Adic algebraic groups, parahoric subgroups and fundamental strata

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March 2001

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A.-M. AUBERT AND C. CUNNINGHAM

ABSTRACT. This paper establishes basic properties of adic algebraic groups, adic parahoric groups and fundamental strata. The latter recovers the notion of K-types for p-adic groups as certain l-adic sheaf complexes on an adic parahoric group.

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INTRODUCTION

It has become common practice to discuss certain unsolved geometric problems in harmonic analysis on p-adic groups by way of an analogy to loop groups: this discussion is predicated on the belief that an analysis of groups of the form $G(F)$, where $F = \mathbb{C}(p)$ or $F = \mathbb{F}_p((t))$ (formal Laurent series), should yield insight concerning the p-adic group $G(\mathbb{Q}_p)$. Ideally, one would like to adapt geometric notions such as affine Springer fibres, ind-schemes and perverse sheaves from affine

The second author would like to thank the École Normale Supérieure, Paris, especially the Département de Mathématiques et Applications for generous support and gracious hospitality during much of the writing of this paper.

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Grassmanians, for example, to an analogous $p$-adic object; however, it has long been recognized that there are serious obstacles to a straightforward adaptation of these geometric objects to the $p$-adic picture. And so, our subject seems in need of new geometric ideas.

On the other hand, the intimate relationship between characteristic functions of character sheaves and characters of certain admissible representations of $p$-adic groups, especially reductive unipotent representations, is plain to see; aspects of this relationship are particularly clear in certain work of George Lusztig [17] and recent work of Jean-Loup Waldspurger [25]. However, no direct relationship between character sheaves and admissible $p$-adic group representations themselves has yet been established.

This paper presents the first step toward a direct, geometric connection between character sheaves and admissible $p$-adic group representations; to establish this connection we have found the theory of $l$-adic cohomology on adic spaces, as defined and studied by Roland Huber, invaluable. In this paper we establish basic properties of adic algebraic groups and parahoric subgroups as they relate to the theory of types for $p$-adic groups. In doing so, we set the framework for an adaptation of the theory of perverse sheaves to $p$-adic groups via adic domains (the tower of adic parahoric groups in an adic algebraic group). In another paper we define the notion of induction of certain $l$-adic sheaves on these adic domains, which is a key ingredient in the definition of character sheaves. We will also show how these sheaves define invariant distributions on $p$-adic groups and Lie algebras; this allows us to interpret the values of certain distributions on $p$-adic groups via Lefschetz fix-point formulas for $l$-adic cohomology on adic domains.

In short, this paper presents the first step toward a direct, geometric connection between character sheaves and admissible $p$-adic group representations; to establish this connection we have found the theory of $l$-adic cohomology on adic spaces, as defined and studied by Roland Huber, invaluable. In this paper we establish basic properties of adic algebraic groups and parahoric subgroups as they relate to the theory of types for $p$-adic groups. In doing so, we set the framework for an adaptation of the theory of perverse sheaves to $p$-adic groups via adic domains (the tower of adic parahoric groups in an adic algebraic group). In another paper we define the notion of induction of certain $l$-adic sheaves on these adic domains, which is a key ingredient in the definition of character sheaves. We will also show how these sheaves define invariant distributions on $p$-adic groups and Lie algebras; this allows us to interpret the values of certain distributions on $p$-adic groups via Lefschetz fix-point formulas for $l$-adic cohomology on adic domains.

We now describe this paper more fully. Let $G_F$ be a connected reductive affine group scheme defined over the unramified closure $F$ of a $p$-adic field. Let $I(G, F)$ be the building for $G(F)$ and let $g_F$ be the Lie algebra for $G_F$.

Section 1 sets certain notation and recalls basic facts about Moy-Prasad filtrations. Section 2 begins with the definition of a formal scheme $\hat{G}_{x,r} := \text{Spf}(A_{x,r}^+)$ associated to any pair $(x, r)$, where $x \in I(G, F)$ and $r \in \mathbb{R}$. We show that $A_{x,r}^+$ is a co-commutative Hopf $\mathcal{O}_F$-algebra with a co-Lie-bracket, where $\mathcal{O}_F$ is the ring of integers of $F$. When $r \in \mathbb{R}_{\geq 0}$ we also define a formal scheme $\hat{G}_{x,r} := \text{Spf}(A_{x,r}^+)$ where $A_{x,r}^+$ is again a Hopf $\mathcal{O}_F$-algebra. Section 3 recalls the definition of the adic group $G^d_F$ associated to the scheme $G_F$ and introduces adic affinoid group objects $G^d_{G,F}$ in $G^d_F$. We also describe the corresponding adic affinoid Lie algebra object $g^d_{G,F}$ in $g^d_F$. Section 4 provides examples of parahoroid groups and Lie algebras and their filtrations. Section 5 defines a continuous morphism $\rho_{x,r} : G^d_{G,F} \to M_{x,r}$ in the category of ringed group objects, which we call the reduction morphism, where $M_{x,r}$ is a reductive group scheme $M_{x,r}$ over $k$. We also define a continuous morphism $\rho_{x,r} : g^d_{G,F} \to m_{x,r}$ of ringed Lie algebra objects, where $m_{x,r}$ is a Lie algebra scheme. Section 6 uses the reduction morphism to define a depth-zero stratification of $G^d_{G,F}$, built from the stratification of the reductive quotient $M_{x}$ into decomposition varieties. Likewise, we use $\rho_{x} : g^d_{G,F} \to m_{x}$ to define a depth-zero stratification of $G^d_{G,F}$ using decomposition varieties in $m_{x}$. Section 7 uses $l$-adic étale sheaves to define fundamental strata for $G^d_{G,F}$; this notion recovers K-types for $p$-adic groups.
in a geometric manner. In an attempt to make this paper relatively self-contained we have included an appendix on adic spaces in section 8.

The theory presented here is easily adapted to loop groups; specifically, we may replace $F$ by the field $\mathbb{C}(q)$ of formal Laurent series in $q$ with complex coefficients, in which case we produce sheaves on adic spaces for loop group representation theory.

1. Preliminary remarks

Let $F$ be an unramified closure of a $p$-adic field and let $\hat{F}$ be the completion of $F$. The ring of integers in $F$ (resp. $\hat{F}$) will be denoted $\mathcal{O}_F$ (resp. $\mathcal{O}_{\hat{F}}$) and the maximal ideal in $\mathcal{O}_F$ (resp. $\mathcal{O}_{\hat{F}}$) will be denoted $p_F$ (resp. $\hat{p}_F$). For each $n \in \mathbb{Z}$ we write $p_F^n$ (resp. $\hat{p}_F^n$) for the fractional ideal $\mathfrak{p}^n\mathcal{O}_F$ (resp. $\mathfrak{p}^n\mathcal{O}_{\hat{F}}$). The residual field for $F$ is $\mathbb{F}_q$, where $q$ is a power of $p$. We choose a valuation $| \cdot |$ on $F$ by fixing a generator $\mathfrak{p}$ for $p_F$ and asking that $|\mathfrak{p}| = q^{-1}$. We will extend $| \cdot |$ from $F$ to $\hat{F}$ without further remark.

Let $(G, \mathcal{O}_G)$ be an affine group scheme over $\mathbb{Z}$ and let $(g, \mathcal{O}_g)$ be the associated Lie algebra scheme. For any extension $R$ of $\mathbb{Z}$, let $G_R$ (resp. $g_R$) denote the scheme over $R$ and let $G(R)$ (resp. $g(R)$) be the group (resp. Lie algebra) of $R$-rational points on the variety for $G_R$ (resp. $g_R$).

We assume throughout that $G_F$ is a connected reductive affine group scheme. Further, we suppose that the group $G(F)$ of $F$-rational points on $G_F$ is a semi-simple group.

Since $G$ is an affine group over $\mathbb{Z}$, there is a surjective group scheme morphism $G \to \text{GL}(n)$ defined over $\mathbb{Z}$; in fact, we will often view $G$ as a subscheme of $\text{GL}(n)$. Let $I(G, F)$ denote the affine Bruhat-Tits building for $G$ over $F$. We assume that $G_F \to \text{GL}(n)_F$ determines a strong descent map $I(G, F) \to I(\text{GL}(n), F)$ (see [13] and [22]). We will often identify $x \in I(G, F)$ with its image under this function, as in definitions 2.3 and 2.7. We also assume that the inverse image of $I(\text{GL}(n), F)$ under the strong descent map contains a fundamental domain for the action of $G(F)$ on $I(G, F)$.

Each point $x$ in $I(G, F)$ determines a parahoric group $G(F)_x$ in $G(F)$ (resp. parahoric algebra $g(F)_x$ in $g(F)$) and a filtration of $G(F)_x$ (resp. $g(F)_x$) by normal subgroups (resp. ideals) $G(F)_{x, r}$ (resp. $g(F)_{x, r}$) where $r \in \mathbb{R}^0$ (resp. $r \in \mathbb{R}$). We refer the reader to [19] for the definition of $G(F)_{x, r}$ (resp. $g(F)_{x, r}$) and the relevant conventions; in particular, $G(F)_x := G(F)_{x, 0}$ (resp. $g(F)_x := g(F)_{x, 0}$). We deviate from the notation of [19] only in the following regard: we write $G(F)_{x, r^+}$ (resp. $g(F)_{x, r^+}$) for the largest proper normal subgroup (resp. Lie algebra) in $G(F)_{x, r}$ appearing in the filtration of $G(F)_x$ (resp. $g(F)_x$) (see remark 7.2).

2. The formal scheme attached to a parahoric subgroup

Let $\text{GL}_n(F)$ (resp. $gl_n(F)$) denote the group (resp. Lie algebra) of $F$-rational points on the variety for $\text{GL}(n)_F$ (resp. $gl(n)_F$).

