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ASYMPTOTIC DESCRIPTION OF
DIRAC MASS FORMATION
IN KINETIC EQUATIONS FOR QUANTUM PARTICLES

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Abstract In this paper we describe in a detailed manner the way in which a dirac mass is generated asymptotically in time for solutions of a kinetic equation for quantum particles.

1.-Introduction. The purpose of this article is to describe in detail the long time asymptotics of the following kinetic equation

\begin{equation}
\frac{\partial f}{\partial t} = \int_0^\infty k^2 k'^2 b(k, k') \left\{ f'(1 + f)e^{-k} - f(1 + f')e^{-k'} \right\} dk'
\end{equation}

where \( f = f(k, t), \ k \geq 0, \ t \geq 0 \) and \( f' \) stands for \( f(k', t) \). Equations of the type (1.1) are a typical example of kinetic equations for quantum particles. In particular, equation (1.1) arises in the description of the approach to thermal equilibrium of a distribution of bosons that are in contact with a bath of fermions in thermal equilibrium (for instance: photons-electrons in a plasma) c.f. [Dr], [LY1], [LY2]. A very interesting feature of equations with the form (1.1) is the fact that they have, for suitable choices of the function \( b(k, k') \), solutions that develop Dirac masses for \( f \) at \( k = 0 \) as \( t \to \infty \). One explanation for this behaviour lies in the fact that the distributions of Bosons in thermal equilibrium are given, by the usual Bose-Einstein distributions

\begin{equation}
\frac{1}{e^{\mu + k} - 1}, \ \mu \geq 0.
\end{equation}

It turns out, however, as it was already observed by Bose and Einstein ([B], [E1], [E2]) that for systems of bosons in thermal equilibrium a careful analysis of the statistical physics of the problem leads to enlarge the class of steady distributions and to include also the solutions containing a Dirac mass at the origin:

\begin{equation}
k^2 f_{BEC}(k) = \frac{\hat{k}^2}{e^k - 1} + \rho \delta(k), \ \rho \geq 0.
\end{equation}
Since the total density of bosons,

\[ N(t) = \int_0^\infty k^2 f(k, t) \, dk \]

remains constant under the evolution by (1.1) it is natural to expect for its solutions a long time asymptotics
given by the steady states (1.3) with the same total density of bosons as the initial data chosen for (1.1). In
particular, if the initial distribution of particles has a density larger than the Planck distribution (i.e. the
distribution in (1.2) with \( \mu = 0 \)), then it would be natural to expect a long time asymptotics given by (1.3)
with \( \rho > 0 \).

To our knowledge, the first results indicating Dirac mass formation for the solutions of (1.1) were given in
\[ [\text{LY1}, \text{LY2}] \] using suitable approximations of the equation for \( k \to 0^+ \). Numerical simulations showing Dirac
mass formation for solutions of equations related to (1.1) have been obtained in \[ [\text{ST1}, \text{ST2}] \]. A rigorous
proof of the fact that solutions of (1.1) behave as steady states containing Dirac masses at the origin, as \( t \to \infty \)
was given in \[ [\text{EM1}, \text{EM2}] \]. The proof given in \[ [\text{EM2}] \] works under rather broad assumptions and
is based on general features of kinetic equations. First, notice that equation (1.1) can be estimated in terms
of linear equations of the variable \( f \) (due to mass conservation). This rules out the possibility of finite time
blow up for the solutions. On the other hand, equation (1.1) satisfies a standard \( \mathcal{H} \)-theorem, or in another
words, a suitable entropy increases along the solutions. Using this entropy as a Lyapunov function it can be
checked that the solutions of (1.1) approach steady states as \( t \to \infty \), and for

\[ N(0) > \int_0^\infty \frac{k^2}{(e^k - 1)} \, dk \equiv N_0 \]

steady states have the form (1.3) with \( \rho > 0 \). This yields Dirac mass formation as \( t \to \infty \). The details of
the argument can be carried out for a large class of kernels \( b(k, k') \) (cf. Theorems A, B later).

In this paper we analyse the precise manner in which the Dirac mass develops as \( t \to \infty \). To this end, we
study in detail the case of constant \( b(k, k') \) that is explicitly solvable using Laplace transforms in spite of
the nonlinearity of the problem. We describe in detail the formation of a Dirac measure as \( t \to \infty \) if initially
this one is completely absent. We also study the effect in the asymptotics of the solutions of the presence at
\( t = 0 \) of a fraction of mass concentrated at \( k = 0 \). Finally, in an Appendix at the end of the paper we show,
using formal arguments, that the results previously obtained in the case of constant \( b \) can be extended to a
large class of kernels \( b(k, k') \).

For technical reasons it is more convenient to make in (1.1) the change of variables:

\[ F(k, t) = k^2 f(k, t) \]

that transforms (1.1) into

\[ \frac{\partial F}{\partial t} = Q(F, F) = \int_0^\infty b(k, k') \left( F'(k^2 + F) e^{-k} - F(k^2 + F') e^{-k'} \right) \, dk'. \]

We will study (1.6) in the class of functions \( F \in \mathcal{M}_0 = \{ F \in M^1([0, \infty)), \, F \geq 0, \, M((1 + k) F) < \infty \} \)
where \( M^1([0, \infty)) \) is the set of Radon measures in \([0, \infty)\) and \( M(g) = \int_{[0, \infty)} g \). We recall that any \( F \in \mathcal{M}_0 \)
can be decomposed as the sum of a measure absolutely continuous with respect to the Lebesgue measure, plus
a singular part

\[ F = \varphi + \Psi \]

where \( \varphi \in L^1([0, \infty)) \) and \( \Psi(A \cap X) = \Psi(A) \) for any Borel set \( A \) and \( X \) such that \( \int_X \, dk = 0 \). Given \( F \in \mathcal{M}_0 \)
we introduce an entropy function by means of

\[ S(F)(k) = \int_0^\infty \left[ (k^2 + \varphi) \ln(k^2 + \varphi) - \varphi \ln \varphi - k^2 \ln k^2 \right] \, dk - \int_0^\infty k \, F \]
Notice that the singular term $\Psi$ does not give any contribution to the entropy in the logarithmic terms of (1.8). Formula (1.8) implies that the singular part of $F$ gives a negative contribution to the entropy as soon it is not supported at $k = 0$.

Formally, the solutions of (1.1) satisfy the following $H$-theorem

$$\frac{d}{dt} S(F) = \frac{1}{2} \int_0^\infty \int_0^\infty b_j \left( (k^2 + F) e^{-k} F', (k'^2 + F') e^{-k'} F' \right) dk' dk,$$

where $j(u, v) = (u - v)(\log u - \log v)$ extended in a suitable limit manner for $u = 0, v = 0$. Taking into account that the logarithmic term acts only on the absolutely continuous part of the measure we can make precise (1.9) as follows. We define for $F \in \mathcal{M}_0$ the dissipation function $D(F)$ as

$$D(F) = D_1(\varphi) + 2D_2(\varphi, \Psi) + D_3(\Psi)$$

where the dissipations of entropy terms $D_i$ are given by

$$D_1(\Psi) = \int_0^\infty \int_0^\infty b j \left( (k^2 + \varphi) e^{-k} \varphi', (k'^2 + \varphi') e^{-k'} \varphi' \right) dk' dk,$$

$$D_2(\varphi, \Psi) = \int_0^\infty \int_0^\infty b \Psi' j \left( (k^2 + \varphi) e^{-k}, \varphi e^{-k'} \right) dk' dk,$$

$$D_3(\Psi) = \int_0^\infty \int_0^\infty b \Psi' j \left( e^{-k}, e^{-k'} \right) dk' dk,$$

Notice that $D_i \geq 0$ for $i = 1, 2, 3$ and therefore, so is $D(F)$. We recall here by convenience the $H$-theorem (1.9) in the precise manner formulated in [EM1] and [EM2].

**Theorem A.** Assume that $b$ satisfies $0 < b \in L^\infty$. Then for any initial datum $F_{in} = \varphi_{in} + \Psi_{in} \in \mathcal{M}_0$ there exists a unique entropy solution $F = \varphi + \Psi \in C([0, \infty), \mathcal{M}_0)$ of (1.6) satisfying

$$\int_0^\infty F(t, k) \phi(t, k) dk = \int_0^\infty F_{in}(k) \phi(0, k) dk + \int_0^t \int_0^\infty Q(F, F) \phi(t, k) dk ds,$$

$\forall \phi \in C_c([0, \infty) \times [0, \infty)),$

$$F(0, .) = \varphi(0, .) + \Psi(0, .) = F_{in} = \varphi_{in} + \Psi_{in}$$

and such that

$$\int_{t_1}^{t_2} D(F(s, .)) ds \leq S(F(t_2, .)) - S(F(t_1, .)) \quad \text{for all } t_2 \geq t_1 \geq 0.$$

Moreover, $F$ satisfies

$$M(F(t, .)) = M(F_{in}) =: M_{in} \quad \text{for all } t \geq 0$$

and is such that

$$\text{supp } \Psi(t, .) \subset \text{supp } \Psi_{in}.$$ 

In particular, if $F_{in} = \varphi_{in} \in L^1(0, \infty)$ then $\Psi(t, .) = 0$ for every $t \geq 0$ and thus $F(t, .) = \varphi(t, .) \in L^1(0, \infty)$ for every $t \geq 0$. 

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Moreover, the long time asymptotics of the solutions of (1.6) was also studied in [EM2]. We introduce the following compact notation to describe all the steady solutions of (1.3) by means of a formula depending on one parameter, namely the mass.

\begin{equation}
B_m = \begin{cases} 
g_{\mu} & \text{with } M(g_{\mu}) = m \text{ if } m \leq N_0 \\
g_0 + \rho_0 \delta_0 & \text{where } \rho = m - N_0 \text{ if } m > N_0. 
\end{cases}
\end{equation}

where

\begin{equation}
g_{\mu}(k) = \frac{k^2}{\rho^2 + k^2 - 1}, \quad \rho \geq 0, \quad N_0 = \int_0^{\infty} \frac{k^2}{\rho^2 + k^2 - 1} dk.
\end{equation}

It can be checked that the functions \(B_m\) are the only steady states of (1.3), since they are the only functions for which the dissipation \(D(\cdot)\) defined in (1.10) vanishes. On the other hand, and related to this, the function \(B_m\) maximises the entropy for a given total density. More precisely

\[ S(B_m) = \max\{S(F); \; F \in \mathcal{M}_0, \; M(F) = m\}. \]

We recall the Theorem describing long time asymptotics for the solutions of (1.1) proved in [EM2].

**Theorem B (Asymptotic behavior).** Assume that \(b\) and \(F_{in}\) satisfy the assumptions of the previous Theorem. Let be \(m = M(F_{in})\), and \(F \in C([0, \infty); \mathcal{M}_0)\) the corresponding solution to (1.6). Then we have

\begin{equation}
\begin{cases} 
F(t, \cdot) \xrightarrow{t \to \infty} B_m(\cdot) & \text{weakly * in } (C_c([0, \infty)))' \\
\lim_{t \to \infty} \|\varphi(t, \cdot) - g_{\mu}\|_{L^1((k_0, \infty))} = 0 & \forall k_0 > 0.
\end{cases}
\end{equation}

Moreover if \(m \leq N_0\) or \(0 \leq \varphi_{in} \leq g_0\) we can take \(k_0 = 0\).

Notice the following consequences of the previous results. Assume first that we start with a regular initial data \(F_{in} \equiv \varphi_{in} \in L^1\). Then, the solution \(F\) remains regular for all time: \(F(t) \equiv \varphi(t) \in L^1\). Moreover, suppose that \(M(F_{in}) = M(\varphi_{in}) = m > N_0\). Then \(F(t, \cdot) \equiv \varphi(t, \cdot) \to B_m\) where \(B_m = g_0 + (m - N_0)\delta_0\). This shows that a regular initial state of total mass greater that \(N_0\) condensates at the origin in infinite time, yielding a Dirac measure.

Suppose now that the initial data is \(F_{in} = \varphi_{in} + \rho_{in}\delta_0\). If \(M(F_{in}) = M(\varphi_{in}) + \rho_{in} \leq N_0\) then, Theorem A implies \(F(\cdot, t) = \varphi(\cdot, t) + \rho(t)\delta_0\) and, by Theorem B, \(\rho(t) \to 0\) and \(\varphi(t) \to g_{\mu}\) for some \(\mu \geq 0\). On the other hand, if \(M(\varphi_{in}) > N_0\), then,

\[ F(t) = \varphi(t) + \rho(t)\delta_0 \to g_0 + (m - N_0)\delta_0. \]

and \(\varphi(t) \to g_0\) in \(L^1((k_0, \infty))\) \(\forall k_0 > 0\).

Therefore, if \(M(\varphi_{in}) > N_0\) part of the mass of \(\varphi(t)\) is transferred to the condensate. A question that arises in a natural manner is to establish how the presence of the condensate influences the mechanism of Dirac formation as \(t \to \infty\). More precisely: Is there any transfer of mass from the regular part to the singular part during the evolution, or on the contrary, does this mass always remains in the \(L^1\) part and eventually concentrates at the origin as \(t \to \infty\)? We will prove in this paper that in the case of a constant \(b\) the first possibility takes place, or more precisely, the whole amount of mass \((M_{in} - N_0)\) flows to the singular part during the evolution.

The main result of this paper is a detailed description of the asymptotics of the solutions of (1.6) as \(t \to \infty\) and for initial data \(F_{in}(k)\) that behaves like a power as \(k \to 0\). The behaviour of the solution depends on three different factors: the value of the initial mass, the precise power law of the initial data as \(k \to 0\) and the presence or not of condensate at \(k = 0\) at the initial time. In a precise manner, the result is the following.
Theorem C. Let be \( F_{in} \in \mathcal{M}_0 \), \( M(F_{in}) = M_{in} \) and let \( g_{\mu} \) be the regular part of \( B_{M_{in}}^* \). Assume that

\[
F_{in} = g_{\mu} + g_{in} + \rho(0)\delta_0
\]

is such that \( \rho(0) \geq 0 \) and \( g_{\mu} + g_{in} \geq 0 \). Suppose that \( g_{in}(k) \) satisfies:

\[
g_{in}(z) = A\alpha z^\alpha + O(z^\beta), \quad z \to 0^+,
\]

\[
g_{in}^{'}(z) = A\alpha z^{\alpha-1} + O(z^{\beta-1}), \quad z \to 0^+
\]

(1.23)

\[
\sup_{0 < R \leq 1} \frac{1}{R^{\alpha-(1+\epsilon)}} \sup_{x, y \in B_{in} \setminus B_{R/3}} \frac{|g_{in}^{'}(x) - g_{in}^{'}(y)|}{|x - y|^\epsilon} \leq C_1,
\]

(1.24)

\[
|g_{in}(z)| \leq e^{-r z} \zeta \to \infty,
\]

for some positive constants \( A, C_1, \Gamma, \beta > \alpha \). We also assume that

\[
\alpha < 2, \quad \text{and:} \quad \beta < 1 \quad \text{if} \quad \alpha < 1, \quad \beta < 2 \quad \text{if} \quad 1 \leq \alpha < 2.
\]

Then the solution \( F \) to (1.6) with \( b(k, k') \equiv 1 \) and initial data \( F_{in} \) is such that, for all \( t > 0 \)

\[
F(t, t) = g_{\mu}(\cdot) + g(\cdot, t) + \rho(t)\delta_0
\]

where \( g \) satisfies the following asymptotic behaviour as \( t \to \infty \), uniformly on sets \( 0 < kt \leq C \):

