HOMOGENEOUS BOLTZMANN EQUATION FOR QUANTUM AND RELATIVISTIC PARTICLES

M. ESCOBEDO
S. MISCHLER
M.A. VALLE

DMA - 01 - 27
HOMOGENEOUS BOLTZMANN EQUATION FOR QUANTUM AND RELATIVISTIC PARTICLES

M. ESCOBEDO*
S. MISCHLER**
M.A. VALLE***

DMA - 01 - 27
November 2001

Département de mathématiques et applications - École normale supérieure
45 rue d’Ulm 75230 PARIS Cedex 05
Tel : (33)(1) 01 44 32 30 00
E-mail : preprints@dma.ens.fr

* Departamento de Matemáticas, Universidad del País Vasco
E-mail : mtpesmam@lg.ehu.es

** Département de Mathématiques et Applications, Ecole Normale Supérieure
E-mail : mischler@dma.ens.fr

*** Departamento de Física teórica e Historia de la Ciencia, Universidad del País Vasco
I. INTRODUCTION.

It is known that when quantum methods are applied to molecular encounters some divergence from the classical results is found. It is then necessary in some cases to modify the classical theory in order to take into account the quantum effects in the collisions (see [CC], §17 where the conditions under which the quantum effects are important are discussed in detail). In spite of their great formal similarity, the equations for quantum and non quantum particles display very different features. Surprisingly, the equations for quantum particles have not been considered so much in the mathematical literature.

We consider in this work some mathematical questions about Boltzmann equations for quantum particles, relativistic or not. Our first motivation is the study of different models involving Boltzmann equations which appear to be related to Bose-Einstein condensation in the more or less recent physical literature. Our purpose is, not only to give some new mathematical results on homogeneous Boltzmann equations but also to present and introduce some interesting open questions. We consider to this end some relevant particular cases (Bose, Bose-Fermi, photon-electron gases), simplifications (spherical symmetry) and asymptotic limits (systems where one of the species is at equilibrium) which are important from a physical point of view and give rise to interesting mathematical questions, some of which we have solved. It turns out that some of the most interesting open questions is to understand the sense of the different quantities appearing in the study of the equations as well as the simplifications and asymptotic limits. Since quantum and classic or relativistic particles are involved, we are lead to consider such a general type of equations.

Before coming to such a complex situation, we first consider the homogeneous Boltzmann equation for a gas constituted by one single species of quantum particles. We solve the maximisation entropy problem under the momentum constraints in the general quantum relativistic case. The question of well posedness, i.e. existence, uniqueness, stability of solutions and the question of long time behavior of the solutions is also treated in some particular relevant cases. One could also consider other qualitative properties such as regularity, positivity, eternal solution in a purely kinetic perspective or study the link between the Boltzmann equation and other level of description of the gas: quantum field theory deduction, particles description (BBGKY asymptotic) or fluid description (hydrodynamical limit), but we do not go further in these directions.

I.1 The Boltzmann equations.

To begin with, we focus our attention on a gas composed of identical and indiscernible particles. When two particles with respective impulsion $p$ and $p_*$ in $\mathbb{R}^3$ encounter each other, they collide and we denote $p'$ and $p'_*$ their new impulsions after the collision. We assume that the collision is elastic, which means that the
total impulse and the total energy of the system constituted by this pairs of particles are conserved. More precisely, denoting by $\mathcal{E}(p)$ the energy of one particle with impulse $p$, we assume that

\[
\begin{cases}
    p' + p'_* = p + p_* \\
    \mathcal{E}(p') + \mathcal{E}(p'_*) = \mathcal{E}(p) + \mathcal{E}(p_*).
\end{cases}
\]

We note $\mathcal{C}$ the set (or manifold) of all 4-upplets of particles $(p, p_*, p', p'_*) \in \mathbb{R}^4$ which are compatible by elastic collision, i.e. satisfying equations (1.1). The expression of the energy $\mathcal{E}(p)$ of a particle in function of its impulse $p$ depends on the type of the particle:

\[
\begin{cases}
    \mathcal{E}(p) = \mathcal{E}_{nr}(p) = \frac{|p|^2}{2m} & \text{for a non relativistic particle}, \\
    \mathcal{E}(p) = \mathcal{E}_r(p) = \gamma mc^2; \gamma = \sqrt{1 + \frac{|p|^2}{c^2m^2}} & \text{for a relativistic particle}, \\
    \mathcal{E}(p) = \mathcal{E}_{ph}(p) = c|p| & \text{for massless particle such as a photon or neutrino}.
\end{cases}
\]

Here, $m$ stands for the mass of the particle and $c$ for the velocity of light. The velocity $v = v(p)$ of a particle with impulse $p$ is defined by $v(p) = \nabla_p \mathcal{E}(p)$, and therefore

\[
\begin{cases}
    v(p) = v_{nr}(p) = \frac{p}{m} & \text{for a non relativistic particle}, \\
    v(p) = v_r(p) = \frac{p}{m\gamma} & \text{for a relativistic particle}, \\
    v(p) = v_{ph}(p) = \frac{c}{|p|} & \text{for a photon}.
\end{cases}
\]

Now we consider a gas constituted by a very large number (of order the Avogadro number $A \sim 10^{23}$/mol) of a single specie of identical and indiscernible particles. The very large number of particles make impossible (or irrelevant) the description of the gas by the knowledge of the position and impulse $(x, p)$ (with $x$ in a domain $\Omega \subset \mathbb{R}^3$ and $p \in \mathbb{R}^3$) of all the particles of the gas. We introduce the gas density distribution $f = f(t, x, p) \geq 0$ of particles which at time $t \geq 0$ have position $x \in \mathbb{R}^3$ and impulse $p \in \mathbb{R}^3$. Under the hypothesis of molecular chaos and of low density of the gas, so that particles collide by pairs (no collision between three or more particles occurs), L. Boltzmann [B] established that the evolution of a classic (i.e. no quantum nor relativistic) gas density $f$ must satisfy

\[
\begin{cases}
    \frac{\partial f}{\partial t} + v(p) \cdot \nabla_x f = Q(f(t, x, .))(p) \\
    f(0, .) = f_{in},
\end{cases}
\]

where $f_{in} \geq 0$ is the initial gas distribution and $Q(f)$ is the so-called Boltzmann collision kernel which describe how particles change their impulse due to the collisions.

A similar equation was proposed by L. W. Nordheim [N] in 1928 and by E.A. Uehling & G.E. Uhlenbeck [UU] in 1933 for the description of a quantum gas, where only the collision term $Q(f)$ had to be changed to take into account the quantum degeneracy of the particles.

The relativistic generalisation of the Boltzmann equation including the effects of collisions was given by A. Lichnerowicz & R. Marrot [LM] in 1940.

