{C}) \) is given in the basis \( \{P^k\} \) by multiplication by \( \exp(-(n+1)2\pi \sqrt{-1}P) \). Notice that the set \( h^{-P(n+1)}P^k \) is a single-valued frame in \( \mathcal{R}_f \).

Let us consider a completion of the space \( \Gamma(\mathbb{A}^1_{\text{can}} \setminus \{0\}, \mathcal{R}_f) \) defined as Hilbert space \( H = \Gamma_L^2(S^1, \mathcal{R}_f|_{S^1}) \) consisting of \( L^2 \) sections of restriction of the sheaf \( \mathcal{R}_f \) to the circle \( S^1 = \{h| h \in \mathbb{C}, |h| = 1\} \). Explicitly, elements of \( H \) can be written as

\[ h^{-P(n+1)} \sum_{i=0}^{n} P^i \xi_i(h), \text{ where } \xi_i(h) \in L^2(S^1, \mathbb{C}) \]

where \( h^{-P(n+1)} := \exp(-P(n+1) \log h) \). Let us consider Segal-Wilson Grassmanian associated with \( H \). We introduce polarization \( H = H^+ \oplus H^- \) where \( H^+ \) and \( H^- \) are the closed subspaces generated by elements of the form \( h^{-P(n+1)+k}P^i \) for \( k \geq 0 \) and \( k < 0 \) respectively. Recall (see [SW]) that Segal-Wilson Grassmanian consists of closed subspaces \( L \subset H \) "comparable" with \( H^+ \): \( pr_+ : L \to H^+ \) is a Fredholm operator and \( pr_- : L \to H^- \) is a Hilbert-Schmidt operator.

The semi-infinite subspace \( L(t) \subset H \) associated with a point \( t \in \mathcal{U} \) is defined as closure of the subspace of oscillating integrals \( L^{an}(t) \), i.e. it is the closed subspace generated by elements

\[ \sum_{k=0}^{n} P^k \int_{\Delta_k(t)} \exp \left( \frac{\tilde{F}(x; t)}{h} \right) \varphi h^i, \quad i \geq 0, \quad \varphi \in \Gamma_{Zar}(X, \Omega^n_{X}) \]

where \( \tilde{\Delta}_k(t, h) \), is covariantly constant family of elements in \( H^n(X, \Re \frac{\tilde{F}(x; t)}{h} \ll 0; \mathbb{C}) \), \( \tilde{\Delta}_k(0, h) = \Delta_k(h) \). The family of subspaces \( L(t) \) has the following principal property.
Proposition 3.1. The family of subspaces $L(t), t \in \mathcal{U}$ satisfies semi-infinite analog of Griffiths transversality condition with respect to the Gauss-Manin connection $\nabla^M$:

\begin{equation}
\forall i \quad \frac{\partial}{\partial t^i} L(t) \subseteq h^{-1} L(t)
\end{equation}

Proof. \[ \frac{\partial}{\partial t^i} \exp \left( \frac{\tilde{F}(t)}{h} \right) = \frac{1}{h} \frac{\partial \tilde{F}(t)}{\partial t^i} \exp \left( \frac{\tilde{F}(t)}{h} \right) \]

The induced map (symbol of the Gauss-Manin connection):

\[ \text{Symbol}(\nabla^M) : L(t)/(h L(t)) \otimes T_t \mathcal{M} \to (h^{-1} L(t))/L(t) \]

can be described explicitly as follows. The space $L(t)/(h L(t)) \simeq (h^{-1} L(t))/L(t)$ is identified naturally with

\[ \Gamma_{Zar}(X, \Omega_{X}^{n+1})/\{d_1 \tilde{F}(x; t) \wedge d_\alpha \mid \alpha \in \Gamma_{Zar}(X, \Omega_{X})\} \]

The tangent space to the moduli space is equal to

\begin{equation}
T_t \mathcal{M} = \Gamma_{Zar}(X, O_X)/\{\text{Lie}_v (\tilde{F}(x; t)) \mid v \in \Gamma_{Zar}(X, T_X)\}
\end{equation}

Then

\[ \text{Symbol}(\nabla^M)(\varphi \otimes \alpha) = [\varphi \cdot \alpha] \]

4. Solutions to WDVV-equation.

In this section we will show how to construct solution to WDVV-equations associated with $(X, f)$ and any element from an open domain in isotropic affine Grassmannian of $H$.

In the next section we single out a solution which coincides with solution defined by Gromov-Witten invariants of $\mathbb{C}P^n$.

