Diophantine Approximation in Negatively Curved Manifolds and in the Heisenberg Group

S. HERSONSKY
F. PAULIN

(with an appendix: Diophantine Approximation on Hyperbolic Surfaces, by J. Parkkonen and F. Paulin)

DMA - 01 - 04
Diophantine Approximation in Negatively Curved Manifolds and in the Heisenberg Group

S. HERSONSKY*
F. PAULIN**

DMA - 01 - 04

February 2001

Département de mathématiques et applications - École normale supérieure
45 rue d’Ulm 75230 PARIS Cedex 05
Tel : (33)(1) 01 44 32 30 00
E-mail : preprints@dma.ens.fr

* Ben Gurion University, Israel
E-mail : saarh@math.bgu.ac.il

** Département de Mathématiques et Applications, École Normale Supérieure
E-mail : Frederic.Paulin@ens.fr
Diophantine Approximation in Negatively Curved Manifolds and in the Heisenberg Group

Sa’ar Hersonsky Frédéric Paulin

Abstract

This paper is a survey of the work of the authors [20, 1, 21], with a new application to Diophantine approximation in the Heisenberg group. The Heisenberg group, endowed with its Carnot Caratheodory metric, can be seen as the space at infinity of the complex hyperbolic space (minus one point). The rational approximation on the Heisenberg group can be interpreted and developed using arithmetic subgroups of \( SU(n, 1) \). In the appendix, the case of hyperbolic surfaces is developed by Jouni Parkkonen and the second author.

1 Introduction

This paper is a survey of the work of the authors [20, 1, 21], with a new application to Diophantine approximation in the Heisenberg group. In the appendix, the case of hyperbolic surfaces is developed by Jouni Parkkonen and the second author.

Let \( M \) be a geometrically finite pinched negatively curved Riemannian manifold with at least one cusp. Inspired by the theory of Diophantine approximation of a real (or complex) number by rational ones, we developed a theory of approximation of geodesic lines starting from a given cusp by ones returning to it. We proved [20] a Dirichlet type theorem, expressing its Hurwitz type constant in terms of the lengths of closed geodesics and their depths outside the cusp neighbourhood. Using the cut locus of the cusp, we defined [20] an explicit approximation sequence for geodesic lines starting from the cusp. We proved [21] a Khinchine-Sullivan type theorem on the Hausdorff measure of the geodesic lines starting from a cusp that are well approximated by cusp returning ones.

We state some of these results here, illustrating them by the well-known examples of the modular curve and the Bianchi orbifolds. We have tried to keep in section 2 the informal presentation of the talk at the March 2000 Cambridge conference. Connections between Diophantine approximation problems and hyperbolic geometry have already been studied a lot (see for instance [8, 12, 13, 16, 17, 22, 30, 35, 33, 36, 38, 40, 24]) and we apologize for the omissions and a very partial presentation of the field. The main point of this work is to work in variable curvature, with surprises coming from the fact that the geometry of the cuspidal ends can be much more complicated. We get new applications even in the constant curvature case. The case when \( M \) is a complex hyperbolic manifold yields new important results about the Diophantine approximation in the Heisenberg group.

The authors thank Stéphane Fischler and Michel Waldschmidt for numerous conversations about the rational approximation in the Heisenberg group.
2 The survey part

Let $M$ be a smooth complete Riemannian manifold, with sectional curvature $-\kappa^2 \leq K \leq -1$ where $1 \leq \kappa < +\infty$. We will assume in this paper that $M$ has finite volume, and has exactly one cusp. See [20, 1, 21] for the general case.

Two geodesic rays are asymptotic if they remain within bounded Hausdorff distance. A cusp $e$ in $M$ is an asymptotic class of minimizing geodesic rays in $M$ along which the injectivity radius converges to 0. We say that a geodesic ray converges to $e$ if some subray belongs to the class $e$.

Since $M$ has finite volume and one cusp, it has exactly one end, whose neighbourhoods are exponentially thin near infinity. More precisely, by Gromov’s theorem on almost flat manifolds (see for instance [5]), there exists an infranilmanifold $N$ and a homeomorphism from a neighbourhood of the end to $N \times [0, +\infty]$, such that the diameter of the image in $M$ of $N \times \{t\}$ is less than $e^{-t}$. (An infranilmanifold is just a quotient of a nilpotent Lie group, endowed with a left-invariant metric, by a discrete torsion-free group of isometries.)

Example 1: Let $\mathbb{H}^2_\mathbb{R}$ be the real hyperbolic plane, seen as the upper halfplane $\{(x, t) \in \mathbb{R}^2 : t > 0\}$ with the Riemannian metric $ds^2 = \frac{dx^2 + dy^2}{t^2}$, that has constant curvature $-1$. We identify with the (orientation preserving) isometry group of $\mathbb{H}^2_\mathbb{R}$ by sending $\pm \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$ to $z \mapsto \frac{az + b}{cz + d}$. Let $\operatorname{PSL}_2(\mathbb{Z})$ be the modular subgroup of $\operatorname{PSL}_2(\mathbb{R})$. Then $\operatorname{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}^2_\mathbb{R}$ is an orbifold, with two orbifold points of order 2 and 3, and one cusp.

Example 2: Let $d$ be a squarefree positive integer. Let $\mathcal{O}_d$ be the ring of integers in the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. Let $\mathbb{H}^3_\mathbb{R}$ be the real hyperbolic space of dimension 3, seen as the upper halfspace $\{(x, y, t) \in \mathbb{R}^3 : t > 0\}$ with the Riemannian metric $ds^2 = \frac{dx^2 + dy^2 + dt^2}{t^2}$, that has constant curvature $-1$. We identify $\operatorname{PSL}_2(\mathbb{C})$ with the (orientation preserving) isometry group of $\mathbb{H}^3_\mathbb{R}$. Let $\operatorname{PSL}_2(\mathcal{O}_d)$ be the Bianchi subgroup of $\operatorname{PSL}_2(\mathbb{C})$. Then $\operatorname{PSL}_2(\mathcal{O}_d) \backslash \mathbb{H}^2_\mathbb{R}$ is an orbifold. Note that this orbifold has one and only one cusp if and only if $d = 1, 2, 3, 7, 11, 19, 43, 67, 163$ (see for instance [37]).

Both examples are orbifolds and not manifolds. But there are only minor modifications, left to the reader, in order for the following arguments to be adapted for them.

The end $e$ of $M$ has a canonical neighbourhood, called the maximal Margulis neighbourhood, constructed as follows (see for instance [5]).

We first recall some elementary facts about the universal covering of negatively curved manifolds, see for instance [4]. Let $\tilde{M}$ be a fixed universal cover of $M$, with covering group $\Gamma$. The boundary $\partial \tilde{M}$ of $M$ is the set of asymptotic classes of geodesic rays in $M$. The class of a geodesic ray is called its point at infinity. The set $\tilde{M} \cup \partial \tilde{M}$ is endowed with the cone topology. It is the unique metrizable compact topology such that $\rho(t)$, as $t$ tends to $+\infty$, converges to the point at infinity of $\rho$, for every geodesic ray $\rho$ in $\tilde{M}$, and such that, for every point $x$ in $\tilde{M}$, the map $T\tilde{M} \to \partial \tilde{M}$, which sends a unit tangent vector to the point at infinity of the geodesic rays it defines, is a homeomorphism.

For $\xi$ in $\partial \tilde{M}$, the Busemann function $\beta_\xi : \tilde{M} \times \tilde{M} \to \mathbb{R}$ is defined by

$$\beta_\xi(x, y) = \lim_{t \to \infty} d(x, \xi_t) - d(y, \xi_t)$$

for any geodesic ray $t \mapsto \xi_t$ converging to $\xi$. The horospheres centered at $\xi$ are the level sets of $x \mapsto \beta_\xi(x, y)$ (for any $y$ in $\tilde{M}$), and the (open) horoballs the (strict) sublevel sets.
Let $\xi_0$ be a point on the boundary $\partial \widetilde{M}$ of $\widetilde{M}$, which is the endpoint of a lift of a geodesic ray converging to $e$. Let $\Gamma_0$ be the stabilizer of $\xi_0$ in $\Gamma$, which is non trivial. Its non trivial elements are parabolic isometries of $\widetilde{M}$, that is, their only fixed point in $M \cup \partial M$ is $\xi_0$, and they preserve each horosphere centered at $\xi_0$. By the Margulis lemma (see for instance [5]), there is a unique maximal open horoball $HB_0$ centered at $\xi_0$, such that $\Gamma_0 \setminus HB_0$ embeds in $M$ under the canonical map $\Gamma_0 \setminus \widetilde{M} \rightarrow M$. This subset of $M$ is called the \textit{maximal Margulis neighbourhood} of the cusp $e$. For every $\gamma \in \Gamma$, note that $\gamma HB_0$ meets $HB_0$ if and only if $\gamma$ belongs to $\Gamma_0$. We denote by $H_0$ the horosphere centered at $\xi_0$ which is the boundary of $HB_0$.

$$
\begin{array}{c}
\text{Maximal Margulis neighbourhood of } e \\
\text{universal cover } \pi \\
\text{height function } \beta
\end{array}
$$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{diagram.png}
\caption{Precisely invariant horoballs and height function.}
\end{figure}

\textbf{Examples:} The boundary of $\mathbb{H}^n_R$ is the horizontal coordinate hyperplane, union the point at infinity $\infty$ of all vertical geodesic rays. The horospheres in $\mathbb{H}^n_R$ are the horizontal hyperplanes and the Euclidean spheres tangent to the horizontal coordinate hyperplane. We will draw pictures using a projective transformation mapping the upper half space to the unit ball; the horospheres become the Euclidean spheres in the unit ball tangent to the unit sphere. For the modular group $PSL_2(\mathbb{Z})$ or the Bianchi groups $PSL_2(\mathbb{O}_d)$, we will take $\xi_0 = \infty$. The parabolic subgroup $\Gamma_0$ is the subgroup of upper triangular matrices. The maximal open horoball as above for the modular group is $HB_0 = \{(x, t) \in \mathbb{R}^2 : t > 1\}$ and for the Bianchi groups is $HB_0 = \{(x, y, t) \in \mathbb{R}^3 : t > 1\}$.