Definition 2.1. Let $A(\text{GL}(n), S, F)$ be the apartment in the Bruhat-Tits building $I(\text{GL}(n), F)$ corresponding to the split maximal torus $S$ of diagonal matrices in $\text{GL}(n)$. Suppose that $x \in A(\text{GL}(n), S, F)$ and $r \in \mathbb{R}$. For each $1 \leq i, j \leq n$ define $r_{x, r}^{i, j} \in \mathbb{R}$ as follows: $r_{x, r}^{i, i} := nr$ if $i = j$ and $r_{x, r}^{i, j} := nr - na_{i, j}(x)$ if $i \neq j$. Here
\{a_{i,j} \mid 1 \leq i \neq j \leq n\} is the root system for \text{GL}(n) chosen in the usual manner. We will also write \(R_{\alpha,r}\) for the matrix \((q^{-r_{i,j}})_{1 \leq i,j \leq n}\).

We recall that Moy-Prasad filtrations for \(gl_n(F)\) are classical lattice-chain filtrations in the following sense.

**Lemma 2.2. (P. Broussous et B. Lemaire)** Let \(x\) and \(r\) be as in definition 2.1. The Moy-Prasad lattice \(gl_n(F)_{x,r}\) consists of matrices \(X = (X_{i,j})_{1 \leq i,j \leq n}\) in \(gl_n(F)\) such that \(|X_{i,j}| \leq q^{-r_{i,j}}\).

**Proof.** This is corollary 4.6 and the appendix of [5].

Recall the definition of the generalized Tate algebra \(T_F^n(R_{\alpha,r})\) from 8.4. The ring of integers \(T_F^n(R_{\alpha,r})^\circ\) in \(T_F^n(R_{\alpha,r})\) is a Banach \(\mathcal{O}_F\)-algebra with respect to the Gauss norm defined in section 8.2. Let \(\text{Spec}(\alpha)\) be the cokernel of the morphism \(g \to gl(n)\) determined by our chosen \(G \to \text{GL}(n)\). Recall that \(gl(n)_F = \text{Spec}(F[T_{i,j}]_{1 \leq i,j \leq n}]\). Now \(\alpha\) generates an ideal in \(T_F^n(R_{\alpha,r})^\circ\) which we will also denote by \(\alpha\), abusing notation somewhat. Since \(\alpha\) is a closed ideal in \(T_F^n(R_{\alpha,r})^\circ\) it is therefore also a Banach \(\mathcal{O}_F\)-algebra in its own right when equipped with the restriction of the Gauss norm for \(T_F^n(R_{\alpha,r})\) (cf: 8.1).

**Definition 2.3.** Let \(x\) be an element of \(I(G,F)\) which is contained in the inverse image of \(A(gl(n),S,F)\) under the descent map \(I(G,F) \to I(gl(n),F)\). Let \(r\) be any real number. Recall the definition of \(R_{\alpha,r}\) from 2.1. Let \(a^+_{x,r}\) be the Banach \(\mathcal{O}_F\)-algebra

\[(2.1) \quad a^+_{x,r} := T_F^n(R_{\alpha,r})^\circ/\alpha.\]

Let \(|\quad|_{x,r}\) denote the residue norm on \(a^+_{x,r}\) (see: 8.1 for the residue norm). Finally, let \(\delta_{x,r}\) denote the formal scheme \(\text{Spf}(a^+_{x,r})\).

**Remark 2.4.** Note that we have identified \(x\) with its image under the descent map \(I(G,F) \to I(gl(n),F)\), as promised in section 1.

**Lemma 2.5.** Let \(x\) and \(r\) be as in definition 2.3. Then \(a^+_{x,r}\) is a Banach co-commutative Hopf \(\mathcal{O}_F\)-algebra with Lie co-bracket.

**Proof.** First observe that, since \(\alpha\) is closed in \(T_F^n(R_{\alpha,r})^\circ\), it follows from 8.1 that \((a^+_{x,r}, |_{x,r})\) is a Banach \(\mathcal{O}_F\)-algebra. It only remains to see that \(a^+_{x,r}\) is a co-commutative Hopf algebra with Lie co-bracket. Because \(G_F\) is an affine algebraic group, and because we have chosen an injective group scheme morphism \(G_F \to gl(n)_F\) defined over \(\mathbb{Z}\), there is a canonical Hopf-algebra isomorphism between the ring of global sections \(\mathcal{O}_{gr}(g_F)\) and \(F[T_{i,j}]_{1 \leq i,j \leq n}/\alpha\). In particular we may assume that the Hopf algebra structure on \(F[T_{i,j}]_{1 \leq i,j \leq n}/\alpha\) factors through co-addition \(T_{i,j} \mapsto T_{i,j} \otimes 1 + 1 \otimes T_{i,j}\), co-bracket \(T_{i,j} \mapsto \sum_{k=1}^n (T_{i,k} \otimes T_{k,j} - T_{k,j} \otimes T_{i,k})\), co-inverse \(T_{i,j} \mapsto -T_{i,j}\) and co-identity \(T_{i,j} \mapsto 0\). Since these morphisms are defined over \(\mathcal{O}_F\) they lift to co-addition, co-bracket, co-inverse and co-identity in \(T_F^n(R_{\alpha,r})^\circ/\alpha\). □

It is a simple matter to adapt definition 2.3 and lemma 2.5 for \(G\). As above, we assume that the image of \(x\) under the descent map \(I(G,F) \to I(gl(n),F)\) is contained in \(A(gl(n),S,F)\). Also, suppose \(r \in \mathbb{R}_{\geq 0}\).
Definition 2.6. Let \( x \) be as in definition 2.3 and let \( r \) be a non-negative real number. Define \( \tilde{R}_{x,r} := (R_{x,r}, 1) \), where \( R_{x,r} \) is given in 2.1. Also, let \( \tilde{e} = (e, 1) \) where \( e = (\delta_{i,j})_{1 \leq i, j \leq n} \). Choose coordinates \((T_{i,j})_{1 \leq i, j \leq n} \). \( T_0 \) for \( T_F^{n^2+1}(e, \tilde{R}_{x,r}) \) and define
\[
S^n_F(e, R_{x,r}) := T_F^{n^2+1}(e, \tilde{R}_{x,r})/\langle T_0 \rangle \text{det} ((T_{i,j})_{1 \leq i, j \leq n} - 1).
\]

Recall that \( \text{GL}(n) = \text{Spec}(F[[T_{i,j}]_{1 \leq i, j \leq n}, T_0^{-1}]) \), where \( T_0 = \text{det} ((T_{i,j})_{1 \leq i, j \leq n}) \).

Let \( \text{Spec}(\mathfrak{A}) \) be the cokernel of \( G_F \to \text{GL}(n)_F \). Note that \( \mathfrak{A} \) generates an ideal in \( S^n_F(e, R_{x,r})^0 \). We write \( \mathfrak{A} \) for the ideal in \( S^n_F(e, R_{x,r})^0 \) also.

Definition 2.7. Let \( x \) be as in definition 2.3 and let \( r \) be a non-negative real number. Recall the definition of \( S^n_F(e, R_{x,r}) \) from 2.6. Let \( A^+_x, r \) denote the quotient Banach \( \mathcal{O}_F \)-algebra
\[
A^+_x, r := S^n_F(e, R_{x,r})^0/\mathfrak{A},
\]
and let \( |.|_{x,r} \) denote the residue norm on \( A^+_x, r \), as defined in 8.1. Let \( \hat{G}_{x,r} \) denote the formal scheme \( \text{Spf}(A^+_x, r) \).

Lemma 2.8. Let \( x \) be as in definition 2.3 and let \( r \) be a non-negative real number. Then \( A^+_x, r \) is a Hopf Banach \( \mathcal{O}_F \)-algebra.

Proof. For each \( i, j = 1, \ldots, n \), let \( \hat{T}_{i,j} = T_{i,j} + \mathfrak{A} \in A^+_x, r \). Define co-multiplication \( \mu : \hat{T}_{i,j} \mapsto \sum_{k=1}^n \hat{T}_{i,k} \otimes \hat{T}_{k,j} \) – inversion \( \iota : \hat{T}_{i,j} \mapsto (-1)^{i+j}\hat{T}_0 \) det \((T_{m,n})_{m \neq i, n \neq j} \) and co-identity \( \epsilon : \hat{T}_{i,j} \mapsto \delta_{i,j} \).

Remark 2.9. For the moment, the Banach algebras \( A^+_x, r \) and \( a^+_x, r \) are only defined under the condition on \( x \) stated in 2.3. In section 3 we will generalise this definition to all \( x \in I(G, F) \).

Remark 2.10. The notion of the formal scheme over \( \mathcal{O}_F \) associated to a \( p \)-adic group \( G(F)_{x,r} \) is a natural consequence of the authors’ efforts to adapt the work of Kazhdan-Lusztig in [12] to the \( p \)-adic context. To see this, let \( A_{x,r} \) denote the \( \mathcal{O}_F \)-algebra formed by replacing \( F \) by \( F \) in the definition of the Tate algebras appearing in the definition of \( A^+_x, r \) above. (Notice that \( A_{x,r} \) is not a Banach algebra.) Let \( A^n_{x,r} := A_{x,r}/\varpi^n A_{x,r} \). Then \( A^+_x, r = \varprojlim_{n} A^n_{x,r} \) and
\[
\hat{G}_{x,r} := \text{Spf}(A^+_x, r) = \varprojlim_{n} \text{Spec}(A^n_{x,r}).
\]

We let the Spaltenstein topology for \( G(F)_{x,r} \) be the coarsest topology such that the canonical map \( G(F)_{x,r} \to \text{Spec}(A^n_{x,r}) \) (of topological spaces) is continuous, for each \( n \in \mathbb{N} \). Thus, the inverse image under \( G(F)_{x,r} \to \text{Spec}(A^n_{x,r}) \) of a Zariski-open set in \( \text{Spec}(A^n_{x,r}) \) is Spaltenstein-open in \( G(F)_{x,r} \). If \( U \) is open in the Spaltenstein topology, then the set of all points fixed by Frobenius in \( U(\mathcal{O}_F) \) is the \( p \)-adic version of certain ‘open’ sets studied in [12]. We will recover these sets in a more geometric manner in [1].

