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on the boundary of a regular tree

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Self-repelling processes on the boundary of a regular tree

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Abstract
We construct the path of a self-repelling version of “stable” Lévy processes on $\mathbb{R}^\infty$. This construction uses the tree representation of the paths of these processes, which was introduced in [19] for the study of multiple points. We compute the Hausdorff dimension of the range and the critical parameter for the self-avoiding property.

1 Introduction

The problem of constructing self-avoiding random objects, which partly originates in some problems of mathematical physics (see for instance the references in [7]), is a challenging problems for probabilists. Among the various models related to this topic are the self-avoiding random walk [13], which has raised further questions related to Brownian motion [15, 16, 17] and the loop-erased random walk, which is connected to the problem of spanning trees of a graph [1, 12, 21, 26].

A natural way to proceed is to try to construct such a self-avoiding random object as the path of a self-repelling process. So far, the literature on self-repelling processes seems to have focused on the one-dimensional case, either by methods of stochastic differential equations [6, 8, 22], or with more original techniques [29]. Let us mention that the case of self-attractive processes has also been considered, in the discrete and continuous settings [5, 23, 27, 28], and here again, mainly in the one-dimensional case.

The goal of this paper is to construct and study the path of a self-repelling process, the state space being the boundary of a regular tree. Processes on such a state space, which can be viewed as $(\mathbb{R}/n\mathbb{Z})^\infty$, are quite natural examples of processes on a self-similar group. In particular, “stable”, i.e. self-similar processes of this kind have already been studied in the literature (see for instance [2, 10]).

In [19], we showed that the paths of these processes have a hidden tree structure that contains all the information on the processes up to time change. We used this to study multiple points by Peres’ method of comparison with a percolation [24] or to establish the existence of points of infinite multiplicity.

We shall see that if we define a self-repelling mechanism that only depends on the set of already visited points (and not, for instance, on the precise occupation

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measure), we can explicitly construct the tree structure of the path of the self-repelling process. However, we will not obtain the time parametrization, in contrast with [20] where we were able to recover it.

Let us give a brief outline of the construction. The state space $(\mathbb{Z}/n_0\mathbb{Z})^N$ is identified as the boundary (that is, the set of lines of descent) of a tree $\Gamma$ where every vertex has $n$ children. If we think of each edge of $\Gamma$ at level $n$ as a segment with length $\alpha^n$, we can construct a Brownian motion $B$ on $\Gamma$. Moreover, if $\alpha < 1$, the boundary of $\Gamma$ can be seen as a set of “ends” at finite distance and it is possible to construct a version of $B$ reflected at the boundary. In that case, the set of points of the boundary visited by $B$ is the same as the path of a stable process on $(\mathbb{Z}/n_0\mathbb{Z})^N$. One can even reconstruct the latter by defining the local time of $B$ at the boundary. The index of the process depends on $\alpha$ and $n_0$. See [20].

The self-repelling mechanism is the following: once a point $x$ of the boundary has been visited, the Brownian motion on any edge lying above $x$ is submitted to a repelling force towards the root, thus decreasing the probability of visiting a point of the boundary near $x$. If such a Brownian motion exists, its trace on the boundary of $\Gamma$ induces a tree structure $T$. We do not actually prove the existence of this Brownian motion but our first result, Theorem 1 in Section 3.1, states the existence and uniqueness of the law of the associated tree structure $T$. We give the construction of $T$, using transfinite induction.

We shall see that a suitable choice of the repelling force is equivalent to changing the parameter $\alpha$ related to the length of the edges above $x$. Therefore, loosely speaking, the process has an index $\nu_1$ on the set of non-visited points and another index $\nu_2 < \nu_1$ on the set of already visited points.

Here again, $T$ contains all the information on the path on the boundary of $\Gamma$, and we can study the problems of Hausdorff dimension of the path and of self-intersections. Recall that the “natural” Hausdorff dimension of $(\mathbb{Z}/n_0\mathbb{Z})^N$ is $d = \log n_0$ and that if a stable Lévy process on $(\mathbb{Z}/n_0\mathbb{Z})^N$ has index $\nu < d$, the Hausdorff dimension of its range is $d_H = \nu$. The process is self-avoiding if and only if $\nu < d/2$ [19]. In the self-repelling case, we determine the Hausdorff dimension of the range, which is a real $\in [\nu_2, \nu_1]$, and we compute, for fixed $\nu_1$, the critical value for $\nu_2$ separating the self-intersecting and self-avoiding behaviours. This is our second result, stated in Theorem 2 (Section 4.1).

The sequel is organized as follows. In the next section, we explain the role of the tree structure for the paths of processes on $(\mathbb{Z}/n_0\mathbb{Z})^N$. The construction of the self-repelling path is described in Section 3. The properties of the paths are analyzed in Section 4.

2 Processes on $(\mathbb{Z}/n_0\mathbb{Z})^N$ and their tree structure

2.1 The state space

The trees we shall consider in the sequel are ordered trees, that is, for every vertex, there exists a total order on the set of its children. This is equivalent to defining a planar embedding of the trees. We define a ray of a rooted tree as an infinite line of descent along the edges of the tree, starting at the root. The set of rays is called the boundary. A ray can be viewed as a “leaf at infinity”.

Remark that we exclude in our definition of the boundary the case of “true”
leaves at finite distance. We say that a vertex is essential if it belongs to a ray.

The group \((\mathbb{Z}/n_0\mathbb{Z})^N\) can be identified with the boundary \(\partial \Gamma\) of an \(n_0\)-regular tree \(\Gamma\), i.e. a tree with a root \(R\) and such that every vertex has \(n_0\) children. Indeed, every vertex at level \(n\) can be represented by a finite sequence of integers \((a_0, a_1, \ldots, a_n)\) where \(a_i \in \{0, 1, \ldots, n_0 - 1\}\) for each \(i\), and every ray can be represented likewise by an infinite sequence \((a_0, a_1, \ldots)\). For convenience, we add a vertex \(\overline{R}\) to \(\Gamma\), \(\overline{R}\) being the father of \(R\). More generally, we shall denote by \(\overline{u}\) the father of a vertex \(u\).