Let $\rho$ be any minimizing geodesic ray in $M$ starting from a point on the boundary of the maximal Margulis neighbourhood of $e$ and converging to $e$. Define the \textit{height function} on $M$ (with respect to $e$) as

$$
\beta(u) = \lim_{t \to +\infty} t - d(u, \rho(t)).
$$

Note that $\beta(u) = \inf_{\gamma \in \Gamma} \beta_{\gamma \xi_0}(\gamma x_0, \tilde{u})$ for any lift $\tilde{u}$ in $\widetilde{M}$ of a point $u$ in $M$ and any $x_0$ in $H_0$. This map is Lipschitz, piecewise smooth and proper on $M$, and positive precisely on the maximal Margulis neighborhood of the cusp. We obviously define the height of a point of $M$ as the value of $\beta$ at that point. If $A$ is a compact subset of $M$, we define its \textit{height} as the maximum of the heights of its points. Here is our first invariant of $M$. 

3
Define $h_M$ as the lower bound of the heights of the closed geodesics in $M$. 

For instance, let $\mathcal{M}_{g,1}$ be the space of isometry classes of complete, finite volume hyperbolic metrics on $\mathcal{S}_{g,1}$, the compact, connected, oriented surface with genus $g$ and one point removed.

Proposition 2.1 [20, Prop. 4.2] The map $h : \mathcal{M}_{g,1} \rightarrow \mathbb{R}$ is continuous and proper.

In fact, $h(X)$ converges to $-\infty$ as $X$ goes out of every compact subset of the moduli space $\mathcal{M}_{g,1}$. Indeed, by Mumford’s lemma, if $X$ goes out of every compact set, then it develops a shorter and shorter closed geodesic. By Margulis’ lemma, such a closed geodesic is simple, and has a tubular neighbourhood whose radius is bigger and bigger, and disjoint from the maximal Margulis neighbourhood of the cusp. Hence the closed geodesic is lower and lower for the height function. We refer to the appendix for speculations about the problem of finding a lowest closed curve, simply stating the following result here.

Theorem 2.2 [16][20, Theo. 4.7] The map $h : \mathcal{M}_{1,1} \rightarrow \mathbb{R}$ is $\mathbb{R}$-analytic, and reaches its maximum log $\frac{\sqrt{3}}{2}$ exactly on the modular hyperbolic once-punctured torus $\text{PSL}_2(\mathbb{Z})\backslash \mathbb{H}^2$.

Note that the derived subgroup $\text{PSL}_2(\mathbb{Z})'$ of $\text{PSL}_2(\mathbb{Z})$ has index 6, and $\mathcal{M}_{\text{mod}} = \text{PSL}_2(\mathbb{Z})\backslash \mathbb{H}^2$ is the hyperbolic surface obtained by gluing isometrically opposite faces of a regular hyperbolic hexagon with three vertices at infinity and three vertices of angle $\frac{2\pi}{3}$. Note that $\mathcal{M}_{\text{mod}}$ is also the element of $\mathcal{M}_{1,1}$ which has a maximal order symmetry group 6, and the longest shortest curve (see for instance [34]). The dotted line represents the boundary of the maximal Margulis neighbourhood of the cusp, which is a circle with three self-tangent points. The three continuous lines represent the three closed geodesics whose heights are minimum on $\mathcal{M}_{\text{mod}}$, hence are equal to $h_{\mathcal{M}_{\text{mod}}}$.

![Figure 2: The modular torus.](image-url)

We now consider the set $Lk(e)$ of oriented geodesic lines starting from the cusp $e$. (When we compactify $M$ by adding a point at its end, we get a CW-complex, and $Lk(e)$ can be identified with the link of the added point.) By lifting to the universal cover, this set $Lk(e)$ identifies with the set of $\Gamma_0$-orbits of oriented geodesic lines starting from $\xi_0$. By taking the unique point of intersection with $H_0$, or the point at infinity, of a geodesic line starting from $\xi_0$, the set $Lk(e)$ also identifies with $\Gamma_0 \backslash H_0$ and with $\Gamma_0 \backslash \partial M - \{\xi_0\}$.

Since $M$ has finite volume and only one cusp, a geodesic line starting from $e$ either converges to $e$, or accumulates inside $M$ (see for instance [4]). We say that the geodesic line is rational in the first case, and irrational otherwise.

The reason for this terminology is the following. Let $G$ be a connected semisimple algebraic group defined over $\mathbb{Q}$, with $\mathbb{R}$-rank and $\mathbb{Q}$-rank one. Let $P$ be a minimal parabolic subgroup of $G$, defined over $\mathbb{Q}$. Let $G = G(\mathbb{R})_0$ be the identity component of the $\mathbb{R}$-points of $G$, and $K$ be a maximal compact subgroup of $G$. Let $P = P(\mathbb{R}) \cap G$. The symmetric
space $X = G/K$, when $G$ is endowed with any left $G$-invariant and right $K$-invariant metric, has pinched negative curvature. The boundary of $X$ identifies with the projective variety $G/P$, which is defined over $\mathbb{Q}$. Let $\zeta_0$ be the trivial coset of $G/P$. Let $\Gamma = G(\mathbb{Z})$ be the lattice of integer points in $G$. The orbifold $\Gamma \backslash X$ has one and only one cusp if and only if there is one and only one $\Gamma$-orbit of $\mathbb{Q}$-points in $G/P$. See for instance [2, 3]. An oriented geodesic line starting from $\zeta_0$ is rational in our sense if and only if its endpoint is a $\mathbb{Q}$-point in $G/P$.

For instance, for the orbifold $\mathrm{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}^2$, a geodesic line starting from the cusp is rational (irrational) if and only if its lifts from $\zeta_0 = \infty$ (in the upper halfplane model) end at a rational (irrational) point on the real line.

In order to be able to effectively approximate irrational lines by rational ones, we need some complexity function on the rational lines and some uniform structure on the set $Lk(e)$ of geodesic lines starting from the cusp.

We define the depth $D(r)$ of a rational line $r$ as the length of the subsegment of $r$ between the first and last intersection point of $r$ with the level set $\beta^{-1}(t)$, minus $2t$, for any $t > 0$. We proved in [20] that the set of depths of rational lines is a discrete subset of $\mathbb{R}$. In particular, $D(r)$ converges to $+\infty$ as $r$ goes out of every finite set of rational lines.

Since $Lk(e)$ identifies with $\Gamma_0 \backslash (\partial \tilde{M} - \{\zeta_0\})$, it is sufficient to define a natural uniform structure on $\partial \tilde{M} - \{\zeta_0\}$. Let $a, b \in \partial \tilde{M}$. Their Gromov product with respect to a base point $x$ in $\tilde{M}$ is defined as the first time the geodesic rays starting from $x$ and converging to $a, b$ really start to diverge one from the other, i.e.

$$
(a, b)_x = \lim_{t \to +\infty} \frac{1}{2} (d(x, a_t) + d(x, b_t) - d(a_t, b_t))
$$

independently of the geodesic rays $a, b : [0, +\infty] \to \tilde{M}$ converging to $a, b$. The visual distance $d_x$ seen from $x$ on $\partial \tilde{M}$ is then defined by

$$
d_x(a, b) = \begin{cases} 
0 & \text{if } a = b \\
 e^{-d_x(a, b)} & \text{otherwise}.
\end{cases}
$$

These visual distances are natural, in the sense that every isometry $\gamma$ of $\tilde{M}$ extends to an homeomorphism of $\partial \tilde{M}$ which is an isometry between $d_x$ and $d_{\gamma x}$. The Hamenstädt distance $d_{\zeta_0}$ on $\partial \tilde{M} - \{\zeta_0\}$, which is invariant under $\Gamma_0$, is defined by scaling the visual distance seen from a point converging to $\zeta_0$ (see [19, Appendix]):

$$
d_{\zeta_0}(a, b) = \lim_{t \to +\infty} e^{-2t}d_x(a, b),
$$

with $a, b \in \partial \tilde{M} - \{\zeta_0\}$, and $r : [0, +\infty] \to \tilde{M}$ a geodesic ray with origin on $H_0$ and converging to $\zeta_0$. Note that (see [19, Appendix])

$$
d_{\zeta_0}(a, b) = \lim_{t \to +\infty} e^{-\frac{1}{2} d(H_0, a_t) + d(H_0, b_t) - d(a_t, b_t)}
$$

where $a, b : \mathbb{R} \to \tilde{M}$ are the geodesic lines starting from $\zeta_0$, passing at time $t = 0$ through $H_0$ and converging to $a, b$. Taking the quotient by $\Gamma_0$, and identifying $\Gamma_0 \backslash (\partial \tilde{M} - \{\zeta_0\})$ with $Lk(e)$, we get a distance $d_e$ on $Lk(e)$, that we also call the Hamenstädt distance on $Lk(e)$. 