In all the following we make the assumption that the density $f$ only depends of the impulse. The collision term $Q(f)$ may then be expressed in all the cases described above as:

\[
\begin{cases}
    Q(f)(p) = \int \int_{\mathbb{R}^3} W(p, p_*, p', p'_*) q(f) \, dp_* \, dp' \, dp'_* \\
    q(f) \equiv q(f)(p, p_*, p', p'_*) = [f' f'_*(1 + \tau f)(1 + \tau f_*) - f f_*(1 + \tau f')(1 + \tau f'_*)] \\
    \tau \in \{-1,0,1\}.
\end{cases}
\]
where as usual, we denote:
\[ f = f(p), \quad f_\ast = f(p_\ast) \quad f' = f(p') \quad f'_\ast = f(p'_\ast), \]
and \( W \) is a non negative measure called transition rate, which may be written in general as:
\[ W(p, p_\ast, p', p'_\ast) = w(p, p_\ast, p', p'_\ast)\delta(p + p_\ast - p' - p'_\ast)\delta(E(p) + E(p_\ast) - E(p') - E(p'_\ast)) \]
where \( \delta \) represents the Dirac measure. The quantity \( W dp'dp'_\ast \) is the transition probability per unit volume and per unit time that two particles with incoming momenta \( p, p_\ast \) are scattered with outgoing momenta \( p', p'_\ast \).

The character relativistic or not, of the particles is taken into account in the expression of the energy of the particle \( E(p) \) given by (1.2). The quantum effect appears in the expression of the term \( q(f) \). The case of classical particles corresponds to the choice \( \tau = 0 \). The quantum effect is taken into account by choosing \( \tau = \pm 1 \). It actually corresponds to the values \( \tau = \pm \hbar \), where \( \hbar \) is the Planck constant. But for the sake of simplicity we have chosen this constant to be one all along this work. Therefore, one takes \( \tau = 1 \) in the case of a gas of Bosons and \( \tau = -1 \) for a gas of Fermions.

The function \( w \) is strongly related with the differential cross section \( \sigma \) (see (5.11)) and is determined by the kind of interaction considered between the particles. To determine the properties of this function \( \sigma \) in terms of the interaction potential is one of the classical problems in mechanics: two body problem in classical mechanics, scattering theory in quantum mechanics. In the context of high energy, the computation of the differential cross section is based on quantum field theory. We give some explicit examples in the Appendix 3 but let us only mention here the case
\[ w = 1 \]
which corresponds to a non relativistic short ranged interaction (see Appendix 3). Since the particles are indistinguishable, the collisions are reversible and the two interacting particles form a closed physical system, we have:
\[
\begin{align*}
W(p, p_\ast, p', p'_\ast) &= W(p_\ast, p, p', p'_\ast) = W(p', p'_\ast, p, p_\ast) \\
&+ \text{Galilean invariance (in the non relativistic case)} \\
&+ \text{Lorentz invariance (in the relativistic case)}. 
\end{align*}
\]

The question (mathematical) of what is the sense of the expression (1.5) under general assumptions on the distribution \( f \) is not a simple one in general. Let us just remark that \( Q(f) \) is well defined as a measure when \( f \) and \( w \) are assumed to be continuous. But we will see below that this is not always a reasonable assumption. It is in part one of the purposes of this work to clarify this question.

The Boltzmann equation reads then very similar, formally at least, in all the different contexts: classic, quantum and relativistic. In particular some of the fundamental physically relevant properties of the solutions \( f \) may be established formally in all the cases in the same way: conservation of the total number of particles, mean impulse and total energy; existence of an “entropy function” which increases along the trajectory (Boltzmann’s H-Theorem). For any \( \psi = \psi(p) \), symmetries (1.7), imply the fundamental and elementary identity
\[
\int_{\mathbb{R}^3} Q(f) \psi \, dp = \frac{1}{4} \int_{\mathbb{R}^1} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} W(p, p_\ast, p', p'_\ast) \, q(f) \left( \psi + \psi_\ast - \psi' - \psi'_\ast \right) \, dp dp_\ast dp' dp'_\ast.
\]

First, taking \( \psi(p) = 1, p_\ast \) and \( E(p) \) one get thanks to the definition of \( C \) that particle number, momentum and energy of a solution \( f \) of Boltzmann equation (1.5) are conserved along the evolution, i.e.
\[
\frac{d}{dt} \int_{\mathbb{R}^3} f(t, p) \left( \begin{array}{c} 1 \\ p \\ E(p) \end{array} \right) \, dp = \int_{\mathbb{R}^3} Q(f) \left( \begin{array}{c} 1 \\ p \\ E(p) \end{array} \right) \, dp = 0.
\]
so that

\begin{equation}
\int_{\mathbb{R}^3} f(t,p) \begin{pmatrix} 1 \\ \frac{p}{\mathcal{E}(p)} \end{pmatrix} dp = \int_{\mathbb{R}^3} f_{\text{in}}(p) \begin{pmatrix} 1 \\ \frac{p}{\mathcal{E}(p)} \end{pmatrix} dp.
\end{equation}

The entropy functional is defined by

\begin{equation}
H(f) := \int_{\mathbb{R}^3} h(f(p)) \, dp, \quad h(f) = \tau^{-1} (1 + \tau f) \ln (1 + \tau f) - f \ln f.
\end{equation}

Taking in (1.8) \( \psi = h'(f) = \ln (1 + \tau f) - \ln f \), we get

\begin{equation}
\int_{\mathbb{R}^3} Q(f) h'(f) \, dp = \frac{1}{4} D(f)
\end{equation}

with

\begin{equation}
D(f) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} W \, e(f) \, dp \, dp \, dp \, d\rho
\end{equation}

\begin{align}
e(f) &= j(f, f_*) (1 + \tau f_*) (1 + \tau f'_*) (1 + \tau f) (1 + \tau f) \\
j(s,t) &= (t-s) (\ln t - \ln s) \geq 0.
\end{align}

We deduce the \( H \)-Theorem: the entropy is increasing along trajectories, i.e.

\begin{equation}
\frac{d}{dt} H(f(t,.)) = \frac{1}{4} D(f) \geq 0.
\end{equation}

The main qualitative characteristics of \( f \) are described by these two properties: conservation (1.9) and decreasing entropy (1.14). It is therefore natural to expect that as \( t \) tends to \( \infty \) the function \( f \) converges to a function \( f_{\infty} \) which realizes the maximum of entropy \( H(f) \) under the momentum constraint (1.10). A first simple and heuristic remark is that if \( f_{\infty} \) solves the maximum entropy problem with constraints (1.10), there exist Lagrange multipliers \( \mu \in \mathbb{R}, \beta^0 \in \mathbb{R} \) and \( \beta \in \mathbb{R}^3 \) such that

\begin{align}
< \nabla H(f_{\infty}), \varphi > &= \int_{\mathbb{R}^3} h'(f_{\infty}) \, \varphi \, dp = < \beta^0 \frac{\mathcal{E}(p)}{p} - \beta \cdot \nu - \mu, \varphi > \quad \forall \varphi,
\end{align}

which implies

\begin{align}
\ln (1 + \tau f_{\infty}) - \ln f_{\infty} &= \beta^0 \frac{\mathcal{E}(p)}{p} - \beta \cdot \nu - \mu
\end{align}

and therefore

\begin{equation}
f_{\infty}(p) = \frac{1}{e^{\nu(p) - \tau}} \quad \text{with} \quad \nu(p) := \beta^0 \frac{\mathcal{E}(p)}{p} - \beta \cdot \nu - \mu.
\end{equation}

The function \( f_{\infty} \) is called a Maxwellian when \( \tau = 0 \), a Bose-Einstein distribution when \( \tau > 0 \) and a Fermi-Dirac distribution when \( \tau < 0 \).