4.1. Picard-Fuchs equation for periods of the semi-infinite Hodge structure. We start from fixing a choice of semi-infinite subspace $S \subset H$, $h^{-1} S \subset S$ transversal to $L(0)$:

\begin{equation}
H = L(0) \oplus S
\end{equation}

Next we would like to choose an element $\Omega_0 \in L(0)$ such that the symbol of Gauss-Manin connection restricted to the class $[\Omega_0] \mod (h L([f]))$ in the quotient $L([f])/(h L([f]))$ gives an isomorphism

\begin{equation}
\text{Symbol}(\nabla^M)([\Omega_0] \otimes \cdot) : T_{[f]} \mathcal{M} \to (h^{-1} L([f]))/L([f])
\end{equation}

is an isomorphism. It is easy to see that

\[ T_{[f]} \mathcal{M} \simeq \mathbb{C}[p]/(p^{n+1} - 1) \quad \text{where} \quad p = [(1/n + 1) \sum_{i=0}^{n} x_i] \]

\[ L([f])/(h L([f])) \simeq \mathbb{C}[p]/(p^{n+1} - 1) \quad [\frac{\partial x_0 \ldots \partial x_n}{\partial (x_0 \ldots x_n)}] \]

So one can take here as $\Omega_0$ for example an arbitrary element of the form

\[ \exp \left( \frac{1}{h} f \right) \left( \sum_{i=0}^{n} x_i \right)^{k} + h \omega_1 + \ldots \left( \frac{\partial x_0 \ldots \partial x_n}{\partial (x_0 \ldots x_n)} \right) = k, \ldots, n \]
The transversality condition (4.1) implies that the intersection of \( L(t) \) for \( t \in \mathcal{U} \) with affine space \( \{ \Omega_0 + \alpha \mid \alpha \in S \} \) consists of a single element. Let us denote it via \( \Psi^S \):

\[
\{ \Psi^S(t, h) \} = L(t) \cap \{ \Omega_0 + \alpha \mid \alpha \in S \}
\]

Notice that the property (4.2) and the mini-versality of the family \( \hat{F}(t) \) implies that \( \{ \partial_t \Psi^S(t) \in h^{-1} L(t) \mod L(t) \} \) is a basis in finite-dimensional vector space \( (h^{-1} L(t))/L(t) \) and, therefore, \( \{ \partial_t \Psi^S(t) \in S \mod h^{-1} S \} \) is a basis in finite-dimensional vector space \( S/(h^{-1} S) \). Hence the map

\[
(4.3) \quad \Psi^S(t, \infty) := [\Psi^S(t, h) - \Omega_0] \in S \mod hS
\]

is a local isomorphism and induces a set of coordinates on \( \mathcal{U} \) which we denote by \( \{ t^{a}_S \} \). Let us denote via \( \Psi^{S,k} = \int \Delta(t; \Omega) \Psi^S \) the components of \( \Psi^S \) with respect to the basis \( P^k \).

**Proposition 4.1.** The periods \( \Psi^{S,k}(t^{a}_S, h) \) satisfy

\[
(4.4) \quad \frac{\partial^2 \Psi^{S,k}}{\partial t^{a}_S \partial t^{b}_S} = h^{-1} \sum_{c} A^{c}_{a}(t) \frac{\partial \Psi^{S,k}}{\partial t^{c}_S}
\]

**Proof.** It follows from the prop. 3.4 that

\[
\frac{\partial^2 \Psi^{S}}{\partial t^{a} \partial t^{b}} \in h^{-2} L(t)
\]

Therefore one has

\[
\frac{\partial^2 \Psi^{S}}{\partial t^{a} \partial t^{b}} - h^{-1} \sum_{m} \left( A^{(m)} \right)_{ij}(t) \frac{\partial \Psi^{S}}{\partial t^{m}} - \sum_{m} \left( A^{(0)} \right)_{ij}(t) \frac{\partial \Psi^{S}}{\partial t^{m}} \in L(t)
\]

for some \( (A^{(m)})_{ij}(t), (A^{(0)})_{ij}(t) \in \mathcal{O}_C^{analytic} \). On the other hand,

\[
\frac{\partial^2 \Psi^{S}}{\partial t^{a} \partial t^{b}}, \frac{\partial \Psi^{S}}{\partial t^{c}} \in S
\]

Therefore

\[
\frac{\partial^2 \Psi^{S}}{\partial t^{a} \partial t^{b}} = h^{-1} \sum_{m} \left( A^{(m)} \right)_{ij}(t) \frac{\partial \Psi^{S}}{\partial t^{m}} + \sum_{m} \left( A^{(0)} \right)_{ij}(t) \frac{\partial \Psi^{S}}{\partial t^{m}}
\]

The coordinates \( \{ t^{a}_S \} \) were chosen so that \( \Psi^S \mod h^{-1} S : \mathcal{U} \rightarrow S/h^{-1} S \) is linear in \( \{ t^{a}_S \} \). Therefore in these coordinates \( (A^{(0)})_{ab} = 0 \). \( \square \)