3. Adic parahoric groups and Lie algebras

Let \( G^n_F \) and \( g^n_F \) be the adic spaces associated to \( G_F \) and \( g_F \); that is, \( G^n_F = G_F \times_{\text{Spec}(F)} \text{Spa}(F, \mathcal{O}_F) \) and \( g^n_F = g_F \times_{\text{Spec}(F)} \text{Spa}(F, \mathcal{O}_F) \) (see definition 8.8). We view \( G^n_F \) as a subspace of \( \text{GL}(n)_{F}^{ad} \), according to the map \( G \to \text{GL}(n) \) chosen in section 1; likewise, we view \( g^n_F \) as a subspace of \( \text{gl}(n)_{F}^{ad} \).
**Definition 3.1.** Let \( x \) be as in definition 2.3. For any non-negative real number \( r \), define
\[
G_{x,r}^{ad} := \text{Spa}(A_{x,r}^+ \hat{\otimes} \mathcal{O}_r \hat{\bar{F}}, A_{x,r}^+).
\]

For any real number \( r \), define
\[
g_{x,r}^{ad} := \text{Spa}(a_{x,r}^+ \hat{\otimes} \mathcal{O}_r \hat{\bar{F}}, a_{x,r}^+).
\]

Although we have introduced the adic ring \( a_{x,r} \) (see: definition 3.6) in order to define \( g_{x,r}^{ad} \), one might take a more geometric point of view, as the following indicates.

**Lemma 3.2.** For any \( x \in I(G, F) \) and \( r \in \mathbb{R}_{\geq 0} \),
\[
G_{x,r}^{ad} = \left\{ g \in G_{x,r}^{ad} \mid |T_{i,j}(g) - \delta_{i,j}| \leq q^{-\epsilon_{i,j}}, \ 1 \leq i, j \leq n \right\},
\]
where \( r_{x,r}^{i,j} \in \mathbb{R} \) is as defined in 2.1. Likewise, for any \( x \in I(G, F) \) and any \( r \in \mathbb{R} \),
\[
g_{x,r}^{ad} = \left\{ X \in g_{x,r}^{ad} \mid |T_{i,j}(X)| \leq q^{-\epsilon_{i,j}}, \ 1 \leq i, j \leq n \right\}.
\]

**Remark 3.3.** Fix a Borel subgroup in \( G_F \) containing a maximal torus \( T \); let \( X(T) \) be the root system defined by this choice. The choice of a Borel subgroup determines an ordering of \( X(T) \). Let \( \{ a \mid a \in X(T) \} \) be the épiplonge for \( G_F \) determined by these choices. The épiplonge determines affine group scheme morphisms \( t_a : GL(1)_F \rightarrow G_F \) and \( u_a : A_1^F \rightarrow G_F \) for each \( a \in X(T) \), in the usual manner. The affine group scheme morphisms \( t_a \) and \( u_a \) induce adic affine group morphisms \( t_a^{ad} : GL(1)_F^{ad} \rightarrow G_F^{ad} \) and \( u_a^{ad} : (A_1^F)^{ad} \rightarrow G_F^{ad} \). For each affine root \( \alpha = (a, \rho) \), let \( U_\alpha \) denote the image of \( B_\rho^F(q^{-\rho}) \) (refer to 8.20 in appendix 8.6; see also example 4.1) under \( u_\alpha^{ad} \). Also, let \( T_\alpha \) denote the image of \( T_\rho^F(q^{-\rho}/q^{-\rho})(T_1 T_2 - 1) \) (c.f.: example 4.3) under \( t_\alpha^{ad} \). If \( x \in I(G, F) \) and \( r \in \mathbb{R}_{\geq 0} \), then \( G_{x,r}^{ad} \) is the smallest adic subgroup of \( G_F^{ad} \) containing \( T_\alpha^{ad} \) for each root \( \alpha \), and also containing \( U_\alpha \) for each affine root \( \alpha \) such that \( \alpha(x) \geq r \).

Let \( x \) be any element of \( I(G, F) \). Recall that in section 1 we assumed that the inverse image of \( A(GL(n), S, F) \) under the descent map \( I(G, F) \rightarrow I(GL(n), F) \) contains a fundamental domain for the action of \( G(F) \) on \( I(G, F) \). Let \( x' \) be the unique element in the intersection of \( G(F) \cdot x \) and that fundamental domain. It follows that there is a unique \( g \in G(F) \) such that \( g \cdot x = x' \). Then \( G(F)_{x',r} = \text{Inn}(g^{-1})G(F)_{x',r} \), where \( \text{Inn} : G_F \times \text{Spec}(F) G_F \rightarrow G_F \) is the morphism of algebraic group schemes corresponding to the action of \( G(F) \) on \( G(F) \) by inner automorphisms. We will use this idea to define \( G_{x,r}^{ad} \).

Let \( \text{Inn}^{ad} : G_F^{ad} \times \text{Spec}(F) \rightarrow G_F^{ad} \) be the morphism of adic spaces induced from \( \text{Inn} : G_F \times \text{Spec}(F) G_F \rightarrow G_F \) and let \( \text{Inn}^{ad}(g^{-1}) : G_F^{ad} \rightarrow G_F^{ad} \) be the adic morphism defined using \( \text{Inn}^{ad} \) and the canonical injection \( G_F \rightarrow G_F^{ad} \) of topological spaces. We may now use definition 3.1 to define \( G_{x',r}^{ad} \), since \( x' \) is contained in the inverse image of \( A(GL(n), S, F) \) under the descent map \( I(G, F) \rightarrow I(GL(n), F) \).

**Definition 3.4.** Let \( x, x' \) and \( gG(F)_x \in G(F)/G(F)_x \) be as in the paragraph above. Define
\[
G_{x',r}^{ad} := \text{Inn}^{ad}(g^{-1})G_{x',r}^{ad}.
\]
In the same manner one defines
\begin{equation}
\mathfrak{g}_{x, r}^{\text{ad}} := \text{Ad}^{-1}(g^{-1}) \mathfrak{g}_{x, r}^{\text{ad}}.
\end{equation}

Let \( \{ r_i \}_{i \in \mathbb{Z}} \) be the increasing sequence of jumping points for \( x \) defined in [20, §3.4], with \( r_0 = 0 \). For any \( r \in \mathbb{R}_{\geq 0} \) we will write \( G_{x, r}^{\text{ad}} \) for \( G_{x, r_i + 1}^{\text{ad}} \), where \( r_{i-1} < r \leq r_i \); likewise, for any \( r \in \mathbb{R} \) \( \mathfrak{g}_{x, r}^{\text{ad}} := \mathfrak{g}_{x, r_i + 1}^{\text{ad}} \).

\textbf{Remark 3.5.} To see that \( G_{x, r}^{\text{ad}} \) is well-defined, it is enough to note that \( \text{Inn}^{\text{ad}}(g_1, g_2) = \text{Inn}^{\text{ad}}(g_1) \circ \text{Inn}^{\text{ad}}(g_2) \) and that \( \text{Inn}^{\text{ad}} \) restricts to \( G_{x, r}^{\text{ad}} \times_{\text{Spa}(F, \mathcal{O}_F)} G_{x, r}^{\text{ad}} \to G_{x, r}^{\text{ad}} \). (It is also worth remarking that the image of \( G(F)_{x, r} \) under \( G(F) \to G_{x, r}^{\text{ad}} \) is precisely the set of type-I points in \( G_{x, r}^{\text{ad}} \).)

\textbf{Definition 3.6.} Suppose \( x \) is any element of \( I(G, F) \). If \( r \in \mathbb{R}_{\geq 0} \), let \( A_{x, r} \) denote the pair \((A_{x, r}^{\text{op}}, A_{x, r}^+) := (\mathcal{O}_{G_{x, r}^{\text{ad}}}(G_{x, r}^{\text{ad}}), \mathcal{O}_{G_{x, r}^{\text{ad}}}(G_{x, r}^{\text{ad}})^+)\). (Refer to section 8.5 for the definition of the sheaves \( \mathcal{O}_{G_{x, r}^{\text{ad}}} \) and \( \mathcal{O}_{G_{x, r}^{\text{ad}}}^+ \).) Likewise, for any \( r \in \mathbb{R} \), let \( a_{x, r} \) denote the pair \((a_{x, r}^{\text{op}}, a_{x, r}^+) := (\mathcal{O}_{\mathfrak{g}_{x, r}^{\text{ad}}}(\mathfrak{g}_{x, r}^{\text{ad}}), \mathcal{O}_{\mathfrak{g}_{x, r}^{\text{ad}}}(\mathfrak{g}_{x, r}^{\text{ad}})^+)\).\( \mathcal{O}_{\mathfrak{g}_{x, r}^{\text{ad}}} \) and \( \mathcal{O}_{\mathfrak{g}_{x, r}^{\text{ad}}}^+ \) are modules over \( \mathcal{O}_{G_{x, r}^{\text{ad}}} \) and \( \mathcal{O}_{G_{x, r}^{\text{ad}}}^+ \), respectively.

\textbf{Remark 3.7.} For any \( x \in I(G, F) \), note that \( A_{x, r}^{\text{op}} = A_{x, r}^+ \odot \mathcal{O}_F \).

\textbf{Lemma 3.8.} Let \( x \) and \( r \) be as in definition 3.4. Then \( G_{x, r}^{\text{ad}} \) is an adic affinoid group object in \( G_{x, r}^{\text{ad}} \) and \( \mathfrak{g}_{x, r}^{\text{ad}} \) is an adic affinoid Lie algebra object in \( \mathfrak{g}_{x, r}^{\text{ad}} \).

\textbf{Proof.} We will prove the lemma for \( G_{x, r}^{\text{ad}} \) only; the argument for \( \mathfrak{g}_{x, r}^{\text{ad}} \) is similar.

Observe that \( A_{x, r} \) is an affinoid adic ring. It follows immediately from definition 3.4 that \( G_{x, r}^{\text{ad}} := \text{Spa}(A_{x, r}) \); thus, \( G_{x, r}^{\text{ad}} \) is an adic affinoid space. The Hopf algebra operations extend from \( A_{x, r}^+ \) to \( A_{x, r} \) and the resulting operations are adic ring homomorphisms. Thus, \( G_{x, r}^{\text{ad}} \) is a group object in the category of adic spaces, as defined in section 8.7.