The classical Gromov distance \(\delta\) on \((\mathbb{Z}/n_0\mathbb{Z})^N\) with parameter \(\beta < 1\) (in general, \(\beta = e^{-1}\)) is given by \(\delta(x, y) = \beta^{l(z)}\), where \(x \wedge y\) is the last common ancestor of \(x\) and \(y\), viewed as leaves at infinity, and \([z]\) is the level of the vertex \(z\) in the tree. This is equivalent to assigning to each edge at level \(n\) a length \(c\beta^n\) (\(c\) being an irrelevant constant) and saying that the distance between \(x\) and \(y\) is the length of the path from \(x\) to \(y\) along the edges. With this latter definition, the metric can be extended to \(\Gamma \cup \partial \Gamma\). One checks that if a sequence \(x_n\) converges to \(x\), either \(x \in \Gamma\) and \(x_n\) is constant for \(n\) sufficiently large, or \(x \in \partial \Gamma\). Note that the Hausdorff dimension of \(\partial \Gamma\) is \(d = -\log n_0/\log \beta\).

### 2.2 Processes on the tree and paths on the boundary

We say that a continuous-time process \(X\) on \(\Gamma \cup \partial \Gamma\) is a process reflected at the boundary if \(X\) is locally a random walk on \(\Gamma\) along the edges and \(X\) can reach the boundary in finite time and go back. More precisely, \(X\) is a process reflected at the boundary if it is almost surely a function \(\mathbb{R}_+ \to \Gamma \cup \partial \Gamma\) satisfying the following conditions:

- (0) The process starts at \(R\) and stops at the hitting time \(\zeta\) of \(\overline{R}\), which is finite.

- (i) For every \(a \in \Gamma\), \(X^{-1}(a)\) is a (possibly empty) union of disjoint intervals of the form \([t, t')\).

- (ii) If \(X([t, t']) = a \in \Gamma\), then there exists a positive \(s\) such that \(X([t', t' + s]) = a' \in \Gamma\) where \(a'\) is adjacent to \(a\) in \(\Gamma\).

- (iii) If \((t_n)\) is a sequence of reals converging to \(t\), and if \(X(t_n)\) is not constant for \(n\) sufficiently large, then \(X(t_n)\) converges to \(X_t\).

The condition (0) is only a technical one. The absorption at \(\overline{R}\) could be dropped without changing the local properties of the process. For (iii), remark that if \(X(t_n)\) is constant for \(n\) sufficiently large, one can be in the case described in (ii). Remark that \(X\) is cadlag.

If we view the edges of \(\Gamma\) as segments with length 1, we can define a Brownian motion on \(\Gamma\), provided that we choose the law when this Brownian motion arrives at a node (i.e. a vertex). If each edge at level \(n\) has length \(\alpha^n\) with \(\alpha < 1\), one can define a version of the Brownian motion reflected at the boundary. Then one can record the successive passages of this Brownian motion on the vertices of \(\Gamma\) and on \(\partial \Gamma\), and this induces a process reflected at the boundary [20]. More general constructions of reflected processes can be found in [3].

We can think of the successive visits of \(\partial \Gamma\) by \(X\) as the trace of a process on \(\partial \Gamma\). More precisely, the existence of a local time at the boundary would
enable us define a process on $\partial \Gamma$ by a time change. In fact, stable processes on $(\mathbb{Z}/n\mathbb{Z})^N$ can be constructed this way [20].

If one does not take into account the possible time changes and only considers the points that are visited and the order in which they are visited, one can set the following definition. A path on the boundary is a couple $(A, g)$ where $A$ is a subset of $\mathbb{R}_+$ and $g$ is a function $A \to \partial \Gamma$, up to the equivalence relation $(A, g) \sim (A', g')$ if there exists an increasing bijection $h : A \to A'$ such that $g \circ h = g$. If $X$ is a process reflected at the boundary, the associated path on the boundary is given by $(X^{-1}(\partial \Gamma), X)$.

2.3 The tree structure of the process

Let $X$ be a process reflected at the boundary. For each vertex $a \in \Gamma$, let $\tau(a)$ be the set of times $t$ at which $X_t$ is either $a$, or a descendant of $a$, or a ray containing $a$. It is easily seen from assumptions (i) (ii) and (iii) that $\tau(a)$ is a (possibly empty) union of intervals

$$\tau(a) = [t_0(a), t'_0(a)) \cup [t_1(a), t'_1(a)) \cup \ldots$$

with

$$t_0(a) < t'_0(a) < t_1(a) < t'_1(a) \ldots$$

For every $a, n$, one has $X_{t_n(a)} = a$, $X_{t'_n(a)-} = a$, $X_{t'_n(a)} = \overline{a}$.

We construct a tree $T$ whose vertices are the intervals of the form

$$[t_n(a), t'_n(a)), \quad n \in \mathbb{N}, \ a \in \Gamma$$

This includes $R' = [0, \zeta)$, which we choose as the root of $T$. We say that an interval $[t_n(a), t'_n(a))$ is an ancestor of $[t_m(b), t'_m(b))$ if and only if $[t_m(b), t'_m(b)) \subset [t_n(a), t'_n(a))$.

We define a function $f : T \to \Gamma$ by $f([t_n(a), t'_n(a))) = a$. Remark that if $[t_n(a), t'_n(a))$ is an ancestor of $[t_m(b), t'_m(b))$, then $a$ is an ancestor of $b$ in $\Gamma$.

Therefore one can extend $f$ to the set of rays: if $r$ is a ray of $T$, viewed as a sequence $(r_n)$ of vertices of $T$, $f(r)$ is the ray of $\Gamma$ defined by the sequence $(f(r_n))$. We denote $(T, f) = F(X)$.

Proposition 1 (a) If $X'$ is a time-changed version of $X$, i.e. if there exists an increasing bijection $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $X'_{g(t)} = X_t$ for every $t$, then $F(X) = F(X')$. As a consequence, the path on the boundary associated with $X$ is entirely determined by $F(X)$.

(b) The set of rays visited by $X$ is the set $A = \{f(r), \ r \in \partial T\}$.

We shall say that a ray $r' \in \partial \Gamma$ is visited by $(T, f)$ if $r \in A$.