\textbf{I.2 The classic case.}

Let us consider for a moment the case \( \tau = 0 \), i.e. the classic Boltzmann equation, which has been the most studied in the mathematical and physical literature.

It is known that for any initial data \( f_{\text{in}} \) there exists a unique distribution \( f_{\infty} \) of the form (1.15) such that

\begin{equation}
\int_{\mathbb{R}^3} f_{\infty}(p) \begin{pmatrix} 1 \\ \frac{p}{\mathcal{E}(p)} \end{pmatrix} dp = \int_{\mathbb{R}^3} f_{\text{in}}(p) \begin{pmatrix} 1 \\ \frac{p}{\mathcal{E}(p)} \end{pmatrix} dp.
\end{equation}
From the point of view of the Cauchy problem (existence, uniqueness, stability of solutions) and the long time behaviour of the solutions of the Boltzmann equation, we may briefly recall the main results which are known up to now. We refer to [C], [L], for a more detailed exposition and their proofs.

**Theorem 1. Stationary solutions.** For any $f \geq 0$ measurable function on $\mathbb{R}^3$ such that

\begin{equation}
\int_{\mathbb{R}^3} f(1, p, \frac{|p|^2}{2}) \, dp = (N, P, E)
\end{equation}

for some $N, E > 0$, $P \in \mathbb{R}^3$, the following assertions are equivalent:

(i) $f$ is the Maxwellian $\mathcal{M}_{N,P,E} = \mathcal{M}[\rho, u, \Theta] = \frac{\rho}{(2\pi \Theta)^{3/2}} \exp\left(-\frac{|p-u|^2}{2\Theta}\right)$ where $(\rho, u, \Theta)$ is uniquely determined by $N = \rho$, $P = \rho u$, $E = \frac{\rho}{2} (|u|^2 + 3 \Theta)$;

(ii) $f$ is the solution of the maximisation problem

$$H(f) = \max\{H(g), \text{g satisfies the momentum equation (1.20)}\},$$

where $H(g) = -\int_{\mathbb{R}^3} g \log g \, dp$ stands for the classical entropy;

(iii) $Q(f) = 0$;

(iv) $D(f) = 0$.

Considering the evolution problem one can prove.

**Theorem 2.** Assume that $w = 1$ (to simplify!). For any initial data $f_{in} \geq 0$ with finite number of particles, energy and entropy there exists a unique global solution $f \in C([0, \infty); L^1(\mathbb{R}^3))$ which conserves particle number, energy and impulsion. Moreover, when $t \to \infty$, $f(t, \cdot)$ converges to the Maxwellian $\mathcal{M}$ with same particle number, impulsion an energy (defined by Theorem 1.1) and more precisely, for any $m > 0$ there exists $C_m = C_m(f_0)$ explicitly computable such that

\begin{equation}
\|f - \mathcal{M}\|_{L^1} \leq \frac{C_m}{(1 + t)^m}
\end{equation}

We refer to [Ar1], [E], [MW], [Lu] for existence, conservations and uniqueness and to [Ar2], [V], [W], [C], [TV] for convergence to the equilibrium. Also note that Theorem 1.2 can be generalized (some time only partially) to a large class of cross-section $W$ we refer to [V] for details and references. Even when one does not know how to establish Theorem 1.2 for other cross-section $w \neq 1$ there is no result which is at variance with this result, and we expect that this result is always true.

**Remark 1.1.** The equivalence (i) - (ii) is simpler to show than the others because it only involves the entropy $H(f)$ and not the collision integral $Q(f)$ itself. The result is also natural from (1.9) and (1.12). It will finally be helpfull to conjecture what are the solutions of the problem in the quatum and relativistic cases.

**Remark 1.2.** To show that (i), (iii) and (iv) are equivalent one has first to define the quantities $Q(f)$ and $D(f)$ for the functions $f$ belonging to the physical functional space. The first difficulty is to define precisely the collision integral $Q(f)$, (see Section 3.2).

1.3 Qantum and/or relativistic gases.

The Boltzmann equation looks formally very similar in the different contexts: classic, quantum and relativistic, but it actually present some very different features in each of these different contexts. To get a first insight on these differences consider the two following remarks.
The natural functional spaces to look for the density \( f \) are the spaces of distribution \( f \geq 0 \) such that the “physical” quantities are bounded:

\[
(1.19) \quad \int_{\mathbb{R}^3} f \, (1 + \mathcal{E}(p)) \, dp < \infty \quad \text{and} \quad H(f) < \infty.
\]

where \( H \) is given by (1.11). This provides very different conditions since we obtain:

\[
\begin{align*}
& f \in L^1 \cap L \log L \quad \text{in non quantum case, relativistic or not} \\
& f \in L^1 \cap L^\infty \quad \text{in the Fermi case, relativistic or not} \\
& f \in L^1 \quad \text{in the Bose case, relativistic or not},
\end{align*}
\]

where

\[
(1.20) \quad L^1_s = \{ f \in L^1(\mathbb{R}^3); \int_{\mathbb{R}^3} (1 + |p|^s) \, d|f|(p) < \infty \}
\]

and \( s = 2 \) in the non relativistic case, \( s = 1 \) in the relativistic case.

Remember that the density entropy \( h \) given by (1.11) is:

\[
h(f) = \tau^{-1} (1 + \tau f) \ln(1 + \tau f) - f \ln f.
\]

Therefore, in the Fermi case where \( \tau = -1 \), \( h(f) = +\infty \) if \( f \notin [0,1] \) and so \( H(f) < \infty \) provides a strong \( L^\infty \) bound on \( f \). While in the Bose case, i.e. for \( \tau = 1 \), \( h(f) \sim \ln f \) when \( f \to \infty \), so that the entropy bound does not give any additional information on the momentum bound.