Since \( G_{x, r}^{\text{ad}} \) is the adic space for the scheme \( G_F \), it follows that \( \mathcal{O}_{G_{x, r}^{\text{ad}}}(G_{x, r}^{\text{ad}}) \) is isomorphic to \( \mathcal{O}_{G_{F}}(G_{F}) \) which in turn is isomorphic to
\begin{equation}
\hat{F}(\mathbb{Z})_{1 \leq i, j \leq n, \det((T_{i, j})_{1 \leq i, j \leq n})} / \mathfrak{A}.
\end{equation}

By 2.3 there is an obvious adic monomorphism \( (\mathcal{O}_{G_{x, r}^{\text{ad}}}(G_{x, r}^{\text{ad}}), \mathcal{O}_{G_{x, r}^{\text{ad}}}(G_{x, r}^{\text{ad}})^+) \to A_{x, r} \).

By lemma 8.19, this corresponds to an injective morphism \( \mathfrak{g}_{x, r}^{\text{ad}} \to \mathfrak{g}_{x, r}^{\text{ad}} \) in category \( (\mathcal{O}) \) which allows us to view \( \mathfrak{g}_{x, r}^{\text{ad}} \) as a subset of \( \mathfrak{g}_{x, r}^{\text{ad}} \). (Here is matters that we work in category \( (\mathcal{O}) \) where objects are isomorphism classes.)

\section{Examples of Adic Parahorics and Lie Algebras}

\textbf{Example 4.1.} Let \( g = gl(1) \). Note that \( g = (\mathbb{A}^1, +) \) (i.e., the Lie bracket is trivial). The adic ring of global sections on \( \mathfrak{g}_{F}^{\text{ad}} \) is \( a := (\hat{F}[T], \mathcal{O}_F[T]) \). The Hopf algebra structure on this adic ring is given by \( F \)-algebra homomorphisms \( \alpha : T \mapsto T \odot 1 + 1 \odot T \) (co-addition), \( \beta : T \mapsto 0 \) (co-bracket), \( \iota : T \mapsto -T \) (co-inversion), and \( \epsilon : T \mapsto 0 \) (co-identity). The affine Bruhat-Tits building \( I(GL(1), F) \) consists of one point, which we denote \( x_0 \). Before describing the parahoroid Lie algebra in \( \mathfrak{g}_{F}^{\text{ad}} \) recall that the parahoric algebra in \( g(F) \) is \( g(F)_{x_0} = g(\mathcal{O}_F) \). More generally, for any \( r \in \mathbb{R} \), \( g(F)_{x_0, r} = g(p_F^{|r|}) \), where \( |r| \) denotes the greatest-integer function. (That is, \( |r| \) is the greatest integer less than or equal to \( r \).)
The Banach $\Omega_F$-algebra for the adic parahoric Lie algebra $g^{ad}_{x_0}$ is $a^{+}_{x_0} = T^1_F(1)^{\times}$; thus, $a_{x_0} = (F(T), \Omega_F(T))$. Geometrically, the adic parahoric Lie algebra $g^{ad}_{x_0}$ is the unit polydisc $B^1_F(1)$, as defined in 8.20; that is,

\begin{equation}
\mathfrak{g}^{ad}_{x_0} = \left\{ X \in \mathfrak{g}^{ad} \mid |T(X)| \leq 1 \right\},
\end{equation}

where $\mathfrak{g}_F = \text{Spec}(F[T])$. To see that $\mathfrak{g}^{ad}_{x_0}$ is a sub-Lie algebra of $\mathfrak{g}_F^{ad}$ it suffices to observe that co-addition and co-inversion extend from $a$ to $a_{x_0}$. In particular, if $f \in F(T)$, then $\alpha(f) \in F(T) \otimes_F F(T)$ and $\iota(f) \in F(T)$; moreover, if $f \in \Omega_F(T)$ then $\alpha(f) \in \Omega_F(T) \otimes_F \Omega_F(T)$ and $\iota(f) \in \Omega_F(T)$. These assertions are very simple to check.

Turning to the adic groups in the filtration of $\mathfrak{g}^{ad}_{x_0}$ we see that $a^{+}_{x_0,r} = T^1_F(q^{-r})$ as defined in 8.4. Geometrically, we see that the adic parahoric Lie algebra $\mathfrak{g}^{ad}_{x_0}$ is filtered by the adic Lie algebras

\begin{equation}
\mathfrak{g}^{ad}_{x_0,r} = \left\{ X \in \mathfrak{g}^{ad} \mid |T(X)| \leq q^{-r} \right\};
\end{equation}

that is, $\mathfrak{g}^{ad}_{x_0,r} = B^1_F(q^{-r})$. To see that $\mathfrak{g}^{ad}_{x_0,r}$ is a sub-Lie algebra in $\mathfrak{g}^{ad}_F$, for any $r$, is it sufficient to observe that $\alpha, \beta, \iota$ and $\epsilon$ extend as adic ring morphisms from $a$ to $a^{+}_{x_0,r}$. The only non-trivial part of this is the following. Suppose $\sum_{n \in \mathbb{N}} a_n T^n \in a^{+}_{x_0,r}$. Thus, $|a_n| q^{-nr} \leq 1$ and $\lim_{n \to \infty} |a_n| q^{-nr} = 0$. Then

\begin{equation}
\alpha \left( \sum_{n \in \mathbb{N}} a_n T^n \right) = \sum_{i,j \in \mathbb{N}} a_{i,j} (T^i \otimes 1)(1 \otimes T^j)
\end{equation}

where $a_{i,j} := \binom{i+j}{i}$ (the binomial coefficient). By the binomial theorem,

$|a_{i,j}| \leq |a_{i+j}|$. Thus, $|a_{i,j}| q^{-(i+j)r} \leq 1$ and $\lim_{i,j \to \infty} |a_{i,j}| q^{-(i+j)r} = 0$. This shows that $\alpha$ maps $a^{+}_{x_0,r}$ to $a_{x_0,r} \otimes \Omega_F a_{x_0,r}$, for every $r \in \mathbb{R}$. This completes the demonstration that $\mathfrak{g}^{ad}_{x_0,r}$ is a sub-Lie algebra in $\mathfrak{g}^{ad}_F$.

**Example 4.2.** Let $g = g(l(n))$, so $g^{ad} = (A^n, +, [\cdot, \cdot])$ The global sections on $\mathfrak{g}^{ad}_F$ are $\hat{F}[T_1, \ldots, T_n]$. With the choices above we see that $\mathfrak{g}^{ad}_{x_0,r} = B^1_F(R)$ where $0 = (0, \ldots, 0)$ and $R = (q^{-r}, \ldots, q^{-r})$. Note also that $\mathfrak{g}^{ad}_{x_0,r} = T^1_F(R)$, where $R = (q^{-r}, \ldots, q^{-r})$. In fact, if $R = (q^{-r_1}, \ldots, q^{-r_n})$, then the polydisc $B^1_F(R)$ is equal to $\mathfrak{g}^{ad}_{x_0,r}$ for some $x \in I(G, F)$ and $r \in \mathbb{R}$. For example, if $\sum_{i=1}^n r_i = 0$ and $R = (q^{-r_1}, \ldots, q^{-r_n})$, then $B^1_F(R) = \mathfrak{g}^{ad}_{x_0}$, for some vertex $x \in I(G, F)$.

**Example 4.3.** Let $G = \text{GL}(1)$, so $G^{ad}_F$ is the adic group associated to $G(L) = \text{Spec}(F[T, T^{-1}])$. The adic ring for $G^{ad}_{x_0}$ is $(F(T, T^{-1}), \Omega_F(T, T^{-1}))$, which is easily seen to be a Hopf adic ring with respect to co-multiplication $\mu : T \mapsto T \otimes T$, co-inversion $\iota : T \mapsto T^{-1}$ and co-identity $\epsilon : T \to 1$. (This is a very special case of lemma 8.2.) The adic parahoric group $G^{ad}_{x_0}$ is all $g \in G^{ad}_F$ such that $|T(g)| \leq 1$ and $|T^{-1}(g)| \leq 1$; thus,

\begin{equation}
G^{ad}_{x_0} = \left\{ g \in G^{ad}_F \mid |T(g)| = 1 \right\}.
\end{equation}
The proper subgroups in the filtration of \(G_{x_0}^{ad}\) are given as follows: if \(r \in \mathbb{R}^{\geq 0}\) then
\[
(4.5) \quad G_{x_0, r}^{ad} = \left\{ g \in G_F^{ad} \mid |T(g) - 1| \leq q^{-r} \right\}.
\]

The ring \(A_{x_0, r}^{p}\) for \(G_{x_0, r}^{ad}\) is the ring of formal power series \(f\) with coefficients in \(F\) such that \(\lim_{i \to \infty} |a_i|_{F}q^{-1}r = 0\), where \(f = \sum_{i \in \mathbb{Z}} a_i(T - 1)^i\). This

**Remark 4.4.** Example 4.3 illustrates an important feature of definition 3.4. Consider the canonical adic space for \(G_F^{ad}\), as in section III.4.5 of [8] for example, given by the family of affinoid adic spaces
\[
(4.6) \quad (G_F^{ad})_n := \text{Spa}(F(\mathbb{T}^{-n}T, \mathbb{T}^{-n}T^{-1})),
\]
where \(n\) ranges over all \(\mathbb{Z}\). To be sure, these affinoid spaces do indeed define a filtration of \(G_{x_0} = (G_F^{ad})_0\) by restricting \(n\) to \(\mathbb{N}\); however, \((G_F^{ad})_n\) is a group object in category \(\mathcal{V}\) (see 8.6) only when \(n = 0\). On the other hand, the spaces in the Moy-Prasad filtration of the adic parahoric space for \(G_F^{ad}\) are all group objects in category \(\mathcal{V}\). In particular, in the example above, if \(r \in \mathbb{R}^{> 0}\) then \(G_{x_0, r}^{ad} = \text{Spa}(S_T(q^{-r}))\), which is a Hopf algebra with co-multiplication \(\mu : T \mapsto T \otimes T\), co-inversion \(\iota : T \mapsto T \otimes T^{-1}\) and co-identity \(e : T \mapsto 1\).