Proof. Assertion (a) follows readily from the definition of $(T, f)$. Let us prove (b). Let $r'$ be a ray of $T$ viewed as a sequence of vertices

$$[t_{n_1}(a_1), t'_{n_1}(a_1)), [t_{n_2}(a_2), t'_{n_2}(a_2)) \ldots$$

so that $f(r') = (a_1, a_2 \ldots)$. Then the sequence $(t_{n_k}(a_k))$ is increasing and bounded above by $t'_{n_1}(a_1)$, and thus converges to some real $t$. Assumption (iii) entails that $X_t = f(r')$.
Conversely, suppose that $X_t = r = (a_1, a_2, \ldots) \in \partial \Gamma$. The continuity assumptions (ii) and (iii) allow us to define $s_n = \sup \{ s \leq t, (X_s)_n \neq (X_t)_n \}$, where $(X_s)_n$ is the n-th coordinate of $X_s$ viewed as an element of $(\mathbb{Z}/n_0 \mathbb{Z})^N$. It is clear that for each $n$, $X_{s_n} = a_n$, and therefore $s_n$ has the form $t_{i_n}(a_n)$. It follows that the ray

$$r = [t_{i_1}(a_1), t'_{i_1}(a_1)], [t_{i_2}(a_2), t'_{i_2}(a_2)] \ldots$$

satisfies $f(r') = r$. □

2.4 Ordering of the tree $T$

One can equip $T$ with the ordering induced by the natural order on $\mathbb{R}_+$. Indeed, if $u$ and $v$ are brothers, then they are disjoint intervals of $\mathbb{R}_+$ and we agree that $u \leq v$ in $T$ if and only if $u \leq v$ in $\mathbb{R}_+$.

Recall that the prefix order on the set of vertices of an ordered tree is defined as follows.

- A father is smaller than any of its children.
- If $u$ and $v$ are brothers and if $u$ lies on the left of $v$, then $u$ is smaller than $v$ and so is every descendent of $u$.

Suppose that $X_t = a \in \Gamma$ with $t \in u = [t_n(a), t'_n(a)) \in \Gamma$, and assume that $u$ has $i$ children

$$[t_{m_1}(a_1), t'_{m_1}(a_1)), \ldots [t_{m_i}(a_i), t'_{m_i}(a_i))$$

with

$$t_n(a) < t_{m_1}(a_1) < \ldots < t_{m_i}(a_i) < t'_n(a)$$

One easily deduces from the definition of the ordering on $T$ the following

**Proposition 2** The set of vertices of $\Gamma$ visited by $X$ before time $t$ has the form $f(T(t))$ where $T(t)$ is a subtree of $T$ given by

- If $t < t_{m_1}(a_1)$, $T(t)$ is the set of vertices smaller than $u$ in the prefix order on $T$.
- If $t_{m_j}(a_j) < t < t_{m_{j+1}}(a_{j+1})$, then $T(t)$ is the set of vertices of $T$ lying on the left of the right-most line of descent containing the $j$-th child of $u$.
- If $t_{m_i}(a_i) < t < t'_{m_i}(a_i)$, then $T(t)$ is the set of vertices of $T$ lying on the left of the right-most line of descent containing the $i$-th child of $u$.

The proof is left to the reader.

2.5 The stable case

Fix a parameter $\beta \in (0, 1)$ for the metric on the tree. In the case of a stable process with index $\nu$, killed at an exponential rate with a suitable parameter, the law of $(T, f)$ can be described as follows [20]:

- $T$ is a Galton-Watson tree, rooted at $R'$, with geometric offspring with mean $m = \beta^{-\nu}$. 

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\begin{itemize}
  \item $f(R) = R$
  \item If $v$ is a child of $v$ in $T$, $f(v)$ is one of the $n_0$ children of $f(u)$ in $\Gamma$, chosen at random, uniformly and independently.
\end{itemize}

If we condition $T$ to have a non-empty boundary and remove from $T$ all the non essential vertices, we obtain a tree $T'$ whose law is characterized by

\begin{itemize}
  \item The tree $T'$ is a Galton-Watson tree, rooted at say $R'$, with offspring distribution $1 + \text{geometric}(m - 1)$.
  \item $f(R') = R$.
  \item If $v$ is a child of $v$ in $T$, $f(v)$ is one of the $n_0$ children of $f(u)$ in $\Gamma$, chosen at random, uniformly and independently.
\end{itemize}

See [19] for a proof and various properties of these processes. Alternatively, one could define the law of $(T, f)$ as follows. The tree $T$ is rooted at $R'$ and $f(R') = R$. If $u \in T$ and $f(u) = a \in \Gamma$, put a conductance $1$ between $a$ and its father and a conductance $C = m/n_0$ between $a$ and each of its children. Choose a vertex adjacent to $a$ with probability proportional to the conductance. If this vertex is the father of $a$, $u$ has no child. If this vertex is a child $b$ of $a$, then $u$ has a first child $v_1$ and $f(v_1) = b$. Then proceed similarly to determine whether $u$ has a second child or not etc. By induction on the level of the vertices, starting at the root, one can construct $(T, f)$.

3 Construction of a self-repelling path

3.1 Choice of the parameters

We would like to construct a process $X$ reflected on the boundary where, at time $t$, the set of points of the boundary visited before $t$ induces a force towards the root along the corresponding edges. The model we shall adopt is the following. The process $X$ will be locally a random walk along the edges of $\Gamma$ induced by an electric network [9], and the resistances will be greater along the edges belonging to some ray that has already been visited. More precisely, $X$ will have to fulfill the following condition.

(R) If $X_t = a \in \Gamma$, then the next vertex visited after time $t$ is chosen with probability proportional to the conductance, given that we put

\begin{itemize}
  \item a conductance $1$ between $a$ and $\overline{a}$
  \item a conductance $C \in (1/n_0, 1)$ between $a$ and each child of $a$ not belonging to a ray visited before time $t$,
  \item a conductance $C' < C$ between $a$ and each child of $a$ belonging to a ray visited before time $t$.
\end{itemize}

Recall that the process is absorbed at $\overline{R}$. The condition $C < 1$ entails that single points are polar [19] and $C > 1/n_0$ is necessary to have a positive probability of reaching the boundary. Furthermore, $C' < C$ means that the
process is self-repelling. Comparing with the case of a stable process, we shall say that the associated path on the boundary has two indices,

\[ \nu = -\log(n_0 C)/\log \beta \]
\[ \nu' = -\log(n_0 C')/\log \beta \]

Remark that we do not exclude the case \( C' \leq 1/n_0 \), in which case \( \nu' < 0 \).

If such a process \( X \) exists, it yields a random tree \((T, f) = F(X)\) and a path on the boundary. In fact, we shall not construct \( X \) itself but the tree \((T, f)\).

**Theorem 1** There exists a unique law \( \mu(\nu, \nu') \) on the set of trees \((T, f)\) such that if a process \( X \) reflected on the boundary satisfies \((R)\) with parameters \( \nu, \nu' \), then \( F(X) \) has the law \( \mu(\nu, \nu') \).