Moreover, and still concerning the Bose case, let \( a \in \mathbb{R}^3 \) be any fixed vector and \( (\varphi_n)_{n \in \mathbb{N}} \) an approximation of the identity:

\[
(\varphi_n)_{n \in \mathbb{N}}; \quad \varphi_n \to \delta_a.
\]

It is shown in [CL], see also Section 2, that for any \( f \in L^1 \) the quantity \( H(f + \varphi_n) \) is well defined by (1.11) for every \( n \in \mathbb{N} \) and moreover,

\[
(1.21) \quad N(f + a\varphi_n) \to N(f) + a, \quad \text{and} \quad H(f + \varphi_n) \to H(f) \quad \text{as} \quad n \to \infty.
\]

This indicates that the expression of \( H \) given in (1.11) may be extended to nonnegative measures and that, moreover, the singular part of the measure does not contributes to the entropy. More precisely, for any non negative measure \( F \) of the form \( F = gdp + G \), where \( g \geq 0 \) is an integrable function and \( G \geq 0 \) is singular with respect to the Lebesgue measure \( dp \), we define the Bose-Einstein entropy of \( F \) by

\[
(1.22) \quad H(F) := H(g) = \int_{\mathbb{R}^3} [(1 + g) \ln(1 + g) - g \ln g] \, dp
\]

The discussion above shows how different is the quantum from the non quantum case, and even the Bose from the Fermi case. Concerning the Fermi gases, the Cauchy problem has been studied by J. Dolbeault [D] and P.L. Lions [L], under the hypothesis (H1) which includes the hard sphere case \( w = 1 \). As it is pointed by the remark above, the estimates at our disposal in this case are even better than in the classical case. In particular the collision term \( Q(f) \) may be defined in the same way as in the classical case. But as far as we know, no analogue of Theorem 1.1 was known for Fermi gases. The problem for Bose gases is essentially open as we shall see below. Partial results for radially symmetric \( L^1 \) distributions have been obtained by X. Lu [Lu] under strong cut off assumptions on the function \( w \).

### 1.3.1 Equilibrium states. Entropy.

As it is formally indicated by the identity (1.9), the particle number, momentum and energy of Boltzmann equation is conserved along the trajectory of the solutions. It is then very natural to consider the so called
momentum equation which may be stated as follows: given \( N > 0, P \in \mathbb{R}^3 \) and \( E \in \mathbb{R} \), find a distribution \( f \) which maximises the entropy \( H \) and whose mommets are \((N, P, E)\). The solution of this problem is well known in the non quantum non relativistic case (and is recalled in Theorem 1.1 above). This problem has been solved by R. T. Glassey [Gi] and R. T. Glassey & W. Strauss [GS] in the relativistic non quantum case. We solve here the general quantum relativistic case and show the following result.

**Theorem 3.** For every possible choice of \((N, P, E)\) such that the set \( K \) defined by

\[
g \in K \quad \text{if and only if} \quad \int_{\mathbb{R}^3} g(1, p, \frac{|p|^2}{2}) \, dp = (N, P, E),
\]

is non empty, there exists a unique solution \( \overline{f} \) to the entropy maximisation problem:

\[
\overline{f} \in K, \quad H(\overline{f}) = \max \{ H(g); \, g \in K \}.
\]

Moreover, \( \overline{f} = f_\infty \) given by (1.15) for the nonquantum and Fermi case, while for the Bose case \( \overline{f} = f_\infty + \alpha \delta_\overline{p} \) for some \( \overline{p} \in \mathbb{R}^3 \).

It was already observed by Bose and Einstein ([B], [E1], [E2]) that for systems of bosons in thermal equilibrium a careful analysis of the statistical physics of the problem leads to enlarge the class of steady distributions to include also the solutions containing a Dirac mass. On the other hand, the strong uniform bound introduced by the Fermi entropy over the Fermi distributions leads to include in the family of Fermi steady states the so called degenerate states. We present in Section 2 a detailed mathematical proof of these two facts both for relativistic and non relativistic particles.

1.3.2 Collision kernel. Entropy dissipation. Cauchy Problem.

Theorem 3 is the natural extension to quantum particles of the results for non quantum particles, i.e. points (i) and (ii) of Theorem 1.1. The extension of the points (iii) and (iv) even for the non relativistic case is more delicate. In the Fermi case it is possible to define the collision integral \( Q(f) \) and the entropy dissipation \( D(f) \) and to solve the problem under some additional conditions (see J. Dolbeault [D] and P.L. Lions [L]). We consider this problem and related questions in Section 3.2. As for the Bosons, the first difficulty is to define the collision integral \( Q(f) \) and the entropy dissipation \( D(f) \) in a sufficiently general setting. This question was treated by X. Lu in [L] and solved under the following additional assumptions:

(i) \( f \in L^1 \) is radially symmetric.

(ii) Strong truncature on \( w \).

These two conditions are introduced in order to give a sense to the collision integral. Unfortunately, the second one is not satisfied by the main physical examples such as \( w = 1 \) (see Appendix 3). Moreover, Theorem 2.2 shows that the natural framework to study the quantum Boltzmann equation, relativistic or not, for Bose gases is the space of non negative measures. This is an additional difficulty with respect to the non quantum or quantum Fermi cases. We partly extend the study of Lu to the case where \( f \) is a non negative radially symmetric measure.

1.4 Two species gases. Towards the Compton-Boltzmann equation.

Two species gases, in particular of bosons and fermions, are interesting by themselves for physical reasons and have thus been considered in the physical literature (see the references below). On the other hand, from a mathematical point of view, they are simplified but still interesting versions of Boltzmann equations for quantum particles. Their study may be then a first natural step to understand the behaviour of that type of equations.
Let us then call \( F(t, p) \geq 0 \) the density of Bose particles and \( f(t, p) \geq 0 \) that of Fermi particles. Under low density assumption, the evolution of the gas is now given by the following system of Boltzmann equations (see \([CC]\)):

\[
\begin{align*}
\frac{\partial F}{\partial t} &= Q_{1,1}(F,F) + Q_{1,2}(F,f) \quad F(0,.) = F_{in} \\
\frac{\partial f}{\partial t} &= Q_{2,1}(f,F) + Q_{2,2}(f,f) \quad f(0,.) = f_{in}.
\end{align*}
\]

The collision terms \( Q_{1,1}(F,F) \) and \( Q_{2,2}(f,f) \) stand for collisions between particles belonging to the same species and therefore are given by (1.2). The collision terms \( Q_{1,2}(F,f) \) and \( Q_{2,1}(f,F) \) stands for collisions between particles of the two different species, where

\[
Q_{1,2}(F,f) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} W_{1,2} q_{1,2} dp dp' dp'_*,
\]

\[
q_{1,2} = F' f'_* (1 + F) (1 - \tau f_* - F f_* (1 + F')) (1 - \tau f_*'),
\]

\( Q_{2,1}(f,F) \) is given by a similar expression and \( W_{1,2} = W_{1,2}(p, p_*, p'_*, p'_*) \) corresponds to the probability transition that in a collision the pair of particles changes the pair of impulsions \((p, p_*) \) to \((p', p'_*)\). It satisfies the micro-reversibility hypothesis

\[
W_{1,2}(p', p'_*, p, p_*) = W_{1,2}(p, p_*, p', p'_*),
\]

but not the indiscernability hypothesis \( W_{1,2}(p, p_*, p', p'_*) = W_{1,2}(p, p_*, p', p'_*) \) as in (1.7) since we have two species which are discernable one of the other.