**Example 4.5.** Let \(G = SL(2)\). The adic parahoric group \(G_{x_0}^{ad}\) corresponding to \(G(F_0)_{x_0} = G(\mathbb{D}_0)\) is
\[
(4.7) \quad G_{x_0}^{ad} = \left\{ g \in G_F^{ad} \mid |T_i(g)| \leq 1, \forall i = 1, \ldots, 4 \right\},
\]
where \(G_F = \text{Spec}(F[T]/(\det T - 1))\) with \(T = (T_1, \ldots, T_4)\) and \(\det T = T_1T_4 - T_2T_3\). Thus, \(A_{x_0}^{p} = F(T)/(\det T - 1)\), and \(A_{x_0}^{c} = \Omega_F(T)/(\det T - 1)\). If \(r \in \mathbb{R}^{> 0}\) then the parahoroid subgroup \(G_{x_0, r}^{ad}\) is the set of all \(g \in G_{x_0}^{ad}\) such that \(|T_i(g) - 1| \leq q^{-r}\) if \(i = 1, 4\) and \(|T_i(g)| \leq q^{-r}\) otherwise. In this case, \(A_{x_0, r}^{p}\) is an \(F\)-algebra of formal power series \(f \in F[[T]]/(\det T - 1)\) such that such that \(|f|_{F, R_{x_0, r}} < \infty\), where \(e = (1, 0, 0, 1)\) and \(R_{x_0, r} = (q^{-r}, q^{-r}, q^{-r}, q^{-r})\).

**Example 4.6.** Let \(G = SL(2)\) and let \(x_1\) be the special versuspecial point in the affine Bruhat-Tits building for \(G\). The maximal parahoric group \(G_{x_1}^{ad}\) is the set of all \(g \in G_{x_0}^{ad}\) such that \(|T_i(g)| \leq 1\), for \(i = 1, 4\) and \(|T_2(g)| \leq q\) and \(|T_3(x)| \leq q^{-1}\). If \(r \in \mathbb{R}^{> 0}\), then the parahoroid subgroup \(G_{x_0, r}^{ad}\) is the set of all \(g \in G_{x_0}^{ad}\) such that \(|T_i(x) - 1| \leq q^{-r}\) for \(i = 1, 4\) and \(|T_2(x)| \leq q^{-1+r}\) and \(|T_3(x)| \leq q^{-r-1}\). Turning to the adics, we have \(A_{x_0, r}^{p} = T_F(e, R_{x_0, r})/(\det T - 1)\) where \(R_{x_0} = (1, q, q^{-1}, 1)\). More generally, \(A_{x_0, r}^{p} = T_F(e, R_{x_0, r})/(\det T - 1)\) where \(R_{x_0, r} = (q^{-r}, q^{-1-r}, q^{-1-r}, q^{-r})\).

**Example 4.7.** Let \(G = SL(2)\) and let \(x_0\) be the midpoint in the standard chamber for \(I(G, F)\). Then \(G(F_0)_{x_0}\) is the standard Iwahori subgroup in the \(p\)-adic group \(G(F_0)\) and the parahoric group \(G_{x_0}^{ad}\) is the set of all \(g \in G_{x_0}^{ad}\) such that \(|T_i(g)| \leq 1\) for \(i = 1, 2, 4\) and \(|T_3(g)| \leq q^{-1}\). The ring of sections on \(G_{x_0}^{ad}\) is \(A_{x_0}^{p} = T_F(R)/(\det T - 1)\) where \(R = (1, 1, q^{-1}, 1)\), with \(T\) and \(\det T\) as above. For \(r \in \mathbb{R}^{> 0}\) the parahoroid subgroup \(G_{x_0, r}^{ad}\) consists of all \(g \in G_{x_0}^{ad}\) such that \(|T_3(g) - 1| \leq q^{-2r}\) for \(i = 1, 2\) and \(|T_2(g)| \leq q^{-2r}\) and \(|T_0(g)| \leq q^{-1-2r}\). The ring of sections on \(G_{x_0, r}^{ad}\) is \(A_{x_0, r}^{p} = T_F(e, R_{x_0, r})/(\det T - 1)\), where \(e = (1, 0, 0, 1)\) and \(R_{x_0, r} = (q^r, q^r, q^{-r}, q^r)\).
5. The reductive quotient for adic parahoric groups and Lie algebras

In this section we look for the cokernel of the inclusion $G_{x,r}^{ad} \hookrightarrow G_{x,r}^{ad}$ in the category of ringed group objects. Although $G_{x,r}^{ad}$ is a group object in category $(\mathcal{V})$, the topological space $|G_{x,r}^{ad}|$ for $G_{x,r}^{ad}$ is not a group. Nevertheless, if the quotient space $|G_{x,r}^{ad}|/|G_{x,r}^{ad+}|$ is well-defined. In this section we define an affine algebraic group scheme $M_{x,r}$ over $k$ and a continuous morphism $\rho_{x,r} : G_{x,r}^{ad} \to M_{x,r}$ in the category of ringed group objects, called a reduction map, with kernel $G_{x,r}^{ad+}$. We also define an affine Lie algebra scheme $m_{x,r}$ over $k$ and a reduction map $\rho_{x,r} : \mathfrak{g}_{x,r}^{ad} \to m_{x,r}$ with kernel $\mathfrak{g}_{x,r}^{ad+}$. In section 7 we use the reduction map to define a functor from the étale site on $M_{x,r}$ to the étale site on $G_{x,r}^{ad}$. This will allow us to inflate sheaves from $M_{x,r}$ to $l$-adic étale sheaves on $G_{x,r}^{ad}$.

Definition 5.1. Let $\mathcal{G}_{x,r}$ denote the formal scheme $\text{Spf}(A_{x,r}^+(r))$ and let $\mathfrak{g}_{x,r}$ denote the reduced fibre of $\mathcal{G}_{x,r}$. Likewise, let $\mathfrak{a}_{x,r}$ denote the formal scheme $\text{Spf}(a_{x,r}^+(r))$ and let $\mathfrak{a}_{x,r}$ denote the reduced fibre of $\mathfrak{a}_{x,r}$.

Lemma 5.2. $G_{x,r}^{ad}$ is the adic space associated to the formal scheme $\mathcal{G}_{x,r}$ and the reduced fibre $\mathcal{G}_{x,r}^r$ is a reductive group scheme over $k$. Likewise, $\mathfrak{g}_{x,r}^{ad}$ is the adic space associated to the formal scheme $\mathfrak{a}_{x,r}$ and the reduced fibre $\mathfrak{a}_{x,r}$ is the Lie algebra of $\mathcal{G}_{x,r}$.

Proof. Recall from lemma 3.4 that $A_{x,r}^+ = (A_{x,r}^o)^c$, where $A_{x,r}$ is the adic ring defining $G_{x,r}^{ad}$. The claims follow immediately from basic definitions in [7] and [9].

Remark 5.3. In fact, $G_{x,r}^{ad}$ (resp. $\mathfrak{g}_{x,r}^{ad}$) is the adic space associated to the rigid analytic space $\text{Sp}(A_{x,r}^+(r))$ (resp. $\text{Sp}(a_{x,r}^+(r))$), where $\text{Sp}()$ indicates the functor taking Banach algebras to their Gel'fand maximal ideal spectra. Since we do not take that perspective here, we will not follow this observation except to say that it leads to the definition of the rigid analytic spaces appearing in [24].

Definition 5.4. Let $\lambda_{x,r} : G_{x,r}^{ad} \to \mathcal{G}_{x,r}$ be the morphism of ringed spaces defined by section 1.9 of [10]; define $\lambda_{x,r} : \mathfrak{g}_{x,r} \to \mathfrak{a}_{x,r}$ likewise.

Lemma 5.5. Suppose $x \in I(G,F)$. If $r \in \mathbb{R}^{\geq 0}$, the cokernel of $\mathcal{G}_{x,r}^{+} \to \mathcal{G}_{x,r}$ exists in the category of group schemes over $\mathbb{F}_q$. Likewise, if $r \in \mathbb{R}$, the cokernel of $\mathfrak{a}_{x,r}^{+} \to \mathfrak{a}_{x,r}$ exists in the category of Lie algebra schemes over $\mathbb{F}_q$.

Definition 5.6. Let $\mathcal{G}_{x,r} \to M_{x,r}$ be the kernel of $\mathcal{G}_{x,r}^{+} \to \mathcal{G}_{x,r}$ as in lemma 5.5; we will refer to $M_{x,r}$ as the reductive quotient for $G_{x,r}^{ad}$. Likewise, let $\mathfrak{a}_{x,r} \to m_{x,r}$ be the kernel from $\mathfrak{a}_{x,r}^{+} \to \mathfrak{a}_{x,r}$; we will refer to $m_{x,r}$ as the reductive quotient for $\mathfrak{g}_{x,r}^{ad}$.

Example 5.7. Let $\mathfrak{g} = gl(1)$, so $\mathfrak{g} \cong (\mathbb{A}^1, +)$. As in example 4.1 we see that $a_{x,0}^+ = F(T)$ and $a_{x,0} = \mathcal{O}_F(T)$; thus, ignoring the Lie algebra structure we have $\mathfrak{g}_{x,0}^{ad} = B_+^1(1)$. It follows that $\mathcal{a}_{x,0} = \text{Spf}(\mathcal{O}_F(T))$ and $\mathfrak{a}_{x,0} = \text{Spec}(\mathbb{F}_q[T])$. In the case at hand, $\mathfrak{g}_{x,0,0}^{ad} = B_+^1(q^{-1})$ and $\mathfrak{a}_{x,0,0} = \text{Spec}(\mathbb{F}_q[T^\pm 1])$, and the cokernel from definition 5.6 is the identity morphism; that is, $m_{x,0} = \mathfrak{a}_{x,0} = \text{Spec}(\mathbb{F}_q[T])$. 

Example 5.8. Let $g = sl(2)$ and let $x$ be the barycentre of the standard chamber for the affine Bruhat-Tits building for $G$. Thus, $x = x_{01}$ and $g(F_0,x)$ is the standard Bruhat Lie algebra in $g(F_0)$, as in example 4.7. Ignoring the Lie algebra structure, $g^a_{x} \cong B_F^1(R_0)$ where $R_0 = (1,1,q^{-1})$. Thus, $a_{x_0,x} = \text{Spa}(T_F^2(R_0),T_F^3(R_0)^*)$.