The remainder of this section is devoted to the proof of Theorem 1. To make our arguments rigorous, we shall use the notion of transfinite induction to construct \((T, f)\). The reader may skip the discussion, which is not necessary to establish the properties of the path stated in Section 4.

### 3.2 Description of the construction

Our construction makes use of transfinite induction (see for instance Bourbaki [4] for a reference). Let us summarize the properties we shall use in the sequel. Transfinite induction is an induction indexed by the ordinals. The ordinals are

\[ 0, 1, 2, \ldots N, N + 1, \ldots 2N, \ldots \]

A fundamental property is that any set of ordinals contains a smallest element. Also, one can find arbitrary large ordinals. Apart from 0, there are two kinds of ordinals: successor ordinals, i.e. ordinals that are the successor of another one (e.g. 1, 2, \( N + 1 \)), and limit ordinals, which are the limit of an increasing family of ordinals (e.g. \( N, 2N \)). To construct a sequence of mathematical objects \( M_k \) by transfinite induction, beginning at \( M_0 \), one has to

- define the procedure to obtain the next object of the sequence, i.e. define \( M_{k+1} \) given \( M_k \)
- define the increasing limit of a sequence of objects, i.e. if \( k \) is a limit ordinal, define \( M_k \), given \( (M_{k'}, k' < k) \).

We first define two construction procedures related to the trees \( T \) and \( \Gamma \).

**Painting Procedure.** Consider a tree \((T, f)\) where, some vertices of \( T \) are painted in red. For every ray \((a_0 = R, a_1, \ldots) \in \partial \Gamma \) satisfying \( \forall n, a_n \in f(T) \), paint all the vertices \( a_n \) in black. Paint the other vertices of \( \Gamma \) in white. Let \( u \in T \) be a non-red vertex. If there exists a non-red vertex \( v \in T \) such that \( f(v) \) is a child of \( f(u) \) and \( f(v) \) is white, paint \( u \) in orange. Otherwise, paint \( u \) in green.

**Extension Procedure.** Suppose we have chosen a vertex \( u \) in \( T \). Put a conductance 1 between \( f(u) \) and its father, a conductance \( C' \) between \( f(u) \) and
each black child of \( f(u) \), and a conductance \( C \) between \( f(u) \) and each white child of \( f(u) \). Choose a vertex \( a \) adjacent to \( f(u) \) with probability proportional to the conductance. If \( a \) is the father of \( f(u) \), then \((T,f)\) remains the same. Otherwise, add a new child \( v \) on the right of the previous children of \( u \) and set \( f(v) = a \).

The transfinite induction is initialized by the tree \( T_0 \), which consists of a single vertex \( R' \) and which we paint in green. We set \( f_0(R') = R \). If, at some ordinal \( k \), all the vertices are painted in red, the construction stops.

If \( k \) is an ordinal with a predecessor \( k - 1 \), one constructs \((T_k, f_k)\) from \((T_{k-1}, f_{k-1})\) in the following way. Let \( n \) be the minimal level at which there is a green vertex in \( T_{k-1} \) and \( v \) be the left-most green vertex at level \( n \) (we say that \( v \) is overgreen). Apply the extension procedure with chosen vertex \( v \) to construct \( T_k \). Then apply the painting procedure, the red vertices of \( T_k \) being the same as in \( T_{k-1} \), plus \( v \) if the extension procedure yields no child.

If \( k \) is a limit ordinal, then \((T_k, f_k)\) is the limit of \((T_{k'}, f_{k'})\), \( k' < k \). That is, the set of vertices of \( T_k \) is the union of all the vertices of the trees \( \{T_{k'}, \; k' < k\} \), and the family relations between these vertices as well as the function \( f_k \) are given by compatibility. Say that a vertex \( u \) is red in \( T_k \) if and only if there exists an ordinal \( k' < k \) such that \( u \) is red in \( T_{k'} \). Then apply the painting procedure.

### 3.3 Proof of the construction

We first prove that the construction is almost surely well-defined and eventually stops. If \( k' < k \) are two ordinals and if \( u \in T_{k'} \), the family relations of \( u \) are preserved in \( T_k \) and \( f_k(u) = f_{k'}(u) \). Thus the compatibility relations enable to define \((T_k, f_k)\) from \((T_{k'}, f_{k'})\), \( k' < k \) if \( k \) is a limit ordinal.

If \( k \) is the successor of \( k - 1 \), and if there is a non-red vertex in \( T_{k-1} \), we have to check that there is at least one green vertex. Suppose that this is not the case. Let \( u_1 \) be an orange vertex. Then there exists a non-red vertex \( u_2 \) such that \( f(u_2) \) is a child of \( f(u_1) \) and \( f(u_2) \) is white. But since \( u_2 \) is not red, it is orange, and there exists \( u_3 \) such that \( f(u_3) \) is a child of \( f(u_2) \) and \( f(u_3) \) is white. Thus we can construct a sequence \((u_n)\) such that for each \( n \), \( f(u_{n+1}) \) is a child of \( f(u_n) \) and \( f(u_{n+1}) \) is white. The sequence of vertices \( f(u_n) \) yields a ray of \( r \in \partial \Gamma \) and, therefore, all the vertices of \( r \) should be black, which yields a contradiction.

Let us prove that the construction eventually stops. Let \( u \) be a given vertex of \( T \). If \( u \) is overgreen at some ordinal \( h \), then applying the extension procedure, the probability that \( u \) becomes red is at least \( 1/(1+ n_0(C)) \). Therefore, the number of ordinals \( h \) at which \( u \) is overgreen is almost surely finite. As the possible children of \( u \) can only be born at such ordinals, it follows that \( u \) has finitely many children, almost surely.

Therefore, we can show by induction on \( n \) that with probability 1, for every \( n \), every vertex of \( T \) at level \( n \) has finitely many children. This entails that \( T \) is almost surely countable, and thus the construction almost surely stops before the first non-countable ordinal. □

We now state three facts that follow easily from the construction of \((T,f)\).

**Lemma 1** For each vertex \( u \in T \), \( u \) is overgreen \( i \) times in the construction of
(T, f) if and only if u has i − 1 children. The j-th time (j < i) at which u is overgreen is the time at which the j-th child of u appears, and the i-th time is the time at which u becomes red.

**Lemma 2** Let u, v be two vertices of T at the same level and such that \( f(u) = f(v) \). Denote by k, k′ respectively the ordinals at which u and v appear in the construction of (T, f). Then u lies on the left of v if and only if k ≤ k′.