**Corollary 4.2.** One has \( [A, A] = 0, dA = 0 \) for \( A = \sum_{a} A^{c}_{ab}(t_S) dt^{c}_S \).

In particular, the following formula defines commutative and associative product \( \circ \) on tangent sheaf of \( \mathcal{U} \):

\[
(4.5) \quad \frac{\partial}{\partial t^{a}} \circ \frac{\partial}{\partial t^{b}} := \sum_{c} A^{c}_{ab}(t_S) \frac{\partial}{\partial t^{c}}
\]

This algebra structure can be written explicitly as follows. The tangent space at \( t \in \mathcal{M}^{(X, t)} \) is given by the quotient 3.5. The algebra structure is induced from the algebra structure on \( \mathcal{O}_X \). Notice that this is defined correctly since \( \{ \text{Lie}_v(\hat{F}(x; t)) \mid v \in \Gamma_{Z_{ab}}(X, T_X) \} \) is an ideal.
4.2. Flat metrics. Next ingredient needed for construction of the solution to WDVV-equation is a flat metric on $\mathcal{M}$ compatible with multiplication (4.5). In this subsection we show that the period map $\Psi^S$ induces such a metric under condition that the semi-infinite subspace $S$ is isotropic with respect to certain pairing.

Let us notice that sheaf $\mathcal{R}_{F(t)}$ is equipped with natural pairing between the opposite fibers:

$$G(h) : \; H^0(X, \text{Re} \frac{\tilde{F}(t)}{h} \ll 0; \mathbb{C}) \otimes H^0(X, \text{Re} - \frac{\tilde{F}(t)}{h} \ll 0; \mathbb{C}) \to \mathbb{C}$$

which is given by the Poincare pairing dual to intersection pairing on relative cycles. Hence by continuity one has natural pairing on elements of $L(t)$, $t \in \mathcal{U}$.

**Lemma 4.3.** For $\alpha(h), \beta(h) \in L(t) \subset H$

$$G(\alpha, \beta) \in h^n \mathbb{C}[[h]]$$

**Proof.** One has for $h$ from any sector $a < \text{arg}(h) < b$, $b - a < \pi$ and $\alpha(h), \beta(h) \in L^n(t)$:

$$G(\alpha, \beta) = \sum_{i,j} \#(\Delta^+_i \cap \Delta^-_j)(\int_{\Delta^+_i} \alpha(h))(\int_{\Delta^-_j} \beta(-h))$$

where $\{\Delta^+_i\}, \{\Delta^-_j\}$ are locally constant frames of relative cycles in $H_n(X, \text{Re} \frac{\tilde{F}(t)}{h} \ll 0; \mathbb{C})$, $H_{n+1}(X, \text{Re} - \frac{\tilde{F}(t)}{h} \ll 0; \mathbb{C})$ correspondingly. Asymptotic expansion for the integrals $\int_{\Delta^+_i} \alpha(h), \int_{\Delta^-_j} \beta(-h)$ at $h \to 0$ can be evaluated using the steepest-descent method. Given a non-degenerate critical point $p$ of $\tilde{F}(t)$ and a metric one has two relative cycles $\Delta^+_p, \Delta^-_p$ from $H_n(X, \text{Re} \frac{\tilde{F}(t)}{h} \ll 0; \mathbb{C})$ and $H_{n+1}(X, \text{Re} - \frac{\tilde{F}(t)}{h} \ll 0; \mathbb{C})$ respectively which are formed by gradient lines $\xi(t)$ of $\text{Re} \tilde{F}(t)$ such that $\xi \to p$ as $t \to +\infty$ ($t \to -\infty$ correspondingly). Then one has the following asymptotic expansions

$$\int_{\Delta^+_p} e^{\frac{\tilde{F}(t)}{h}} (\varphi_0 + h\varphi_1 + \ldots) \sim e^{\frac{\tilde{F}(p)}{h}}(\text{const}^{(+)} h^\frac{2}{2+n} + O(h^\frac{n+1}{2+n}))$$

$$\int_{\Delta^-_p} e^{-\frac{\tilde{F}(t)}{h}} (\varphi_0 - h\varphi_1 + \ldots) \sim e^{-\frac{\tilde{F}(p)}{h}}(\text{const}^{(-)} h^\frac{2}{2+n} + O(h^\frac{n+1}{2+n}))$$