5
In order to interpret constants correctly, we modify a bit this distance.

For \( a, b \in \partial \tilde{M} - \{ \xi_0 \} \), let \( L_a, L_b \) be the oriented geodesic lines from \( \xi_0 \) to \( a, b \). For \( r > 0 \), let \( H_r \) be the horosphere centered at \( a \), meeting \( L_a \) at a point at signed distance \(-\log 2r \) of \( H_0 \cap L_a \) along \( L_a \). Define \( d^e_{\xi_0}(a, b) \) to be the infimum of all \( r > 0 \) such that \( H_r \) meets \( L_b \). Note that \( d^e_{\xi_0} \) is invariant under \( \Gamma_0 \). Taking the quotient by \( \Gamma_0 \), and identifying \( \Gamma_0 \backslash (\partial M - \{ \xi_0 \}) \) with \( Lk(e) \), we get a map \( d^e_e \) on \( Lk(e) \backslash Lk(e) \), that we call the \textit{cuspidal distance} on \( Lk(e) \).

Figure 3 : Lift of the cuspidal distance.

In constant curvature, both the Hamenstädt distance and the cuspidal distance coincide with the induced Riemannian metric on \( \Gamma_0 \backslash H_0 \), which is flat. In general, and contrarily to the Hamenstädt distance, the cuspidal distance might not be a distance. But since there exists a constant \( c > 0 \) such that \( \frac{1}{c}d_e \leq d^e_e \leq cd_e \) (see [20]), this doesn’t make a big difference.

Example: We summarize in the following table the values of the corresponding notions that have just been introduced in our two test cases.

<table>
<thead>
<tr>
<th>Finite volume one-cusped orbifold</th>
<th>( \text{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}^2 )</th>
<th>( \text{PSL}_2(\mathcal{O}_d) \backslash \mathbb{H}_R^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Space ( Lk(e) ) of geodesics starting from the cusp</td>
<td>( \mathbb{R}/\mathbb{Z} )</td>
<td>( \mathbb{C}/\mathcal{O}_d )</td>
</tr>
<tr>
<td>subset of rational lines</td>
<td>( \mathbb{Q}/\mathbb{Z} )</td>
<td>( \mathbb{Q}(\sqrt{-d})/\mathcal{O}_d )</td>
</tr>
<tr>
<td>depth of ( \frac{p}{q} )</td>
<td>( \log</td>
<td>q</td>
</tr>
<tr>
<td>cuspidal distance</td>
<td>Euclidean distance</td>
<td>Euclidean distance</td>
</tr>
<tr>
<td>Hurwitz constant (see below)</td>
<td>( \frac{1}{\sqrt{3}} )</td>
<td>( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, 1, ?, ?, ? )</td>
</tr>
</tbody>
</table>

Our first approximation result is the following, which for \( M = \text{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}^2 \) is simply the Dirichlet theorem on Diophantine approximation of a real number by a rational one. See [39, 40] for the extension to the case of constant curvature.

**Theorem 2.3** ([20, Theo. 1.1]) There exists a constant \( c > 0 \) such that for every irrational line \( \alpha \), there exist infinitely many rational lines \( r \) such that

\[
    d^e_e(r, \alpha) \leq ce^{-D(r)}.
\]
This result allows us to define our second invariant for $M$.

Define the Hurwitz constant $K_M$ of $M$ as the lower bound of all such constants $c$.

The terminology again comes from the fact that the Hurwitz constant of the modular curve $M = \text{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}_\mathbb{R}^2$ is the classical Hurwitz constant for the Diophantine approximation for a real number by rational ones. Few values of the Hurwitz constant are known. For instance, it seems that the Hurwitz constant for the Bianchi groups for $d = 49, 67, 163$ are still unknown, as well as the Hurwitz constants at the cusp corresponding to $\infty$ for $\Gamma(N) \backslash \mathbb{H}_\mathbb{R}^2$ (which has more than one cusp if $N \neq 1$) where $\Gamma(N) = \ker(\text{PSL}_2(\mathbb{Z}) \to \text{PSL}_2(\mathbb{Z}/N\mathbb{Z}))$ is the $N$-th principal modular subgroup. Note that the behaviour of the Hurwitz constant under covering is unknown.

There is an exact relationship between our two invariants. This formula is due to H. Cohn [7] for $M = \text{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}_\mathbb{R}^2$.

**Theorem 2.4** [20, Theo. 1.2] \[ \frac{1}{2K_M} = e^{h_M}. \]

Let us give an idea of the proof, see [20, Theo. 1.2] for a complete one. First, by definition of the cuspidal distance, an irrational line $\alpha$ which has the biggest approximation constant $c$ (as in the statement of Theorem 2.3) is precisely one which asymptotically stays as far as possible from the cusp.

Now, if $\tau$ is a closed geodesic whose height is almost minimal, let $\alpha$ be a geodesic line starting from the cusp $e$ that spirals around $\tau$. Then $\alpha$ is irrational and, after some time, is not much higher than $\tau$.

**Figure 4:** Spiraling geodesics.

Conversely, let $\alpha$ be an irrational geodesic line whose asymptotic height $\limsup_{t \to +\infty} \beta(\alpha(t))$ is almost the lowest possible. In particular, after some time, $\alpha$ is not much higher than the lower bound. After that time, $\alpha$ accumulates both in space and direction, that is in $T^1M$. By Anosov’s closing lemma, there will be a closed geodesic $\tau$ contained in the $\epsilon$-neighbourhood of $\alpha$. In particular the height of this closed geodesic is also not much higher than the lower bound.

**Figure 5:** Accumulating geodesics.
Let us give a corollary of Theorem 2.4 in constant curvature, which appears to be new. Assume that $M$ is a finite volume hyperbolic orbifold of dimension $n = 2$ or $3$. Identify $M$ with $\Gamma \backslash \mathbb{H}^n$, with $\Gamma$ a discrete subgroup of $\text{PSL}_2(\mathbb{R})$ if $n = 2$ and $\text{PSL}_2(\mathbb{C})$ if $n = 3$. Write every element $\gamma$ of $\Gamma$ as $\gamma = \pm \begin{pmatrix} a(\gamma) & b(\gamma) \\ c(\gamma) & d(\gamma) \end{pmatrix}$. After normalization, assume that $\infty$ is a parabolic fixed point in $\Gamma$, and assume that the quotient, by the stabilizer of $\infty$ in $\Gamma$, of the open horoball, centered at $\infty$, with equation $t > 1$, is the maximal Margulis neighbourhood of the corresponding cusp.

**Corollary 2.5 [20, Theo. 1.4]** With the above notations,

$$\frac{1}{K_M} = \inf_{\gamma \in \Gamma : |\text{tr } \gamma| > 2} \frac{|\sqrt{\text{tr}^2 \gamma - 4}|}{\min_{\alpha \in \Gamma} |d(\alpha \gamma \alpha^{-1})|}.$$ 

Since the set of depths of rational lines in $M$ is a discrete subset of $\mathbb{R}$, we may define the depth counting function $N_\epsilon(t)$ as the number of rational lines whose depth is less than $t$. If $r$ is a rational line in $M$, any lift of $r$ to $\widetilde{M}$ starting from $\xi_0$ ends at $\gamma_0 \xi_0$ for some $\gamma$ in $\Gamma$. Note that by definition, $D(r) = d(H_0, \gamma \gamma_0)$. The double class of $\gamma$ in $\Gamma_0 \backslash \Gamma / \Gamma_0$ is well defined, and gives a bijection from the set of rational lines in $M$ with the set of non trivial double cosets $\Gamma_0 \backslash (\Gamma / \Gamma_0)$/$\Gamma_0$. Hence $N_\epsilon(t)$ is the number of $[\gamma]$ in $\Gamma_0 \backslash (\Gamma / \Gamma_0)$/$\Gamma_0$ such that $d(H_0, \gamma \gamma_0) \leq t$.

Before giving the asymptotics of the depth counting function, we need some definitions. The *Poincaré series* of a group $G$ of isometries of $M$ is

$$P_G(x, s) = \sum_{g \in G} e^{-s d(x, gx)}$$

for $x$ in $\widetilde{M}$ and $s$ in $\mathbb{R}^+$. This series converges if $s > \delta_G$ and diverges if $s < \delta_G$ for some $\delta_G$ in $[0, +\infty]$, called the *critical exponent* of $G$, which is independent of $x$.

Let $\delta_\Gamma$ and $\delta_{\Gamma_0}$ be the critical exponents of $\Gamma$ and $\Gamma_0$ respectively. Note that $0 < \delta_{\Gamma_0} \leq \delta_\Gamma < +\infty$ (see for instance [4]). For $n$ in $\mathbb{N}$, let $f_{\Gamma_0}(n)$ be the number of elements $\gamma$ in $\Gamma_0$ such that $d(x, \gamma x) \leq n$, where $x$ is any base point in $\widetilde{M}$.