**Lemma 3** Let α be a vertex of \( \Gamma \) at level l, u a vertex of T at level l such that \( f(u) = α \). \( r = (r₀, r₁, \ldots) \) a ray of \( \partial T \) such that \( f(r₁) = α \), and k an ordinal at which u is overgreen. Then if \( r \in \partial T_k \), u lies on the right of \( r₁ \), with possibly \( u = r₁ \).

To prove Lemmas 2 and 3, remark that if v, w are two vertices of T at the same level such that v is on the left of w and \( f(v) = f(w) \), and if v and w are not red, then they have the same color. Thus v becomes red before w is overgreen for the first time. In particular, v becomes red before the appearance of the first child of w. □

We now want to show that if (T, f) is constructed as in Section 3.2 and if \( X \) is a process such that \( (T, f) = F(X) \), then \( X \) satisfies the rule \((R)\). Choose some \( t ≥ 0 \) and suppose that \( X_t = a \), with \( a \) lying at level l in \( \Gamma \). We have to show that, conditionally on the path up to time \( t \), the next vertex \( b \) visited after time \( t \) is chosen according to the distribution prescribed by \((R)\).

Let us use the same notations as in Section 2.4. There exists \( n \in \mathbb{N} \) such that \( t \in u = [t_n(a), t'_n(a)] \) and \( i ≥ 0 \) such that the first \( i \) children of \( u \) are

\[
[t_{m₁}(a₁), t'_{m₁}(a₁)), \ldots [t_{m_i}(a_i), t'_{m_i}(a_i))
\]

with

\[
t_n(a) < t_{m₁}(a₁) < \ldots < t_{m_i}(a_i) < t
\]

Then \( b = π \) if \( u \) has exactly \( i \) children. If \( u \) has an \((i + 1)\)-th child

\[
[t_{m_i+₁}(a_{i+₁}), t'_{m_i+₁}(a_{i+₁})]
\]

then \( b = a_{i+₁} \). Finally, the information on the past of the process at time \( t \) is given by the subtree \( \mathcal{T}(t) \) defined in Proposition 2.

Using Lemma 1, we see that \( b \) is chosen by the application of the extension procedure at the \((i + 1)\)-th time at which \( u \) is overgreen. Let \( k \) be the ordinal at which \( u \) is overgreen for the \((i + 1)\)-th time. Remark that Lemma 1 entails that \( u \) has \( i \) children in \( T_k \). We have to prove that at \( k \), the choice of conductances for the application of the extension procedure corresponds to the rule \((R)\). This amounts to proving that at ordinal \( k \), for every child \( a' \) of \( a \), \( a' \) is black if and only if some ray \( r \in \partial \Gamma \) containing \( a' \) has been visited before time \( t \).

Suppose that \( a' \) is black at ordinal \( k \). Then there exists a ray \( r = (r₀, r₁, \ldots) \in \partial \Gamma \) satisfying \( r_n ∈ f(T_k) \) for every \( n \) and \( f(r_{i+₁}) = a' \). We claim that there is a ray \( r' = (r'₀, r'₁, \ldots) ∈ \partial T \), such that \( f(r') = r \). Indeed, one can construct \( (r_n) \) by induction, beginning with \( r'_0 = R' \). As \( r'_0 \) has finitely many children almost surely, at least one of these children has infinitely many descendants whose image by \( f \) belongs to \( \{r₀, r₁, \ldots\} \). Such a child \( v \) satisfies of course \( f(v) = r_{n+₁} \) and we set \( r'_{n+₁} = v \).
According to Lemma 3, \( r'_1 \) lies on the left of \( u \) in the tree \( T_k \). Therefore, as \( u \) has \( i \) children at \( k \), \( r'_{i+1} \) lies on the left of the \( i \)-th child of \( u \), which entails that \( r' \in \partial T(t) \). Thus, according to Proposition 2, \( r \) has been visited before time \( t \).

Conversely, suppose that a ray \( r \in \partial T \) containing \( v' \) has been visited before time \( t \). Then by Proposition 2, there exists a ray \( r' = (r'_{i+1}, \ldots) \in \partial T(t) \) satisfying \( f(r'_{i+1}) = a' \). Let \( V \) be the (non-empty) set of vertices \( v \in T(t) \) at level \( l+1 \) such that \( f(v) = a' \). Lemmas 1 and 2 entail that for every \( v \in V, v \in T_k \). If \( a' \) were white at \( k \), then \( r' \) would not belong to \( \partial T_k \). Moreover, the fact that \( u \) is green at \( k \) would entail that all the vertices in \( V \) are red at \( k \). But this would mean that \( r' \) cannot appear after ordinal \( k \), and this would henceforth contradict the existence of \( r' \). Therefore, \( a' \) is black at ordinal \( k \). □

Finally, we have to prove the uniqueness of the measure \( \mu(\nu, \nu') \). Consider a fixed tree \( (T, f) \) with a root \( R' \). Then one can define a sequence of painted subtrees \( (T_k, f_k) \) indexed by the ordinals as follows.

For each \( k \), \( f_k \) is the restriction of \( f \) to \( T_k \). We construct the sequence \( T_k \) by transfinite induction, starting with \( T_0 = \{ R' \} \), \( R' \) being painted in green. If \( k \) is the successor of \( k - 1 \), one can determine an overgreen vertex \( u \) in \( T_{k-1} \) as in Section 3.2. Then if \( u \) has a child in \( T - T_{k-1} \), we obtain \( T_k \) by adding to \( T_{k-1} \) the left-most child of \( u \) in \( T - T_{k-1} \), and we use the painting procedure on \( T_k \). Otherwise, \( T_k = T_{k-1} \) and we paint \( u \) in red. If \( k \) is a limit ordinal, we define \( T_k \) as the increasing limit of \( (T_{k'}, k' < k) \), as in Section 3.2.

Now if \( (T, f) \) is a random tree derived from a process \( X \) satisfying the rule \( (R) \), one easily checks, using the same arguments as above, that the sequence \( (T_k, f_k) \) has the same law as the sequence constructed in Section 3.2. Moreover, the assumption that the hitting time of \( R \) is finite entails that the sequence becomes stationary when all the vertices of \( T \) are red. Hence the measure \( \mu(\nu, \nu') \) is unique. □

4 Some properties of the path

4.1 Statement of the results

The aim of this section is to establish some properties of the paths defined previously. Let us first set a definition.