(1) \( M_{in} > N_0 \) and \( \rho(0) \neq 0 \),

\[
g(k, t) = \begin{cases} 
C_1 k^\alpha e^{-M_{in}(1-e^{-k})t} + O(k^\beta + k^\alpha/t^\alpha + 1) & \text{if} \quad 0 < \alpha < 1, \\
C_2 k e^{-M_{in}(1-e^{-k})t} + O(k^\beta + k^\alpha/t^\alpha) & \text{if} \quad \alpha = 1, \\
C_3 e^{-M_{in}(1-e^{-k})t} + O(k^\alpha) & \text{if} \quad \alpha > 1.
\end{cases}
\]

(2) \( M_{in} > N_0 \) and \( \rho(0) = 0 \),

\[
g(k, t) = \begin{cases} 
C_4 k^{\alpha+1} e^{-M_{in}(1-e^{-k})t} + O(k^\beta t^{\alpha+1} + k^\alpha t^{2\alpha}) & \text{if} \quad 0 < \alpha < 1, \\
C_5 k t^2 e^{-M_{in}(1-e^{-k})t} + O(k^\beta t^2 + kt + kt^{3-\beta}) & \text{if} \quad \alpha = 1, \\
C_6 k t^2 e^{-M_{in}(1-e^{-k})t} + O(k^\alpha) & \text{if} \quad \alpha > 1.
\end{cases}
\]

(3) \( M_{in} = N_0 \) and \( \rho(0) = 0 \),

\[
g(k, t) = \begin{cases} 
C_7 k^\alpha e^{-M_{in}(1-e^{-k})t} + O(k^\beta + k^\alpha/t^\alpha + 1) & \text{if} \quad \alpha < 1, \\
C_8 k e^{-M_{in}(1-e^{-k})t} + O(k^\beta + k^\alpha/t^2) & \text{if} \quad \alpha = 1, \\
C_9 k e^{-M_{in}(1-e^{-k})t} + O(k^\alpha + k^\alpha/t) & \text{if} \quad \alpha > 1.
\end{cases}
\]
(4) If \( M_{in} = N_0 \) and \( \rho(0) > 0 \),

\[
g(k, t) = -\frac{1}{M_{in}t} + \frac{\pi^2 A}{3\rho(0)} k^3 e^{-M_{in}(1-e^{-k})t} + O\left(\frac{\log^2 t}{t}\right).
\]

(5) If \( M_{in} < N_0 \), there exists \( \delta > 0 \) such that

\[
g(k, t) = O(e^{-M_{in}\delta t}).
\]

where \( C_1, \cdots, C_9 \) are positive constants, depending only on \( N_0 \) and \( g_{in} \) and whose precise values are given in Section 4.

**Remark 1.** The restriction \( 0 \leq \alpha < 2 \) does not suppose a loss of generality. It is easily checked that if \( \alpha \geq 2 \) the leading terms in the asymptotics of \( g \) is given by the corresponding equilibrium \( g_{eq} \) and only the higher order terms would then be affected.

The condition (1.25) is introduced only to keep the presentation shorter. If \( \rho \geq 2 \), the leading order of the asymptotics of the solution \( g \) would be given by \( g_0 \) and only the higher order terms would be changed.

We remark that the profile of the solution near the origin is very sensitive to the particular behaviour near \( k = 0 \) of the considered initial data \( F_{in} \) (cf. (1.27)). The behaviour of the solution could be rather pathological under more general choices of the initial data and. For that reason we have restricted ourselves to these simple initial data \( F_{in} \) that behaves basically as power laws as \( z \to 0^+ \) and decay exponentially as \( z \to \infty \).

It is also interesting to understand the behaviour of the “condensate”, i.e. of \( \rho(t) \) as \( t \to \infty \). Of course, as we have seen, if \( \rho(0) = 0 \) then \( \rho(t) = 0 \) for all time \( t > 0 \). But if \( \rho(0) > 0 \), then \( \rho(t) > 0 \) for all \( t > 0 \) and its asymptotic behaviour is given in the following result.

**Theorem D.** Under the same assumptions as in Theorem C, assume that \( \rho(0) > 0 \). Then,

(i) If \( M_{in} > N_0 \),

\[
\lim_{t \to \infty} \rho(t) = M_{in} - N_0.
\]

(ii) If \( M_{in} = N_0 \),

\[
\rho(t) = \frac{\pi^2}{M_{in}t} + O\left(\frac{\log^2 t}{t^2}\right) \quad \text{as} \; t \to \infty,
\]

(iii) If \( M_{in} < N_0 \), and \( \mu > 0 \) is such that \( M_{in} = M[g_{eq}] \), then for \( \delta \) satisfying \( M_{in}(e^\mu - 1) < \delta < 2M_{in}(e^\mu - 1) \) we have:

\[
\rho(t) = \rho(0) - \frac{e^{-M_{in}(e^\mu - 1)t}}{1 + 4 \int_0^\infty \frac{G_{in}(k)}{e^\mu - e^{-k}} dk \int_0^\infty \frac{k^2 e^{-k}}{\sinh^2\left(\frac{k}{2}\right)} dk} + O(e^{-\delta t}) \quad \text{as} \; t \to \infty
\]

**Remark 2.** When \( M_{in} > M_0 \), the behaviour of the solution \( F \) to (1.6), or equivalently of the solution \( (g, \rho) \) to (2.6) strongly depends on three conditions:

(i) the behaviour of the initial data \( g_{in} \) near the origin (i.e. \( A \) and \( \alpha \),
(ii) the value of $M_{in}$ with respect to $N_0$.

(ii) the presence or not of condensate at the initial stage (the value of $\rho(0)$).

Assume in particular that $M_{in} > N_0$. When there is no condensate initially, $\rho(0) = 0$, it was known that the solution $F$ develops a Dirac mass at the origin as $t \to \infty$. This is explicitly seen in the expressions (4.26) where the function $g(k, t)$ increases like $t$ on the region $0 < k \leq C/t$ as $t \to \infty$. On the other hand, the presence of condensate at the initial time (i.e. $\rho(0) \neq 0$) has some regularising effect on the solution since, in that case, the function $g$ does not develop any Dirac delta at the origin and behaves essentially, either as the initial data $g_{in}$ (up to a multiplicative constant) or as the distribution $g_0$.

**Remark 3.** Suppose that $F_{in} = g_0 + g_{in} + \rho(0)\delta_0$ with $\rho(0) > 0$ and $g_{in} \geq 0$ in such a way that $\int_0^\infty (g_0 + g_{in})dk > N_0$. Then, of course, $M_{in} > N_0$ and we already knew that as $t \to \infty$, the solution $F$ to (1.6) converges to the Bose equilibrium state $B_{M_{in}} = g_0 + (M_{in} - N_0)\delta_0$. By (1.25) we know moreover that all the condensate part $(M_{in} - N_0)\delta_0$ of this Bose equilibrium state $B_{M_{in}}$ proceeds from the singular part $\rho(t)$. In other words, no part of the condensate is due to the aggregation of mass from the regular part $g(t)$. The presence of condensate at the initial stage prevents the “condensation” of the regular part $g(t)$ as $t \to \infty$.

Finally the plan of the paper is the following. In Section 2 we obtain a formal explicit representation of the solution $F$ to (1.6) when $b \equiv 1$. In Section 3 we start analyzing the representation formulae under some general assumptions on the initial data $F_{in}$. In Section 4 we deduce the asymptotic behaviour of the solutions. Finally in the Appendix we study the extension of these results to more general functions $b$ by means of formal asymptotics.

2. **Solving (1.6) with $b = \text{constant}$.** In this Section we derive representation formulae for the solutions of (1.6) under general assumptions on $g_{in}$ and for $b$ equal to a constant, that we may choose to be one without loss of generality. Using these representation formulae we will be able to obtain detailed information for the long time asymptotics of the solution.

Let $F$ be the solution to equation (1.6)

\begin{align}
\begin{cases}
\frac{\partial F}{\partial t} = Q(F, F) = \int_0^\infty \left( F'(k^2 + F) e^{-k} - F(k^2 + F) e^{-k'} \right) dk', \\
F(k, 0) = F_{in}(k).
\end{cases}
\end{align}

We decompose $F$ as:

\begin{align}
F = G + g_\mu, \quad G = g + \rho(t)\delta_0, \quad F_{in} = g_{in} + g_\mu + \rho(0)\delta_0,
\end{align}

where $g_\mu$ is the regular part of $B_{M_{in}}$ as given in (1.18). Notice that

\begin{align}
M_{in} = M(F) = M(g)(t) + M(g_\mu) + \rho(t), \quad t > 0
\end{align}

The function $G$ then satisfies:

\begin{align}
\frac{\partial G}{\partial t} = \int_0^\infty \left( F'(k^2 + G + g_\mu) e^{-k} - (G + g_\mu)(k^2 + G' + g'_\mu) e^{-k'} \right) dk'.
\end{align}

Taking into account that $(k^2 + g_\mu) e^{-k} = e^u g_\mu$ and $M(F)(t) = M_{in}$ we obtain

\begin{align}
\begin{cases}
\frac{\partial G}{\partial t} = (G + g_\mu) \varepsilon(t) - M_{in} (e^u - e^{-k}) G, \\
G(0) = G_{in} = g_{in} + \rho(0)\delta_0
\end{cases}
\end{align}
where

\begin{equation}
(2.5) \quad \varepsilon(t) = \int_0^\infty (e^\mu - e^{-k^\mu}) G' dk^\mu = \int_0^\infty (e^\mu - e^{-k^\mu}) g' dk' + \rho(t)(e^\mu - 1)
\end{equation}

Using the decomposition of \( G \) in (2.2) we can write (2.4) as:

\begin{equation}
(2.6) \quad \begin{cases}
\frac{\partial g}{\partial t} = (g + g_\mu) \varepsilon(t) - M_{in}(e^\mu - e^{-k^\mu}) g, \\
\frac{\partial \rho}{\partial t} = [\varepsilon(t) - M_{in}(e^\mu - 1)] \rho.
\end{cases}
\end{equation}

The first equation of (2.6) is linear in \( g \) and it can be reduced to a constant coefficient one by the change of variables

\begin{equation}
(2.7) \quad h = g e^{-\int_0^t \varepsilon(s) \, ds}, \quad \omega(t) = \rho(t) e^{-\int_0^t \varepsilon(s) \, ds}
\end{equation}

\begin{equation}
(2.8) \quad \lambda(t) = \varepsilon(t) e^{-\int_0^t \varepsilon(s) \, ds}.
\end{equation}

Whence, \((h, \omega)\) solve the linear equations:

\begin{equation}
(2.9) \quad \begin{cases}
\frac{\partial h}{\partial t} = \lambda(t) g_\mu - M_{in}(e^\mu - e^{-k}) h, \\
\frac{\partial \omega}{\partial t} = -M_{in}(e^\mu - 1) \omega.
\end{cases}
\end{equation}

The two linear equations in (2.9) can be solved explicitly in terms of the (still unknown) function \( \lambda(t) \) using the classical Laplace transform. We define:

\begin{equation}
(2.10) \quad H(z, k) = \int_0^\infty h(t, k) e^{-zt} \, dt, \quad \Omega(z) = \int_0^\infty \omega(t) e^{-zt} \, dt.
\end{equation}

\begin{equation}
(2.11) \quad \Lambda(z) = \int_0^\infty \lambda(t) e^{-zt} \, dt.
\end{equation}

Standard computations yield

\begin{equation}
\begin{cases}
z H - g_{in} = g_\mu \Lambda - M_{in}(e^\mu - e^{-k}) H \\
\varepsilon \Omega - \rho(0) = -M_{in}(e^\mu - 1) \Omega.
\end{cases}
\end{equation}

Thus:

\begin{equation}
(2.12) \quad H(k, z) = \frac{g_{in}}{z + M_{in}(e^\mu - e^{-k})} + \frac{g_\mu \Lambda(z)}{z + M_{in}(e^\mu - e^{-k})}.
\end{equation}

where \( \Lambda \) is the only quantity that remains to be computed. To this end we plug (2.8) and (2.5) in (2.11) and use (2.7) to obtain:

\begin{equation}
\Lambda(z) = \int_0^\infty (e^\mu - e^{-k^\mu}) H(k', z) \, dk' + \Omega(z)(e^\mu - 1).
\end{equation}

Whence,

\begin{equation}
(2.13) \quad \Lambda(z) = \int_0^\infty \frac{(e^\mu - e^{-k^\mu}) G_{in}}{z + M_{in}(e^\mu - e^{-k^\mu})} \, dk \left[ 1 - \int_0^\infty \frac{(e^\mu - e^{-k}) g_\mu}{z + M_{in}(e^\mu - e^{-k})} \, dk \right]^{-1}.
\end{equation}
Notice moreover that, by (2.8):

\[ \lambda(t) = e(t) e^{-\int_0^t \varepsilon(s) ds} = -\frac{d}{dt} e^{-\int_0^t \varepsilon(s) ds} \]

In particular, formally:

\[ \Lambda(0) = \int_0^\infty \lambda(t) \, dt = 1 - e^{-\int_0^\infty \varepsilon(s) ds}. \]

Notice that (2.10) indicates that \( \Lambda(0) \) is finite for any value of \( M_{in} \). Recalling (2.3), and using (2.10), we easily deduce:

\[ 1 - \Lambda(0) = e^{-\int_0^\infty \varepsilon(s) ds} = \begin{cases} 1 - \frac{M_{(g_{in})}}{M_{in} - N_0} = \frac{\varrho(0)}{M_{in} - N_0}, & \text{if } M_{in} > N_0 \\ 1 + 3 \left( \frac{\pi}{2} \right) \int_0^\infty \frac{g_{in}(k)}{1 - e^{-k}} \, dk & \text{if } M_{in} = N_0, \\ 1 + 4 \int_0^\infty \frac{G_{in}(k) \, dk}{e^\mu - e^{-k}} + \int_0^\infty \frac{k^2 e^{-\mu} \, dk}{\sinh^2 \left( \frac{k + \mu}{2} \right)} & \text{if } M_{in} = M[g_{\mu}], \mu > 0. \end{cases} \]

Combining (2.15) and the first identity in (2.14) we readily obtain:

\[ \varepsilon(t) = \frac{\lambda(t)}{1 - \Lambda(0) + \int_0^\infty \lambda(s) \, ds} \]

Let us write now:

\[ \zeta = \frac{z}{M_{in}} \]

we then have:

\[ \Lambda(z) = \Lambda(\zeta) = \frac{\Phi(\zeta)}{S(\zeta)}, \]

where from now on

\[ \Phi(\zeta) = \frac{1}{M_{in}} \int_0^\infty \frac{(e^\mu - e^{-k}) G_{in}}{\zeta + e^\mu - e^{-k}} \, dk \]

and

\[ S(\zeta) = 1 - \frac{1}{M_{in}} \int_0^\infty \frac{(e^\mu - e^{-k}) g_{in}}{\zeta + e^\mu - e^{-k}} \, dk. \]

Using the classical inversion formula of the Laplace transform (see for instance [D]) we obtain:

\[ \lambda(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{zt} \Phi(\zeta) \, S(\zeta) d\zeta, \]

and

\[ h(k, t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{zt} H(k, z) \, dz, \quad \omega(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{zt} \Omega(z) \, dz \]

where \( \gamma \) is greater than the real part of all the singularities of \( \Phi(\zeta)/S(\zeta) \) and \( H(k, z) \) given by (2.12). On the other hand, using (2.7), (2.15) we arrive at

\[ g(k, t) = \frac{h(k, t)}{1 - \int_0^t \lambda(s) \, ds} \]

where
The computations above are a bit formal. However the results in [EM2] quoted in the Introduction show that functions \(g(k, t)\) and \(\rho(t)\) are well defined for \(t > 0\) \(k \geq 0\). Actually it will be seen later that the available estimates for the functions \(\Phi(\zeta), S(\zeta), \Lambda(\zeta)\) imply that the functions \(g(k, t)\) and \(\rho(t)\) defined above provide the required solution of (1.6) in the classical sense.