If $f, g$ are maps from $\mathbb{N}$ to $\mathbb{R}^+$, write $f \asymp g$ if there is a constant $c > 0$ such that $\frac{1}{c} f(n) \leq g(n) \leq c f(n)$ for every $n$ in $\mathbb{N}$.

If $M$ is locally symmetric (for instance with constant curvature $-1$), then we have a strict inequality $\delta_{\Gamma_0} < \delta_\Gamma$ and $f_{\Gamma_0}(n) \asymp e^{\delta_{\Gamma_0} n}$. These properties may be false in the case of variable curvature (see [11]), due to the possibly much more complicated structure of the cusp neighbourhoods in variable curvature.

**Theorem 2.6 [1, Theo. 1.1]** If $\delta_{\Gamma_0} < \delta_\Gamma$, then

$$\limsup_{n \to +\infty} \frac{\log N_\epsilon(t)}{n} = \delta_\Gamma.$$ 

Under the same hypothesis, this asymptotic of the depth counting function has been improved by [8] in constant curvature and very recently by [32] in general to $N_\epsilon(t) \sim c e^{\delta_{\Gamma} n}$ for some constant $c > 0$ depending on $M$ and the base point $x$.

We end with a measurable result for the Diophantine approximation in negatively curved manifolds.
Let $\mu_e$ be the Hausdorff measure (in the Hausdorff dimension of $Lk(e)$, which is $\delta_\Gamma$),
defined by the Hamenstädt distance $d_e$ on the set $Lk(e)$ of geodesic lines starting from the
cusp $e$. For instance, in constant curvature, $\mu_e$ is just the Lebesgue measure on the flat
manifold $Lk(e) = \Gamma_0 \backslash H_0$.

Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a slowly varying map, in the sense that there exists a constant
$c > 0$ such that, if $|y - x| \leq 1$, then $\psi(y) \leq c\psi(x)$. For instance, one can take $\psi(t) = e^{-\alpha t}$
where $\alpha \geq 0$ is some constant.

**Theorem 2.7** [21, Theo. 1.3] Let $E_\psi$ be the set of irrational lines $\alpha$ in $Lk(e)$ for which
there exist infinitely many rational lines $r$ with $d_e(\alpha, r) \leq \psi(D(r))e^{-D(r)}$. If $\delta_{\Gamma_0} < \delta_\Gamma$
and $f_{\Gamma_0}(n) = e^{\epsilon_0 n}$, then $\mu_e(E_\psi)$ is zero if and only if the integral $\int_1^{+\infty} \psi(t)^{2(\delta - \delta_0)}dt$
converges.

This theorem is exactly Khintchine’s theorem [23] for the Diophantine approximation
of a real number by rational ones, when one takes $M$ to be the modular curve $PSL_2(\mathbb{Z}) \backslash \mathbb{H}_2$.
It is due to Sullivan [36] in constant curvature, see also [24].

In the next section, we will apply these theorems to get new results for the Diophantine
approximation on the Heisenberg group.

### 3 Diophantine approximation in the Heisenberg group

Let $n \geq 1$ be an integer. Let $\omega$ be the standard symplectic form on the affine $2n$-space
$\mathbb{A}^{2n}$, defined by $\omega(X, X') = \sum_{i=1}^n (x_i y'_i - y_i x'_i)$ if $X = (x_1, y_1, \ldots, x_n, y_n)$ and
$X' = (x'_1, y'_1, \ldots, x'_n, y'_n)$. The Heisenberg group $H_{2n+1}$ is the nilpotent connected
algebraic group, defined over $\mathbb{Q}$, which is the set of points $(X, t)$ in the affine $2n + 1$ space
$\mathbb{A}^{2n} \times \mathbb{A}^1$, endowed with the multiplication

$$(X, t)(X', t') = (X + X', t + t' + \omega(X, X')).$$

The identity is $(0, 0)$ and the inverse of $(X, t)$ is $(-X, -t)$. The space $H_{2n+1}(\mathbb{R})$ of $\mathbb{R}$-points
of $H_{2n+1}$ is a simply connected nilpotent Lie group, in which the set $H_{2n+1}(\mathbb{Q})$ of $\mathbb{Q}$-points
is a dense subgroup, and the set $H_{2n+1}(\mathbb{Z})$ a discrete subgroup. Note that the algebraic
group of unipotent upper triangular matrices is $\mathbb{Q}$-isomorphic (over fields of characteristic
different from 2) to $H_3$ by the map \( \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto (x, y, 2t - xy) \).

If $r$ belongs to $H_{2n+1}(\mathbb{Q})$, define the height $h(r)$ of $r$ as the absolute value of the
lowest common multiple of the denominators of the components of $r$. Note that if
$r = (\frac{p_i}{q_i})_{i=1,2n+1}$ with $(p_i, q_i) = 1$ and $|p_i| \leq q_i$, then $h(r) = \text{lcm}(q_1, \ldots, q_{2n+1})$ is
the (multiplicative) height of the $\mathbb{Q}$-point $r$ in the projective variety $H_{2n+1}$ over $\mathbb{Q}$ (see
for instance [27, page 52]).

The Lie group $H_{2n+1}(\mathbb{R})$ has a natural distance, which does not come from a
left-invariant Riemannian metric, but is a sub-Riemannian metric, called the Carnot-Carathéodory distance (see for instance [15]). This distance $d_{CC}$ is constructed as follows. The tangent space $T_e H_{2n+1}(\mathbb{R})$ at the identity contains an hyperplane $V_0 = \{(X, 0) : X \in \mathbb{R}^{2n}\}$. Endow $V_0$ with the standard Euclidean norm. The images of $V_0$ by the left translations define a (non-integrable) distribution of Euclidean hyperplanes on $H_{2n+1}(\mathbb{R})$.

For any two points $x, y$ in $H_{2n+1}(\mathbb{R})$, there exists (see for instance [15]) a $C^1$ path from
$x$ to $y$ which is tangent at each point to the hyperplane distribution. The Euclidean
structure of the hyperplane distribution defines a length for each such path. The Carnot-Caratheodory distance between $x$ and $y$ is the lower bound of the lengths of $C^1$ paths from $x$ to $y$ tangent to the hyperplane distribution.

Our first result is analogous to Dirichlet’s theorem for the Diophantine approximation of a real number by rational ones. After we wrote the first version of this paper, Stephane Fischler gave us a short elementary proof of it, that we give in the next section.

**Theorem 3.1** There exists a constant $c > 0$ such that for every $\alpha$ in $H_{2n+1}(\mathbb{R}) - H_{2n+1}(\mathbb{Q})$, there exists infinitely many $r$ in $H_{2n+1}(\mathbb{Q})$ such that $d_{CC}(\alpha, r) \leq c/h(r)$.

Note that a ball of radius $\epsilon$ at the origin for $d_{CC}$ looks like a Euclidean ball of radius $\epsilon$ in the direction of $V_0$ and like a Euclidean ball of radius $\sqrt{\epsilon}$ in the direction of $\{0\} \times \mathbb{R}$ (see the next subsection). This last fact explains the absence of a power 2 to $h(r)$ in the above result.

The Hausdorff dimension of the Carnot-Caratheodory distance is $2n + 2$ (see for instance [15]), though its topological dimension is only $2n + 1$. (See for instance [28] for the definition of the Hausdorff dimension and the Hausdorff measure of a metric space.) Let $\mu_{CC}$ be the Hausdorff measure of the Carnot-Caratheodory distance.

Recall that a map $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is slowly varying if there exists a constant $c > 0$ such that, if $|y - x| \leq 1$, then $\frac{1}{c}\psi(x) \leq \psi(y) \leq c\psi(x)$. Our last result is analogous to Khintchine’s theorem for the Diophantine approximation of a real number by rational ones. It also follows from the work of Kleinbock-Margulis [24], but our proof is quite different.

**Theorem 3.2** Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a slowly varying map. Let $E_f$ be the set of points $\alpha$ in $H_{2n+1}(\mathbb{R}) - H_{2n+1}(\mathbb{Q})$ such that there exists infinitely many $r$ in $H_{2n+1}(\mathbb{Q})$ with $d_{CC}(\alpha, r) \leq \psi(h(r))/h(r)$.

Then $\mu_{CC}(E_\psi) = 0$ if and only if the integral $\int_{1}^{+\infty} \psi(t)^{2n+2} dt$ converges.

### 3.1 The Heisenberg group

In this subsection, we recall some properties of the Heisenberg group (see for instance [14, sect. 2.6]), and we restate the above two theorems in a form which is more appropriate for our setting.

From now on, $n$ is an integer at least 2. On $\mathbb{C}^{n-1}$, we use the standard hermitian product $\zeta \cdot \zeta' = \sum z_i \bar{z}'_i$ if $\zeta = (z_1, \cdots, z_{n-1})$ and $\zeta' = (z'_1, \cdots, z'_{n-1})$.