Say that the path of the boundary is self-intersecting if there exist two rays \( r \neq r' \) such that \( f(r) = f(r') \). If \( X \) is a process reflected on the boundary such that \( F(X) = (T, f) \), this is equivalent to saying that there exist \( t < t' \in \mathbb{R} \) such that \( X_t = X_{t'} \in \partial T \). If the path is not self-intersecting, we say that it is self-avoiding.

Theorem 2 (i) Let \( A \) be the subset of \( \partial T \) hit by a random tree \( (T, f) \) with distribution \( \mu(\nu, \nu') \). Then conditionally on the event \( A \neq \emptyset \), \( A \) has Hausdorff dimension

\[
d_H = \frac{\log \left( \frac{n_0 \beta^{-\nu} - \beta^{-\nu'}}{n_0 + \beta^{-\nu} - \beta^{-\nu'}} \right)}{\log \beta}
\]

almost surely.

(ii) There exists a positive constant \( c_1 \) such that if \( A_1 \) is the subset of \( \partial T \) surviving a Bernoulli percolation on \( T \) with parameter \( p = \beta^{-d_H} / n_0 \), then for
any compact subset \( B \subset \partial \Gamma \),
\[
P(A_1 \cap B \neq \emptyset) \leq c_1 P(A \cap B \neq \emptyset)
\]

(iii) Fix \( \nu > 0 \) and let \( \nu'_{cr} \) be the only positive solution of
\[
\frac{n_0 \beta^{-\nu} - \beta^{-\nu'}}{n_0 + \beta^{-\nu} - \beta^{-\nu'} - 1} = n_0 \beta^{-\nu'}
\]

Then if \( \nu' > \nu'_{cr} \), the path on the boundary is self-intersecting almost surely and if \( \nu' < \nu'_{cr} \), the path on the boundary is self-avoiding almost surely.

**Remark 1.** Finer estimates on the hitting probabilities for the self-repelling process would be needed to obtain, using Lyons’ theorem for dependent percolation, a double inequality in (ii). Then further results on multiple points could be obtained by the method of comparison with a percolation (this is a general demonstration pattern introduced by Peres [24], see [25] for references of various applications of this method).

**Remark 2.** It is natural to conjecture that the process is self-avoiding if \( \nu' = \nu'_{cr} \).

**Remark 3.** The existence of the absorption at \( \overline{R} \) amounts to killing the process at a certain rate that depends on the conductance between \( R \) and \( \overline{R} \). It should become clear from the proof that when this killing rate varies, (i) and (iii) remain true (but the constant \( c_1 \) in (ii) depends on the killing rate).

**Remark 4.** In contrast with the case of Markovian stable processes on \((\mathbb{Z}/n_0\mathbb{Z})^N[19]\), the results here depend on the model. That is, if we change \( n_0 \) and \( \beta \) accordingly so that \( \nu \) remains constant, \( d_H \) and \( \nu'_{cr} \) will change.

### 4.2 Upper bound for the Hausdorff dimension

Let \((T, f)\) be a tree constructed as in Section 3 and \( X \) a process such that \( F(X) = (T, f) \). We say that a vertex \( a \in \Gamma \) is hit by the path on the boundary if some ray containing \( a \) belongs to this path. Remark that if two vertices at the same level are hit by the path on the boundary, one of both is hit first. If \( a \) is at level \( n \), we would like to compute the probability \( P_n \) that \( a \) is hit. This probability only depends on \( n \).

Let \( \Gamma(a) \) be the set of ancestors of \( a \), with \( a \in \Gamma(a) \), to which we add an element \( a' \) standing for the set of rays of \( \partial \Gamma \) containing \( a \). We consider the random walk \( S \) on \( \Gamma(a) \) obtained by recording the successive passages of \( X \) on \( \Gamma(a) \), a passage on \( a' \) meaning a visit of a ray containing \( a \). In a formal way, set

- \( S_0 = R \), \( U_0 = 0 \)
- \( S_{n+1} \) is the first vertex of \( \Gamma(a) \) different from \( S_n \) visited by \( X \) after time \( U_n \), and \( U_{n+1} \) is the first visit time of \( S_{n+1} \) after time \( U_n \).

Here we agree that \( a' \) is visited by \( X \) each time a ray containing \( a \) is visited by \( X \). Note that if we take a time-changed version of \( X \), this will not affect the random walk \( S \).
As $X$ follows the rule $(\mathcal{R})$, the law of $S$, up to the hitting time of $a'$, can be described as follows. First, assume $S_n = b \neq a$. If the (unique) child of $b$ in $\Gamma(a)$ belongs to a ray already visited at time $U_n$, put a conductance 1 between $b$ and its father, and a conductance $C'$ between $b$ and its child. Otherwise, put a conductance 1 between $b$ and its father, and a conductance $C$ between $b$ and its child. Choose $S_{n+1}$ according to the conductances. Note that the set of rays hit by $X$ at time $U_n$ changes according to $n$, and that the choice of the conductances will change accordingly.

If $S_n = a$, put a conductance 1 between $a$ and its father and a conductance $K$ between $a$ and $a'$, $K$ corresponding to the total conductance between $a$ and the boundary under $a$. The point is that if we consider $S$ up to the hitting time $\tau$ of $a'$, no ray containing $a$ is visited before $\tau$, which entails that $K$ does not need to be updated at each step (in contrast to the other conductances).

It is easy to compute $K$: $a$ is connected to each of its $n_0$ children via a conductance $C$, and the total conductance between each of these children and the boundary is $CK$. Hence

$$K = \frac{n_0}{1/C + 1/KC}$$

$$K = n_0C - 1$$

We compute $P_n$ by induction. Remark that if $a$ is hit by the path on the boundary, so is its father $\overline{a}$. Conversely, if $\overline{a}$ is hit by the path on the boundary, so is at least one of its children, and one of these children, say $b$, is hit first. The probability that $b = a$, conditionally on the event that $\overline{a}$ is hit, equals $1/n_0$.