The hypothesis (1.21)-(1.25) on the initial data \(g_{in}\) presented in the Introduction ensure, as we prove later, that the inverse Laplace transforms (2.21), (2.22) are well defined. We have not intended to reach the greater generality. Our goal is just to provide general conditions that allow us to compute the long time asymptotics for the solutions of the problem, avoiding very technical assumptions on the initial data. On the other hand, it will be seen from the forthcoming arguments that the detailed asymptotics of the Dirac mass could be rather bizarre with suitable choices of \(g_{in}\).

Our purpose is now to study the representation formula (2.20)-(2.22) and the long time behaviour of the solution \(g\) under the assumptions (1.21)-(1.25). Let us summarise the result that describes in a detailed manner the long time asymptotics of \(h(k, t)\) on the region \(k \to 0^+\), and that will be proved in the next Section.

**Theorem 2.1.** Assume (1.21)-(1.25). Then the function \(h(k, t)\), given by (2.22) is defined for all \(t > 0\), an all \(k > 0\) and every \(k \geq 0\). Moreover,

(i) if \(M_{in} \geq N_0\), and \(\rho(0) = 0\) if \(M_{in} = N_0\), then:

(2.24) \[ h(k, t) = [g_{in}(k) + \Lambda(0)g_0(k)]e^{-M_{in}(1-e^{-k})t} + g_0(k)[A(k)e^{-M_{in}(1-e^{-k})t} + B(k, t)] + O(e^{-M_{in} \delta t}) \]

as \(t \to \infty\) and uniformly in \(0 < k < C/t\), where

(2.25) \[ A(k) = \frac{1}{2}[\Lambda(e^{-k} - 1 + i0) - \Lambda(0)] + \Lambda((e^{-k} - 1 - i0) - \Lambda(0)] \]

and,

(2.26) \[ B(k, t) = \text{pv} \int_{-\delta}^{0} \frac{\Lambda(\zeta - i0) - \Lambda(0)}{\zeta + 1 - e^{-\zeta}} - \frac{(\Lambda(\zeta + i0) - \Lambda(0))}{\zeta + 1 - e^{-k}}]e^{M_{in} \zeta t} d\zeta. \]

Moreover:

(2.27) \[ A(k) = \Lambda'(0)(e^{-k} - 1) + O(k^\alpha + k \log k) \quad \text{as} \quad k \to 0 \]

and,

(2.28) \[ |B(k, t)| \leq Ck^{\gamma+1}, \quad \text{where} \quad \gamma = \min(\alpha', 1) \quad \text{as} \quad t \to \infty. \]

uniformly on \(0 < kt \leq C\) and \(\alpha' < \alpha\) arbitrarily close to \(\alpha\).

(ii) If \(M_{in} < N_0\) and \(\mu\) is such that \(M_{in} = M[g_{in}]\), then:

(2.29) \[ h(k, t) = g_{in}(k)e^{-M_{in}(1-e^{-k})t} + O(e^{-M_{in} \delta t}), \quad \text{as} \quad t \to \infty \]

uniformly on \(0 < k \leq C/t\) with \(0 < \delta < e^\mu - 1\).

(iii) If \(M_{in} = N_0\) and \(\rho(0) > 0\),

(2.30) \[ h(k, t) = g_{in}(k)e^{-M_{in}(1-e^{-k})t} - \frac{3\rho(0)}{\pi^2} + O(k|\log t) \quad \text{as} \quad t \to \infty \]

uniformly on \(0 < k \leq C/t\).
3. Proof of Theorem 2.1.

We start studying the analyticity properties of the function \( \Lambda(\zeta) = \Phi(\zeta)/S(\zeta) \).

**Proposition 3.1.** The functions \( S(\zeta) \) and \( \Phi(\zeta) \) are analytic on \( \mathbb{C} \setminus \{\mathbb{R} \} \). Moreover \( S \) has the following properties in \( \zeta \in \mathbb{R} \cap \{\mathbb{R} \} \):

1. \( S \) is increasing in \( (-\infty,-e^\mu) \cup (-e^\mu + 1, +\infty) \).
2. \( S(-\infty) = 1, S(+\infty) = 1. \)
3. \( S((-e^\mu + 1)^+) = 1 - \frac{N_0}{M_{in}}. \)
4. \( S((-e^\mu)^-) = +\infty. \)

5. If \( M_{in} > N_0 \), and then \( \mu = 0 \), there are no roots of \( S \) in \( (-\infty,-1) \cup [0, +\infty) \). The functions \( [S(\zeta)]^{-1} \) and \( \Lambda(\zeta) \) are analytic on \( \mathbb{C} \setminus \{\mathbb{R} \} \).

6. If \( M_{in} < N_0 \), or if \( M_{in} = N_0 \) and \( \phi(0) = 0 \), then \( \zeta = 0 \) is the only zero of the function \( S(\zeta) \) in \( (-\infty,-e^\mu) \cup [-e^\mu + 1, +\infty) \), \( \phi(0) = 0 \), and \( \Lambda(\zeta) \) is analytic on \( \mathbb{C} \setminus \{\mathbb{R} \} \).

7. If \( M_{in} = N_0 \) and \( \phi(0) > 0 \), then \( \zeta = 0 \) is still the only zero of \( S(\zeta) \) in \( (-\infty,-1) \cup [0, +\infty) \), but \( \phi(0) = -\phi(0)/M_{in} \). \( \Lambda(\zeta) \) is analytic on \( \mathbb{C} \setminus \{\mathbb{R} \} \).

**Proof.** Formulas (2.19), (2.20) and general properties of analytic functions imply that the functions \( S \) and \( \Phi \) are analytic for \( \zeta \in \mathbb{C} \setminus \{\mathbb{R} \} \). Let us write \( \zeta = \zeta_1 + i\zeta_2, \zeta_i \in \mathbb{R}, i = 1, 2 \). Using (2.19) we obtain:

\[
S(\zeta) = 1 - \frac{1}{M_{in}} \int_0^\infty \frac{(e^\mu - e^{-k}) g_\mu}{\zeta_1 + i\zeta_2 + e^\mu - e^{-k}} dk
\]

\[
= 1 - \frac{1}{M_{in}} \int_0^\infty \frac{(e^\mu - e^{-k}) g_\mu(\zeta_1 - i\zeta_2 + e^\mu - e^{-k})}{(\zeta_1 + e^\mu - e^{-k})^2 + \zeta_2^2} dk.
\]

Therefore,

\[
\text{Im}(S(\zeta)) = \frac{\zeta_2}{M_{in}} \int_0^\infty \frac{(e^\mu - e^{-k}) g_\mu}{(\zeta_1 + e^\mu - e^{-k})^2 + \zeta_2^2} dk = 0 \quad \text{if} \quad \zeta_2 \neq 0.
\]

Since \( g_\mu > 0 \), all the zeros of the function \( S \) lie in the real line \( \zeta \in \mathbb{R} \). Moreover notice that for \( \zeta \in \mathbb{C} \setminus \{\mathbb{R} \} \):

\[
S'(\zeta) = \frac{1}{M_{in}} \int_0^\infty \frac{g_\mu(k)(e^\mu - e^{-k})}{(e^\mu - e^{-k} + \zeta)^2} dk > 0
\]

and

\[
S(\zeta) = 1 - \frac{1}{\zeta} \cdot \frac{1}{M_{in}} \int_0^\infty g_\mu(k)(e^\mu - e^{-k}) dk + o(\frac{1}{|\zeta|}) \quad \text{as} \quad |\zeta| \to \infty.
\]

Since,

\[
\int_0^\infty g_\mu(k)(e^\mu - e^{-k}) dk = 2,
\]

we obtain

\[
S(\zeta) = 1 - \frac{2}{M_{in}|\zeta|} + o(\frac{1}{|\zeta|}), \quad \text{as} \quad |\zeta| \to \infty.
\]

On the other hand,

\[
\lim_{\zeta \to (-e^\mu + 1)^+} S(\zeta) = 1 - \frac{1}{M_{in}} \int_0^\infty \frac{(e^\mu - e^{-k}) g_\mu(k)}{(1 - e^{-k})} dk = 1 - \frac{N_0}{M_{in}}.
\]
Let us evaluate the limit of $S(\zeta)$ as $\zeta$ approaches $-e^\mu$, $\zeta \in \mathbb{R}$, $\zeta < -e^\mu$. To this end we set $\zeta = -e^\mu - \theta$, $\theta > 0$. Then

$$S(\zeta) = 1 + \frac{1}{M_{in}} \int_0^\infty \frac{(e^{\mu} - e^{-k})g_k}{\theta + e^{-k}}dk = 1 + \frac{1}{M_{in}} \int_0^\infty \frac{e^{-k}}{\theta + e^{-k}}k^2dk.$$ 

and it follows that

$$\lim_{\zeta \to -e^\mu} S(\zeta) = +\infty.$$ 

Finally, if $M_{in} < N_0$, or $M_{in} = N_0$ and $\rho(0) = 0$, then it is straightforward from (2.19), (2.20) and (2.2) that $S(0) = \Phi(0) = 0$. Nevertheless:

$$S'(0) = \frac{e^{-\mu}}{4M_{in}} \int_0^\infty \frac{k^2}{\sinh^2(\frac{k+\mu}{2})}dk > 0$$

whence $\Lambda$ is still analytic in a neighbourhood of $\zeta = 0$ and then in all $\mathbb{C} \setminus \{-e^\mu, -e^\mu + 1\}$. 

**Lemma 3.2.** Assume (1.21)-(1.25), $M_{in} \geq N_0$ and $\rho(0) = 0$ if $M_{in} = N_0$. Then

$$h(k, t) = g_{in}(k)e^{-M_{in}(1-e^{-k})t} + \frac{g_0}{2} [\Lambda(\zeta_0 + i0) - \Lambda(0) + \Lambda(\zeta_0 - i0) - \Lambda(0)]e^{-M_{in}(1-e^{-k})t}$$

$$+ g_0 B(k, t) + J_1(k, t)$$

where $\zeta_0 = e^{-k} - 1$, $B(k, t)$ is defined in (2.26) and $J_1$ is such that, for any $\delta > 0$ there exists $C_\delta > 0$ for which for all $k > 0$ and $t > 0$

$$|J_1(k, t)| \leq C_\delta e^{-\delta M_{in}t}.$$ 

**Proof.** By simplicity we write $H(k, \zeta) = H(k, z)$ with $\zeta$ as in (2.17). Then (2.12) becomes

$$H(k, \zeta) = \frac{1}{M_{in}} \left( \frac{g_{in}(k)}{\zeta + 1 - e^{-k}} + \frac{g_0\Lambda(\zeta)}{\zeta + 1 - e^{-k}} \right) = G_1(k, \zeta) + G_2(k, \zeta),$$

and by the inversion formula for the Laplace transform (c.f. (2.22)):

$$h(k, t) = \frac{M_{in}}{2\pi i} \int_{\gamma} (G_1(k, \zeta) + G_2(k, \zeta)) e^{M_{in}\zeta}d\zeta \equiv h_1(k, t) + h_2(k, t).$$

The first term on the right hand side of (3.3) may be calculated explicitly:

$$h_1(k, t) = \frac{g_{in}(k)}{2\pi i} \int_{\gamma} \frac{1}{\zeta + 1 - e^{-k}}e^{M_{in}\zeta}d\zeta$$

$$= g_{in}(k)e^{-M_{in}(1-e^{-k})t}$$

The second term is slightly more delicate. We first remark that, as $\zeta \to 0^+$, $\Phi(\zeta) \to \Phi(0) = 1$, so that

$$\Lambda(\zeta) \to \Lambda(0) \equiv \frac{M(g_{in})}{M_{in} - N_0} = \frac{M_{in} - (N_0 + \rho(0))}{M_{in} - N_0} \quad \text{if } M_{in} \neq 0$$

and

$$\Lambda(\zeta) \to \Lambda(0) = 1 - \frac{3}{\pi^2} \int_0^\infty \frac{e^{-k}g_{in}(k)}{(1-e^{-k})}dk, \quad \text{if } M_{in} = N_0.$$
We may then write:

\[ h_2(k, t) = \frac{g_0(k)}{2\pi i} \int_{\gamma} \frac{\Lambda(\zeta)e^{M_\gamma \zeta t}}{\zeta + 1 - e^{-k}} d\zeta \]

\[ = \frac{g_0(k)}{2\pi i} \int_{\gamma} \frac{(\Lambda(\zeta) - \Lambda(0))e^{M_\gamma \zeta t}}{\zeta + 1 - e^{-k}} d\zeta + \frac{\Lambda(0)g_0(k)}{2\pi i} \int_{\gamma} \frac{e^{M_\gamma \zeta t}}{\zeta + 1 - e^{-k}} d\zeta. \]

and arguing as in (3.4):

\[ \frac{\Lambda(0)g_0(k)}{2\pi i} \int_{\gamma} \frac{e^{M_\gamma \zeta t}}{\zeta + 1 - e^{-k}} d\zeta = \Lambda(0)g_0(k)e^{-M_\gamma (1 - e^{-k})t}. \]

Combining (3.4)-(3.6) we obtain

\[ h(k, t) = g_\infty(k)e^{-M_\gamma (1 - e^{-k})t} + \Lambda(0)g_0(k)e^{-M_\gamma (1 - e^{-k})t} + \frac{g_0(k)}{2\pi i} \int_{\gamma} \frac{(\Lambda(\zeta) - \Lambda(0))e^{M_\gamma \zeta t}}{\zeta + 1 - e^{-k}} d\zeta. \]

The asymptotic behaviour as \( t \to \infty \) of the last term in (3.7) depends of the behaviour of the initial data \( g_\infty(k) \) near the origin. We need to analyse the integral:

\[ I = \frac{1}{2\pi i} \int_{\gamma} \frac{\Lambda(\zeta) - \Lambda(0)}{\zeta + 1 - e^{-k}} e^{M_\gamma \zeta t} d\zeta \]

where \( \gamma \) is any closed path surrounding the interval \([-1, 0]\). We may for instance take

\[ \gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 \]

\[ \gamma_1 = \{\zeta_1 + i \zeta_2 \in \mathbb{C}; (\zeta_1 + 1)^2 + \zeta_2^2 = 1, \z_1 \leq -1\} \]

\[ \gamma_2 = \{\zeta_1 + i \zeta_2 \in \mathbb{C}; (\zeta_1)^2 + \zeta_2^2 = 1, \z_1 \geq 0\} \]

\[ \gamma_3 = \{\zeta_1 + i \in \mathbb{C}; -1 \leq \z_1 \leq 0\} \]

\[ \gamma_4 = \{\zeta_1 - i \in \mathbb{C}; -1 \leq \z_1 \leq 0\} \]

Given now \( \delta > 0 \) arbitrarily, we write

\[ I = I_1(\delta) + I_2(\delta) \]

with

\[ I_1(\delta) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{\Lambda(\zeta) - \Lambda(0)}{\zeta + 1 - e^{-k}} e^{M_\gamma \zeta t} d\zeta \]

and

\[ \Gamma_1 = \gamma \cap \{Re(\zeta) \geq \delta\} \]

\[ \Gamma_2 = \gamma \cap \{Re(\zeta) < \delta\} \]

Consider first \( I_1 \). Since \(|\zeta + 1 - e^{-k}|\) is bounded below over \( \gamma \) we trivially have

\[ |I_1| \leq Ce^{-M_\gamma \delta t} \]

where \( C \) is independent \( t \), and for all \( k \) in a fixed neighbourhood of zero.
We now calculate $I_2$ given by

$$I_2(\delta) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{\Lambda(\zeta) - \Lambda(0)}{\zeta + 1 - e^{-k} e^{M_{in}\zeta}} d\zeta$$

and

$$\Gamma_2 = \gamma_2 \cup \{ \zeta_1 + i \in \mathbb{C}; -\delta \leq \zeta_1 \leq 0 \} \cup \{ \zeta_1 - i \in \mathbb{C}; -\delta \leq \zeta_1 \leq 0 \} \cup \{ -\delta + i\zeta_2, -1 \leq \zeta_2 \leq 1 \}.$$

Since the function $\frac{\Lambda(\zeta) - \Lambda(0)}{\zeta + 1 - e^{-k} e^{M_{in}\zeta}}$ is analytic in the region inside $\Gamma_2 \setminus [-1, 0]$ we have for all $\varepsilon > 0$:

$$I_2(\delta) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{\Lambda(\zeta) - \Lambda(0)}{\zeta + 1 - e^{-k} e^{M_{in}\zeta}} d\zeta$$

$$\Gamma_2 = \gamma_2 \cup \{ \zeta_1 + i \in \mathbb{C}; -\delta \leq \zeta_1 \leq 0 \} \cup \{ \zeta_1 - i \in \mathbb{C}; -\delta \leq \zeta_1 \leq 0 \} \cup \{ -\delta + i\zeta_2, -1 \leq \zeta_2 \leq 1 \},$$

and passing to the limit as $\varepsilon \to 0$ by means of the classical Plemej-Sjojtski formule (cf. Ahlfors [A]):

$$I_2(\delta) = \frac{1}{2}(\Lambda(\zeta_0 + i0) - \Lambda(0) + \Lambda(\zeta_0 - i0) - \Lambda(0)) e^{M_{in}\zeta_0 t}$$

$$+ \frac{1}{2\pi i} [\text{pv} \int_{-\delta}^{0} \frac{(\Lambda(\zeta - i0) - \Lambda(0))}{\zeta + 1 - e^{-k_0} e^{M_{in}\zeta}} d\zeta - \text{pv} \int_{-\delta}^{0} \frac{(\Lambda(\zeta + i0) - \Lambda(0))}{\zeta + 1 - e^{-k_0} e^{M_{in}\zeta}} d\zeta]$$

$$+ \frac{1}{2\pi} \int_{-1}^{1} \frac{\Lambda(-\delta + i\zeta_2) - \Lambda(0)}{-\delta + i\zeta_2 + 1 - e^{-k_0} e^{M_{in}(-\delta + i\zeta_2)t}} d\zeta_2,$$

where $\zeta_0 = e^{-k} - 1 > -\delta$.