Define the (real) **Heisenberg group** $H_{2n-1}(\mathbb{R})$ as the manifold $\mathbb{C}^{n-1} \times \mathbb{R}$, with coordinates $(\zeta, v)$, endowed with the multiplication (with the conventions of Koranyi-Reimann [26], Goldman [14], and [18])

$$(\zeta, v)(\zeta', v') = (\zeta + \zeta', v + v' + 2 \ \text{Im} \ \zeta \cdot \zeta').$$

Note that $\text{Im} \ \zeta \cdot \zeta'$ is the standard sympletic form on $\mathbb{C}^{n-1}$. There is a factor 2 appearing here, but $H_{2n-1}(\mathbb{R})$ is also the set of $\mathbb{R}$-points of a connected algebraic group $H_{2n-1}$ defined over $\mathbb{Q}$, whose set of $\mathbb{Q}$-points is $H_{2n-1}(\mathbb{Q}) = \mathbb{Q}[i]^{n-1} \times \mathbb{Q}$, and which is isomorphic over $\mathbb{Q}$ (in characteristic different from 2) to $H_{2n-1}$ by the map $(\zeta, v) \mapsto (\zeta, \frac{v}{2})$. This map is an isometry between $H_{2n-1}(\mathbb{R})$ and $H_{2n-1}(\mathbb{R})$ for the Carnot-Caratheodory metrics (defined
on $\mathcal{H}_{2n-1}(\mathbb{R})$ in a similar way), and changes the heights (defined on $\mathcal{H}_{2n-1}(\mathbb{R})$ in a similar way) only up to a factor 2.

The Cygan distance on $\mathcal{H}_{2n-1}(\mathbb{R})$ is defined as follows (see [9, 14]):

$$d_{\text{Cyg}}((\zeta, v), (\zeta', v')) = (|\zeta - \zeta'|^4 + |v - v'|^2)^{\frac{1}{4}}.$$  

Note that the Cygan distance on $\mathcal{H}_{2n+1}(\mathbb{R})$ is equivalent to the Carnot-Caratheodory distance. Moreover, $d_{\text{CC}}$ is the length metric induced by $d_{\text{Cyg}}$ (see for instance [14, page 161]).

Hence, to prove Theorems 3.1 and 3.2, we only have to prove the following results.

**Theorem 3.3** There exists a constant $c > 0$ such that for every $\alpha$ in $\mathcal{H}_{2n-1}(\mathbb{R})-\mathcal{H}_{2n-1}(\mathbb{Q})$, there exists infinitely many $r$ in $\mathcal{H}_{2n-1}(\mathbb{Q})$ such that $d_{\text{Cyg}}(\alpha, r) \leq c/h(r)$.

**Theorem 3.4** Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a slowly varying map. Let $E_f$ be the set of points $\alpha$ in $\mathcal{H}_{2n-1}(\mathbb{R})-\mathcal{H}_{2n-1}(\mathbb{Q})$ such that there exists infinitely many $r$ in $\mathcal{H}_{2n-1}(\mathbb{Q})$ with $d_{\text{Cyg}}(\alpha, r) \leq \psi(h(r))/h(r)$. Then $\mu_{\text{Cyg}}(E_f) = 0$ if and only if the integral $\int_1^{+\infty} \psi(t)^{2n} \frac{dt}{t}$ converges.

The next result gives a (probably non sharp) lower bound on the Hurwitz constant for the Diophantine approximation in the Heisenberg group.

**Theorem 3.5** The lower bound of the constants $c$ as in Theorem 3.3 is at least $\frac{1}{\sqrt{5}}$.

The next result is a counting result for $\mathbb{Q}$-points modulo $\mathbb{Z}$-points with bounded height.

**Theorem 3.6** Let $N(t)$ be the number of points $r$ in $\mathcal{H}_{2n-1}(\mathbb{Q})/\mathcal{H}_{2n-1}(\mathbb{Z})$ with $h(r) \leq t$. Then there exists a constant $c > 0$ such that $N(t) \sim ct^{2n}$ as $t$ tends to $\infty$.

All these results will be proven in subsection 3.8, as an application of the results of section 2. As said in the previous subsection, here is a simple proof of 3.3, due to Stéphane Fishler.

Let $\alpha = (\zeta_1, \zeta_2, \ldots, \zeta_{2n-2}, \nu)$ be an irrational point in $\mathcal{H}_{2n-1}(\mathbb{R})$. By the classical Dirichlet theorem, there exists infinitely many rational numbers $p/q$ such that $|\nu - p/q| \leq 1/q^2$. For $i = 1, \ldots, 2n-2$, let $p_i$ be the integer part of $q\zeta_i$. Then the Cygan distance between $\alpha$ and the rational point $(p_1/q, p_2/q, \ldots, p_{2n-2}/q, p/q)$ is at most $(2n-1)/q$. This proves the result.

### 3.2 The complex hyperbolic space

In this section, we recall some properties of the complex hyperbolic $n$-space $\mathbb{H}^n_\mathbb{C}$, and in particular the fact that its boundary $\partial \mathbb{H}^n_\mathbb{C}$ is the one-point compactification of the Heisenberg group. The main reference is [14], though as in [18], we will use a different Hermitian form, better suited for our purpose.

Let $q = -(z_0\overline{z}_1 + z_1\overline{z}_0) + z \cdot \overline{z}$ be our chosen hermitian form of signature $(n, 1)$, defined on $\mathbb{C}^{n+1} = \mathbb{C} \times \mathbb{C} \times \mathbb{C}^n$, with coordinates $(z_0, z_1, z)$, where $z \cdot \overline{z}$ is the standard Hermitian form on $\mathbb{C}^{n-1}$.
We will use two models for $\mathbb{H}_C^n$. The first one is the *Siegel domain*. It is better suited for the understanding of the boundary with a chosen point at infinity and is analogous to the upper halfspace model for $\mathbb{H}_R^n$. This is the domain

$$\mathbb{H}_C^n = \{(w_1, w) \in \mathbb{C} \times \mathbb{C}^{n-1} : 2 \Re w_1 - |w|^2 > 0\}$$

endowed with the Riemannian metric

$$ds^2 = \frac{4}{(2 \Re w_1 - |w|^2)^2} ((dw_1 - dw \cdot \overline{w})(d\overline{w_1} - w \cdot \overline{dw}) + (2 \Re w_1 - |w|^2)dw \cdot \overline{dw})$$

which has constant holomorphic sectional curvature $-1$, hence sectional curvatures between $-1$ and $-\frac{1}{4}$.

The second model is the *projective model*, obtained by mapping $\mathbb{H}_C^n$ into the complex projective space $\mathbb{P}^n(\mathbb{C})$ (with its standard homogeneous coordinates) by the map $(w_1, w) \mapsto [1, w_1, w]$. The image of this embedding is the open cone defined by $q < 0$. In particular, $PU(q)$ acts naturally on $\mathbb{H}_C^n$. It is well known (see for instance [14]) that $PU(q)$ is the group of orientation preserving isometries of $\mathbb{H}_C^n$.

The subspace of $\mathbb{H}_C^n$ defined by the equation $w = 0$ is totally geodesic, and isometric to the real hyperbolic plane with sectional curvature $-1$. In particular, the map $c : \mathbb{R} \to \mathbb{H}_C^n$ defined by $t \mapsto (e^{-t}, 0)$ is a unit speed geodesic line. Denote by $\infty$ the point of $\partial \mathbb{H}_C^n$ corresponding to the limit of $c(t)$ as $t$ tends to $-\infty$. In the projective model, $\infty$ corresponds to the point $[0, 1, 0]$.

The horospheres centered at $\infty$ are

$$H_t = \{(w_1, w) \in \mathbb{C} \times \mathbb{C}^{n-1} : 2 \Re w_1 - |w|^2 = t\}$$

for $t > 0$, which bound the open horoballs

$$HB_t = \{(w_1, w) \in \mathbb{C} \times \mathbb{C}^{n-1} : 2 \Re w_1 - |w|^2 > t\}.$$

The Heisenberg group acts on $\mathbb{H}_C^n$ by

$$(\zeta, v)(w_1, w) = (w_1 + w \cdot \overline{\zeta} + \frac{1}{2} |\zeta|^2 - \frac{i}{2} v, w + \zeta).$$

This action is isometric, and preserves $\infty$ and the horospheres centered at $\infty$. Furthermore, $\mathcal{H}_{2n-1}(\mathbb{R})$ acts simply transitively on each horosphere $H_t$ for $t > 0$. In particular, the unit speed geodesic lines starting from $\infty$ are the paths

$$t \mapsto (e^{t_0 - t} + \frac{1}{2} |\zeta|^2 - \frac{i}{2} v, \zeta)$$

for $t_0$ in $\mathbb{R}$ and $(\zeta, v)$ in $\mathbb{C}^{n-1} \times \mathbb{R}$. Hence the boundary of $\mathbb{H}_C^n$ is:

$$\partial \mathbb{H}_C^n = \{(w_1, w) \in \mathbb{C} \times \mathbb{C}^{n-1} : 2 \Re w_1 - |w|^2 = 0\} \cup \{\infty\}.$$  

Note that the continuous extension to $\partial \mathbb{H}_C^n$ of the isometric action of $\mathcal{H}_{2n-1}(\mathbb{R})$ on $\mathbb{H}_C^n$ is also simply transitive on $\partial \mathbb{H}_C^n - \{\infty\}$. In particular every non trivial element of $\mathcal{H}_{2n-1}(\mathbb{R})$ is a parabolic isometry of $\mathbb{H}_C^n$. 