If $b \neq a$, then after the first hitting time of $b$, the random walk $S$ is induced by the following electric network: one puts a conductance $C'$ on each edge between levels $k - 1$ and $k$, $k \leq n - 1$, a conductance $C^{n-1}C$ between $a$ and $\overline{a}$ and a conductance $C^{n-1}CK$ between $a$ and $a'$. The probability $\tilde{P}_n$ that a ray containing $a$ is hit by the path on the boundary is the probability that $S$, started at $\overline{a}$, hits $a'$ before $\overline{R}$. The total conductance between $\overline{a}$ and $\overline{a}'$ equals

$$C_1 = \frac{1}{1 + C^{-1} + \ldots + C^{-(n-1)}} = \frac{C^{n-1} - C^n}{1 - C^n}$$

Between $\overline{a}$ and $a'$, the total conductance equals

$$C_2 = \frac{1}{(C^{n-1}CK)^{-1} + (C^{n-1}C)^{-1}} = C^{n-1}C(1 - \frac{1}{n_0C})$$

By the classical link between electric networks and random walks [9],

$$\tilde{P}_n = \frac{C_2}{C_1 + C_2} = \frac{C - 1/n_0}{1 - C' + C - 1/n_0}(1 + O(C^n))$$

Thus we have the recursion formula

$$P_n = P_{n-1} \left( \frac{1}{n_0} + \frac{(n_0 - 1)\tilde{P}_n}{n_0} \right)(1 + O(C^n))$$

$$= P_{n-1} \left( \frac{n_0C - C'}{n_0 - n_0C' + n_0C - 1} \right)(1 + O(C^n))$$

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Set

\[ D = \frac{n_0C - C'}{n_0 - n_0C' + n_0C - 1} \]

Then there exists a positive constant \( c \) such that

\[ P_n \sim cD^n \]

Hence the number of vertices of \( \Gamma \) hit by the path on the boundary at level \( n \) has expected value \( \sim c(Dn_0)^n \). It follows that the path on the boundary has box dimension (and Hausdorff dimension) less than

\[ -\log(Dn_0)/\log \beta \]

### 4.3 Lower bound and the link with percolation

Define a percolation on \( \Gamma \) by saying that a vertex is kept if and only if it is hit by the path on the boundary. Associated with every percolation on a tree is an electric network, and Lyons’ theorem [18] states that under certain conditions, the hitting probabilities for the percolation can be compared with a certain capacity.

Recall that if \( h \) is a positive, decreasing function \( \mathbb{R}_+ \to \mathbb{R}_+ \) and if \( \theta \) is a finite measure on \( \partial \Gamma \), the \( h \)-energy of \( \theta \) \((x > 0)\) is defined by

\[ E_h(\theta) = \int_{\partial \Gamma} h(\delta(y, z))\theta(dy)\theta(dz). \]

The \( h \)-capacity of a closed subset \( K \) of \( \partial \Gamma \) is given by

\[ \text{Cap}_h(K) = \sup(E_h(\theta)^{-1}), \]

where the supremum it taken over all the probability measures \( \theta \) supported by \( K \) with total mass 1. In particular, for \( x > 0 \), the \( x \)-energy \( E_x \) of \( \theta \) and the corresponding \( x \)-capacity are associated to the function \( t \to t^{-x} \).

Here the capacity is induced by an electric network. That is, every measure on \( \partial \Gamma \) can be seen as a current flow and the energy is given by the electric energy of the flow, see [18]. The conductances are given by \( K_n = (1/P_{n+1} - 1/P_n)^{-1} \), \( K_n \) being the conductance on each edge between levels \( n \) and \( n + 1 \).

To apply the general version of Lyons’ theorem, we have to find an upper bound for the correlation of the percolation probabilities.

**Proposition 3** If \( a \) and \( b \) are two vertices of \( \Gamma \) and \( a \land b \) is their last common ancestor,

\[ P(a \text{ and } b \text{ percolate} | a \land b \text{ percolates}) \leq 2P(a \text{ percolates} | a \land b \text{ percolates})P(b \text{ percolates} | a \land b \text{ percolates}) \]

**Proof.** The case when \( a \) is an ancestor of \( b \), or conversely, is trivial. So we suppose that \( a \land b \) is different from \( a \) and from \( b \). In that case, either \( a \) or \( b \) is hit first by the path of the boundary. It is sufficient to prove

\[ P(a \text{ and } b \text{ percolate, } a \text{ is hit first} | a \land b \text{ percolates}) \leq P(a \text{ percolates} | a \land b \text{ percolates})P(b \text{ percolates} | a \land b \text{ percolates}) \]
The left member of this inequality can be bounded by

\[ P(a \text{ and } b \text{ percolate}, \text{a is hit first } | a \land b \text{ percolates}) \]
\[ = P(a \text{ is hit before } b | a \land b \text{ percolates})P(b \text{ is hit after } a | a \text{ is hit before } b) \]
\[ \leq P(a \text{ percolates} | a \land b \text{ percolates})P(b \text{ is hit after } a | a \text{ is hit before } b) \]

Therefore, the proof of Proposition 3 amounts to the following inequality

\[ P(b \text{ is hit after } a | a \text{ is hit before } b) \leq P(b \text{ percolates} | a \land b \text{ percolates}) \quad (1) \]

Let us consider the left member of (1). The process \( X \) hits a ray containing \( a \) for the first time at, say \( t \). The first vertex of \( \Gamma(b) \) visited by \( X \) after \( t \) is \( a \land b \). Let \( t' \) be the first hitting time of \( a \land b \) after time \( t \). We can construct a random walk \( S \) on \( \Gamma(b) \) as in Section 4.2, started at \( t' \). Then the probability \( P \) we are considering is the probability that \( S \) hits \( b' \) before \( R \).

The law of \( S \) depends on the lowest vertex \( c \) of \( \Gamma(b) \) hit by the path on the boundary at time \( t' \), because this determines the initial choice of the conductances. Remark that at least \( a \land b \) has been hit at time \( t' \). Moreover, the self-repelling mechanism entails that \( P \) is maximal if \( c = a \land b \). Hence \( P \leq P_1 \), where \( P_1 \) is the probability that \( S \) hits \( b' \) before \( R \), conditionally on the event that \( c = a \land b \).

We use the same arguments to evaluate the probability in the right member of (1). Let \( t \) be the hitting time of a ray containing \( a \land b \). Conditioning on the lowest vertex \( b_1 \) of \( \Gamma(b) \) hit by the path on the boundary at time \( t \), we construct a random walk \( S' \) on \( \Gamma(b) \), starting at \( b_1 \). The probability \( P' \) we are considering is the probability that \( S' \) hits \( b' \) before \( R \). Plainly, \( P' \) is minimal if \( b_1 = a \land b \). So we have \( P' \geq P'_1 \), where \( P'_1 \) is the probability that \( S' \) hits \( b' \) before \( R \), conditionally on the event that \( b_1 = a \land b \). Finally, we have \( P_1 = P'_1 \).