Arguing as before:

$$|\frac{1}{2\pi i} \int_{-1}^{1} \frac{\Lambda(-\delta + i\zeta_2) - \Lambda(0)}{-\delta + i\zeta_2 + 1 - e^{-k_0} e^{M_{in}(-\delta + i\zeta_2)t}} d\zeta_2| \leq C e^{-\delta M_{in} t}$$

where $C$ is independent of $k$ in a fixed neighbourhood of zero. Combining (3.7)-(3.11), Lemma 3.2 follows.

Our next goal is to show estimate (2.28). This would conclude the proof of Theorem 2.1 in the case $M_{in} \geq N_0$ with $\rho(0) = 0$ if $M_{in} = N_0$.

**Proposition 3.3.** Assume (1.21)-(1.25). $M_{in} \geq N_0$ and $\rho(0) = 0$ if $M_{in} = N_0$. Then, the function $B$ is well defined by (2.26) for all $t > 0$, and every $k > 0$. Moreover, there exists $k_0 > 0$ sufficiently small and $T > 0$ sufficiently large such that, for all positive constant $C_0$ there exists a positive constant $C$ for which:

$$|B(k, t)| \leq C k^{\gamma + 1}$$

for all $0 < k < k_0$, $t > T$ and $0 < kt < C_0$,

where $\gamma = \min(1, \alpha')$, $\alpha' < \alpha$ arbitrarily close to $\alpha$.

**Proof.** We may write $B$ as

$$B(k, t) = \frac{2\pi i}{M_{in}} S(0) \text{pv} \int_{0}^{\delta} e^{-M_{in}\zeta t} \frac{1}{1 - \zeta - e^{-k}} \varphi(\zeta) d\zeta,$$

where

$$\varphi(\zeta) = [(\Lambda(\zeta - i0) - \Lambda(0)) - (\Lambda(\zeta + i0) - \Lambda(0))]$$

From (1.21)-(1.25) the right hand side of (3.13a) converges for all $k > 0$ and every $t > 0$. 

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Proposition 3.3 is a consequence of the following lemmata.

**Lemma 3.4.** Suppose that the function \( \varphi \) in (3.13) is a \( C^1 \) function on \([0, \delta]\) such that:

\[
|\varphi(\zeta)| \leq C|\zeta|^\gamma + 1 \quad \text{for } \zeta \in [0, \delta]
\]

\[
|\varphi'(\zeta)| \leq C|\zeta|\gamma \quad \text{for } \zeta \in [0, \delta],
\]

for some \( \gamma \in (0, 1] \). Then for \( T > 0 \) sufficiently large and for all \( C_0 > 0 \) there exists a positive constant \( C > 0 \) such that

\[
|B| \leq C(|k|^{\gamma+1} + \frac{1 + (kt)^\gamma}{t^{\gamma+1}} e^{-2M_{ix}^k t}) \quad \text{for all } t > T \text{ and } 0 < kt < C_0.
\]

**Proof.** We first decompose \( B \) as:

\[
B = J_{11} + J_{12}
\]

with

\[
J_{11} = \frac{2\pi i}{M_{ix}S(0)} \text{PV} \int_{0}^{\frac{\pi}{2}} e^{-M_{ix} \zeta t} \frac{e^{-M_{ix} \zeta t}}{(1 - \zeta - e^{-k})^{\gamma}} \varphi(\zeta) d\zeta
\]

\[
J_{12} = \frac{2\pi i}{M_{ix}S(0)} \text{PV} \int_{\frac{\pi}{2}}^{\frac{\pi}{k}} e^{-M_{ix} \zeta t} \frac{e^{-M_{ix} \zeta t}}{(1 - \zeta - e^{-k})^{\gamma}} \varphi(\zeta) d\zeta.
\]

We begin estimating \( J_{12} \) in the following Lemma.

**Lemma 3.5.** Assume that \( k_0 > 0 \) is small enough and \( 0 \leq k \leq k_0 \). Then:

\[
|J_{12}(k, t)| \leq C \frac{1 + (kt)^\gamma}{t^{\gamma+1}} e^{-2M_{ix}^k t}.
\]

**Proof of Lemma 3.5.** Since

\[1 - \zeta - e^{-k} = k - \zeta + O(k^2) \quad \text{as } k \to 0\]

we deduce that, for \( k \) small enough and \( \zeta \geq \frac{3}{2}k \)

\[|1 - \zeta - e^{-k}| = \zeta - k + O(k^2) \geq \zeta - k \geq \frac{3}{2}k \]

and therefore:

\[
|1 - \zeta - e^{-k}| \leq \frac{3}{2\zeta}
\]

We deduce:

\[
|J_{12}| \leq C \int_{\frac{\pi}{2k}}^{\delta} e^{-M_{ix} \zeta t} |\zeta|^{\gamma} d\zeta \leq C \frac{1}{t^{\gamma+1}} \int_{\frac{\pi}{2k}}^{\infty} e^{-M_{ix} u} |u|^{\gamma} du.
\]

Since

\[
\int_{\pi}^{\infty} e^{-M_{ix} u} |u|^{\gamma} du \leq C(1 + x^{\gamma}) e^{-M_{ix} x},
\]

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End of the proof of Lemma 3.4  We have to estimate now $J_{11}$. To this end we write:

$$\begin{align}
\text{pv} \int_0^{\frac{2}{\epsilon}} \frac{e^{-M_{\infty} \zeta t}}{(1 - \zeta - e^{-k})} \varphi(\zeta) d\zeta &= \text{pv} \int_0^{\frac{2}{\epsilon}} \frac{1}{(1 - \zeta - e^{-k})} [e^{-M_{\infty} \zeta t} \varphi(\zeta) - e^{-M_{\infty} (1 - e^{-k}) t} \varphi(1 - e^{-k})] d\zeta \\
&+ e^{-M_{\infty} (1 - e^{-k}) t} \varphi(1 - e^{-k}) \text{pv} \int_0^{\frac{2}{\epsilon}} \frac{1}{(1 - \zeta - e^{-k})} d\zeta \\
&= A_1 + A_2.
\end{align}$$

Now,

$$\begin{align}
\text{pv} \int_0^{\frac{2}{\epsilon}} \frac{1}{(1 - \zeta - e^{-k})} d\zeta &= \text{pv} \int_0^{\frac{2}{\epsilon}} \frac{dy}{1 - y} = \int_0^{1/2} \frac{dy}{1 - y} + \int_{3/2}^{2(1 - e^{-k})} \frac{dy}{1 - y}
\end{align}$$

We deduce that the left hand side of (3.20) is uniformly bounded for $k$ small enough. Since on the other hand, by (3.14), we have

$$|\varphi(1 - e^{-k})| \leq C|k|^{1 + \gamma}$$

we deduce:

$$|A_2| = |e^{-M_{\infty} (1 - e^{-k}) t} \varphi(1 - e^{-k}) \text{pv} \int_0^{\frac{2}{\epsilon}} \frac{1}{(1 - \zeta - e^{-k})} d\zeta| \leq C|k|^{1 + \gamma} e^{-M_{\infty} (1 - e^{-k}) t}.$$

In order to estimate the last remaining term $A_1$ of (3.20) we define:

$$H(\zeta, t) = e^{-M_{\infty} \zeta t} \varphi(\zeta) - e^{-M_{\infty} (1 - e^{-k}) t} \varphi(1 - e^{-k}).$$

Observe that

$$H(1 - e^{-k}, t) = 0$$

and

$$\frac{\partial H}{\partial \zeta} = -M_{\infty} t e^{-M_{\infty} \zeta t} \varphi(\zeta) + e^{-M_{\infty} \zeta t} \varphi'(\zeta).$$

so that by (3.14) and (3.15)

$$|\frac{\partial H}{\partial \zeta}| \leq C \{\zeta^{\gamma + 1} + t \zeta^{\gamma + 1} \} e^{-M_{\infty} \zeta t}.$$

Writing:

$$H(\zeta, t) = \int_{1 - e^{-k}}^{\zeta} \frac{\partial H}{\partial \zeta} (\lambda, t) d\lambda$$

we then easily obtain:

$$\frac{|H(\zeta, t)|}{|1 - \zeta - e^{-k}|} \leq \frac{C}{|1 - \zeta - e^{-k}|} \int_{1 - e^{-k}}^{\zeta} \{\lambda^{\gamma} + t \lambda^{\gamma + 1} \} e^{-M_{\infty} \lambda t}. $$

Since,

$$|\lambda^{\gamma} + (t \lambda^{\gamma + 1})| e^{-M_{\infty} \lambda t} \leq C \lambda^{\gamma} ,$$

(3.24)

$$\frac{C}{|1 - \zeta - e^{-k}|} \int_{1 - e^{-k}}^{\zeta} \{\lambda^{\gamma} + t \lambda^{\gamma + 1} \} e^{-M_{\infty} \lambda t} d\lambda \leq C (1 - e^{-k})^{\gamma} \leq C k^{\gamma} ,$$

and, integrating over $[0, 3k/2]$ it follows that

(3.25)

$$|A_1| \leq C k^{\gamma + 1}.$$
In particular, by (3.18), (3.22) and (3.25) imply, for all $k > 0$:

(3.26) \[ |J_{11}| \leq Ck^{1+\gamma}. \]

Combining (3.26) and Lemma 3.5, Lemma 3.4 follows. \[ \square \]

To conclude the proof of Lemma 3.3 we have to show that

\[ \varphi(\zeta) = [(\Lambda(\zeta - i0) - \Lambda(0)) - (\Lambda(\zeta + i0) - \Lambda(0))] \]

satisfies the hypothesis of Lemma 3.4. First, we prove the following auxiliary result:

**Lemma 3.6.** Assume that the function $q$ satisfies (1.21)-(1.25) and define, for all $\zeta \in \mathbb{C}$

(3.27) \[ K(\zeta) = \int_0^\infty \frac{\psi(k)}{\zeta + 1 - e^{-k}} dk, \quad \psi(k) = (1 - e^{-k})q(k). \]

(3.28) \[ R(\zeta) = \begin{cases} \zeta^\alpha & \text{if } 0 < \alpha < 1 \\ \zeta \log \zeta & \text{if } \alpha = 1 \\ \zeta & \text{if } 1 < \alpha < 2. \end{cases} \]

Then, for $\delta$ small enough, and every $\zeta \in \mathbb{C}$ such that $|\zeta| \leq \delta$:

(3.29) \[ |K'(\zeta) - K'(0)| \leq C|R(\zeta)| \quad \text{and} \quad |K(\zeta) - K(0) - K'(0)\zeta| \leq C|\zeta R(\zeta)|. \]

**Proof.** By definition,

\[ K'(\zeta) - K'(0) = \frac{\zeta}{\zeta + 1} \int_0^\infty (\psi'(k) + \psi(k)) \frac{2 + \zeta - e^{-k}}{\zeta + 1 - e^{-k}(1 - e^{-k})} dk. \]

Now, if $\delta < 1/2$, $2|\zeta| < 1$ and

\[ \left| \int_0^{\log \frac{1}{|\zeta|^2}} (\psi'(k) + \psi(k)) \frac{2 + \zeta - e^{-k}}{\zeta + 1 - e^{-k}(1 - e^{-k})} dk \right| \leq \frac{C}{|\zeta|^2} \int_0^{\log \frac{1}{|\zeta|^2}} k^{\alpha-1} dk \leq C|\zeta|^\alpha. \]

Let us fix $r$ small such that for some positive constant $C$:

\[ |\psi(k)| \leq Ck^{\alpha+1}, \quad |\psi'(k)| \leq Ck^\alpha \quad \text{for all } 0 < k < r \]

and

\[ 1 - e^{-k} \geq \frac{3k}{4} \quad \text{for } 0 < k < r. \]

Then, if $\alpha < 1$,

\[ \left| \int_{\log \frac{1}{|\zeta|^2}}^r (\psi'(k) + \psi(k)) \frac{2 + \zeta - e^{-k}}{\zeta + 1 - e^{-k}(1 - e^{-k})} dk \right| \leq C \int_{\log \frac{1}{|\zeta|^2}}^r k^{\alpha-2} dk \leq C|\zeta|^\alpha. \]

On the other hand, since $\alpha < 1$,

\[ \left( \int_r^\infty (\psi'(k) + \psi(k)) \frac{2 + \zeta - e^{-k}}{\zeta + 1 - e^{-k}(1 - e^{-k})} dk \right) \leq \frac{\int_r^\infty |\psi'(k) + \psi(k)| \frac{dk}{|\zeta| + 3r/4(1 - e^{-k})}} \leq C(r) < |\zeta|^{\alpha-1}. \]
if $|\zeta| \leq \delta$ and $\delta$ is small enough.