Choose coordinates by viewing $a_{x_0,x}^b$ as a subalgebra of $F[[T_0,T_1,T_2]]$. Thus, $\tilde{g}_x,0 = \text{Spec}(\mathbb{F}_q[T_0,T_1,\omega^{-1}T_2])$. On the other hand, $g^a_{x_0,x}^d$ is isomorphic to $B_F^1(R_0^*)$, where $R_0^* = (\omega^{-1},1,q^{-1})$.

In this situation, $\tilde{g}_x,0 = \text{Spec}(\mathbb{F}_q[\omega^{-1}T_0,T_1,\omega^{-1}T_2])$. Thus, $m_x = \text{Spec}(\mathbb{F}_q[T_0]) = \mathbb{A}_q^1$.

Definition 5.9. Let $\rho_{x,r}: G^a_{x,r} \rightarrow M_x,r$ be the composition of $\lambda_{x,r}: G^a_{x,r} \rightarrow \hat{G}_{x,r}$ and the cokernel $\bar{G}_{x,r} \rightarrow M_x,r$, where we identify $\bar{G}_{x,r}$ with $G_{x,r}$ (cf. remark 7.4).

Likewise, let $\varphi_{x,r}: g^a_{x,r} \rightarrow m_x,r$ be the composition of $\lambda_{x,r}: g^a_{x,r} \rightarrow \tilde{g}_x,0$ and the cokernel $\bar{g}_x,0 \rightarrow m_x,r$. The functions $\rho_{x,r}$ and $\varphi_{x,r}$ will be called reduction maps.

We will set $\rho_x := \rho_{x,0}$ and $\varphi_x := \varphi_{x,0}$.

Remark 5.10. In fact, the reductive quotient $M_{x,r}$ is the Galois group of $G^a_{x,r}$ over $G^a_{x,s}$. More generally, if $0 \leq r < s$, the Galois group $M_{x,n_s}$ of $G^a_{x,s}$ over $G^a_{x,r}$ is an algebraic group over $k$ which may be given a $k$-rational structure as in section 4 of [18].

6. Depth-zero stratification of adic parahoric groups and Lie algebras

Because we have assumed $G(F)$ is semisimple, the reductive group scheme $M_x$ is connected. We now briefly recall the stratification of $M_x$ used by Lusztig in 3.1 of [15] and used to define character sheaves on $M_z$. Let $L$ be a Levi subgroup of a parabolic subgroup of $M_z$ and let $\Sigma$ be the image of an isolated class in $L/Z^{\circ}_{M_z}(L)$ under the map $L \rightarrow L/Z^{\circ}_{M_z}(L)$. Let $\Sigma_{\text{reg}}$ be the set of all $l \in L$ such that $Z^{\circ}_{M_z}(l) \subset L$, where $l$ is the semi-simple part of $l$. The $M_z$-orbit of $\Sigma_{\text{reg}}$ is denoted by $Y_{[L,\Sigma]}$. From 3.1 of [15] we also recall that $Y_{[L,\Sigma]}$ is a smooth, constructible, irreducible variety in $M_x$ and that $M_x$ is the disjoint union of the $Y_{[L,\Sigma]}$ as $(L,\Sigma)$ runs over all such pairs up to $M_z$-conjugacy. We will refer to each $Y_{[L,\Sigma]}$ as a decomposition class for $M_x$ (see: [4]). Likewise one defines decomposition classes in $m_x$.

Lemma 6.1. If $Y$ is a decomposition class for $M_x$, then $\rho_x^{-1}(Y)$ is a rational domain and is constructible in $G^a_{x,r}$. Likewise, if $Y$ is a decomposition class for $m_x$, then $\varphi_x^{-1}(Y)$ is a rational domain and is constructible in $g^a_{x,r}$.

Definition 6.2. The depth-zero stratification for $G^a_x$ consists of sets $\rho_x^{-1}(Y)$, as $Y$ ranges through all decomposition classes for $M_x$. Likewise, the depth-zero stratification for $g^a_x$ consists of sets $\varphi_x^{-1}(Y)$, as $Y$ ranges through all decomposition classes for $m_x$.

Example 6.3. Let $g = sl(2)$. The restriction of the stratification of $m_{x_0}$ to the variety $m^\text{nill}_{x_0}$ of nilpotent elements in $m_{x_0}$ yields the decomposition $m^\text{nill}_{x_0} = Y_{[2]} \cup Y_{[3]}$, where $Y_{[2]} = \{0\}$ (the trivial nilpotent orbit in $m_{x_0}$) and where $Y_{[2]}$ is the open variety of regular nilpotent elements in $m_{x_0}$. Pick coordinates by writing $m_{x_0} = \text{Spec}(k[T_0,T_1,T_2])$ (ignoring the Lie algebra structure); thus, $\mathcal{O}_{m_{x_0}}(m^\text{nill}_{x_0}) = k[T_0,T_1,T_2]/(T_0^2 + T_1T_2)$. The inverse image of $m^\text{nill}_{x_0}$ under $\varphi_x: g^a_{x_0} \rightarrow m_{x_0}$ is the
rational Weierstrass domain

\[ (g^{ad}_{x_0})^{in} := \left\{ X \in g^{ad}_{x_0} \left| |(T_0^2 + T_1 T_2)(X)| < 1 \right. \right\}. \]

Thus, the depth-zero stratification of the adic space of topologically nilpotent elements in $g^{ad}_{x_0}$ is

\[ (g^{ad}_{x_0})^{in} = g^{ad}_{x_0} (Y_{1|1}) \cup g^{ad}_{x_0} (Y_{2|1}). \]

Note that $g^{ad}_{x_0} (Y_{1|1}) = g^{ad}_{x_0, 0+}$.

Remark 6.4. In fact, it will not be enough to consider each adic parahoric group alone; in [1] we will make use of all the depth-zero stratification of all adic parahoric groups simultaneously.

7. Fundamental strata for adic algebraic groups

It is time to turn our attention to adic groups and adic parahoric groups over $\text{Spa}(\hat{F}_a)$, where $\hat{F}_a$ denotes the completion of the algebraic closure $F_a$ of $F$. Recall that $\hat{F}_a$ is itself algebraically closed.

Let $G^{ad}_{F_a}$ be the adic space associated with $G_{F_a}$; that is, $G^{ad}_{F_a} = G_{F_a} \times \text{Spec}(F_a)$ $\text{Spa}(\hat{F}_a, \Omega_{F_a})$.

Definition 7.1. Let $G^{ad}_{x, r}$ be the adic parahoric group over $F_a$ for the pair $(x, r)$, where $x \in I(G, F)$ and $r \in \mathbb{R}_{\geq 0}$; that is,

\[ G^{ad}_{x, r} := C^{ad}_{x, r} \times \text{Spa}(\hat{F}_a, \Omega_{F_a}). \]

Also, let $A_{x, r}$ be the adic ring $(A^{x, r}, A^{x, r}_+) := (A_{x, r} \hat{\otimes}_{\hat{F}_a} \hat{F}_a, A_{x, r} \hat{\otimes}_{\hat{F}_a} \hat{F}_a).$ Likewise, let $g^{ad}_{x, r}$ be the adic parahoric Lie group over $F_a$ for the pair $(x, r)$, where $x \in I(G, F)$ and $r \in \mathbb{R}$; that is,

\[ g^{ad}_{x, r} := g^{ad}_{x, r} \times \text{Spa}(\hat{F}_a, \Omega_{F_a}). \]

Let $a_{x, r}$ be the adic ring $(a^{x, r}, a^{x, r}_+) := (a_{x, r} \hat{\otimes}_{\hat{F}_a} \hat{F}_a, a^{x, r} \hat{\otimes}_{\hat{F}_a} \hat{F}_a).$

Remark 7.2. Note that $G^{ad}_{x, r} = \text{Spa}(A_{x, r})$ and $g^{ad}_{x, r} = \text{Spa}(a_{x, r})$. Moreover, these are continuously varying families of affinoid adic spaces in $r$; in particular, notice that $G^{ad}_{x, r} \neq \bigcup_{s \geq r} G^{ad}_{x, t}$.

Definition 7.3. Let $g^* := \text{Hom}_{F_a}(g, \hat{F}_a)$. For every point $x \in I(G, F)$, we set for any $r \in \mathbb{R}$

\[ g^*_{x, r} := \left\{ \beta \in g^* : \beta(X) \in p_{F_a} \text{ for all } X \in g_{x, r+} \right\}. \]

(see [20, §3.5 (*)]). We denote by $\{r_i\}_{i \in \mathbb{N} \cup \{0\}}$ the increasing sequence of jumping points defined in [20, §3.4] (cf. definition 3.4). We also define $g^*_{x, r+} = g^*_{x, r_i-}$ where $r_{i-1} < r \leq r_i$.

For $i \geq 1$, the $\hat{F}_q$-bilinear map

\[ g^*_{x, r_i}/g^*_{x, r_i} \times g^*_{x, r_{i+1}} \to \hat{F}_q \]

\[ (\beta, X) \mapsto \beta(X) \mod p_F \]

is a nondegenerate $M_2$-invariant pairing. This nondegenerate pairing composed with a fixed nontrivial character of the prime field of $\hat{F}_q$ provides an $M_2$-equivariant isomorphism $J_{x, r_i}$ of the Pontrjagin dual of $g^*_{x, r_i}/g^*_{x, r_{i+1}}$ onto $g^*_{x, r_i}/g^*_{x, r_{i-1}}$. Then
for $r > 0$, the natural isomorphism of $G_{x,r}/G_{x,r}^+$ with $g_{x,r}/g_{x,r}^+$ gives an isomorphism of $G_{x,r}/G_{x,r+1}$ onto $g_{x,r}/g_{x,r+1}$ which is $M_x$-equivariant (see [21, 2.24]). Thus the above also provides an isomorphism $J_{x,r}$ of the Pontrjagin dual of $M_{x,r} = G_{x,r}/G_{x,r+1}$ onto $g_{x,r}/g_{x,r+1}$. We will say that an element $\beta$ of $g^*$ is nilpotent if there is a 1-parameter subgroup $\lambda: GL_1 \to G$ defined over $F$ such that $\lim_{t \to 0} \lambda(t)\beta = 0$.