In the Section 4 we consider some mathematical questions about these systems. We do not perform the general study of the steady states in detail since that would be in great part a repetition of what has been done in Section 3. Here again, the problem of the sense of the integral collision \( Q_{1,2} \) and \( Q_{2,1} \) is previous to any other question. We focus in the collision terms \( Q_{1,2}(F,f) \) and \( Q_{2,1}(f,F) \), since the kernel \( Q_{1,1}(F,F) \) and \( Q_{2,2}(f,f) \) have been treated in the precedent section. Let us remark that a more favourable case is when \( \tau > 0 \) since this gives a \( L^\infty \) a priori bound on the Fermi density \( f \). Nevertheless, even in that case, the problem of existence of solutions and their asymptotic behaviour for generic interactions, even with strong unphysical truncation kernel and for radially symmetric distributions \( f \) remains an open question.

In order to get some insight on these problems, we consider two simpler situations which are important from a physical point of view and still mathematically interesting. These are the equations describing boson-fermion interactions with electrons at equilibrium, and photon-electron Compton scattering. Of course the deductions of these two reduced models are well known in the physical literature but we believe nevertheless that it may be interesting to sketch them here. In the first one, considered still in Section IV, we suppose that the Fermi particles are at rest at isothermal equilibrium. This is nothing but to fix the distribution of Fermi particles \( f \) in the system to be a Maxwelian or a Fermi state. Without any loose of generality, this may be chosen to be centered at the origin, so that it is radially symmetric. Moreover, the Boson-Fermion interaction is a short range and we may consider the “slow particle interaction” approximation of the differential cross section \( \omega = 1 \) (see Appendix 3). The system reduces then to a single equation which moreover is quadratic and not cubic. Namely:

\[
\frac{\partial F}{\partial t} = \int_0^\infty S(\varepsilon, \varepsilon') \left[ F' (1 + F) e^{-\varepsilon} - F (1 + F') e^{-\varepsilon'} \right] d\varepsilon',
\]

for some kernel \( S \) (see Section 4).

We prove in Section 4 the following result about existence, uniqueness and asymptotic behaviour of global solutions for the Cauchy problem associated to (1.26).

**Theorem 4.** For any initial datum \( F_{in} \in L^1_+ (\mathbb{R}^+), F \geq 0 \), there exists a solution \( F \in C([0, \infty), L^1_+) \) to the equation (1.26) and such that

\[
\lim_{t \to 0} ||F(t) - F_{in}||_{L^1_+} = 0.
\]
Moreover, if $\overline{f} = B_N$ is the unique solution to the maximisation problem

$$H(\overline{f}) = \max \{ H(g); \int g(\varepsilon) \varepsilon^2 \, d\varepsilon = \int F_m(\varepsilon) \varepsilon^2 \, d\varepsilon =: N\}, \quad H(F) = \int_0^\infty h(f, \varepsilon) \varepsilon^2 \, d\varepsilon$$

with $h(x, \varepsilon) = (1 + x) \ln(1 + x) - x \ln x - \varepsilon x$,

$$(4.51) \quad \begin{cases} F(t, \cdot) \rightharpoonup \overline{f} \text{ weakly * in } (C_\varepsilon(\mathbb{R}_+))' \quad \text{weakly * in } (C_\varepsilon(\mathbb{R}_+))' \quad \text{weakly * in } (C_\varepsilon(\mathbb{R}_+))' \quad \text{weakly * in } (C_\varepsilon(\mathbb{R}_+))' \quad \text{weakly * in } (C_\varepsilon(\mathbb{R}_+))' \\
\lim_{t \to \infty} \|F(t, \cdot) - \overline{f}\|_{L^1([k_0, \infty])} = 0 \quad \forall k_0 > 0.\end{cases}$$

This result shows that the density of bosons $F$ underlies a Bose Einstein condensation asymptotically in infinite time if its initial value is large enough. The phenomena was already predicted by Levich & Yakhot in [LY1], [LY2], and was described as condensation driven by the interaction of bosons with a cold bath (of fermions) (see also D.V. Semikoz & I.1. Tkachev [ST1], [ST2]).

**Compton scattering.**

In Section 5 we consider the equation describing the photon-electron interaction by Compton scattering. This equation, that we call Boltzmann-Compton equation, has been extensively studied in the physical and mathematical literature (see in particular the works by A.S. Kompaneets [K], Ya. B. Zel’dovich and E.V. Levich [ZL], H. Dreicer [D], R. Weymann [W], G. Chapline, G. Cooper and S. Slutz [CCS]). From our point of view, this equation is interesting not only on itself as an important case of quantum Boltzmann equation, but also as modelling a very basic quantum phenomena. We show how it can be derived starting from the system which describes the photon electron interaction via Compton scattering. This interaction is described, in the non relativistic limit, by the Thomson cross section, (see Appendix 3).

It is important in this case to start with the full relativistic quantum formulation since photons are relativistic particles. Even if, later on, the electrons are considered at non relativistic classical equilibrium. Finally, The equation has the same form as in (1.26) where the only difference lies in the kernel $S$.

The possibility of some kind of “condensation” for this Compton Boltzmann equation was already considered in physical litterature by G. Chapline, G. Cooper and S. Slutz in [CCS], and R. Caflisch & C.D. Levermore [CL].

The plan of the paper is the following:

**II. The Maximum entropy problem.**

- II.1 Relativistic non quantum gas.
- II.2 Bose gas.
- II.3 Fermi-Dirac gas.

**III. The Boltzmann equation for one single specie of quantum particles.**

- III.1 The Boltzmann equation for Fermi-Dirac particles.
- III.2 Bose collision operator for isotopic densities.

**IV. Boltzmann equation for two species.**

- IV.1 Second specie at thermodinamical equilibrium.
- IV.2 Isotropic distribution and second specie at thermodinamical equilibrium.

**V. The collision integral for relativistic quantum equations.**

- V.1 Parametrisations.
II. THE MAXIMUM ENTROPY PROBLEM.