If $1 < \alpha < 2,$

$$
\left| \int_{\log \frac{1}{1 - \delta}}^{\infty} (\psi'(k) + \psi(k)) \frac{2 + \zeta - e^{-k}}{(\zeta + 1 - e^{-k})(1 - e^{-k})} \, dk \right| \leq C \int_{\log \frac{1}{1 - \delta}}^{\infty} \frac{\psi'(k) + \psi(k)}{(1 - e^{-k})^2} \, dk \leq C,
$$

And, if $\alpha = 1,$

$$
\left| \int_{\log \frac{1}{1 - \delta}}^{\infty} (\psi'(k) + \psi(k)) \frac{2 + \zeta - e^{-k}}{(\zeta + 1 - e^{-k})(1 - e^{-k})} \, dk \right| \leq C \int_{\log \frac{1}{1 - \delta}}^{\infty} \frac{\psi'(k) + \psi(k)}{(1 - e^{-k})^2} \, dk \leq C \int_{\log \frac{1}{1 - \delta}}^{\infty} k^{-1} \, dq \leq C \log |\zeta|,
$$

and

$$
\int_{r}^{\infty} (\psi'(k) + \psi(k)) \frac{2 + \zeta - e^{-k}}{(\zeta + 1 - e^{-k})(1 - e^{-k})} \, dk \leq \int_{r}^{\infty} \frac{\psi'(k) + \psi(k)}{|\zeta| + 3r/4(1 - e^{-k})} \, dq \leq \frac{4}{r} \int_{r}^{\infty} \frac{\psi'(k) + \psi(k)}{(1 - e^{-k})} \, dq \leq C.
$$

We define:

$$
\Phi(y) = \frac{1}{(1 - y)} (\psi'(\log \frac{1}{1 - y}) + \psi(\log \frac{1}{1 - y})),
$$

then,

$$
\int_{\log \frac{1}{1 - \delta}}^{\infty} (\psi'(k) + \psi(k)) \frac{2 + \zeta - e^{-k}}{(\zeta + 1 - e^{-k})(1 - e^{-k})} \, dq = \int_{|\zeta|/2}^{2} \Phi(y) \frac{dy}{(\zeta + y)\eta} + \int_{|\zeta|/2}^{2} \Phi(y) \frac{dy}{\eta}
$$

(3.30)

$$
= \frac{1}{|\zeta|} \int_{1/2}^{2} [\Phi(|\zeta|\eta) - \Phi(|\zeta|)] \frac{dy}{\eta(e^{\eta} + \eta)} + \frac{\Phi(|\zeta|)}{|\zeta|} \int_{1/2}^{2} \frac{dy}{\eta(e^{\eta} + \eta)} + \frac{1}{|\zeta|} \int_{1/2}^{2} [\Phi(|\zeta|\eta) - \Phi(|\zeta|)] \frac{dy}{\eta} + \frac{\Phi(|\zeta|)}{|\zeta|} \int_{1/2}^{2} \frac{dy}{\eta}.
$$

Now, by (1.23), for all $0 < R \leq 1$:

$$
\sup_{x, y \in B_{3R} \setminus B_{R/3}} \frac{|\psi'(x) - \psi'(y)|}{|x - y|^s} \leq CR^{-s}.
$$

where $\gamma = \min(\alpha, 1)$. Therefore:

$$
\sup_{y_1, y_2 \in B_{2R} \setminus B_{R/2}} \frac{|\psi'(\log \frac{1}{1 - y_1}) - \psi'(\log \frac{1}{1 - y_2})|}{|y_1 - y_2|^s} \leq C \sup_{x, y \in B_{3R} \setminus B_{R/3}} \frac{|\psi'(x) - \psi'(y)|}{|x - y|^s} \leq CR^{-s}.
$$

We easily deduce that

$$
\sup_{y_1, y_2 \in B_{2R} \setminus B_{R/2}} \frac{|\Phi(y_1) - \Phi(y_2)|}{|y_1 - y_2|^s} \leq CR^{-s}
$$

for every $0 < R < \delta$ and $\delta$ small enough. In particular, if we choose $R = |\zeta|$, and make the change of variables $x = |\zeta| \eta_1$, $y = |\zeta| \eta_2$, we obtain:

$$
\sup_{\eta_1, \eta_2 \in B_{2\delta} \setminus B_{\delta/2}} \frac{|\Phi(|\zeta| \eta_1) - \Phi(|\zeta| \eta_2)|}{|\eta_1 - \eta_2|^s} \leq C |\zeta|^\gamma.
$$
from where,
\[
\left| \frac{1}{|\zeta|} \int_{1/2}^{2} \left[ \Phi(|\zeta| \eta) - \Phi(|\zeta|) \right] \frac{d\eta}{\eta(e^{i\theta} + \eta)} \right| \leq \frac{C|\zeta|^\gamma}{|\zeta|} \int_{1/2}^{2} \frac{|\eta - 1|^\alpha}{|\eta + e^{i\theta}|} d\eta \leq C|\zeta|^{-1},
\]

since:
\[
\frac{|\eta - 1|^\alpha}{|\eta + e^{i\theta}|} = \frac{|\eta - 1|^\alpha}{\sqrt{(\eta + \cos \theta)^2 + \sin^2 \theta}} = \frac{|\eta - 1|^\alpha}{\sqrt{\eta^2 + 2\eta \cos \theta}} 
\leq \frac{1}{|\eta - 1|^{1-\alpha}}.
\]

On the other hand,
\[
\int_{1/2}^{2} \frac{d\eta}{\eta(e^{i\theta} + \eta)} = e^{-i\theta} \int_{1/2}^{2} \frac{1}{\eta} - \frac{1}{\eta + e^{i\theta}} d\eta = e^{-i\theta} \log 4 - \log(2 + e^{i\theta}) + \log\left(\frac{1}{2} + e^{i\theta}\right).
\]

The two remaining terms in (3.30) can be computed by similar arguments. Summarizing:

\[
K'(\zeta) - K'(0) = O(|R(\zeta)|) \quad \text{as} \quad |\zeta| \to 0.
\]

Then
\[
K(\zeta) - K(0) - K'(0)\zeta = O(|\zeta R(\zeta)|) \quad \text{as} \quad |\zeta| \to 0.
\]

As a consequence of the previous results we deduce the following.

**Lemma 3.7.** Assume (1.21)-(1.25) and let \( \Lambda(\zeta) \) be the function defined by (2.13). Then, as \( |\zeta| \to 0 \),

\[
(3.31) \quad \Lambda'(\zeta) = \Lambda'(0) + O(1)|R(\zeta)|
\]

\[
(3.32) \quad \Lambda(\zeta) = \Lambda(0) + \Lambda'(0)\zeta + O(|\zeta R(\zeta)|)
\]

where
\[
\overline{R}(\zeta) = \begin{cases} 
|\zeta|^\alpha & \text{if } 0 < \alpha < 1, \\
|\zeta|\log|\zeta| & \text{if } 1 \leq \alpha < 2.
\end{cases}
\]

**Proof.** By formula (2.13) we have,

\[
\Lambda(z) = \frac{1}{M_{in}} \int_0^\infty \frac{(1 - e^{-k}) g_{in}(k)}{\zeta + (1 - e^{-k})} dk \left[ 1 - \frac{1}{M_{in}} \int_0^\infty \frac{(1 - e^{-k}) g_0(k)}{\zeta + (1 - e^{-k})} dk \right]^{-1}
= K_1(\zeta) \left[ 1 - K_2(\zeta) \right]^{-1}
\]

where
\[
K_1(\zeta) = \int_0^\infty \frac{\varphi_1(k)}{\zeta + 1 - e^{-k}} dk, \quad \varphi_1(k) = (1 - e^{-k}) g_{in}(k),
\]
\[
K_2(\zeta) = \int_0^\infty \frac{\varphi_2(k)}{\zeta + 1 - e^{-k}} dk, \quad \varphi_2(k) = (1 - e^{-k}) g_0(k).
\]

The function \( g_{in} \) satisfies (1.21)-(1.25) with the exponent \( \alpha \in (0, 2] \) and \( g_0 \) satisfies (1.21)-(1.25) with \( \alpha = 1 \).

Applying Lemma 3.6 to the functions \( K_1, K_2 \) we obtain

\[
K_1'(\zeta) = K_1'(0) + O(|R(\zeta)|) \quad \text{as} \quad |\zeta| \to 0.
\]

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\[ K_1(\zeta) = K_1(0) + K_1'(0)\zeta + O(|\zeta R(\zeta)|) \quad \text{as} \quad |\zeta| \to 0, \]
\[ K_2(\zeta) = K_2(0) + O(|\zeta| \log |\zeta|) \quad \text{as} \quad |\zeta| \to 0. \]
\[ K_3(\zeta) = K_3(0) + O(|\zeta|^2 \log |\zeta|) \quad \text{as} \quad |\zeta| \to 0. \]

And, since \( K_2(0) < 1 \):
\[ 1 - K_2(\zeta) = 1 - K_2(0) - \frac{\lambda_2(0)\zeta}{1 - K_2(0)} + O(|\zeta|^2 \log |\zeta|) \quad \text{as} \quad |\zeta| \to 0. \]

Therefore,
\[ \frac{1}{1 - K_2(\zeta)} = \frac{1}{1 - K_2(0)} \left[ 1 + \frac{K_2'(0)\zeta}{1 - K_2(0)} + O(|\zeta|^2 \log |\zeta|) \right] \quad \text{as} \quad |\zeta| \to 0. \]

whence, (3.32) follows. Then (3.31) follows by integration. \( \Box \)

**Lemma 3.8.** Under the same assumptions as in Lemma 3.7,

(3.33) \[ \Lambda'(\zeta - i0) - \Lambda'(0) - (\Lambda'\zeta + i0) - \Lambda'(0) = O(|\overline{\Gamma}(\zeta)|) \quad \text{as} \quad |\zeta| \to 0, \]

(3.34) \[ \Lambda(\zeta - i0) - \Lambda(0) - (\Lambda + i0) - \Lambda(0) = O(|\overline{\Gamma}(\zeta)|) \quad \text{as} \quad |\zeta| \to 0. \]

(3.35) \[ \Lambda'(\zeta - i0) - \Lambda'(0) + (\Lambda'(\zeta + i0) - \Lambda'(0) = O(|\overline{\Gamma}(\zeta)|) \quad \text{as} \quad |\zeta| \to 0. \]

(3.36) \[ \Lambda(\zeta - i0) - \Lambda(0) + (\Lambda(\zeta + i0) - \Lambda(0) = 2\Lambda'(0)\zeta + O(|\overline{\Gamma}(\zeta)|) \quad \text{as} \quad |\zeta| \to 0. \]

In particular the function \( \varphi \) defined in (3.13) satisfies the hypothesis of Lemma 3.4 with \( \gamma = \min(\alpha', 1) \) and \( \alpha' < \alpha \) arbitrarily close to \( \alpha \).

**Proof.** We have
\[ \Lambda(0) = \lim_{\zeta \to 0} \Lambda(\zeta) = \frac{M(g_m)}{M_{in} - N_0}, \quad \text{since} \quad M_{in} > N_0 \]
and
\[ \Lambda'(0) = \lim_{\zeta \to 0} \Lambda'(\zeta) = -\frac{1}{M_{in} - N_0} \int_{0}^{\infty} \frac{g_{in}(k)dk}{1 - e^{-k}} - \frac{M(g_m)}{(M_{in} - N_0)^2} \int_{0}^{\infty} \frac{g_0(k)dk}{1 - e^{-k}}, \]
where we have used the notation,
\[ \Lambda(\zeta \pm i0) = \lim_{\xi \to 0^+} \Lambda(\zeta \pm i\xi). \]

Then (3.33)-(3.36) follow using Lemma 3.7.

Notice also that for any \( \alpha' < \alpha \) and arbitrarily close to \( \alpha \) we have
\[ \Lambda'(\zeta - i0) - \Lambda'(0) - (\Lambda'\zeta + i0) - \Lambda'(0) = O(|\zeta|^\gamma) \quad \text{as} \quad |\zeta| \to 0 \]
\[ \Lambda(\zeta - i0) - \Lambda(0) - (\Lambda(\zeta + i0) - \Lambda(0) = O(|\zeta|^\gamma+1) \quad \text{as} \quad |\zeta| \to 0, \]
with \( \gamma = \min(\alpha', 1) \) and this ends the proof of Corollary 3.11. \( \Box \)

This ends the proof of Proposition 3.3 and then Theorem 2.1 in the case \( M_{in} \geq N_0 \) with \( \rho(0) = 0 \) if \( M_{in} = N_0 \).

We now prove (2.29) in the case \( M_{in} < N_0 \).
Lemma 3.9. Assume (1.21)-(1.25), $M_{in} < N_0$ and $\rho(0) \geq 0$. Then if $\mu$ is such that $M_{in} = M_\mu$ the function $h$ defined by (2.22) is such that (2.29) holds.

Proof of Lemma 3.9. By (3.3) again:

$$h(k,t) = \frac{M_{in}}{2\pi i} \int_\gamma (G_1(k,\zeta) + G_2(k,\zeta)) e^{M_{in} \zeta t} d\zeta \equiv h_1(k,t) + h_2(k,t).$$

The first term in the right hand side of (3.37) may be calculated explicitly:

$$h_1(k,t) = \frac{g_{in}(k)}{2\pi i} \int_\gamma \frac{1}{\zeta + e^{\mu} - e^{-k} e^{M_{in} \zeta t}} d\zeta = g_{in}(k) e^{-M_{in}(e^{\mu} - e^{-k})t}.
$$

As for the second term, we remember that

$$h_2(k,t) = \frac{g_{\nu}(k)}{2\pi i} \int_\gamma \frac{\Lambda(\zeta) e^{M_{in} \zeta t}}{\zeta + e^{\mu} - e^{-k}} d\zeta,
$$

By Proposition 3.1, $\Lambda$ is analytic at $\zeta = 0$. Moreover, we can deform the contour $\gamma$ in (3.38) to be a line included in $Re(\zeta) < -\delta < 0$, with $0 < \delta < e^{\mu} - 1$. Finally, since by (2.19), (2.2) and (2.3), $\Phi(0) = 0$, we deduce

$$|h_2(k,t)| \leq C e^{-M_{in} \delta t}.
$$

We are left finally with the last case, $M_{in} = N_0$ and $\rho(0) > 0$.

Lemma 3.10. Assume (1.21)-(1.25), $M_{in} = N_0$ and $\rho(0) > 0$. Then the function $h$ defined by (2.22) is such that

$$h(k,t) = g_{in}(k) e^{-M_{in}(1 - e^{-k})t} - \frac{3 \rho(0)}{\pi^2} t + O(|k| \log t),
$$

as $t \to \infty$, uniformly on $0 < k < C/|t|.$

Proof. We compute $h(k,t)$ as usual, using (3.3). The function $h_1(k,t)$ is computed using (3.4). The only difference arises in the term $h_2(k,t)$. In order to determine its asymptotics we need to study the function $\Lambda(\zeta)$ as $\zeta \to 0$. In this case, the function $\Lambda$ has a pole at $\zeta = 0$ which coalesces with its region of non-analyticity. Arguing as in the previous computations we obtain the following behaviour for $\Lambda$,

$$\Lambda(\zeta) = -\frac{3 \rho(0)}{\pi^2} \frac{1}{\zeta} + O(|\log |k||), \quad as \ z \to 0,$$

$$\Lambda'(\zeta) = \frac{3 \rho(0)}{\pi^2} \frac{1}{\zeta^2} + O\left(\frac{1}{|k|}\right), \quad as \ z \to 0.$$

We can then estimate $h_2(k,t)$ as follows,

$$h_2(k,t) = -\frac{3 \rho(0)}{\pi^2} \frac{g_{\nu}(k)}{1 - e^{-k}} + \frac{g_{\nu}(k)}{2\pi i} \int_\gamma \frac{R(\zeta) e^{M_{in} \zeta t}}{\zeta + 1 - e^{-k}} d\zeta$$

$$|R(\zeta) = O(|\log |\zeta||), \quad |R'(\zeta)| = O\left(\frac{1}{|k|}\right), \quad as \ z \to 0.$$
Using the Plemej Sojoltsky formula as in the previous case $M_{in} > N_0$, it can easily be computed that,

$$\frac{g_0(k)}{2\pi i} \int_{\gamma} \frac{R(\zeta)e^{M_{in}\zeta t}}{\zeta + 1 - e^{-k}} d\zeta = O(|k| \log t) \quad \text{as } t \to \infty, \text{ uniformly in } 0 < k < C/t,$$

and Lemma 3.10 follows.