12
Define as usual $M^* = \overline{M}$ for any $m \times m'$ complex matrix $M$. Let $I$ be the identity matrix in any dimension. We denote by

$$X = \begin{pmatrix} a & b & \gamma^* \\ c & d & \delta^* \\ \alpha & \beta & A \end{pmatrix},$$

a generic matrix in $U(q)$. If $Q$ is the matrix representing $q$ in the canonical basis of $\mathbb{C}^{n+1}$, one has

$$X^{-1} = Q^{-1} X^* Q = \begin{pmatrix} \bar{d} & \bar{b} & -\beta^* \\ \bar{c} & \bar{a} & -\alpha^* \\ -\delta & -\gamma & A^* \end{pmatrix}.$$

The isomorphism between the Heisenberg group $\mathcal{H}_{2n-1}(\mathbb{R})$ and a subgroup of $PU(q)$ (which is the full orientation preserving isometry group of $\mathbb{H}^n_\mathbb{C}$) is given by

$$(\zeta, v) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2}|\zeta|^2 - \frac{i}{2}v & 1 & \zeta^* \\ \zeta & 0 & 1 \end{pmatrix}.$$ 

An element of $PU(q)$ preserving each horosphere centered at $\infty$ is of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2}|\zeta|^2 - \frac{i}{2}v & 1 & \zeta^* \\ A\zeta & 0 & A \end{pmatrix},$$

with $v \in \mathbb{R}$, $\zeta \in \mathbb{C}^{n-1}$ and $A \in U(n-1)$. Denote by $PU(q)_\infty$ the subgroup of such elements in $PU(q)$.

The following lemma is in [14, page 77], with a different Hermitian form of signature $(n, 1)$, but the proof is the same.

**Lemma 3.7** If $x, y$ are points in $\mathbb{H}^n_\mathbb{C}$, denote by $V, W$ points in $\mathbb{C}^{n+1}$ whose images in $\mathbb{P}^n(\mathbb{C})$ correspond to $x, y$. Then

$$\cosh^2 \frac{d(x, y)}{2} = \frac{|(V, W)|^2}{q(V)q(W)},$$

where $(\cdot, \cdot)$ is the hermitian bilinear form on $\mathbb{C}^{n+1}$ associated to $q$. \hfill \Box

### 3.3 The rational lines are $Q$-points

Consider the group $\Gamma = PU(q)(\mathbb{Z}[i])$ of elements in $PU(q)$ with coefficients in $\mathbb{Z}[i]$. Then $\Gamma$ is a discrete subgroup of $PU(q)$. It is well known that $\Gamma \backslash \mathbb{H}^n_\mathbb{C}$ has finite volume (see for instance [31, page 214]) and, at least if $n = 2$, has one and only one cusp (see for instance [41]).

The point $\infty$ in $\partial \mathbb{H}^n_\mathbb{C}$ is a parabolic point, since for instance the element of $PU(q)$ corresponding to the element $(0, 2)$ of the Heisenberg group is in $\Gamma$. Let $e_\infty$ be the cusp of $\Gamma \backslash \mathbb{H}^n_\mathbb{C}$ corresponding to $\infty$ (see section 2 for definitions).

Recall from section 2 that the rational lines in $\Gamma \backslash \mathbb{H}^n_\mathbb{C}$ are the (orbifold) geodesic lines starting from the cusp $e_\infty$ and converging to it. A geodesic line in $\mathbb{H}^n_\mathbb{C}$ starting from $\infty$ is the lift of a rational line if and only if its endpoint belongs to the orbit of $\infty$ by $\Gamma$.  

13
Let $X$ be an element in $U(q)$, with the notations as above. Let $\Omega = (0, 1, 0)$ in $\mathbb{C}^{n+1}$. Since $X\Omega = (b, d, \beta)$, the isometry $\gamma$ of $\mathbb{H}^n_C$ defined by $X$ fixes $\infty$ if and only if $b = 0$. If $b \neq 0$, then $\gamma$ sends the point $\infty$ to the point $\left(\frac{d}{\beta}, \frac{1}{\beta}\right)$ in $\partial\mathbb{H}^n_C$. In particular, the orbit of $\infty$ by $\Gamma$ is contained (except $\infty$) in the set of $\mathbb{Q}$-points in the real affine algebraic set $\partial\mathbb{H}^n_C - \{\infty\}$. (In fact, when $\Gamma \backslash \mathbb{H}^n_C$ has only one cusp, then $\Gamma \infty$ is equal to the set of $\mathbb{Q}$-points (see for instance [3]; if $n = 2$, this also follows from an elementary but tedious argument).)

Let $O$ be the point $(0, 0)$ in $\partial\mathbb{H}^n_C - \{\infty\}$. The map $\phi : (\zeta, v) \mapsto (\zeta, v)O = \left(\frac{1}{2} |\zeta|^2 - \frac{1}{2}v, \zeta\right)$ is a diffeomorphism from $\mathcal{H}_{2n-1}(\mathbb{R})$ to $\partial\mathbb{H}^n_C - \{\infty\}$ such that $\phi^{-1}(\Gamma \infty - \{\infty\})$ is contained in $\mathcal{H}_{2n-1}(\mathbb{Q})$.

### 3.4 The Hamenstädt distance is a multiple of the Cygan distance

In this section, we determine the maximal Margulis neighbourhood of the cusp $e_\infty$ of $\Gamma \backslash \mathbb{H}^n_C$, and we compute the Hamenstädt distance on $\partial\mathbb{H}^n_C - \{\infty\}$.

Let $(0, v)$ be a non trivial element in $\mathcal{H}_{2n-1}(\mathbb{R})$ with smallest $|v|$ such that the corresponding element in $PU(q)$ belongs to $\Gamma$. Such an element exists since the commutator subgroup of $\mathcal{H}_{2n-1}(\mathbb{R})$ is equal to $\{0\} \times \mathbb{R}$, and since $\Gamma$ is discrete. Let $\Gamma_\infty = \Gamma \cap PU(q)_\infty$ be the stabilizer of $\infty$ in $\Gamma$.

Using Kamiya’s discreteness criterion [25, Theo. 3.2] (see also [29, Prop. 5.2]), we proved in [18, Prop. 5.7] that the horoball $HB_{|v|}$ is precisely invariant under $\Gamma$. This means that if $\gamma$ belongs to $\Gamma$, then $\gamma HB_{|v|} \cap HB_{|v|}$ is non empty if and only if $\gamma$ is in $\Gamma_\infty$.

In our situation, $(0, v)$ corresponds to an element of $\Gamma$ if and only if

$$
\begin{pmatrix}
1 & 0 & 0 \\
-\frac{1}{2}v & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

has coefficients in $\mathbb{Z}[i]$, that is, if and only if $v$ is in $2\mathbb{Z}$. Hence the horoball $HB_2$ is precisely invariant. Let $\gamma_0 = \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}$, which is an element in $U(q)$ with coefficients in $\mathbb{Z}[i]$.

Then $\gamma_0$ preserves the totally geodesic plane $P$ defined by the equation $w = 0$. Note that $P \cap HB_2$ is the horoball in the real hyperbolic plane $P$ defined by the equations $w = 0$ and $2 \Re w_1 > 2$, that is $\Re w_1 > 1$. Hence $\gamma_0 HB_2 \cap HB_2$ contains the point $(1, 0)$. Therefore no horoball strictly containing $HB_2$ is precisely invariant under $\Gamma$. This implies that $\Gamma \backslash HB_2$ is the maximal Margulis neighbourhood of the cusp $e_\infty$ of $\Gamma \backslash \mathbb{H}^n_C$.

Recall that $\mathbb{H}^n_C$ has sectional curvatures between $-1$ and $-\frac{1}{4}$. Hence, if $d$ is the Riemannian distance on $\mathbb{H}^n_C$, then $d' = \frac{1}{2}d$ is the Riemannian distance on $\mathbb{H}^n_C$ normalized to have curvature at most $-1$.

Recall that (see section 2) the Hamenstädt distance $d_\infty$ on $\partial\mathbb{H}^n_C - \{\infty\}$ is defined as follows. For every $\xi, \xi'$ in $\partial\mathbb{H}^n_C - \{\infty\}$,

$$
d_\infty(\xi, \xi') = \lim_{t \to +\infty} e^{-\frac{1}{2}(d(H_2, \xi_t) + d'(H_2, \xi'_t) - d(\xi_t, \xi'_t))}
$$

where $t \mapsto \xi_t, t \mapsto \xi'_t$ are the geodesic lines, with unit speed for the distance $d'$, starting from $\infty$, passing at time $t = 0$ through the horosphere $H_2$, and with endpoints $\xi, \xi'$.

The following result tells us that the Hamenstädt distance coincides with the Cygan distance (up to a multiplicative constant).
Proposition 3.8 For every $\xi, \xi'$ in $\partial \mathbb{H}^n_C - \{\infty\}$, we have

$$d_\infty(\xi, \xi') = \frac{1}{\sqrt{2}} d_{\text{Cyg}}(\xi, \xi').$$

Proof. To compute $d_\infty$, we use the invariance of both distances under $\mathcal{H}_{2n-1}(\mathbb{R})$ and the transitivity of the action of $\mathcal{H}_{2n-1}(\mathbb{R})$ on $\partial \mathbb{H}^n_C - \{\infty\}$. Hence we only have to prove the result for $\xi = O = (0,0)$ and $\xi' = (\zeta, v)O$ for every $(\zeta, v)$ non trivial in $\mathcal{H}_{2n-1}(\mathbb{R})$. Recall that (see subsection 3.2) the geodesic line with unit speed for the distance $d'$, starting from $\infty$, passing at time $t = 0$ through the horosphere $H_2$, and with endpoint $(\zeta, v)O$ is the path $t \mapsto \xi'_t = (e^{-2t} + \frac{1}{2}|\zeta|^2 - \frac{1}{2}v, \zeta)$.