In this part we are interested in the maximisation problem for the entropy function under the momentum constraint. We consider one of the entropies $H$ defined in the introduction and one of the energies $E$. Given three quantities $N > 0$, $E > 0$ and $P \in \mathbb{R}^3$, we look for $F > 0$ such that

\begin{equation}
\int_{\mathbb{R}^3} \left( \frac{1}{p} \right) F(p) \, dp = \left( \begin{array}{c} N \\ P \\ E \end{array} \right)
\end{equation}

and

\begin{equation}
H(F) = \max_{g \text{ satisfying } (2.1)} H(g).
\end{equation}

We will treat successively the case of a relativistic non quantum gas, then the case of a Bose-Einstein gas, and last the case of a Fermi-Dirac gas. For each of these two kind of gases, we first consider in detail the relativistic case, where the energy is given by

\begin{equation}
E(p) = \sqrt{1 + \frac{|p|^2}{c^2 m^2}}.
\end{equation}

Then we treat the non relativistic case, $E(p) = |p|^2/2m$, which is simplest since by Galilean invariance it can be reduced to $P = 0$.

The relativistic non quantum case was completely solved by R. Glassey and W. A. Strauss in [GS], see also R. Glassey in [Gl]. However, we present here another proof, which uses in a crucial way, the Lorentz invariance and may be adapted to the quantum relativistic case. Finally, notice that we do not consider this entropy problem for a gas of photons ($E(p) = |p|$ and $H$ the Bose-Einstein entropy) since it would not have physical meaning. In Section 5 we discuss the entropy problem for a gas constituted of electrons and photons.

II.1. Relativistic non quantum gas.

In this subsection, we consider the Maxwell-Boltzmann entropy

\begin{equation}
H(g) = -\int_{\mathbb{R}^3} g \ln g \, dp.
\end{equation}

of a non quantum gas. From the heuristic argument presented in the introduction, the solution to (2.1)-(2.2) is expected to be a relativistic Maxwellian distribution, which means that it is expected to be of the form

\begin{equation}
\mathcal{M}(p) = e^{-\beta^0 p^0 + \beta \cdot p - \mu}.
\end{equation}

Our result is the following.
Theorem 2.1.  

(i) Given $E, N > 0$, $P \in \mathbb{R}^3$, there exists a least one function $g \geq 0$ which realize the momentum equation (2.1) if, and only if, 

\begin{equation}
  m^2 c^2 N^2 + |P|^2 < E^2.
\end{equation}

When (2.6) holds we will say that $(N, P, E)$ is admissible.

(ii) For any admissible $(N, P, E)$ there exists at least one relativistic Maxwellian distribution $\mathcal{M}$ corresponding to these momentum, i.e. satisfying (2.1).

(iii) Let $\mathcal{M}$ be a relativistic Maxwellian distribution. For any function $g \geq 0$ with same momentum, i.e. satisfying

\begin{equation}
  \int_{\mathbb{R}^3} \left( \frac{1}{p} \right) g(p) dp = \int_{\mathbb{R}^3} \left( \frac{1}{p^0} \right) \mathcal{M}(p) dp,
\end{equation}

one has

\begin{equation}
  H(g) - H(\mathcal{M}) = H(g|\mathcal{M}) := \int_{\mathbb{R}^3} [g \ln \frac{g}{\mathcal{M}} - g + \mathcal{M}] dp.
\end{equation}

Moreover, $H(g|\mathcal{M}) \leq 0$ and vanishes if, and only if, $g = \mathcal{M}$.

(iv) As a conclusion, for any admissible $(N, P, E)$, the entropy problem (2.1)-(2.2)-(2.3)-(2.4) has an unique solution, and this one is the relativistic Maxwellian constructed just above.

Remark 2.2. The relativistic Maxwell distribution $\mathcal{M}$ belongs to $L^1(\mathbb{R}^3)$ if, and only if, $\beta^0 > 0$ and $|\beta| < \beta^0$. In this case, all the momentum of $\mathcal{M}$ are well defined and $\mathcal{M}$ take is maximum at the point $p_{\text{max}}$ such that

\begin{equation}
  \frac{p_{\text{max}}}{p^0_{\text{max}}} = \frac{\beta}{\beta^0}, \quad \text{and thus} \quad p_{\text{max}} := \frac{mc}{\sqrt{\beta^0^2 - |\beta|^2}}.
\end{equation}

We do not prove this claim, since we do not need it in the sequel and its proof is the same that the one of Lemma 2.3. that we present in the next subsection.

We split the proof of Theorem 2.1 in three parts corresponding to the three first intermediary results (assertions (i), (ii), (iii) of Theorem 2.1). The last one (assertions (iv) of Theorem 2.1) is then an immediate consequence of the preceding intermediary results.

Proof of Theorem 2.1.  

(i) Since that $F(p)/N dp$ is a probability measure whose support is not a single point, and because the function $r \mapsto C(r) = \sqrt{m^2 c^2 + r^2}$ is strictly convex, the Jensen inequality (which is therefore a strict inequality) implies

\[ \sqrt{m^2 c^2 + \frac{P^2}{N}} = C\left( \int_{\mathbb{R}^3} p \frac{F(p) dp}{N} \right) < \int_{\mathbb{R}^3} C(p) \frac{F(p) dp}{N} = \frac{E}{N} \]

\[ \square \]

Proof of Theorem 2.1.  

(ii) Given $(N, P, E)$ admissible, we look for a relativistic Maxwellian $\mathcal{M}$ (which means that we look for $\beta^0 > 0$, $\beta \in \mathbb{R}^3$ and $\mu \in \mathbb{R}$) such that

\begin{equation}
  N(\mathcal{M}) = N, \quad P(\mathcal{M}) = P, \quad E(\mathcal{M}) = E.
\end{equation}

By symmetry the second equation implies that $\beta$ must be collinear to $P$. Therefore, in all the sequel, $P$ will denote the norm $|P|$ instead of a vector, so that the second equation as to be understand as a scalar equation.

The idea is now to reduce the system of three scalar equations (2.9) to a system of equations with less unknowns. A first way is to eliminate the $\mu$ dependence since the term $e^{-\mu}$ can be factorised. This is the
most classic way, which is used for instance by Glasey [Gl], but we do not follow it, since it can not be
generalized to the quantum case.