We refer the reader to section 2.7 of [10] for the definition of constructible sheaves on the étale site of an adic space, and to section 1 of [11] for the definition of compactly supported cohomology of $l$-adic sheaves on adic spaces. Let $\mathcal{S}_c(X; \mathcal{B})$ denote the category of constructible $l$-adic sheaves on an adic space $X$.

We say that a morphism $\phi$ of $l$-adic sheaves in $\mathcal{S}_c(G_{x,r}^d, \mathcal{B})$ is defined over $F$ if $\phi = \beta_x^* \phi'$, for some morphism $\phi'$ of $l$-adic in $\mathcal{S}_c(G_{x,r}^d, \mathcal{B})$, where $\beta_x^*: G_{x,r}^d \to G_{x,r}^d$ is the base-change morphism induced from the adic morphism $A_{x,r} \to A_{x,r}$ defined by $f \mapsto f \otimes 1$.

The abelian category $\mathcal{S}_c(X; \mathcal{B})$ of constructible $l$-adic sheaves on an adic space $X$ contains enough injectives, and so we may consider the derived category $D^b(X; \mathcal{B})$ of bounded constructible $l$-adic sheaf complexes on $X$. We will say that $\mathcal{A} \in D^b_c(G_{x,r}^d, \mathcal{B})$ is defined over $F$ if $\mathcal{A}$ is quasi-isomorphic over $F$ to $R\beta^* \mathcal{A}'$, for some $\mathcal{A}' \in D^b_c(G_{x,r}^d, \mathcal{B})$.

**Remark 7.4.** It follows from theorem 18.1.2 of [7] that there is an equivalence of categories between the étale site of $G_{x,r}$ and the étale site of $\mathcal{G}_{x,r}$. We identify these in the following definition.

**Definition 7.5.** Let

\begin{equation}
\rho^*_{x,r}: \mathcal{S}_c(M_{x,r}; \mathcal{B}) \to \mathcal{S}_c(G_{x,r}^d, \mathcal{B})
\end{equation}

be the functor induced from the reduction map $\rho_{x,r}: G_{x,r}^d \to M_{x,r}$, as defined in definition 5.9, mut. mut. We will refer to the derived functor

\begin{equation}
R\rho^*_{x,r}: D^b_c(M_{x,r}; \mathcal{B}) \to D^b_c(G_{x,r}^d, \mathcal{B})
\end{equation}

as an inflation functor.

The following definition is inspired by [6, (1.5), (2.3)], [14] and [19].

**Definition 7.6.** A fundamental stratum is a triple $[x, r, \mathcal{A}]$ consisting of a point $x \in I(G, F)$, a real $r$, and an object $\mathcal{A}$ of $D^b_c(G_{x,r}^d, \mathcal{B})$ defined over $F$ and quasi-isomorphic over $F$ to $R\rho^*_{x,r,s,A}$, where

- if $r > 0$ then the coset $J_{x,r}(A) + g_{x,-r}^*$ does not contain any nilpotent elements,
- if $r = 0$ then $\mathcal{A}$ is a cuspidal character sheaf of $M_x$ (we then call $[x, r, \mathcal{A}]$ a depth-zero fundamental stratum).

**Remark 7.7.** If $[x, r, \mathcal{A}]$ is a depth-zero fundamental stratum for $G_{x}^d$, then $\mathcal{A}$ is constructible with respect to the depth-zero stratification of $G_{x}^d$, for some $x \in I(G, F)$.

8. **Appendix on Adic Spaces**

8.1. **Banach rings.** Recall that a Banach ring is a pair $(E, |E|)$ where $E$ is a commutative ring with identity and $E$ is complete with respect to a norm $|E|$. A
closed subalgebra of a Banach rings is again a Banach ring when equipped with the restriction of the norm (see [3], for example).

**Definition 8.1.** Suppose that $a$ is a closed ideal in a Banach ring $(E, | |_E)$. The residue norm $| |_{E/a}$ on the quotient ring $E/a$ is defined by

\begin{equation}
| f + a |_{E/a} := \inf \left\{ | g |_E \mid g \in f + a \right\}.
\end{equation}

Then $(E/a, | |_{E/a})$ is a Banach ring.

For the remainder of this appendix we suppose that $F$ is a field with norm $| |_F$ and integers $\mathbb{N}_F$. We write $\tilde{F}$ for the completion of $F$ with respect to $| |_F$.

The category of Banach $F$-algebras is closed under direct sums and complete tensor products, with the later defined as follows.

**Definition 8.2.** Let $(E_1, | |_{E_1})$ and $(E_2, | |_{E_2})$ be two Banach $F$-algebras; define a valuation $| |_E$ on $E := E_1 \otimes_F E_2$ by

\begin{equation}
| h |_E := \inf \{ \max_i | f_i |_{E_1} | g_i |_{E_2} \},
\end{equation}

where the infimum is taken over all possible representations $h = \sum_i f_i \otimes g_i$. The completion of $E_1 \otimes_F E_2$ with respect to $| |_E$ is denoted $E_1 \hat{\otimes}_F E_2$.

Then $E_1 \hat{\otimes}_F E_2$ is a Banach $F$-algebra. For this and related facts, the reader is referred to section 2.1 of [3].

### 8.2. Tate algebras and Gauss norms.

**Definition 8.3.** For any positive integer $n$ let $F(T_1, \ldots, T_n)$ denote the $F$-algebra

\begin{equation}
F(T_1, \ldots, T_n) := \left\{ \sum_{i \in \mathbb{N}^n} a_i T^i \in F[[T_1, \ldots, T_n]] \mid \lim_{| i | \rightarrow \infty} | a_i |_F = 0 \right\}.
\end{equation}

Here we employ standard multi-index notation: $i = (i_1, \ldots, i_n)$, $| i | = i_1 + \cdots + i_n$ and $T^i = T_1^{i_1} \cdots T_n^{i_n}$. The $F$-algebra $F(T_1, \ldots, T_n)$ is complete with respect to $| f | := \max_{i \in \mathbb{N}^n} | a_i |_F$, where $f = \sum_{i \in \mathbb{N}^n} a_i T^i$.

**Definition 8.4.** Let $n$ be a positive integer and suppose $R = (R_1, \ldots, R_n)$ where $R_i \in \mathbb{R}^{>0}$. Let $T^n_F(R)$ be the Tate algebra defined by

\begin{equation}
T^n_F(R) := \left\{ \sum_{i \in \mathbb{N}^n} a_i T^i \in F[[T_1, \ldots, T_n]] \mid \lim_{| i | \rightarrow \infty} | a_i |_F R^i = 0 \right\}.
\end{equation}

Here we use standard multi-index notation: $R^i := R_1^{i_1} \cdots R_n^{i_n}$ and $T^i := T_1^{i_1} \cdots T_n^{i_n}$. Also, let $| |_R$ be the Gauss norm on $T^n_F(R)$ defined by $| f |_R := \sup_{i \in \mathbb{N}^n} | a_i |_F R^i$.

**Definition 8.5.** More generally, if $a = (a_1, \ldots, a_n)$ with $a_i \in \mathcal{O}_F$, define $T^n_F(a, R)$ by

\begin{equation}
T^n_F(a, R) := \left\{ f \in F[[T_1, \ldots, T_n]] \mid f(T + a) \in T^n_F(R) \right\}.
\end{equation}

Define $| |_{(a,R)}$ on $T^n_F(a, R)$ by $| f |_{(a,R)} := | f(T + a) |_R$.

The $F$-algebra $T^n_F(a, R)$ is complete with respect to $| |_{(a,R)}$.

Note that the condition $f(T + a) \in T^n_F(R)$ is equivalent to saying that if $f = \sum_{i \in \mathbb{N}^n} a_i (T - a)^i$ is the expansion for $f$ at $a$, then $\lim_{| i | \rightarrow \infty} | a_i |_F R^i = 0$. (Here,
\((T - a)^i := (T_1 - a_1)^i \cdots (T_n - a_n)^i.\) Note that we recover the \(\hat{F}\)-algebra \(\hat{F}(T)\) from \(T^i_F(a, R)\) when \(a = (0, \ldots, 0)\) and \(R = (1, \ldots, n)\). Moreover,

\[
\hat{F}(T_1, \ldots, T_n) \otimes F(T_1, \ldots, T_m) \cong \hat{F}(T_1, \ldots, T_{n+m}).
\]

### 8.3. Affinoid rings

This section, taken almost directly from [10], introduces the notion of an affinoid ring.

Let \(A\) be a commutative ring with multiplicative identity. In the following it is to be understood that if \(A\) is a topological ring, then any subring of \(A\) is given the subring topology. Let \(A^\circ\) denote the set of \(a \in A\) such that \(\{a^n\}_{n \in \mathbb{N}}\) is bounded in \(A\).

The ring \(A\) is adic if there is an ideal \(a\) in \(A\) such that \(\{a^n\}_{n \in \mathbb{N}}\) is a neighbourhood basis for \(0 \in A\). Such an ideal is called an ideal of definition. The ring \(A\) is \(f\)-adic if there is an open subring which is adic and has a finitely generated ideal of definition.

An open subring \(B\) of an \(f\)-adic ring \(A\) is called the a ring of definition of \(A\) if \(B\) is adic.

The ring \(A\) is a Tate ring if it is \(f\)-adic and has a topologically nilpotent unit.

**Definition 8.6.** An affinoid ring is a pair \((A^p, A^+)\) where \(A^p\) is an \(f\)-adic ring and \(A^+\) is a subring in \(A^p\) which is open, integrally closed, and contained in \((A^p)^\circ\).