In fact it is easy to see that $f$ has $(n + 1)$ critical points $p$: $x_0 = \ldots = x_n = \exp \frac{2\pi n}{n+1}$ and that the relative cycles $\{\Delta^+_p\}$, (resp. $\{\Delta^-_p\}$) form basis in $H_n(X, \text{Re} \frac{\tilde{F}(t)}{h} \ll 0; \mathbb{C})$, (resp. $H_{n+1}(X, \text{Re} - \frac{\tilde{F}(t)}{h} \ll 0; \mathbb{C})$). Therefore,

$$G(\alpha, \beta) \sim \text{const} h^n + O(h^{n+1})$$


Let us assume now that $S$ is isotropic with respect to $G$ in the following sense:

$$\forall \; \alpha(h), \beta(h) \in S \; \; G(\alpha, \beta) \in h^{n-2} \mathbb{C}[[h^{-1}]]$$

An example of such subspace is considered in the next section. It is the subspace generated by elements $(hP)^k h^{m-(n+1)p}$, $k \in \{0, \ldots, n\}$, $m \in \mathbb{N}$.

The period map $\Psi^S$ induces a pairing on $T\mathcal{M}$:

$$\frac{\partial}{\partial t^a} \otimes \frac{\partial}{\partial t^b} \to G(\frac{\partial \Psi^S}{\partial t^a} \cdot \frac{\partial \Psi^S}{\partial t^b})$$
Proposition 4.4. For isotropic $S$ the pairing induced by $\Psi^S$ satisfies

$$G(\frac{\partial \Psi^S}{\partial t^a}, \frac{\partial \Psi^S}{\partial t^b}) = h^{n-2} g_{ab}$$

with $g_{ab}(t)$ symmetric and non-degenerate. In the coordinates $\{t^S_S\}$ induced by the map $\Psi^S(t, h = \infty)$ the 2-tensor $g_{ab}$ is constant.

Proof. The property (4.9) follows from $\partial_a \Psi^S(t, h) \in S \cap h^{-1}L(t)$. Notice that for any $\alpha(h), \beta(h) \in H$:

$$G(\alpha, \beta)(h) = (-1)^n G(\beta, \alpha)(-h)$$

It follows that $g_{ab}$ is symmetric. Non-degeneracy of $g_{ab}$ follows from the analogous property of $G$.

Notice that $G$ is obviously $\mathbb{C}[h^{-1}]$-linear and therefore one has induced pairing $G_0$ with values in $\mathbb{C}$: $h^{n-2}$ on elements of $S/h^{-1}S$. It follows from (4.9) that

$$G(\frac{\partial \Psi^S}{\partial t^a}, \frac{\partial \Psi^S}{\partial t^b}) = G_0(\frac{\partial \Psi^S}{\partial t^a}, \frac{\partial \Psi^S}{\partial t^b})$$

By definition of the coordinates $\{t^S_S\}$ one has $\frac{\partial \Psi^S}{\partial t^S} = \text{constant}$ as an element of $S/h^{-1}S$. \qed

Proposition 4.5. The metric $g_{ab}$ is compatible with multiplication defined by $A^c_{ab}(t_S)$:

$$\forall a, b, c \quad g(a \circ b, c) = g(a, b \circ c)$$

Proof. Notice that $G$ is locally constant with respect to the Gauss-Manin connection $\nabla^M$. Therefore

$$\partial_b G(\partial_a \Psi, \partial_c \Psi) = G(\partial_b \partial_a \Psi, \partial_c \Psi) + G(\partial_a \Psi, \partial_b \partial_c \Psi)$$

In the coordinates $\{t^S_S\}$ one has

$$\partial_b G(\partial_a \Psi, \partial_c \Psi) = 0, \quad \partial_a \partial_b \Psi = h^{-1} \sum_d A^d_{ab}(t_S) \partial_d \Psi$$

The pairing $G$ satisfies $G(h^{-1}a, \beta) = -G(a, h^{-1}\beta) = h^{-1}G(a, \beta)$ \qed

Let us denote

$$A_{abc}(t_S) := g(\partial_a \circ \partial_b, \partial_c)$$

Corollary 4.2 and proposition 4.5 imply now that

$$A_{abc}(t_S) = \partial_a \partial_b \partial_c \Phi^S(t_S)$$

for some function $\Phi^S$ defined up to addition of terms of order $\leq 2$, and that the function $\Phi^S(t_S)$ satisfy the system of WDVV-equations:

$$\forall a, b, c, d \quad \sum_{c, f} \partial_a \partial_b \partial_c \Phi^S g^{cf} \partial_f \partial_c \partial_d \Phi^S = \sum_{c, f} \partial_a \partial_d \partial_c \Phi^S g^{cf} \partial_f \partial_b \partial_d \Phi^S$$
4.3. **Symmetry vector field.** Let us notice that the subspace \( L(t) \) satisfies
\[
\mathcal{D} L(t) \subseteq h^{-2} L(t)
\]
Assume now that the subspace \( S \subset H \) satisfies the condition
\[
\mathcal{D} S \subseteq h^{-1} S
\]
Notice that this condition is enough to check only on finite set of elements from \( S \) which form a basis of \( S/h^{-1} S \).