Therefore

\[ P \leq P_1 = P'_1 \leq P' \]

which concludes the proof of Proposition 3. \( \square \)

**Remark.** We could define as well a self-attracting process by taking \( C' > C \). We would have the same proof for the construction and the upper bound of the Hausdorff dimension. However, for the lower bound, the inequality \( P \leq P_1 \) would no longer be true (in fact, the converse would hold).

**Corollary 1** There exists a positive constant \( c_2 \) such that for every compact subset \( K \) of \( \partial \Gamma \),

\[ c_2 \text{Cap}_{d-d_H}(K) \leq P(K \cap A \neq \emptyset) \]

**Proof.** According to the computation of Section 4.2, \( K_n \sim c'D^n \) for some positive constant \( c' \), with \( D = \beta^{-d}/n_0 \). The related capacity is comparable with the one induced by the new choice of conductances \( K'_n = D^n \). “Comparable” means that the ratio of the two capacities is bounded above and below by two positive constants. But the capacity induced by the \( K'_n \)'s is the \( x \)-capacity with \( x = d - d_H \). Using Proposition 3 and the dependent version of Lyons’ theorem, we get the result. \( \square \)
Now remark that Lyons’ theorem also enables us to express the hitting probability of a compact set $K$ by a Bernoulli percolation with parameter $p = \beta^{-d}/n_0$ in terms of the same capacity. This, together with the corollary, proves (ii) in Theorem 2.

On the other hand, one can apply the corollary with $K$ being the subset surviving a Bernoulli percolation with parameter $p' > \beta^{d-d_H}$. Lyons’ theorem entails

$$P(K \cap A \neq \emptyset) \leq 2\text{Cap}_x(A)$$

where $x = \log p'/\log \beta$ and $\text{Cap}_x(A)$ is a random variable. Together with the corollary, this yields

$$\text{Cap}_{d-d_H}(K) \leq c_3\text{Cap}_x(A)$$

where both sides of the inequalities are random variables. On the event $\{K \neq \emptyset\}$, which has positive probability, $\text{Cap}_{d-d_H}(K) > 0$, see [18]. This being true for every $p' > \beta^{d-d_H}$, we deduce that for every $x < d - d_H$, $\text{Cap}_x(A) > 0$ with positive probability. Frostman’s lemma [11] entails that for every $d' < d_H$, $\dim A \geq d'$ with positive probability.

Let us condition $A$ to be non-empty. Then using the local self-similarity and independence properties of the process, one can show that almost surely, for every $d' < d_H$, $\dim A \geq d'$. Indeed, this is a general fact that some properties, having positive probability, actually occur with probability 1. We skip the discussion here but we shall give in the next section a rigorous proof using the same kind of argument. As $\dim A \geq d'$ for every $d' < d_H$, we conclude that $\dim A \geq d_H$, almost surely.

### 4.4 The self-avoiding property

Suppose that for some $t > 0$, $X_t = a \in \Gamma$. The set of vertices of $\Gamma$ belonging to a ray visited before $t$ yields the subtree $\Gamma_t = f(T(t))$. We would like to determine whether $X$ hits $\partial \Gamma_t$ after time $t$.

By the same method as in Section 4.2, we can construct a random walk $S$ on $\Gamma_t$, starting at $a$, up to the hitting time of $\partial \Gamma_t$. Then $X$ hits $\partial \Gamma_t$ after time $t$ if and only if $S$ hits $\partial \Gamma_t$ before $\overline{\Gamma}$.

The law of $S$ is induced by the electric network where one assigns to each edge of $\Gamma_t$ between levels $n$ and $n + 1$ a conductance $C^{n+1}$. Suppose that $\partial \Gamma_t$ is not empty. Then according to (i) in Theorem 2, the Hausdorff dimension of $\partial \Gamma_t$ is $d_H$ almost surely. As a consequence of Lyons [18], the total conductance between $a$ and the boundary is positive if $\nu' > \nu'_c$ and 0 if $\nu' < \nu'_c$. Furthermore, the probability that $S$ hits $\partial \Gamma_t$ before $\overline{\Gamma}$ is positive if and only if the total conductance between $a$ and the boundary is positive. Thus this probability is positive if $\nu' > \nu'_c$, and equals zero if $\nu' < \nu'_c$.

Assume first that $\nu' < \nu'_c$. Denote by $I(a, n)$ the event that $t_n(a)$ exists and that, after time $t_n(a)$, $X$ visits a ray already visited before time $t_n(a)$. Then $\nu' < \nu'_c$ entails that $P(I(a, n)) = 0$. It follows that the countable union

$$\bigcup_{(a, n) \in \Gamma \times \mathbb{R}} I(a, n)$$

also has probability zero. Hence the path of the boundary is self-avoiding almost surely if $\nu' < \nu'_c$. 
Suppose now that \( \nu' > \nu_{cr} \). We know that \( X \) is self-intersecting with positive probability, and we want to show that there are self-intersections with probability 1. We shall use here a scheme of proof that could also be used for the lower bound of the Hausdorff dimension, as already mentioned. Let \( t = \min \{ s, X_s \in \partial \Gamma \} \) and denote \( X_t = r', f(X_t) = r = (a_0, a_1, \ldots) \). Let \( t_n \) be the first time after \( t \) at which \( X_t = a_n \). Let \( \partial_n \) be the subset of rays of \( \partial \Gamma \) containing \( a_n \) but not \( a_{n+1} \). Then \( \partial_n \) is the boundary of a subtree \( \Gamma_n \) of \( \Gamma \). Let \( \Sigma_n = \{ s \in [t_{n+1}, t_n], X_s \in \Gamma_n \} \). We can reconstruct from \( (X_t, t \in \Sigma_n) \) a tree \( (T_n, f_n) \) rooted at \( R_n \) with \( f_n(R_n) = a_n \), as in Section 2.3.

We claim that the trees \( (T_n, f_n) \), \( n \geq 1 \), are independent. Indeed, in the construction of \( (T, f) \) of Section 3.2, for each ordinal \( k \) at which a vertex of \( (T_n, f_n) \) is overgreen, the choice of conductances for the extension procedure does not depend on \( (T_k, f_k), k \neq n \). Moreover, by self-similarity, the probability that \( (T_n, f_n) \) is self-intersecting is positive if \( \nu' > \nu_{cr} \). Hence with probability 1, at least one of the \( (T_n, f_n) \) is self-intersecting, and this entails that \( (T, f) \) itself is self-intersecting, almost surely.

References