We begin introducing two new unknowns \((\vec{\beta}, u)\) with \(\vec{\beta} > 0, u \in \mathbb{R}^3, |u| < c\), which correspond to the
4-vector \((\beta^0, \beta) \in \mathbb{R}^4\) with \(|\beta| < \beta^0\), by the following bijection
\[
(2.10) \quad u := c \beta / \beta^0, \quad \vec{\beta}^2 := (\beta^0)^2 - |\beta|^2,
\]
and thus
\[
\beta^0 = \gamma \vec{\beta}, \quad \beta = \gamma \vec{\beta} u/c, \quad \text{where} \quad \gamma := \frac{1}{\sqrt{1 - (u/c)^2}}.
\]

From the results recalled in the Appendix 1, we know that there is a Lorentz transformation \(\Lambda_u\) associated
to the velocity \(u\) such that
\[
\left( \begin{array}{c} \beta^0 \\ \beta \end{array} \right) = \Lambda_u \left( \begin{array}{c} \vec{\beta} \\ 0 \end{array} \right) .
\]

In the initial frame, the identities \((A.12), (A.13), (A.14)\) written for \(\mathcal{M}\) give
\[
(2.11) \quad \left\{ \begin{array}{l} N(\mathcal{M}) = A_1 \gamma \vec{\beta} \\ E(\mathcal{M}) = A_2 + A_3 (\gamma \vec{\beta})^2 \\ P(\mathcal{M}) = A_3 (\gamma \vec{\beta}) (\gamma \vec{\beta} u/c) \end{array} \right.,
\]
where
\[
N(\phi) = \int_{\mathbb{R}^3} \phi \, dp, \quad P(\phi) = \int_{\mathbb{R}^3} \phi \, dp, \quad E(\phi) = \int_{\mathbb{R}^3} \phi \, dp, \quad G(\phi) = m^2 c^2 \int_{\mathbb{R}^3} \phi \, dp,
\]
and where \(A_i\) are invariant by change of frame under a Lorentz transform.

Let us now introduce the Maxwellian \(\tilde{\mathcal{M}}\) in the rest frame, i.e. we define
\[
(2.12) \quad \tilde{\mathcal{M}}(p) = e^{-\vec{\beta}^2/\beta^2}.
\]

In the rest frame, identities \((A.12), (A.13), (A.15)\) written for \(\tilde{\mathcal{M}}\) are
\[
(2.13) \quad \left\{ \begin{array}{l} N(\tilde{\mathcal{M}}) = A_1 \vec{\beta} \\ E(\tilde{\mathcal{M}}) = A_2 + A_3 \vec{\beta}^2 \\ G(\tilde{\mathcal{M}}) = 4 A_2 + 4 A_3 \vec{\beta}^2 \end{array} \right.,
\]

Inverting the two systems, we get
\[
\left\{ \begin{array}{l} A_1 \vec{\beta} = N(\mathcal{M}) / \gamma \\ A_2 = E(\mathcal{M}) - P(\mathcal{M}) c/u \\ A_2 + A_3 \vec{\beta}^2 = E(\mathcal{M}) - P(\mathcal{M}) u/c \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} A_1 \vec{\beta} = N(\tilde{\mathcal{M}}) \\ A_2 = -H(\tilde{\mathcal{M}}) \\ A_2 + A_3 \vec{\beta}^2 = E(\tilde{\mathcal{M}}) \end{array} \right.
\]
where we have defined
\[
(2.14) \quad H(\phi) := (E(\phi) - G(\phi)) / 3 = \int_{\mathbb{R}^3} \phi \frac{|p|^2}{3 \, p^0} \, dp
\]

Therefore, the fundamental relation between the relativistic Maxwellian \(\mathcal{M}\) in the initial frame and the
reduced relativistic Maxwellian \(\tilde{\mathcal{M}}\) in the rest frame is
\[
(2.15) \quad \left\{ \begin{array}{l} N(\mathcal{M}) = N(\tilde{\mathcal{M}}) \gamma(u) \\ P(\mathcal{M}) c/u - E(\mathcal{M}) = H(\tilde{\mathcal{M}}) \\ E(\mathcal{M}) - P(\mathcal{M}) u/c = E(\tilde{\mathcal{M}}) \end{array} \right.
\]
To find \( \mathcal{M} \) such that equation (2.9) holds is therefore equivalent to find \( \beta > 0, \mu \in \mathbb{R} \) and \( u \geq 0 \) such that

\[
\begin{align*}
E &= E(\beta, \mu) + P \frac{u}{c} \quad (=: \Sigma_E(\beta, \mu)) \\
N &= N(\beta, \mu) \gamma(u), \quad (=: \Sigma_N(\beta, \mu))
\end{align*}
\]  

where \( u \) is given by

\[
u = u(\beta, \mu) := \frac{P c}{E + H(\beta, \mu)}
\]  

and where we have defined \( L(\beta, \mu) := L(\mathcal{M}) = L(e^{-\beta \rho^0}) e^{-\mu} \) for \( L = N, E, G, H \).

Existence of a solution to (2.16), (2.17) is given by the following.

**Lemma 2.3.**

1. For all \( \mu \in \mathbb{R} \) the function \( \Sigma_E(\cdot, \mu) : \mathbb{R}^+_+ \to \mathbb{R} \) is continuous, decreasing and such that \( \Sigma_E(\tilde{\beta}, \mu) \to +\infty \) when \( \tilde{\beta} \to 0 \) and \( \Sigma_E(\tilde{\beta}, \mu) \to 0 \) when \( \tilde{\beta} \to +\infty \). Therefore, for any \( \mu \) there exists a unique \( \beta = \beta(\mu) > 0 \) such that \( \Sigma_E(\beta, \mu) = E \).

2. The function \( \tilde{\beta} : \mathbb{R} \to \mathbb{R} \) is continuous, decreasing, \( \tilde{\beta}(\mu) \to +\infty \) when \( \mu \to -\infty \) and \( \tilde{\beta}(\mu) \to 0 \) when \( \mu \to +\infty \).

3. The function \( \tilde{\Sigma}_N : \mathbb{R} \to \mathbb{R}, \tilde{\Sigma}_N(\mu) := \tilde{\Sigma}_N(\beta(\mu), \mu) \) is continuous and decreasing. Moreover, \( \tilde{\Sigma}_N(\mu) \to \sqrt{E^2 - P^2/mc} \) when \( \mu \to -\infty \) and \( \tilde{\Sigma}_N(\mu) \to 0 \) when \( \mu \to +\infty \). In particular, under the admissible condition (2.6), there exists \( \mu^* \in \mathbb{R} \) such that \( \tilde{\Sigma}(\mu^*) = N \). As a conclusion, setting \( \mu = \mu^*, \beta = \beta(\mu^*), u = u(\beta(\mu^*), \mu^*) \), we have constructed a solution \( (\tilde{\beta}, \mu, u) \) to (2.16), (2.17).

Assertion 3 of Lemma 2.3 states precisely that there exists a relativistic Maxwellian \( \mathcal{M} \) satisfying (2.9), and this concludes the proof of assertion (ii) of Theorem 2.1.