Notice that expressions (2.29) and (2.30) actually give the precise behaviour of $h$ near the origin as $t \to \infty$ in the cases $M_{in} < N_0$ or $M_{in} = N_0$ and $\rho(0) > 0$. When $M_{in} \geq N_0$ the behaviour of $h$ is deduced straightforwardly from (2.24)-(2.30). That is made in the following Section.

4. Asymptotics of $g$.

In this Section we obtain the precise asymptotic behaviour of $g(k, t)$, and hence of $F(k, t)$, for $k$ near the origin and $t$ large. Taking into account (2.7) (2.8) the problem reduces basically to analyze the asymptotics of $\lambda(t)$, that is the main goal of this Section. As a first step we summarize the information so far obtained for $h(k, t)$.

**Proposition 4.1.** Assume (1.21)-(1.25), $M_{in} \geq N_0$ and $\rho(0) = 0$ if $M_{in} = N_0$. Then

$$h(k, t) = \begin{cases} [g_{in}(k) + \Lambda(0)g_0(k) - 2\Lambda'(0)kg_0(k)]e^{-M_{in}(1-\epsilon^k)t} + O(k^{\alpha + 2}) & \text{if } 0 < \alpha < 1, \\ [g_{in}(k) + \Lambda(0)g_0(k) - 2\Lambda'(0)kg_0(k)]e^{-M_{in}(1-\epsilon^k)t} + O(k^3 \log k) & \text{if } \alpha \geq 1. \end{cases}$$

as $t \to \infty$ uniformly in $0 < k < C/t$.

**Proof.** By (2.28) in Theorem 2.1:

$$h(k, t) = [g_{in}(k) + \Lambda(0)g_0(k)]e^{-M_{in}(1-\epsilon^k)t} + g_0(k)[A(k)e^{-M_{in}(1-\epsilon^k)t} + B(k, t)] + O(e^{-M_{in}\delta t}).$$

Then, by (2.31),

$$h(k, t) = [g_{in}(k) + \Lambda(0)g_0(k)]e^{-M_{in}(1-\epsilon^k)t} + g_0(k)[(2\Lambda'(0)(-\epsilon^k - 1)e^{-M_{in}(1-\epsilon^k)t} + O(k^2 R(e^{-k-1}) + k^{\gamma + 2})]

= [g_{in}(k) + \Lambda(0)g_0(k) - 2\Lambda'(0)kg_0(k)]e^{-M_{in}(1-\epsilon^k)t} + O(k^2 R(e^{-k-1}) + k^{\gamma + 2})$$

and Proposition 4.1 follows.

In order to determine the behaviour of $g$ from the behaviour of $h$ we need to know the behaviour of the function $\lambda(t)$ (cf. (2.23)).

**Lemma 4.2** Assume that $g_{in}$ satisfies (1.21)-(1.25), $M_{in} \geq N_0$ and $\rho(0) = 0$ if $M_{in} = N_0$. Then the function $\lambda(t)$ defined by (2.21) satisfies:

$$\lambda(t) = \frac{2}{M_{in}^3 S(0)t^3} + \frac{\Phi(0)}{S(0)} \frac{\Gamma(\alpha + 2)}{M_{in}^{\alpha + 2} t^{\alpha + 2}} + O(\frac{1}{t^4} + \frac{1}{t^{\alpha + 3}} + \frac{1}{t^{\beta + 2}}) \quad \text{as } t \to \infty.$$

If $M_{in} = N_0$ and $\rho(0) > 0$,

$$\lambda(t) = \frac{-3\rho(0)}{\pi^2 M_{in}} + O\left(\frac{\log t}{t}\right) \quad \text{as } t \to \infty.$$
If \( M_{in} < N_0 \),

\begin{equation}
\lambda(t) = O(e^{-\delta t}), \quad \text{as } t \to \infty, \quad \text{for any } \delta \in (0, e^\mu - 1).
\end{equation}

**Proof.** Using (2.21) we deform contours as above and obtain:

\begin{equation}
\lambda(t) = \frac{M_{in}}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{M_{in}\zeta t} \Lambda(\zeta) d\zeta
\end{equation}

\begin{equation}
= \frac{M_{in}}{2\pi i} \int_{-\delta}^{0} [(\Lambda(\zeta - i0) - \Lambda(0)) - (\Lambda(\zeta + i0) - \Lambda(0))] e^{M_{in}\zeta t} d\zeta + O(e^{-M_{in}\delta t})
\end{equation}

\begin{equation}
= -\frac{1}{S(0)} \int_{-\delta}^{0} \frac{\zeta (1 + O(|\zeta|))}{(1 + \zeta)} e^{M_{in}\zeta t} [g_{in}(-\log(1 + \zeta)) + \frac{\Phi(0)}{S(0)} g_{in}(-\log(1 + \zeta))] d\zeta + O(e^{-M_{in}\delta t})
\end{equation}

\begin{equation}
= J_1 + J_2 + O(e^{-M_{in}\delta t}) \quad \text{as } t \to \infty.
\end{equation}

We compute now the two terms \( J_1 \) and \( J_2 \). Let us start with \( J_1 \).

\begin{equation}
J_1 = -\frac{1}{S(0)} \int_{-\delta}^{0} \frac{\zeta (1 + O(|\zeta|))}{(1 + \zeta)} e^{M_{in}\zeta t} g_{in}(-\log(1 + \zeta)) d\zeta
\end{equation}

\begin{equation}
= \frac{1}{S(0)} \int_{0}^{\delta} z^2(1 + O(z))(1 - z) e^{-M_{in}zt} dz
\end{equation}

\begin{equation}
= \frac{1}{M_{in}^3 S(0) t^3} \int_{0}^{M_{in} t^3} y^2 e^{-y} dy + O\left(\frac{\pi i}{M_{in}^3 S(0) t^4} \int_{0}^{M_{in} t^3} y^2 e^{-y} dy\right)
\end{equation}

\begin{equation}
= \frac{2}{M_{in}^3 S(0) t^3} + O\left(\frac{1}{t^4}\right) \quad \text{as } t \to \infty.
\end{equation}

As for \( J_2 \), observe first of all that by (2.24),

\begin{equation}
g_{in}(-\log(1 - z)) = A(-\log(1 - z))^\alpha + O((-\log(1 - z))^{\beta})
\end{equation}

\begin{equation}
= A z^\alpha + O(z^{\alpha + 1} + z^{\beta}) \quad \text{as } z \to 0
\end{equation}

Then,

\begin{equation}
J_2 = -\frac{\Phi(0)}{S(0)^2} \int_{-\delta}^{0} \frac{\zeta (1 + O(|\zeta|))}{(1 + \zeta)} e^{M_{in}\zeta t} g_{in}(-\log(1 + \zeta)) d\zeta
\end{equation}

\begin{equation}
= \frac{\Phi(0)}{S(0)^2} \int_{0}^{\delta} (z^\alpha + O(z^{\alpha + 1} + z^{\beta + 1})) e^{-M_{in}zt} dz
\end{equation}

\begin{equation}
= \frac{\Phi(0)}{S(0)^2} \frac{\Gamma(\alpha + 2)}{M_{in}^{\alpha + 2} t^{\alpha + 2}} + O\left(\frac{1}{t^4} + \frac{1}{t^{\alpha + 3}} + \frac{1}{t^{\beta + 2}}\right) \quad \text{as } t \to \infty.
\end{equation}

Formula (4.3) follows from (4.6)-(4.8).

We now study the case \( M_{in} = N_0 \) and \( \rho(0) > 0 \). The interesting feature of that case is the presence of a pole for \( \Lambda \) at \( \zeta = 0 \). That means that the integral \( \int_{a}^{\infty} \lambda(t) dt \) is divergent. Arguing as in Lemma 3.10 it follows that

\begin{equation}
\lambda(t) = -\frac{3\rho(0)}{\pi^2} M_{in} + O\left(\frac{\log t}{t}\right) \quad \text{as } t \to \infty.
\end{equation}

If \( M_{in} < N_0 \), by Proposition 3.1 the function \( \Lambda \) is analytic on \( \mathbb{C} \setminus \{-e^\mu, -e^\mu + 1\} \). Then, (4.5) follows as in the proof of Proposition 3.4. \( \square \)

Once the behaviour of \( \lambda(t) \) is known, the behaviour of \( \varepsilon(t) \) (cf. (2.15), (2.16)) follows.

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Corollary 4.4. Assume (1.21)-(1.25) and $M_{in} > N_0$.

(i) If $\rho(0) \neq 0$, then:

$$
\varepsilon(t) = \left( \frac{\rho(0)}{M_{in} - M_0} + O\left(\frac{1}{t^{\gamma+1}}\right) \right) \lambda(t) \quad \text{as} \quad t \to \infty.
$$

(ii) If $\rho(0) = 0$, then (assuming without any loss of generality that when $0 < \alpha \leq 1$, $\alpha < \beta < 2$)

$$
\varepsilon(t) = \begin{cases} 
2 + \frac{2(\alpha - 1)\Phi(0)}{M_{in}^{\alpha-1}S(0)\Gamma(3)t^\alpha} + O\left(\frac{1}{t^{2\alpha-1}} + \frac{1}{t^2}\right) & \text{if } \alpha > 1 \\
\frac{2}{t} + O\left(\frac{1}{t^{\beta}}\right) & \text{if } \alpha = 1 \\
\frac{\alpha + 1}{t} + \frac{(2 - \alpha)S(0)}{\Phi(0)M_{in}^{\alpha-\alpha}T(\alpha + 1)t^{2-\alpha}} + O\left(\frac{1}{t^{3-2\alpha}} + \frac{1}{t^2}\right) & \text{if } \alpha < 1,
\end{cases}
$$

as $t \to \infty$.

If $M_{in} = N_0$ and $\rho(0) = 0$, then $\Lambda(0) \neq 1$ and

$$
\varepsilon(t) = \left( \frac{-3}{\pi^2} \int_0^\infty \frac{e^{-k}g_{in}(k)}{1 - e^{-k}} dk + O\left(\frac{1}{t^{\gamma+1}}\right) \right) \lambda(t) \quad \text{as} \quad t \to \infty.
$$

If $M_{in} < N_0$

Proof. Suppose first that $M_{in} > N_0$. By (2.15) and (2.16), when $\rho(0) = 0$ we have

$$
\varepsilon(t) = \lambda(t)\left\{ \int_t^\infty \lambda(s)ds \right\}^{-1}
$$

and formula (4.10) follows then from (4.2) by a straightforward and simple calculation. If $\rho(0) \neq 0$, formula (4.11) follows from (2.15), (2.16) and (4.2).

Consider now the case $M_{in} = N_0$. By (2.15) we deduce that $1 - \Lambda(0) > 0$. Notice indeed that if $M_{in} = N_0$,

$$
1 - \Lambda(0) = 1 + 4\int_0^\infty \frac{g_{in}(k)dk}{1 - e^{-k}} \int_0^\infty \frac{k^2dk}{\sinh^2\left(\frac{k}{2}\right)} \left[ \int_0^\infty \frac{k^2dk}{1 - e^{-k}} \right]^{-1} = 4\int_0^\infty \frac{F_{in}(k)dk}{1 - e^{-k}} \left[ \int_0^\infty \frac{k^2dk}{\sinh^2\left(\frac{k}{2}\right)} \right]^{-1} > 0
$$

since $F_{in} \geq 0$. Then, formula (4.11) also follows from (2.15), (2.16) and (4.2).

Notice that the same argument as above also shows that $1 - \Lambda(0) > 0$ when $M_{in} = M[g_0] < N_0$ for some $\mu > 0$.

Finally we have the asymptotic behaviour of the function $g$ for $k \to 0$, $t \to \infty$ in bounded sets of $kt$. This is done in the Theorem C already stated in the introduction. Let us state it again, and give in particular the precise values of the constants $C_1, \ldots, C_5$ which appear in the statement of Theorem C.

Theorem 4.5. Assume (1.21)-(1.25). Then, as $t \to \infty$ and, in regions $0 < kt \leq C$ one has:

1. If $M_{in} > N_0$ and $\rho(0) \neq 0$, then

$$
g(k, t) = \begin{cases} 
A(M_{in} - N_0)\frac{k^\alpha e^{-M_{in}^{1-e^k}t}}{\rho(0)} + O(k\beta + \frac{k}{t^{\alpha+1}}) & \text{if } 0 < \alpha < 1, \\
(A + 1)(M_{in} - N_0)\frac{ke^{-M_{in}^{1-e^k}t}}{\rho(0)} + O(k\beta + \frac{k}{t^2}) & \text{if } \alpha = 1, \\
M_{in} - N_0 - \rho(0)\frac{k^\alpha e^{-M_{in}^{1-e^k}t}}{\rho(0)} + O(k\beta + \frac{k}{t^2} + k^\alpha) & \text{if } \alpha > 1.
\end{cases}
$$

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(2) If $M_\infty > N_0$ and $\rho(0) = 0$,

\[
g(k, t) = \begin{cases} 
(M_\infty - N_0)M_\infty^2 \frac{k^2t e^{-M_\infty(1-e^{-k})t}}{2} + O(k^\alpha) & \text{if } \alpha > 1, \\
(A + 1) \frac{M_\infty^2 (M_\infty - N_0)}{2} \frac{k^2t e^{-M_\infty(1-e^{-k})t}}{2} + O(k^\beta + kt + kt^2) & \text{if } \alpha = 1, \\
\frac{M_\infty^{\alpha+1} (M_\infty - N_0)}{\Gamma(\alpha + 1)} A k^{\alpha+1} e^{-M_\infty(1-e^{-k})t} + O(k^\beta t^{\alpha+1} + k^\alpha t^{2\alpha}) & \text{if } 0 < \alpha < 1.
\end{cases}
\]

(3) If $M_\infty = N_0$ and $\rho(0) = 0$,

\[
g(k, t) = \begin{cases} 
\frac{A}{1 - \Lambda(0)} k^{\alpha} e^{-M_\infty(1-e^{-k})t} + O(k^\beta + \frac{k^\alpha}{t^{\alpha+1}}) & \text{if } \alpha < 1, \\
\frac{A + \Lambda(0)}{1 - \Lambda(0)} k e^{-M_\infty(1-e^{-k})t} + O(k^\beta + \frac{k^\alpha}{t^2}) & \text{if } \alpha = 1, \\
\frac{\Lambda(0)}{1 - \Lambda(0)} k e^{-M_\infty(1-e^{-k})t} + O(k^\beta + \frac{k}{t^2}) & \text{if } \alpha > 1,
\end{cases}
\]

with $\Lambda(0) = -\frac{3}{\pi^2} \int_0^\infty g_in(k) (1 - e^{-k})^{-1} dk$.