By Lemma 3.7, as $d(\xi_t, \xi'_t)$ tends to $+\infty$, we have, when $t \to +\infty$

$$e^{d(\xi_t, \xi'_t)} \sim 4 \cosh^2 \frac{d(\xi_t, \xi'_t)}{2} = 4 \left| \left( 1, e^{-2t}, 0 \right), \left( 1, e^{-2t} + \frac{1}{2}|\zeta|^2 - \frac{1}{2}v, \zeta \right) \right|^2,$$

$$= \frac{4}{4e^{-4t}} \left[ 2e^{-2t} + \frac{1}{2}|\zeta|^2 + \frac{1}{2}v \right]^2 \sim \frac{e^{4t}}{4} (|\zeta|^4 + v^2).$$

Hence

$$d_\infty(O, (\zeta, v)O) = \lim_{t \to +\infty} e^{-t + \frac{1}{2}d(\xi_t, \xi'_t)} = \frac{1}{\sqrt{2}} (|\zeta|^4 + v^2)^{\frac{1}{8}} = \frac{1}{\sqrt{2}} d_{\text{Cyg}}(O, (\zeta, v)O).$$

The result follows. \hfill \Box

3.5 The depth is the logarithmic height

In this section, we compute the depth $D(r)$ of a rational line $r$ in $\Gamma \backslash \mathbb{H}^n_C$.

Proposition 3.9 If $\Gamma_\infty \gamma \Gamma_\infty$ is the non trivial double coset corresponding to $r$, if $\gamma$ is the image of $X = \begin{pmatrix} a & b & \gamma^* \\
 & d & \delta^* \\
 \alpha & \beta & A \end{pmatrix}$ in $PU(q)$, then $D(r) = \log |b|$.

Proof. Since $XX^{-1} = I$, we have the following set of identities

\[
\begin{align*}
\bar{a}d + b\bar{c} - \gamma^* \delta &= 1 \quad (i) \\
\bar{a}b + b\bar{a} - \gamma^* \gamma &= 0 \quad (ii) \\
\bar{a}d + d\bar{c} - \delta^* \delta &= 0 \quad (iii) \\
\bar{a}\beta + b\alpha - A\gamma &= 0 \quad (iv) \\
\bar{c}\beta + d\alpha - A\delta &= 0 \quad (v) \\
\alpha \beta^* + \beta \alpha^* - AA^* &= I \quad (vi)
\end{align*}
\]

Note that by definition of $\gamma$, there exists a lift of $r$ to $\mathbb{H}^n_C$ starting from $\infty$ and ending at $\gamma \infty$. In particular, $\gamma \infty \not= \infty$, which implies that $b \not= 0$.

Recall that (see section 2)

$$D(r) = d'(H_2, \gamma H_2) = \frac{1}{2} d(H_2, \gamma H_2).$$
The point at infinity of the horosphere \( \gamma H_2 \) is \( \gamma \infty \). We may assume, up to multiplying on the left \( \Gamma \) by an element of \( H_{2n-1}(\mathbb{R}) \), that \( \gamma \infty = O \). Indeed, the Heisenberg group acts transitively on \( \partial \mathbb{H}^n_{\mathbb{C}} - \{ \infty \} \) and preserves the horosphere \( H_2 \). Furthermore, the left multiplication of \( X \) by an element of \( H_{2n-1}(\mathbb{R}) \) leaves \( b \) unchanged.

Since \( \gamma \infty = \left( \frac{a}{b}, \frac{\beta}{\bar{b}} \right) \), we have \( d = 0, \beta = 0 \). Equation (ii) implies that \( A \) is in \( U(n-1) \) and Equation (iii) implies that \( \delta = 0 \).

Up to multiplying \( X \) on the left by \( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & A^{-1} \end{pmatrix} \), which belongs to \( U(q) \) and does not change the value of \( b \), we may assume that \( A = I \).

Note that
\[
\begin{pmatrix}
  a & b & \gamma^* \\
  c & 0 & 0 \\
  \alpha & 0 & I
\end{pmatrix}
\begin{pmatrix}
  1 & 0 & 0 \\
  1 & 0 & \frac{1}{2} |\zeta|^2 - \frac{i}{2} v \\
  \zeta & 0 & I
\end{pmatrix}
= \begin{pmatrix}
  a + b(\frac{1}{2} |\zeta|^2 - \frac{i}{2} v) + \gamma^* \zeta & b & b \zeta^* + \gamma^* \\
  c & 0 & 0 \\
  \alpha + \zeta & 0 & I
\end{pmatrix}.
\]

It follows from Equation (i) that \( c = \frac{1}{b} \), from Equation (iv) that \( \gamma = \frac{\alpha}{b} \), and from Equation (iii) that \( 2 \text{ Re } (ab) = |\gamma|^2 \), which implies that \( 2 \text{ Re } (\frac{\alpha}{b}) = |\frac{\alpha}{b}|^2 \).

The right multiplication of \( X \) by an element of \( H_{2n-1}(\mathbb{R}) \) does not change \( d(H_2, \gamma H_2) \) nor \( b \). Hence, multiplying \( X \) on the right by \( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} |\zeta|^2 - \frac{i}{2} v \\
  \zeta & 0 & I\end{pmatrix} \) with \( \zeta = -\alpha \) and \( v = 2 \text{ Im } \frac{\alpha}{b} \), we may assume that \( \alpha = \gamma = 0 \) and \( a = 0 \).

Summarizing the above computations, we may assume that \( \gamma \) is the image in \( PU(q) \) of \( \begin{pmatrix} 1 & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{pmatrix} \). This element preserves the non oriented geodesic line \( \rho \) between \( \infty \) and \( O \), as well as the totally geodesic plane \( P \) in \( \mathbb{H}^n_{\mathbb{C}} \) defined by the equation \( w = 0 \). If \( x = (1,0) \) and \( y \) are the intersection points of \( \rho \) with \( H_2 \) and \( \gamma H_2 \) respectively, then \( d(H_2, \gamma H_2) = d(x,y) \). A simple computation in the constant curvature \(-1\) plane \( P \) shows that \( d(x,y) = \log |b|^2 \). This proves the result. \( \square \)

Note that \( \gamma \infty \) is the \( \mathbb{Q} \)-point \( \left( \frac{d}{b}, \frac{\beta}{b} \right) \) in the real affine algebraic set \( \partial \mathbb{H}^n_{\mathbb{C}} - \{ \infty \} \). The coefficients of \( \beta \) and \( b, d \) are relatively prime, since \( \gamma \) belongs to \( U(q)(\mathbb{Z}[i]) \). Up to acting on the left by an element of \( \Gamma_\infty \), we may assume that \( d \) and the coefficients of \( \beta \) have absolute value at most \( |b| \). In particular, \( D(r) \) is then the logarithmic height of the \( \mathbb{Q} \)-point corresponding to \( \left( \frac{d}{b}, \frac{\beta}{b} \right) \) in \( \mathbb{P}^n(\mathbb{C}) \) (see for instance [27, page 52]).

### 3.6 Computations of the critical exponents

In this section, we compute the critical exponent \( \delta \) of \( \Gamma \) and the critical exponent \( \delta_0 \) of \( \Gamma_\infty \) (see also [6]).

Recall that (see section 2) the Poincaré series of a discrete set of isometries \( G \) of \( \mathbb{H}^n_{\mathbb{C}} \) is
\[
P_G(s) = \sum_{g \in G} e^{-sd(x_0, gx_0)}
\]
where \( x_0 \) is any base point in \( \mathbb{H}^n_{\mathbb{C}} \). This series diverges for \( s < \sigma \) and converges for \( s > \sigma \) for some \( \sigma \) in \([0, +\infty]\), which does not depend on \( x_0 \), and is called the critical exponent of \( G \). Take \( x_0 = \left( \frac{1}{4}, 0 \right) \) in what follows.

16
Since $\Gamma \backslash \mathbb{H}^n_\mathbb{C}$ has finite volume, its critical exponent is equal to the Hausdorff dimension of $\partial \mathbb{H}^n_\mathbb{C}$ endowed with the Carnot-Carathéodory distance (see for instance [6, Theo. 6.1]). Hence $\delta = 2n$.

Note that $\Gamma_\infty$ is commensurable with $\mathcal{H}_{2n-1}(\mathbb{Z})$. Indeed, $\Gamma_\infty \cap \mathcal{H}_{2n-1}(\mathbb{R})$ has finite index in $\Gamma_\infty$, and consists of the $(\zeta, v)$ in $\mathcal{H}_{2n-1}(\mathbb{Z})$ such that $v$ and $|\zeta|^2$ belongs to $2\mathbb{Z}$. Hence $\Gamma_\infty \cap \mathcal{H}_{2n-1}(\mathbb{R})$ has finite index in $\mathcal{H}_{2n-1}(\mathbb{Z})$.