### 8.4. Valuations on affinoid rings

Let \(\Gamma\) be a totally ordered, abelian group written multiplicatively. Extend the order on \(\Gamma\) to \(\Gamma \cup \{0\}\) by \(0 < \gamma \leq \Gamma\) for all \(\gamma \in \Gamma\). Let \(\Gamma \cup \{0\}\) be the monoid defined by \(0 \cdot \gamma = \gamma = 0 = 0\) for all \(\gamma \in \Gamma\). Equip \(\Gamma \cup \{0\}\) with the topology generated by the open sets \(\{\gamma' \in \Gamma \cup \{0\} \mid \gamma' < \gamma\}\).

A valuation of a commutative ring \(A\) (with identity 1) is a function \(x : A \to \Gamma \cup \{0\}\), where \(\Gamma\) is as above and such that:

- 
  \(x(a \cdot b) = x(a) + x(b)\) for all \(a, b \in A\);
  \(x(a + b) \leq \max\{x(a), x(b)\}\) for all \(a, b \in A\); and \(x(0) = 0\) and \(x(1) = 1\).
- For historical reasons, it is common in the literature to write \(|a(x)|\) for \(x(a)\); we will use these interchangeably.

For each valuation of \(A\), \(x^{-1}(0)\) is a prime ideal in \(A\). For any valuation \(x\) of \(A\), let \(A(x)\) denote the residue class field \(A/x^{-1}(0)\) and let \(v_x\) denote the valuation of \(A(x)\) corresponding to \(x\).

Two valuations \(x_1 : A \to \Gamma \cup \{0\}\) and \(x_2 : A \to \Gamma' \cup \{0\}\) are equivalent if there is an isomorphism of ordered monoids \(c : \Gamma_1 \to \Gamma_2\) such that \(c \circ x_1 = x_2\), where \(\Gamma_2\) is the image of \(x_1\). There is a bijection between equivalence classes of valuations of \(A\) and pairs \((p, V)\), where \(p\) is a prime ideal of \(A\) and \(V\) is a valuation ring of the residue class field of \(A/p\).

**Lemma 8.7.** Suppose \(A\) is a topological ring and suppose \(x\) is a continuous valuation of \(A\). Equip \(A(x)\) with the topology induced by the valuation \(v_x\) of \(A(x)\) corresponding to \(x\). Then the mapping \(A \to A(x)\) defined by \(a \mapsto |a(x)|\) is continuous.

Let \(A\) be a topological ring and let \(\text{Spv}(A)\) denote the set of equivalence classes of continuous valuations of \(A\), equipped with the topology generated sets of the form \(\{x \in \text{Spv}(A) \mid x(a) \leq x(b) \neq 0\}\), for \(a, b \in A\).

**Definition 8.8.** For any affinoid ring \((A^p, A^+)\), let \(\text{Spv}(A^p, A^+)\) denote the valuations \(x \in \text{Spv}(A^p)\) such that \(x(a) \leq 1\) for all \(a \in A^+\). Equip \(\text{Spv}(A^p, A^+)\) with the subspace topology from \(\text{Spv}(A^p)\).
8.5. A sheaf on \( \text{Spa}(A) \). It is common to write \( A \) for an affinoid ring \((A^p, A^+)\) and \( \text{Spa}(A) \) for \( \text{Spa}(A^p, A^+) \). We follow that convention for the remainder of this appendix.

**Definition 8.9.** A subspace \( R \) in \( \text{Spa}(A) \) is rational if there are \( f_1, \ldots, f_n, g \in A^p \) such that the ideal \((f_1, \ldots, f_n)\) in \( A^p \) is open and \( R \) is equal to the set of \( x \in \text{Spa}(A) \) such that \(|f_i(x)| \leq |g(x)| \neq 0 \) for all \( i = 1, \ldots, n \).

Rational subsets form a basis for the topology on \( \text{Spa}(A) \).

**Definition 8.10.** Let \( R \) be a rational subset of \( \text{Spa}(A) \) which has a representation as in definition 8.9. Let \( O_A(R) \) be the complete topological ring

\[
A^p(\overline{f_1, \ldots, f_n}) := A^p(\overline{T_1, \ldots, T_n})/(gT_1 - f_1, \ldots, gT_n - f_n).
\]

**Lemma 8.12.** If \( R \) is a rational subset of \( \text{Spa}(A) \) then the ring \( O_A(R) \) is a complete topological ring.

If \( U \) and \( V \) are rational subsets of \( \text{Spa}(A) \) and \( R_1 \subseteq R_2 \), then there is a natural continuous homomorphism \( O_A(R_2) \to O_A(R_1) \), called the restriction homomorphism.

**Definition 8.13.** For any open subset \( U \) in \( \text{Spa}(A) \), define

\[
O_A(U) = \lim_{R \subseteq U} O_A(R),
\]

where the limit is taken over all rational subsets \( R \) contained in \( U \).

Then \( O_A \) is a sheaf of complete topological rings on \( \text{Spa}(A) \), and the ring \( O_A(\text{Spa}(A)) \) of global sections is isomorphic to the completion \( \hat{A}^p \) of \( A^p \); moreover, the subring of global sections \( s \) such that \( v_x(s) \leq 1 \) for all \( x \in \text{Spa}(A) \) is isomorphic to the completion \( \hat{A}^+ \) of \( A^+ \).

Any \( x \in \text{Spa}(A) \) extends from a valuation on \( A^p \) to a valuation \( v_x \) on the stalk \( O_A, x \) in a natural way. Let \( \nu \) denote \( \{v_x \mid x \in \text{Spa}(A)\} \).

**Definition 8.14.** The adic space associated with the affinoid ring \( A \) is the triple \( (\text{Spa}(A), O_A, \nu) \).

**Lemma 8.15.** For any \( x \in \text{Spa}(A) \), the stalk \( O_A, x \) is a local ring with maximal ideal \( \nu_x^{-1}(0) \).

8.6. Adic spaces.

**Definition 8.16.** Let \( \mathcal{V} \) denote the category defined as follows. The objects of \( \mathcal{V} \) are triples \((X, O_X, v)\) where \( X \) is a topological space, \( O_X \) is a sheaf of topological rings on \( X \) and \( v = \{v_x \mid x \in X\} \) where \( v_x \) is an equivalence class of valuations on the stalk \( O_{X, x} \). A morphism in \( \mathcal{V} \) is a pair \((f, \varphi) : (X, O_X, v) \to (Y, O_Y, w)\) where \( f : X \to Y \) is a continuous function, \( \varphi : O_Y \to f_* O_X \) is a morphism of sheaves of topological rings such that \( w_{f(x)} : O_{Y, f(x)} \to \Gamma \cup \{0\} \) is equal to \( v_x \circ \varphi \).

**Definition 8.17.** Define the subsheaf \( O_X^+ \) of \( O_X \) as follows: for \( U \) open in \( X \), let \( O_X^+(U) \) be the ring of all \( s \in O_X(U) \) such that \( v_x(s) \leq 1 \) for all \( x \in U \).
For any adic ring $A$, the triple $(\text{Spa}(A), \mathcal{O}_A, 
u)$ is an object in category $(\mathcal{V})$.

**Definition 8.18.** An **affinoid adic space** is an object in category $(\mathcal{V})$ which is isomorphic to the adic space associated with an affinoid ring. An **adic space** is an object in category $(\mathcal{V})$ which locally is isomorphic to the adic space associated with an affinoid ring.

**Lemma 8.19.** (Cf: [10], p 39) For any adic space $X$ and any affinoid ring $A$, there is a bijection between $\text{Hom}(((\mathcal{O}_X(X), \mathcal{O}_X^2(X)), A)$ (continuous adic ring homomorphisms) and $\text{Hom}_{\mathcal{V}}(\text{Spa}(A), X)$ (morphisms of adic spaces).

One family of affinoid adic spaces deserve special attention.

**Definition 8.20.** For each $a = (a_1, \ldots, a_n)$ with $a_i \in F$ and $R = (R_1, \ldots, R_n)$ with $R_i \in \mathbb{R}^{>0}$, let

$$B^a_F(R) := \left\{ X \in (\mathbb{A}^n_F)^{ad} \mid |(T_i - a_i)(X)| \leq R_i, \forall i = 1, \ldots, n \right\},$$

where $\mathbb{A}^n_F := \text{Spec}(F[T_1, T_2, \ldots, T_n])$. We will often write $B^a_F(R)$ for $B^a_F(0, R)$. The reader should compare this with definition 8.4.

8.7. Adic groups and Lie algebras.

**Definition 8.21.** An adic space $(G, \mathcal{O}_G, \nu)$ over $\text{Spa}(F)$ is an **adic group over $\text{Spa}(F)$** if there are morphisms $m_G : G \times_{\text{Spa}(F)} G \to G$ (multiplication), $i_G : G \to G$ (inversion) and $e_G : \text{Spa}(F) \to G$ (identity) in the category $(\mathcal{V})$ satisfying the usual properties.

**Remark 8.22.** An adic group is a group object in $(\mathcal{V})$; however, the topological space underlying an adic group need not be a group.

An adic space $H$ in $G$ is an adic subgroup if the restriction of $m_G$ (resp. $i_G$) to $H \times_{\text{Spa}(F)} H$ (resp. $H$) defines a morphism into $H$, and if the image of $e_G$ is contained in $H$.

**Lemma 8.23.** If $G_F$ is a group scheme over $F$, then $G_F^{ad}$ is an adic group.

If $G$ is an affinoid adic group over $\text{Spa}(F)$, then the adic ring $A$ for $G$ is a Hopf algebra in the following sense: there are adic ring homomorphisms $\mu_G : A \to A \otimes_F A$ (co-multiplication), $\iota_A : A \to A$ (co-inversion) and $\epsilon_A : A \to \text{Spa}(F)$ (co-identity) satisfying the usual properties and which induce $m_G$, $i_G$ and $e_G$ via the functor $\text{Spa}()$.

**Definition 8.24.** An adic space $(G, \mathcal{O}_G, \nu)$ over $\text{Spa}(F)$ is an **adic Lie algebra over $\text{Spa}(F)$** if there are morphisms $a_G : G \times_{\text{Spa}(F)} G \to G$ (addition), $b_G : G \times_{\text{Spa}(F)} G \to G$ (Lie bracket), $i_G : G \to G$ (inversion) and $e_G : \text{Spa}(F) \to G$ (identity) in the category $(\mathcal{V})$ satisfying the usual properties.

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