(4) If $M = N_0$ and $\rho(0) > 0$,

\[
g(k, t) = -\frac{1}{M_\infty} + \frac{\tau^2 A}{3\rho(0)} k^\alpha e^{-M_\infty(1-e^{-k})t} + O\left(\frac{\log^2 t}{t}\right).
\]

(5) If $M_\infty < N_0$, there exists $\delta > 0$ such that

\[
g(k, t) = O(e^{-M_\infty \delta t}).
\]

**Proof.** If $M_\infty > N_0$ and $\rho(0) = 0$, by (2.23):

\[
g(k, t) = \frac{h(k, t)}{\int_t^\infty \lambda(s) ds}.
\]

By direct calculation from (4.2) we obtain:

\[
\int_1^\infty \lambda(s) ds = M_\infty S(0) t^2 \left[1 + \frac{\Phi(0) \Gamma(\alpha + 1)}{M_\infty^\alpha S(0) t^{\alpha-1}} + O\left(\frac{1}{t^{2(\alpha-1)}}\right)\right], \quad \alpha > 1,
\]

\[
\int_1^\infty \lambda(s) ds = M_\infty^{\alpha+2} S(0)^2 t^{\alpha+1} \left[1 - \frac{S(0)}{\Phi(0) \Gamma(\alpha + 1) M_\infty^{\alpha-1} t^{\alpha-1}} + O\left(\frac{1}{t^{2(\alpha-1)}}\right)\right], \quad \alpha < 1,
\]

\[
\int_1^\infty \lambda(s) ds = M_\infty^2 S(0)^2 t^2 \left[1 + O\left(\frac{1}{t^{\beta-1}}\right)\right], \quad \alpha = 1,
\]

from which (4.13) follows, using that

\[
\Lambda(0) = 1, \quad \Phi(0) = \frac{M_\infty}{M_\infty} = \frac{M_\infty - N_0}{M_\infty}, \quad \frac{S(0)}{S(0) + \Phi(0)} = \frac{M_\infty - N_0}{2M_\infty} \quad \text{and} \quad \frac{S(0)^2}{\Phi(0)} = \frac{M_\infty - N_0}{M_\infty}.
\]

If $M_\infty > N_0$ and $\rho(0) \neq 0$ or if $M_\infty = N_0$ and $\rho(0) = 0$, by (2.23):

\[
g(k, t) = \frac{h(k, t)}{1 - \int_0^t \lambda(s) ds}.
\]
Moreover,
\[
\int_0^t \lambda(s)ds = \int_0^\infty \lambda(s)ds - \int_t^\infty \lambda(s)ds = \Lambda(0) - \int_t^\infty \lambda(s)ds
\]
\[
= \Lambda(0) + O\left(\frac{1}{t^{\gamma+1}}\right) \quad \text{as } t \to \infty.
\]
Since, as we have seen in the proof of (4.11), \(\Lambda(0) \neq 1\), we deduce
\[
g(k,t) = \frac{h(k,t)}{1 - \Lambda(0)}\left(1 + O\left(\frac{1}{t^{\gamma+1}}\right)\right)
\]
from where (4.12) follows using
\[
\Lambda(0) = \frac{M_{in} - N_0 - \rho(0)}{M_{in} - N_0},
\]
and (4.14) follows using (2.15) to obtain \(\Lambda(0)\).

If \(M_{in} = N_0\) and \(\rho(0) > 0\), by (4.5) and (2.14):
\[
e^{-\int_0^t \epsilon(s)ds} = \frac{3\rho(0)}{\pi^2} M_{in} t + O(\log^2 t) \quad \text{as } t \to \infty,
\]
from where (4.15) follows.

When \(M_{in} < N_0\), we know by (2.33) that if \(\mu\) is such that \(M_{in} = M[g_\mu]\), then:
\[
h(k,t) = g_{in}(k)e^{-M_{in}(e^\mu - e^{-\mu})t} + O(e^{-M_{in}t}) \quad \text{as } t \to \infty
\]
with \(0 < \delta < e^\mu - 1\). Moreover, by the remark at the end of the proof of Corollary 4.4, \(1 - \Lambda(0) > 0\) and then, by (2.23),
\[
g(k,t) = \frac{h(k,t)}{1 - \Lambda(0)}\left(1 + O\left(\frac{1}{t^{\gamma+1}}\right)\right).
\]
In the region \(0 < kt \leq C\) we the have:
\[
g(k,t) = O(e^{-M_{in}t\delta}) \quad \text{as } t \to \infty.
\]

It is now possible to study in detail the behaviour of the condensate as \(t \to 0\) and its relations with the regular part of the solution as time grows.

**Proof of Theorem D.** If \(M_{in} > N_0\) and \(\rho(0) > 0\). By the second equation in (2.6) we have:
\[
\rho(t) = \rho(0)e^{\int_0^t \epsilon(s)ds},
\]
and therefore by (2.15):
\[
\lim_{t \to \infty} \rho(\rho t) = \rho(0)e^{\int_0^\infty \epsilon(s)ds} = M_{in} - N_0.
\]

If \(M_{in} < N_0\) and \(\rho(0) > 0\), by (2.7) and (2.19)
\[
\rho(t) = \rho(0)e^{-M_{in}(e^\mu - 1)t}e^{\int_0^t \epsilon(s)ds}.
\]

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On the other hand,
\[ e^\int_0^t \varepsilon(s)ds = (1 - \Lambda(0))^{-1} e^{-\int_0^\infty \varepsilon(s)ds} = (1 - \Lambda(0))^{-1} + O(\int_t^\infty \varepsilon(s)ds) \quad \text{as } t \to \infty. \]

By Lemma 4.2, and (2.16), \( \varepsilon(t) = O(e^{-\delta t}) \) as \( t \to \infty \) for any \( \delta > 0 \) such that \( \delta < e^\mu - 1 \) and (1.27) follows.

Assume now \( M_{in} = N_0 \) and \( \rho(0) > 0 \). By (2.16),
\[ \rho(t) = \rho(0)e^\int_0^t \varepsilon(s)ds. \]
By (2.14) and (4.5),
\[ e^{-\int_0^t \varepsilon(s)ds} = \frac{3\rho(0)}{\pi^2} M_{in} t + O(\log^2 t) \]
and (1.26) follows.

**Remark 4.6.** When \( M_{in} > N_0 \), the presence of condensate at the initial stage has also a stabilizing effect in the following sense. By the definitions (2.2) and (2.5):
\[ F = g + g_0 + \rho_0, \quad \text{and} \quad \varepsilon(t) = \int_0^\infty (1 - e^{-x^2}) g'(dk). \]
The function \( \varepsilon(t) \) is then a measure of the distance from the regular part of the solution \( F \) to the regular part of the Bose equilibrium \( B_{M_{in}} \). It is interesting to point out the different rate of convergence to zero of that function, depending again on the presence or not of condensate at the initial stage. This can be explicitly seen in the expressions (4.9) and (4.10) of the Corollary 4.4. But more generally, we may do the following. If \( \rho(0) \neq 0 \), we deduce by (2.14) that
\[ \int_0^\infty \varepsilon(t) dt = -\ln \left( \frac{\rho(0)}{M_{in} - N_0} \right) = \ln \left( 1 + \frac{M_{in} - \rho(0) - N_0}{\rho(0)} \right) < +\infty. \]
Therefore
\[ \int_0^\infty \lambda(s)ds = e^{-\int_0^\infty \varepsilon(s)ds} - \frac{\rho(0)}{M_{in} - M_0} < \infty. \]
We deduce that
\[ \int_t^\infty \lambda(s)ds \to 0 \quad \text{as} \quad t \to \infty. \]
Since:
\[ \int_0^t \lambda(s)ds < \int_0^\infty \lambda(s)ds = 1 - e^{-\int_0^\infty \varepsilon(s)ds} < 1, \]
we deduce, by (2.15), that
\[ \varepsilon(t) \sim \lambda(t) \quad \text{as} \quad t \to \infty, \]
and in particular
\[ \int_0^\infty \varepsilon(s)ds < \infty. \]
But on the other hand, if \( \rho(0) = 0 \), then by (2.14),
\[ e^{-\int_0^\infty \varepsilon(s)ds} = 0 \]
and so
\[ \int_0^\infty \varepsilon(t) dt = +\infty. \]
In other words, the rate of convergence as $t \to \infty$ of the regular part of the solution $F$ to $g_0$ is faster (it stabilises faster) in presence of condensate.

**Appendix.** In this Appendix we show by means of formal asymptotic analysis that the results obtained in the previous set of pages for the case $b$ constant actually hold for more general kernels $b(k, k')$. Some of the results of this Appendix are reminiscent from [LY1], [LY2] and provide a more precise description for the asymptotics of the solutions than those provided by some of the computations therein.

We consider kernels $b$ approaching a nonzero constant as $(k', k) \to (0, 0)$. Without loss of generality we can therefore assume that

$$b(0, 0) = 1.$$  

For the sake of brevity we only consider the cases $M_{in} > N_0$ and $M_{in} < N_0$ but not the limit case $M_{in} = N_0$. Finally, we shall impose on the kernel $b$ the following conditions:

(i) In the case $M_{in} > N_0$:

there exists $0 < C_1 < C_2 < \infty$ such that:

$$C_1 \int_0^\infty g_0(k') b(k, k') dk' \leq b(k, 0) \leq C_2 \int_0^\infty g_0(k') b(k, k') dk'.$$

(ii) In the case $M_{in} < N_0$:

$$b(k, k') = b(0, k') + \xi(k, k') \quad \text{with} \quad ||\xi||_\infty \text{ small}.$$

We notice that (A.2) is much more stringent than (A.1). However, it provides a general class of functions for which the type of asymptotics previously derived can be obtained.

**A.1 The case with formation of Dirac mass.** We begin with the case $M_{in} > N_0$. Assume that $F$ solves (1.6). We write

$$F = g_0 + g.$$  

Then $g$ solves:

$$\frac{\partial g}{\partial t} = \varepsilon(k, t) g_0 + g(e^{-k} - 1) \int_0^\infty b(k, k')(g_0' + g') dk' + \varepsilon(k, t) g,$$

where,

$$\varepsilon(k, t) = \int_0^\infty b(k, k') g'(1 - e^{-k'}) dk'.$$

Function $g$ behaves in a very different manner in two scales of $k$. It turns out that for $k \sim 1$, $g$ varies in a smooth manner and it contains a small amount of mass. But on the other hand, in the region $k \to 0^+$, $g$ contains a large amount of mass. This suggest to decompose the function $g$ in two pieces

$$g = g_{\text{sing}}(k, t) + g_{\text{reg}}(k, t)$$

where $g_{\text{sing}}$ provides the concentration of mass in the region $k \to 0^+$, and $g_{\text{reg}}$ contains a small amount of mass and is uniformly distributed in the region $k > 0$.

Since $g_{\text{reg}}$ does contain a small amount of mass it is natural to neglect it to the leading order in the integral term in which $g_0$ also appears in the right hand side of (A.3). We then approximate (A.3) as:

$$\frac{\partial g}{\partial t} = \varepsilon(k, t) g_0 + g(e^{-k} - 1) \int_0^\infty b(k, k')(g_0' + g'_{\text{sing}}) dk' + \varepsilon(k, t) g.$$
In the outer region, \( k \sim 1 \), we neglect \( g \) compared to \( g_0 \). Moreover, assuming by analogy with the case \( b \equiv 1 \), that \( g_{\text{reg}} \) behaves algebraically in time as \( t \to \infty \) (something that will be checked “a posteriori”), we could neglect the term \( \partial g_{\text{reg}}/\partial t \). With these two approximations, it would follow from (A.6) that

\[
(A.7) \quad g_{\text{reg}}(k, t) \sim \frac{\varepsilon(k, t)g_0(k)}{(1 - e^{-k}) \int_0^\infty [g_0(k') + g_{\text{sing}}(k')] b(k, k')dk'}
\]

On the other hand, we can use the approximation \( g_{\text{sing}}(k, t) \sim (M_{in} - N_0) \delta(k) \) in (A.7). Then

\[
(A.8) \quad g_{\text{reg}}(k, t) \sim \frac{\varepsilon(k, t)g_0(k)}{(1 - e^{-k}) \int_0^\infty g_0(k') b(k, k')dk' + (M_{in} - N_0) b(k, 0)}
\]

In order to complete the description of the asymptotics of \( g(k, t) \) we need to precise the function \( \varepsilon(k, t) \) in (A.8). At a first glance one could expect to obtain the leading order behaviour of \( \varepsilon(k, t) \) approximating \( g \) by \( g_{\text{sing}} \sim (M_{in} - N_0) \delta(k) \) in (A.4), since the mass concentrated concentrated in \( g_{\text{reg}} \) is much smaller that the one in \( g_{\text{sing}} \). Notice however that the term \( (1 - e^{-k}) \) in (A.4) vanishes at \( k = 0 \). As a consequence the detailed structure of \( g_{\text{sing}}(k, t) \) and also \( g_{\text{reg}}(k, t) \) play a role to determine \( \varepsilon(k, t) \). We then proceed to compute more precisely the asymptotics of \( g_{\text{sing}}(k, t) \) as \( k \to 0^+ \).

We approximate (A.3) as \( k \to 0^+ \) to the leading order by

\[
\frac{\partial g_{\text{sing}}}{\partial t} = \varepsilon(0, t)k - kg_{\text{sing}} \int_0^\infty b(0, k')g_0'(k') + (M_{in} - N_0) \delta(k') dk' + \varepsilon(0, t)g_{\text{sing}}
\]

Define:

\[
h(k, t) = \exp(-\int_0^t \varepsilon(0, s)ds)g_{\text{sing}}(k, t)
\]

and

\[
\Gamma = \int_0^\infty b(0, k')g_0(k') dk' + (M_{in} - N_0), \quad \lambda(t) = \varepsilon(0, t)e^{-\int_0^t \varepsilon(0, s)ds}
\]

we arrive at

\[
(A.10) \quad \frac{\partial h}{\partial t} = \lambda(t)k - \Gamma h,
\]

whose solution is given by:

\[
(A.11) \quad h(k, t) = e^{-\Gamma t}[h_0(k) + k \int_0^t e^{\Gamma s} \lambda(s)ds].
\]

For \( kt = O(1) \) and assuming that \( \lambda(t) \to 0 \) as \( t \to \infty \) (to be checked “a posteriori”), it would follow from (A.11) the approximation

\[
(A.12) \quad h(k, t) = e^{-\Gamma t}[h_0(k) + k \int_0^t \lambda(s)ds].
\]

Notice that we could expect \( \lambda(t) \) to be integrable since,

\[
\int_0^\infty \lambda(t)dt = \int_0^\infty \varepsilon(0, t)e^{-\int_0^t \varepsilon(0, s)ds}dt
\]

\[
= -\int_0^\infty \frac{d}{dt}(e^{-\int_0^t \varepsilon(0, s)ds})dt = (1 - e^{-\int_0^\infty \varepsilon(0, s)ds}) = 1.
\]

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The convergence of the integral appearing in the last formula as well as the integrability of $\lambda(t)$ will be checked later “a posteriori”. Notice that the required integrability took place in the case $b \equiv 1$.

The precise asymptotics for $h(k, t)$ in (A.12) depends in a very sensitive manner on the asymptotics of $h_0(k)$ as $k \to 0^+$, something that could be expected for a hyperbolic problem. In particular if $h_0(k) \ll k$ as $k \to 0^+$, the dynamics of $h(k, t)$ is determined by the last term in the sum of (A.12), and for $h_0(k) \gg k$ as $k \to 0^+$, the term $h_0(k)$ is the dominant one. Notice that this was exactly the situation in the explicitly solvable case $b \equiv 1$. Of course, if $h_0(k)$ competes with $k$ as $k \to 0^+$ all kinds of intermediate cases could take place.

Approximating $g_{\text{sing}}$ by means of (A.9), (A.12) and $g_{\text{reg}}$ using (A.8), we would obtain, from (A.4) the following equation for $\varepsilon(k, t)$

\begin{equation}
\varepsilon(k, t) = \int_0^\infty \frac{b(k, k')\varepsilon(k', t)g_0(k')dk'}{\int_0^\infty g_0(k'')b(k'', k)dk'' + (M_{in} - N_0)b(k', 0)} + \gamma(t)e^{\int_0^t \varepsilon(0, s)ds}b(k, 0).
\end{equation}

\begin{equation}
\gamma(t) = \beta(t) + \frac{2}{1 + 3b^3} \int_0^\infty \lambda(s)ds,
\end{equation}

where $\beta(t)$ is the dominant term in $\gamma(t)$.

If $h_0(k') \ll k'$ as $k' \to 0^+$, $\beta(t) \ll 1/t^3$ and the term $\beta(t)$ is negligible as expected. If on the contrary $h_0(k') \gg k'$, $\beta(t)$ is the dominant term in $\gamma(t)$.