**Proposition 4.6.** The periods \( \Psi^{S,k}(t_S) \) satisfy
\[
\frac{\partial \Psi^{S,k}}{\partial h} = h^{-1} E^a(t_S) \frac{\partial \Psi^{S,k}}{\partial t_S^a}
\]
for some vector field \( E = \sum_a E^a(t_S) \partial_a \)

**Proof.** The arguments are similar to the arguments from the proof of the proposition 4.1. \( \square \)

**Proposition 4.7.** The vector field \( E \) acts as a conformal symmetry on the multiplication \( \circ \) and the metric \( g_{ab} \), that is, for any vector fields \( u, v \in T_M \)
\[
[E, u \circ v] - [E, u] \circ v - u \circ [E, v] = u \circ v
\]

(4.15) \( \text{Lie}_E g(u, v) - g([E, u], v) - g(u, [E, v]) = (2 - n) g(u, v) \)

**Proof.** It follows from \([\nabla^M, \mathcal{D}^R \rho \circ] = 0 \) and \( \mathcal{D}^R \rho \circ G = 0 \) respectively. \( \square \)

4.4. **Flat identity vector field.** Assume that the subspace \( S \) satisfies the condition
\[
h^{-1} \Omega_0 \in S
\]
Let \( e \) denotes vector field on \( \mathcal{M} \) which is affine in the coordinates \( \{t_S\} \) and which corresponds to the class of \( h^{-1} \Omega_0 \) in the quotient \( S/h^{-1} S \). We can assume that \( e = \frac{\partial}{\partial t_S^a} \) where \( t_S^a \) is one of the affine coordinates.

**Proposition 4.8.** The vector field \( e = \frac{\partial}{\partial t_S^a} \) is the identity with respect to the multiplication \( \circ \).

**Proof.** One has \( h^{-1} \Psi^S \in h^{-1} L(t) \), \( h^{-1} \Psi^S \in S \). Therefore
\[
h^{-1} \Psi^S(t_S) = \frac{\partial \Psi^S(t_S)}{\partial t_S^a}
\]
Differentiating this equality with respect to \( \frac{\partial}{\partial t_S^a} \) and using eq.(4.4) we see that
\[
\frac{\partial}{\partial t_S^a} \circ \frac{\partial}{\partial t_S^b} = \frac{\partial}{\partial t_S^a}
\]
In summary, we have

**Proposition 4.9.** For any semi-infinite subspace satisfying the properties (4.1), (4.7), (4.12), (4.16) we have constructed Frobenius manifold structure on the moduli space \( \mathcal{M}(X, f) \).

For the definition of Frobenius structure see \([D],[KM]\).
Remark 4.10. In fact the same statement holds true for arbitrary pair \((X, f)\) satisfying homological Calabi-Yau type condition: there exists \(\Omega_0 \in \Gamma_{Zar}(X, \Omega^*)\) such that the restriction on \([\Omega_0]\) of the symbol of Gauss-Manin condition induces isomorphism \(T_{[f]} M \simeq (\hbar^{-1} L([f]))/L([f])\). The arguments are the same.

5. Rational Gromov-Witten invariants of \(\mathbb{CP}^n\) and oscillating integrals.

Consider the subspace \(S_0 \subset H\) generated by elements
\[
(hP)^k \hbar^{n-(n+1)P}, \quad k \in \{0, \ldots, n\}, \quad m \in \mathbb{N} \cup \{-1, 0\}
\]
Let us also put \(\Omega_0 = [e^{\frac{\partial}{\partial \Omega_0}} \frac{dx_0}{dz_0} \cdots \frac{dx_n}{dz_n}] \in L_0\). In this section we show that for \(S = S_0\) the quasi-homogeneous solution to WDVV equation \(\partial^2 \Phi^{S_0}(t_{S_0})\) constructed in the previous section coincides with the generating function of Gromov-Witten invariants of \(\mathbb{CP}^n\). The idea is to identify some numerical invariants of these two solutions and then use reconstruction theorem - I from [KM].

Proposition 5.1. The subspace \(S_0\) has the properties (4.1), (4.7), (4.12), (4.16).