The critical exponent of $\Gamma_\infty$ is then the same as the critical exponent of the series

$$Q(s) = \sum_{g \in \mathcal{H}_{2n-1}(\mathbb{Z})} \left( \cosh \frac{d(x_0, gx_0)}{2} \right)^{-s}.$$ 

By Lemma 3.7, we have

$$Q(s) = \sum_{(\zeta, v) \in \mathcal{H}_{2n-1}(\mathbb{Z})} \left( \frac{1 + \frac{1}{2}|\zeta|^2 + \frac{1}{2}|v|^2}{(\frac{1}{2})^2} \right)^{-\frac{s}{2}} = \sum_{(\zeta, v) \in \mathcal{H}_{2n-1}(\mathbb{Z})} ((1 + |\zeta|^2)^2 + |v|^2)^{-\frac{s}{2}}.$$ 

By comparison, this sum converges if and only if the following integral converges:

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^+} \frac{d\zeta \, dv}{(1 + |\zeta|^4 + |v|^2)^\frac{s}{2}} = 2v_{2n-3} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{x^{2n-3} \, dx \, dv}{(1 + x^4 + v^2)^\frac{s}{2}} =$$

$$2v_{2n-3} \int_{\mathbb{R}^+} \frac{x^{2n-3} \, dx}{(1 + x^4)^\frac{s}{2}} \int_{\mathbb{R}^+} \frac{dy}{(1 + y^2)^\frac{s}{2}},$$

where $v_k$ is the area of the unit sphere in $\mathbb{R}^{k+1}$ and by setting $v = y\sqrt{1 + x^4}$. The last line converges if and only if $s > n$. Hence $\delta_0 = n$.

### 3.7 The height of closed geodesics

In this section, we compute the “distance to the cusps” of the closed geodesics in $\Gamma \backslash \mathbb{H}^n_\mathbb{C}$. More generally, let $G$ be a discrete non elementary group of isometries of $\mathbb{H}^n_\mathbb{C}$. Assume that $\infty$ is a parabolic fixed point, with $G_\infty$ its stabilizer in $G$. Assume that $G \backslash \mathbb{H} \mathbb{B}_2$ is the maximal Margulis neighbourhood of the cusp corresponding to $\infty$.

Let $\tau$ be a geodesic line in $\mathbb{H}^n_\mathbb{C}$, whose points at infinity differ from $\infty$. Define the height $ht(\tau)$ of $\tau$ as the maximum of $\frac{1}{2} \log \frac{1}{t}$ for $t > 0$ such that $\tau$ meets $\mathbb{H}B_2$. The factor $\frac{1}{2}$ comes from the fact that we need to normalize so that the sectional curvature is at most $-1$; the last part of the formula comes from the fact that the signed distance between $H_2$ and $H_1$ is $\log \frac{1}{s}$.

An element $g$ of $PU(q)$ is loxodromic if it has a unique pair of fixed points in $\mathbb{H}^n_\mathbb{C} \cup \partial \mathbb{H}^n_\mathbb{C}$, that are in $\partial \mathbb{H}^n_\mathbb{C}$. The geodesic line between these points is called the translation axis of $g$, and will be denoted by $A_g$. Note that $g$ is loxodromic if and only if its translation length $\ell(g) = \inf_{x \in X} d(x, g x)$ is non zero. The isometry $g$ preserves $A_g$, and acts by a translation of length $\ell(g)$ on it. The translation length is a conjugation invariant.

**Lemma 3.10** For every loxodromic element $g$ in $PU(q)$, whose fixed points are different from $\infty$, one has

$$ht(A_g) = \frac{1}{2} \log \frac{\sinh \frac{\ell(g)}{2}}{|b(g)|},$$

where $b(g)$ is the coefficient $1-2$ of any matrix $X$ in $U(q)$ mapping to $g$ in $PU(q)$.
Proof. Up to conjugating \( g \) by an element of \( PU(q)_{\infty} \), which does not change \( \ell(g) \) nor \( |b(g)| \), we may assume that the fixed points of \( g \) are \( O = (0,0) \) and \( (u,0) \) with \( \Re u = 0 \) and \( \Im u > 0 \).

Set \( X = \begin{pmatrix} a & b & \gamma^* \\ c & d & \delta^* \\ \alpha & \beta & A \end{pmatrix} \). By the assumptions on \( g \), together with the equations (i)-(vi) in the proof of Prop. 3.9, we obtain

\[ b \neq 0, c = 0, \alpha, \beta, A \in U(n-1), \delta = 0, \beta = 0, \gamma = 0, \Re \overline{a} b = 0, \alpha \overline{d} = 1, u = \frac{a - d}{b}. \]

If \( a = |a|e^{i \theta} \), then \( d = \frac{|d|}{|b|}e^{i \theta} \), \( u = \frac{|b|-|d|}{|b|e^{-i \theta}} \) and \( \Re be^{-i \theta} = 0 \). The element \( \gamma' \) of \( PU(q) \), which is the projection of \( X' = \begin{pmatrix} |a| & be^{-i \theta} & 0 \\ 0 & |d| & 0 \\ 0 & 0 & I \end{pmatrix} \), fixes \((0, 0) \) and \((u, 0) \). Hence \( \gamma' \) is a loxodromic element with the same translation axis as \( \gamma \). Since \( \gamma' \gamma^{-1} \) fixes the point \((\frac{|b|}{2}, 0)\), the isometries \( \gamma, \gamma' \) also have the same translation length. Since \( \gamma' \) sends \( \infty \) to \((\frac{|b|}{2}, 0) \) whose first component is purely imaginary, it preserves the hyperbolic plane \( P \) defined by the equation \( w = 0 \). In particular, \( \ell(\gamma') = \log |a|^2 \). Up to replacing \( \gamma \) by its inverse, we may assume that \( |a| \geq 1 \). Then \( |a| = e^{\frac{\delta(q)}{2}} \). Hence

\[ \frac{|u|}{2} = \frac{\sinh \frac{\delta(q)}{2}}{|b|}. \]

An easy computation in the real hyperbolic plane \( P \) shows that \( \text{ht}(\gamma) = \frac{1}{2} \log \frac{|b|}{2} \). Hence the result follows.

Recall from section 2 that the invariant \( h_M \) of \( M = G \setminus \mathbb{H}^n_{\mathbb{C}} \) is defined as

\[ h_M = \inf_{g \in G : \ell(g) > 0} \max_{h \in G} \text{ht}(A_{hgh^{-1}}). \]

**Corollary 3.11** The Hurwitz constant \( K_M \) of \( M = G \setminus \mathbb{H}^n_{\mathbb{C}} \) satisfies

\[ K_M = \frac{1}{2} \sup_{g \in G : \ell(g) > 0} \left( \frac{\sinh \frac{\ell(q)}{2}}{\min_{h \in G} |b(hgh^{-1})|} \right)^{-\frac{1}{2}}. \]

**Proof.** This follows from the previous lemma and from Theorem 2.4. \( \square \)

### 3.8 Proofs of the results

Theorem 3.3 is a corollary of Theorem 2.3 applied to \( M = \Gamma \setminus \mathbb{H}^n_{\mathbb{C}} \), with the help of the subsections 3.3, 3.4, 3.5.

Theorem 3.4 is a corollary of Theorem 2.7 applied to \( M = \Gamma \setminus \mathbb{H}^n_{\mathbb{C}} \), with the help of the subsections 3.3, 3.4, 3.5 and 3.6.

Theorem 3.5 follows from Corollary 3.11 applied to \( G = \Gamma \), from the fact that \( d_{CG} = \sqrt{2} \ell_{\infty} \) and the fact that the matrix \( \begin{pmatrix} 2 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & I \end{pmatrix} \) is in \( \Gamma \), has coefficient 1-2 with absolute
value 1 (which is the least possible since the coefficients are in \( \mathbb{Z}[i] \)), and has translation length \( 2 \log \frac{3+\sqrt{5}}{2} \).

Theorem 3.6 follows from the paragraph after 2.6 with the computations of subsection 3.5.

We have concentrated on one particular finite volume discrete subgroup of \( PU(q) \), but the results of section 2 apply more generally to other geometrically finite discrete subgroups of isometries of \( \mathbb{H}^n_\mathbb{C} \) with cusps, for instance the groups of integer points of any split \( \mathbb{Q} \)-forms of \( PU(q) \).

Note that by a theorem of Feustel and Zink [41], if \( n = 2 \), if \( d \) is a squarefree positive integer, then the number of cusps of \( PU(q)(\mathbb{O}_d) \) is the class number of the imaginary quadratic fields \( \mathbb{Q}(\sqrt{-q}) \), and in particular is 1 if and only if \( d = 1, 2, 3, 7, 11, 19, 43, 67, 163 \).

Note that similar results apply for the quaternionic hyperbolic space \( \mathbb{H}^n_\mathbb{H} \) and the octonion hyperbolic plane \( \mathbb{H}^2_\mathbb{O} \), the computations of the Hamenstädt distance, of the depth and of the critical exponents being similar to what we have done.

References


20
[34] Schmutz-Schaller, P. *The modular torus has maximal length spectrum*, GAFA 6 (1996) 1057-1073.


Department of Mathematics
Ben Gurion University
BEER-SHEVA, Israel
e-mail: saarah@math.bgu.ac.il

Département de Mathématique et Applications, UMR 8553 CNRS
Ecole Normale Supérieure
45 rue d’Ulm
75230 PARIS Cedex 05, FRANCE
e-mail: Frederic.Paulin@ens.fr