The structure of (A.13) suggests to look for solutions of (A.13) in the form

\begin{equation}
\varepsilon(k, t) = \gamma(t)e^{\int_0^t \varepsilon(0, s)ds}H(k),
\end{equation}

with $H(k)$ solution of

\begin{equation}
H(k) = T[H](k) + b(k, 0)
\end{equation}

\begin{equation}
T[H](k) = \int_0^\infty \frac{b(k, k')g_0(k')H(k')dk'}{\int_0^\infty g_0(k'')b(k'', k')dk'' + (M_{in} - N_0)b(k', 0)}.
\end{equation}

A formal solution of (A.16) can be obtained by means of the classical Neumann series:

\begin{equation}
H(k) = \sum_{n=0}^\infty T^n[b(\cdot, 0)](k).
\end{equation}

It turns out that the series in (A.18) converges under rather general hypothesis on the kernel $b(k, k')$. Suppose, for instance that $b$ satisfies (A.1) and let us briefly indicate how to use it in order to obtain the convergence of (A.18). Assume that

\[ f(k') \leq C \int_0^\infty g_0(k'')b(k', k'')dk''. \]

Then using (A.18), we immediately verify that the following inequality holds

\[ T[f](k) \leq \frac{C}{1 + (M_{in} - N_0)C_1} \int_0^\infty b(k, k')g_0(k')dk' \]

or, by iteration:

\begin{equation}
T^n[f](k) \leq \frac{C}{(1 + (M_{in} - N_0)C_1)^n} \int_0^\infty b(k, k')g_0(k')dk'.
\end{equation}
We deduce now the asymptotics for integrability assumptions which were made above for \( t \to 1 \).

Notice that, for the solution of \((A.20)\), turns out to be:

\[
(A.20) \quad \varepsilon(0, t) = \gamma(t)e^{\int_0^t \varepsilon(0, s)ds} H(0).
\]

Since \( b(0, 0) = 1 \) and \( T^n[b(\cdot, 0)](0) > 0 \) for all \( n \geq 1 \), it follows from \((A.19)\) that \( H(0) > 1 \). Under the assumption \( e^{-\int_0^t \varepsilon(0, s)ds} = 0 \) (that occurs if \( b \equiv 1 \), and will be checked in the general case “a posteriori”) the solution of \((A.20)\) turns out to be:

\[
(A.21) \quad \varepsilon(0, t) = \frac{\gamma(t)}{\int_t^\infty \gamma(s)ds}.
\]

Notice that, for \( h_0(k) \) behaving algebraically as \( k \to 0^+ \), the function \( \gamma(t) \) also behaves algebraically as \( t \to \infty \) (cf. \((A.14)\)) and therefore, by \((A.21)\) \( \varepsilon(0, t) \sim 1/t \) as \( t \to \infty \). This implies in particular all the integrability assumptions which where made above for \( \lambda(t) \) and \( \varepsilon(0, t) \) as \( t \to \infty \).

We deduce now the asymptotics for \( g_{\text{sing}} \) and \( g_{\text{reg}} \). Suppose first that \( h_0(k) \ll k \) as \( k \to 0 \). Then, by \((A.14)\), \( \gamma(t) \sim 1/t^3 \) as \( t \to \infty \) and from \((A.21)\),

\[
\int_0^t \varepsilon(0, s)ds = \log \frac{\int_0^t \gamma(s)ds}{t}.
\]

and

\[
e^{\int_0^t \varepsilon(0, s)ds} \sim \left[ \int_0^\infty \gamma(s)ds \right]^{-1} t^2 \quad \text{as} \quad t \to \infty,
\]

from where,

\[
(A.22) \quad g_{\text{sing}}(k, t) \sim \frac{\int_0^\infty \lambda(s)ds}{\int_0^\infty \gamma(s)ds} e^{-\Gamma k t^2} k \quad \text{for} \quad k = 0(1), \ t \to \infty,
\]

\[
(A.23) \quad g_{\text{reg}}(k, t) \sim \frac{\int_0^\infty \gamma(s)ds H(k)}{t(1 - e^{-k}) [\int_0^\infty g_0(k')b(k, k')dk' + (M_n - N_0)b(k, 0)]} \quad \text{for} \quad k = 0(1), \ t \to \infty.
\]

If on the other hand, \( h_0(k) \sim k^\alpha \) with \( \alpha < 1 \) then \( \gamma(t) \sim 1/t^{\alpha+2} \). In that case,

\[
e^{\int_0^t \varepsilon(0, s)ds} \sim \left[ \int_0^\infty \gamma(s)ds \right]^{-1} t^{1+\alpha} \quad \text{as} \quad t \to \infty
\]

and then

\[
(A.24) \quad g_{\text{sing}}(k, t) \sim \frac{\int_0^\infty \lambda(s)ds}{\int_0^\infty \gamma(s)ds} e^{-\Gamma k t^{1+\alpha}} k^\alpha \quad \text{for} \quad k = 0(1), \ t \to \infty,
\]

and

\[
(A.25) \quad g_{\text{reg}}(k, t) \sim \frac{\int_0^\infty \gamma(s)ds H(k)}{t(1 - e^{-k}) [\int_0^\infty g_0(k')b(k, k')dk' + (M_n - N_0)b(k, 0)]} \quad \text{for} \quad k = 0(1), \ t \to \infty.
\]

A.2 The case \( M_{in} < N_0 \). We now describe the asymptotics of \( g(k, t) \) in the absence of condensation at \( k = 0 \), at least for a particular class of kernels \( b(k, k') \neq 1 \). Assume that \( M_{in} < N_0 \), and let us select \( \mu > 0 \)
such that \( M[g] = M_{in} \). We then define \( g = F - g_\mu \). By analogy with the case \( b \equiv 1 \), one could expect \( g \) exponentially small as \( t \to \infty \). We will verify “a posteriori” that this is actually the case in this more general case. We then approximate (1.6) keeping only linear terms on \( g 
:
(A.26) \quad \frac{\partial g}{\partial t} = M(k)g + \Phi[g](k, t)e^\mu g_\mu,

where,
\[
M(k) = Q(k)e^{-k(1 - e^{\mu+k})}, \quad \Phi[g](k, t) = \int_0^\infty b(k, k')g'(1 - e^{-(k'+\mu)})dk',
\]
(A.27) \quad Q(k) = \int_0^\infty b(k, k')g'_\mu dk'.
Notice that equation (A.26) at \( k = 0 \) reduces to
\[
\frac{\partial g}{\partial t} = Q(0)(1 - e^\mu)g(0, t),

and this shows that \( g(0, t) \) decreases exponentially as \( t \to \infty \). For \( k \neq 0 \) the exponential decay for the homogeneous part of the equation is even faster since the function \( Q(k)e^{-k(1 - e^{\mu+k})} \) decreases on \( k \). Then, the behaviour of \( g(k, t) \) for \( k \neq 0 \) is driven by the term \( \Phi[g](k, t)e^\mu g_\mu(k) \) in (A.26). We can then expect \( g \) to be dominated by an exponential factor on \( t \) as \( t \to \infty \). In order to work with functions that behave algebraically in time we eliminate the exponential decay of \( g \) defining a new variable:
\[
(A.28) \quad h(k, t) = e^{-M(0)t}g(k, t).

Hence
\[
(A.29) \quad \frac{\partial h}{\partial t} = [M(k) - M(0)]h + \varepsilon(k, t)e^\mu g_\mu(k),

where from now on we write by convenience \( \varepsilon(k, t) = \Phi[h](k, t) \). As in the case \( M_{in} > N_0 \), the function \( h \) will behave in a different manner for \( k \sim 1 \), and for \( k \to 0^+ \), a large portion of the mass of \( h \) being in the region \( k \to 0^+ \). We begin describing the outer region \( k \sim 1 \). Since \( h \) behaves algebraically in time, we can neglect the term \( \partial h / \partial t \) in (A.38) and approximate \( h \) there as
\[
(A.30) \quad h(k, t) \sim \frac{\varepsilon(k, t)g_\mu(k)}{M(0) - M(k)} \quad \text{as} \quad t \to \infty, \quad k \sim 1.

We now analyse the region \( k \to 0^+ \). Approximating to the leading order (A.39) as \( k \to 0^+ \), we arrive at:
\[
(A.31) \quad \frac{\partial h}{\partial t} = -\Gamma kh + \varepsilon(0, t)k^2,

where \( M'(0) = Q'(0)(1 - e^\mu) - Q(0) \equiv -\Gamma \). By assumption \( M'(0) < 0 \), something that occurs, for instance, if \( b(k, k') \) is close to one in a suitable way.
Solving (A.30) we obtain
\[
(A.32) \quad h(k, t) = h_0(k)e^{-\Gamma kt} + k^2\int_0^t e^{-\Gamma k(t-s)}\varepsilon(0, s)ds.

Arguing as in Section A.1, we can verify that, under general assumptions on \( \varepsilon(0, s) \) (to be verified “a posteriori”), we can approximate \( h(k, t) \) in the region \( k = 0(1/t) \) as:
\[
(A.33) \quad h(k, t) = e^{-\Gamma kt}\{h_0(k) + k^2\int_0^t \varepsilon(0, s)ds\} \quad \text{as} \quad t \to \infty, \quad k \to O(1/t).

32
We can obtain a global approximation for $h$ taking (A.30) for $k \sim 1$ and (A.33) for $k \to 0^+$. Using these approximations as well as (A.27) we then derive the following integral equation for $\varepsilon(k,t)$

(A.34) \[ \Phi[h](k, t) \equiv \varepsilon(k, t) = \gamma(t) b(k, 0) + \int_0^\infty (e^{\mu} - e^{-k'}) \frac{b(k, k') g_\mu(k')}{M(0) - M(k')} \varepsilon(k', t) dk', \]

where

(A.35) \[ \gamma(t) = \beta(t) + (1 - e^{\mu}) \int_0^\infty e^{-\gamma t} k^2 \int_0^t \varepsilon(0, s) ds dk', \]

(A.36) \[ \beta(t) = (1 - e^{-\mu}) \int_0^\infty e^{-\gamma t} b_0(k') dk'. \]

As in the case $M_{in} > N_0$, the precise asymptotics of $\gamma(t)$ depends on the behaviour of $h_0(k)$ as $k \to 0^+$. The structure of the integral equation (A.34) suggests looking for solutions in the form $\varepsilon(k, t) = \gamma(t) H(k)$. Then, $H(k)$ solves:

(A.37) \[ H(k) = \int_0^\infty (e^{\mu} - e^{-k'}) \frac{b(k, k') g_\mu(k')}{M(0) - M(k')} H(k') dk' + b(k, 0). \]

We have not attempted to solve the equation (A.37) in the most general case, but we will be satisfied finding a sufficient condition for its solvability in the case of functions $b(k, k')$ not too far from $b \equiv 1$. In particular, let us assume $b(k, k')$ satisfies (A.2). Then, (A.37) becomes

(A.38) \[ H(k) = \int_0^\infty \frac{(e^{\mu} - e^{-k'}) g_\mu(k') b(k, k') H(k') dk'}{Q(0)(1 - e^{-k'}) + (e^{\mu} - e^{-k'}) \int_0^\infty \xi(k', k'') g_\mu(k'') dk''} + b(k, 0). \]

In particular, using (1.18), (A.2) and after straightforward computations we will arrive at:

(A.39) \[ H(k) = \frac{1}{Q(0)} \int_0^\infty g_0(k') b(0, k') H(k') dk' + \psi(k), \]

where

(A.40) \[ \psi(k) = \frac{1}{Q(0)} \int_0^\infty g_0(k') R(k, k') H(k') dk' + b(k, 0), \]

(A.41) \[ R(k, k') = \xi(k, k') - \frac{b(k, k')(e^{\mu} - e^{-k'}) \int_0^\infty \xi(k', k'') g_\mu(k'') dk''}{Q(0)(1 - e^{-k'}) + (e^{\mu} - e^{-k'}) \int_0^\infty \xi(k', k'') g_\mu(k'') dk''}. \]

If $\xi = 0$, $\psi$ reduces to $b(k, 0) = 1$, and the integral equation (A.39) becomes trivial, having as solution $H(k) = M_{in}/(M_{in} - N_0)$. In the case of $\xi$ small, we solve (A.39)-(A.41) in a perturbative manner. Notice that (A.39) implies $H(k) = \psi(k) + A$, where,

\[ A = A[\psi] = \frac{\int_0^\infty g_0(k') b(0, k') \psi(k') dk'}{Q(0) - \int_0^\infty g_0(k') b(0, k') dk'}. \]

Plugging this expression in (A.40), we arrive to the integral equation:

(A.42) \[ \psi(k) = \frac{1}{Q(0)} \int_0^\infty g_0(k') R(k, k') \psi(k') dk' + \frac{A[\psi]}{Q(0)} \int_0^\infty g_0(k') R(k, k') dk' + b(k, 0) \]
and, assuming $\xi(k,k')$ uniformly small we obtain solvability of (A.42) using Neumann series or a standard contractive fixed point argument. This provides solvability for (A.38) in the case of $b$ not far from 1.

Having $H(k)$ we can now determine $\varepsilon(0,t)$ as in Section A.1. Notice that

\begin{equation}
(A.43) \quad \varepsilon(0,t) = \gamma(t) H(0) = H(0)(1-e^{-t})\{\beta(t) + \frac{2}{t^2} \int_0^t \varepsilon(0,s) ds\}.
\end{equation}

The asymptotics of $\beta(t)$, defined in (A.35), depends on the behaviour of $h_0(k)$ as $k \to 0^+$. If $h_0(k) \ll k^2$ as $k \to 0^+$, then $\beta(t)$ is negligible in (A.43) and $\varepsilon(0,t) \sim A/t^3$ as $t \to \infty$. If on the contrary $h_0(k) \gg k^2$ and we assume that $h_0(k) \sim k^\alpha$ as $k \to 0^+$, with $\alpha < 2$, then, $\beta(t) \sim \Gamma(\alpha-1)(\Gamma t)^{1-\alpha}$ as $t \to \infty$ is then the dominant factor in the right hand side of (A.43) and it would follow that $\varepsilon(0,t) \sim H(0)(1-e^{-t})\beta(t)$ as $t \to \infty$. Notice in particular that $H(0) \neq 0$ for $b$ close to one. Also, $\int_0^\infty \varepsilon(0,s) ds$ exists under general assumptions on $h_0(k)$ (say $h_0(k) = O(k^\alpha)$, $\alpha > 0$ as $k \to 0^+$). We then have,

\begin{equation}
g(k,t) \sim e^{M(0)t} e^{-\Gamma M}\{h_0(k) + k^2 \int_0^\infty \varepsilon(0,s) ds\} \quad \text{as} \quad t \to \infty, \quad k \to 0^+,
\end{equation}

where

\begin{equation}
g(k,t) \sim e^{M(0)t} \frac{\gamma(t) H(k) g_k(k)}{M(0) - M(k)}, \quad \text{as} \quad t \to \infty, \quad k \sim 1,
\end{equation}

and

\begin{equation}
\gamma(t) \sim \begin{cases} \frac{1}{t^\alpha} & \text{if} \quad h_0(k) \sim k^\alpha, \quad \alpha \geq 2 \quad \text{as} \quad k \to 0^+ \\ \frac{1}{t^{1+\alpha}} & \text{if} \quad h_0(k) \sim k^\alpha, \quad \alpha < 2 \quad \text{as} \quad k \to 0^+ 
\end{cases}
\end{equation}

$M(0)$ and $M(k)$ defined in (A.37), and $\Gamma = -M'(0) = Q'(0)(e^\mu - 1) + Q(0)$.

As we indicated before, the analysis made in Section A.2 assumes severe restrictions on $b(k,k')$. It seems likely that the asymptotics could be driven not by the region $k \to 0^+$ but by some region $k \sim k_0 > 0$ if $M(k)$ is not decreasing. We will not pursue here a more detailed study of all the possible dynamics that could arise in the description of the long time behaviour of the solutions of this problem.

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**References**