Proof. Properties (4.12) (4.16) are easy to check. In order to check the property (4.1) let us notice that the subspace \(L(0)\) is generated by elements
\[
h^{2k+m} \frac{\partial k \xi}{\partial h^k} \mod P^{n+1}, \quad k \in \{0, \ldots, n\}, \quad m \in \mathbb{N} \cup \{0\}
\]
and that
\[
h^{(n+1)P} \frac{\partial k \xi}{\partial h^k} \mod P^{n+1} = \text{const} \ h^{-k} P^k + O(h^{-(k+1)})
\]
To check the property (4.7) let us notice that since \(\{P^k\}\) is dual to locally constant basis \(\{\Delta_k\}\) and \(P^i h^{-(n+1)P}\) are single-valued sections therefore for any \(i, j\) the pairing \(G(P^i h^{-(n+1)P}, P^j h^{-(n+1)P})\) does not depend on \(h\):

\[
\forall \ i, j \ G(P^i \exp(-(n+1) \log(h)P), P^j \exp(-(n+1) \log(h)P)) = \text{const}
\]
Differentiating this equality with respect to \(\frac{\partial}{\partial h}\) one gets
\[
G(P^j h^{-(n+1)P}, P^j h^{-(n+1)P}) = G(P^i h^{-(n+1)P}, P^{i+j} h^{-(n+1)P})
\]
Therefore \(G(P^i h^{-(n+1)P}, P^{i+j} h^{-(n+1)P})\) for \(i + j > n\). \(\square\)

It follows from lemmas

Let us consider system of local coordinates \(\{t^k_{S_0}\}, k = 0 \ldots n\), induced by the map (4.3) from linear coordinates on \(S/h^{-1} S\) corresponding to the basis given by classes of elements \(P^k h^{k-1-(n+1)P}\) modulo \(h^{-1} S\). Notice that this notation is compatible with notation for the flat identity from §4.4 since \([h^{-1} \Omega_0] = [h^{-1-(n+1)P}].\) We put below \(y^k = t^k_{S_0}\) to simplify notations.

Lemma 5.2. In coordinates \(y^k\) the symmetry vector field is written as
\[
E(y^k) = \left( \sum_{k=0}^{n} (k - 1) y^k \frac{\partial}{\partial y^k} \right) - (n + 1) \frac{\partial}{\partial y^1}
\]
Proof. One has
\[ \Psi^S_0(y) = \Omega_0 + \sum_{k=0}^n h^{k-1-(n+1)P} \frac{P^k i^k}{\partial y^k} \mod h^{-1}S_0 \]

Therefore
\[ \frac{\partial \Psi^S_0(y)}{\partial h} = - (n+1)P h^{-1-(n+1)P} + \sum_{k=0}^n h^{k-2-(n+1)P} \frac{P^k (k-1)y^k}{\partial y^k} \mod h^{-2}S_0 \]

and
\[ \frac{\partial \Psi^S_0(y)}{\partial y^k} = \sum_{k=0}^n h^{k-1-(n+1)P} \frac{P^k}{\partial y^k} \mod h^{-1}S_0 \]

This is an exact equality by proposition 4.6.

Lemma 5.3. \[ \frac{\partial \Psi^{S_0+i}(y)}{\partial y^k} \bigg|_{y=0} = \frac{1}{h(n+1)^k} \int_{\Delta_i} f^k \exp \left( \frac{1}{h} f \right) \frac{dx_0...dx_n}{d(x_0...x_n)} \mod hL(0) \]

In other words at \( y = 0 \) the elements \( \frac{\partial}{\partial y^k} \) correspond to elements \( \sum_{n+1}^n \) \( \in \Gamma_{zar}(X, O_X) / \{ Lie_v(f) \mid v \in \Gamma_{zar}(X, T_X) \} \) from \( T \{ f \}, M(X) \)

Proof. One has
\[ h^{k-1} \left( \frac{\partial}{\partial h} \right)^k \Psi^S_0 \bigg|_{y=0} = h^{k-1}(- (n+1)P)^k h^{-(n+1)P} \mod h^{-1}S_0 = \]
\[ = (- (n+1))^k \frac{\partial \Psi^S_0(y)}{\partial y^k} \bigg|_{y=0} \mod h^{-1}S_0 \]

we see that
\[ h^{k-1} \left( \frac{\partial}{\partial h} \right)^k \Psi^S_0 \bigg|_{y=0} = - (n+1))^k \frac{\partial \Psi^S_0(y)}{\partial y^k} \bigg|_{y=0} \in h^{-1}L(0) \cap h^{-1}S_0 \]

Therefore
\[ h^{k-1} \left( \frac{\partial}{\partial h} \right)^k \Psi^S_0 \bigg|_{y=0} = (- (n+1))^k \frac{\partial \Psi^S_0(y)}{\partial y^k} \bigg|_{y=0} \]

and
\[ \frac{\partial \Psi^{S_0+i}(y)}{\partial y^k} \bigg|_{y=0} = \frac{1}{h(n+1)^k} \int_{\Delta_i} f^k \exp \left( \frac{1}{h} f \right) \frac{dx_0...dx_n}{d(x_0...x_n)} \mod hL(0) \]

Corollary 5.4. At \( y = 0 \) the multiplication (4.5) is given by
\[ A^k_{ij}(0) = \delta_{i+j-k mod n,0} \]

in the basis \( \{ \frac{\partial}{\partial y^k} \} \).

Proof. \( \Gamma_{zar}(X, O_X) / \{ Lie_v(f) \mid v \in \Gamma_{zar}(X, T_X) \} \simeq \mathbb{C}[p]/(p^{n+1} - 1) \) with \( p = \sum_{n+1}^n \).
Let us calculate now the pairing \((4.8)\) written in coordinates \(y^k\). By proposition 4.4 and lemma 5.3 the value of \(h^{-n+2}G(\frac{\partial \Phi^S_n}{\partial y^i}, \frac{\partial \Phi^S_n}{\partial y^j})\) is equal to the coefficient in front of \(h^n\) in
\[
(5.1) \quad G(\frac{1}{n+1} \sum_{i=0}^n f^i \exp(\frac{1}{h} f) \frac{dx_0 \ldots dx_n}{d(x_0 \ldots x_n)}, \frac{1}{(n+1)} \sum_{i=0}^n f^i \exp(\frac{1}{h} f) \frac{dx_0 \ldots dx_n}{d(x_0 \ldots x_n)}) =
\]
\[
= \sum_{m=0}^n \frac{1}{(n+1)^{i+j}} \left( \int_{\Delta_{pm}^+} f^i e^{\frac{x}{h}} \frac{dx_0 \ldots dx_n}{d(x_0 \ldots x_n)} \right) \left( \int_{\Delta_{pm}^-} f^j e^{\frac{x}{h}} \frac{dx_0 \ldots dx_n}{d(x_0 \ldots x_n)} \right)
\]
where \(\{p_m\}\) is the set of critical points of \(f\), \(p_m : x_0 = \ldots = x_n = \zeta^m, \zeta = \exp(\frac{2\pi \sqrt{-1}}{n+1})\), and \(\Delta_{pm}^+, \Delta_{pm}^-\) are the associated relative cycles defined in the proof of lemma 4.3. The lowest order coefficient in \((5.1)\) can be calculated by expanding \(f\) near critical points up to second order:
\[
(5.2) \quad \frac{1}{(n+1)^{i+j}} \sum_{m=0}^n \int_{\Delta_{pm}^+} (f(p_m))^i \exp(\frac{1}{h} (f(p_m) + \zeta^m \sum_{1 \leq k \leq n} (1 - \frac{x_k}{\zeta^m})(1 - \frac{x_l}{\zeta^m})) \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n})
\]
\[
\int_{\Delta_{pm}^-} (f(p_m))^j \exp(\frac{1}{h} (f(p_m) + \zeta^m \sum_{1 \leq k \leq n} (1 - \frac{x_k}{\zeta^m})(1 - \frac{x_l}{\zeta^m})) \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n}) \sim \frac{h^n}{(n+1)} \sum_{m=0}^n \zeta^m (i+j-n) + O(h^{n+1}) = h^n \delta_{i+j-n,0} + O(h^{n+1})
\]

**Corollary 5.5**. The pairing on \(TU\) given by
\[
< \frac{\partial}{\partial \Phi^S_n}, \frac{\partial}{\partial \Phi^S_n} > = \delta_{i+j,n}
\]
satisfies the conditions \((4.10)\) and \((4.15)\).

Let us consider Taylor expansion of our solution \(\Phi^S(y)\), where \(\frac{\partial^j}{\partial y^j} \Phi^S(y) = \sum_m g_{ijm} A^m_j(y)\) and we assume that the Taylor expansion of \(\Phi^S(y)\) has no terms of order \(\leq 2\).

**Lemma 5.6**. The Taylor expansion of \(\Phi^S(y)\) has the form
\[
(5.3) \quad \Phi^S = \frac{1}{2} \sum_{ij} y^i y^j \delta_{i+j,n} + \sum_{m_n} \frac{\sigma(m_1, \ldots, m_n)}{m_1! \ldots m_n!} (y^1)^{m_1} \ldots (y^n)^{m_n}
\]
with
\[
(5.4) \quad \sigma(m_1 + 1, m_2, \ldots, m_n) = \frac{3-n + \sum_{k=2}^n (k-1)m_k}{n+1} \sigma(m_1, m_2, \ldots, m_n)
\]
and if \(\frac{3-n + \sum_{k=2}^n (k-1)m_k}{n+1} \not\in \mathbb{N} \cup \{0\}\) then
\[
(5.5) \quad \sigma(m_1, m_2, \ldots, m_n) = 0
\]

**Proof**. The form of the term which contains \(y^0\) follows from lemma 5.4. The equation \((5.4)\) follows from \((4.14)\),\((4.15)\). Therefore the expansion \((5.3)\) can be rewritten as an expansion in powers of \((y^0, e^{\frac{y^1}{n+1}} = (y^1)^0 + \frac{y^1}{n+1} + \ldots, y^2, \ldots, y^n)\). To prove that \(\partial^a \Phi^S(y)\) can be expanded in integral powers of \(e^{y^1}\) let us introduce an additional parameter. Namely, let us consider family \((X^r, f)\) where \(f = x_0 + \ldots + x_n\).
and $X_q \subset \mathbb{C}^n$ is defined by equations $x_0 \cdots x_n = q$, $q \in \mathbb{C}$, $|q - 1| < 1$. Identifying $X_q$ with $X$ via multiplication of every coordinate by $q^{\frac{1}{P(n+1)}}$ we see that $(X_q, f) \sim (X, q^{\frac{1}{P(n+1)}} f)$ as deformations of $(X, f)$. It follows from (3.3) that

$$\sum_{k=0}^{n} P^k \int_{\Delta_k} \exp(\frac{q^{\frac{1}{P(n+1)}}}{h} \frac{dx_0 \cdots dx_n}{d(x_0 \cdots x_n)}) = h^{-(n+1)P} q^P (1 + O(\frac{1}{h}))$$

we see that $[\exp(\frac{q^{\frac{1}{P(n+1)}}}{h} \frac{dx_0 \cdots dx_n}{d(x_0 \cdots x_n)})] \in S_0 + [\Omega_0]$ and that

$$[\exp(\frac{q^{\frac{1}{P(n+1)}}}{h} \frac{dx_0 \cdots dx_n}{d(x_0 \cdots x_n)}) - [\Omega_0] = \ln q P h^{-(n+1)P} \mod h^{-1} S$$

Therefore $(X_q, f)$ corresponds to $(0, y_1, 0, \ldots, 0) \in \mathcal{U}$, $q = e^{y_1}$, in the moduli space of deformations of $(X, f)$. One can now repeat the above story starting with $(X_q, f)$, $q \in \mathbb{C} \setminus \{0\}$ and consider element $\Psi^{S_0}(y, h; q)$ representing periods of the variation of semi-infinite Hodge structures associated with moduli space of deformations of $(X_q, f)$. The normalization condition is given by

$$\Psi^{S_0}(y, h; q) = [\exp(\frac{q^{\frac{1}{P(n+1)}}}{h} \frac{dx_0 \cdots dx_n}{d(x_0 \cdots x_n)})] \in S_0$$

and the parameters $\{y^i\}$ are specified via

$$\Psi^{S_0}(y, h; q) = [\exp(\frac{q^{\frac{1}{P(n+1)}}}{h} \frac{dx_0 \cdots dx_n}{d(x_0 \cdots x_n)}) + \sum_{k=0}^{n} P^k h^{k-1-(n+1)P} y^k \mod h^{-1} S_0$$

Notice that this is correctly defined since $\exp(2\pi i P) S_0 = S_0$ and therefore $S_0$ is invariant under monodromy around $q = 0$. It follows from (5.6) that

$$\Psi^{S_0}(y_0, y_1, \ldots, y_n, h; q) = \Psi^{S_0}(y_0, 0, \ldots, y_n, h; e^{y_1} q)$$

Therefore expansion of $\partial^\Psi S_0(y)$ can contain only integral powers of $e^{y_1}$ which gives equality (??).

**Theorem 5.7.** The Taylor expansion of the function $\Phi^{S_0}(y)$ coincides with generating series for Gromov-Witten invariants of $\mathbb{C}^{P^n}$:

$$\Phi^{S_0}(y) = \mathcal{F}^{\mathbb{C}^{P^n}}(y)$$

**Proof.** It follows from lemmas 5.2, 5.6, corollaries 5.4, 5.5 and theorem 3.1 from [KM] \qed

**Remark 5.8.** Similar arguments as above prove analogous result in the case of complete intersection of $r$ hypersurfaces of degrees $l_1, \ldots, l_r$ in $\mathbb{C}^{P^{n+r}}$ with $\sum_i l_i \leq n+r$. The details will appear elsewhere.

**Remark 5.9.** It is an interesting open question whether a homological mirror symmetry conjecture can be formulated in this case. A natural variant of twisted derived category of coherent sheaves associated with $(X, f)$ seems to have no nontrivial objects (this was communicated to me by M. Kontsevich).
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