**Proof of Theorem 2.1.-(iii)** Since that \( \ln \mathcal{M} = -\beta^0 p^0 + \beta \cdot p - \mu \), using the momentum equation (2.7), we get

\[
H(g|\mathcal{M}) = -\int_{\mathbb{R}^3} g \ln g \, dp + \int_{\mathbb{R}^3} g \ln \mathcal{M} \, dp + \int_{\mathbb{R}^3} (\mathcal{M} - g) \, dp \\
= -\int_{\mathbb{R}^3} g \ln g \, dp + \int_{\mathbb{R}^3} \mathcal{M} \ln \mathcal{M} \, dp = H(g) - H(\mathcal{M}).
\]

Furthermore, the function \( h_s(t) := t \ln(t/s) + s - t \) satisfies \( h'_s(t) = \ln(t/s) \) and \( h''_s(t) = 1/t \) so that \( h_s \) is strictly concave and \( h_s(t) < h_s(s) = 0 \) for any \( t \neq s \).

**Proof of Lemma 2.3.-(1)** For \( L = N, E, G, H \) the function \( L : \mathbb{R}^+_+ \to \mathbb{R}^+_+ \), \( \tilde{\beta} \to L(\tilde{\beta}) \) is \( C^1 \), decreasing. \( L(\tilde{\beta}) \to +\infty \) when \( \tilde{\beta} \to 0 \) and \( L(\tilde{\beta}) \to 0 \) when \( \tilde{\beta} \to +\infty \). Then, \( u(\beta, \mu) : \mathbb{R}^+_+ \times \mathbb{R} \) is continuous, increasing with respect to the two variables, \( u(\beta, \mu) \to 0 \) when \( \beta \to 0 \) (for fixed \( \mu \)) and when \( \mu \to -\infty \) (for fixed \( \beta \)), \( u(\beta, \mu) \to cP/E \) (\( < c \)) when \( \beta \to +\infty \) (for fixed \( \mu \)) and when \( \mu \to +\infty \) (for fixed \( \beta \)). Last, \( \gamma(\beta, \mu) := \gamma(u(\beta, \mu)) : \mathbb{R}^+_+ \times \mathbb{R} \) is continuous, increasing with respect to the two variables, \( \gamma(\beta, \mu) \to 1 \) when \( \beta \to 0 \) (for fixed \( \mu \)) and when \( \mu \to -\infty \) (for fixed \( \beta \)), \( \gamma(\beta, \mu) \to \gamma(cP/E) \) (\( < +\infty \)) when \( \beta \to +\infty \) (for fixed \( \mu \)) and when \( \mu \to +\infty \) (for fixed \( \beta \)).

We now come to study the function \( \Sigma_E \). We clearly have \( \Sigma_E : \mathbb{R}^+_+ \times \mathbb{R} \to \mathbb{R} \) is \( C^1 \) and the derivates \( \Sigma'_E \) with respect to the variables \( \beta \) or \( \mu \) satisfies

\[
\Sigma'_E(\beta, \mu) = \frac{P}{c} u'(\beta, \mu) + E'(\beta, \mu) = \frac{P}{c} (-P c) \frac{H'(\beta, \mu)}{(E + H(\beta, \mu))^2} + E'(\beta, \mu)
\]

\[
= \left( \frac{u(\beta, \mu)}{c} \right)^2 \left[ G'(\beta, \mu) + E'(\beta, \mu) \left( \frac{E}{H(\beta, \mu)} \frac{1}{H(\beta, \mu)} - \frac{1}{3} \right) \right] < 0
\]
since $E/P > 1$, $E(\bar{\beta}, \mu) < 0$ and $G'(\bar{\beta}, \mu) < 0$. This implies that $\Sigma_E$ is decreasing in both $\bar{\beta}$ and $\mu$. Moreover $\Sigma_E(\bar{\beta}, \mu) \to +\infty$ when $\bar{\beta} \to 0$ (for fixed $\mu$) or when $\mu \to -\infty$ (for fixed $\bar{\beta}$), $\Sigma_E(\bar{\beta}, \mu) \to P^2/E$ when $\bar{\beta} \to +\infty$ (for fixed $\mu$) or when $\mu \to +\infty$ (for fixed $\bar{\beta}$). Since $E > P$ and then $E > P^2/E$, for any $\mu \in \mathbb{R}$ there exists an unique $\bar{\beta} = \bar{\beta}(\mu)$ such that $\Sigma_E(\bar{\beta}, \mu) = E$.

**Proof of Lemma 2.3.- (2)** By the implicit function theorem, we know that $\mu \mapsto \bar{\beta}(\mu)$ is $C^1$ and moreover

$$\bar{\beta}'(\mu) = -\frac{\partial \Sigma_E}{\partial \mu} \bigg/ \frac{\partial \Sigma_E}{\partial \bar{\beta}} < 0,$$

so that $\bar{\beta}$ is decreasing. Assuming, by contradiction, that there is $\bar{\beta}_* > 0$ such that $\bar{\beta}(\mu) \geq \bar{\beta}_*$ for any $\mu \in \mathbb{R}$. Then $L(\bar{\beta}(\mu), \mu) \leq L(\bar{\beta}_*, \mu) \to 0$ when $\mu \to +\infty$ and therefore $\Sigma_E(\bar{\beta}(\mu), \mu) \to P^2/E < E$ which is in contradiction with the definition of $\bar{\beta}(\mu)$. Therefore $\bar{\beta}(\mu) \to 0$ when $\mu \to +\infty$. Next, assuming, again by contradiction, that there is $\bar{\beta}^* < +\infty$ such that $\bar{\beta}(\mu) \leq \bar{\beta}^*$ for any $\mu \in \mathbb{R}$. Then $\Sigma_E(\bar{\beta}(\mu), \mu) \geq E(\bar{\beta}(\mu), \mu) \geq E(\bar{\beta}^*, \mu) \to +\infty$ when $\mu \to -\infty$ which is in contradiction with the definition of $\bar{\beta}(\mu)$. Therefore $\bar{\beta}(\mu) \to +\infty$ when $\mu \to -\infty$.

**Proof of Lemma 2.3.- (3)** It is clear that $\Sigma_N$ is continuous as a composition of continuous functions. Since $|p| \leq p^0 \leq |p| + mc$, we have the estimates

$$N(\bar{\beta}, \mu) \leq e^{-\mu} \int_{\mathbb{R}^3} e^{-\bar{\beta}|z|^4} \, dz = C_N \frac{e^{-\mu}}{\bar{\beta}^4},$$

$$E(\bar{\beta}, \mu) \geq e^{-\mu} \int_{\mathbb{R}^3} |p| e^{-\bar{\beta}(|p| + mc)} \, dp = C_E \frac{e^{-\mu - mc}}{\bar{\beta}^4},$$

where $C_N = \int_{\mathbb{R}^3} e^{-|z|^4} \, dz$ and $C_E = \int_{\mathbb{R}^3} |z| e^{-|z|^4} \, dz$. Then, using that $E \geq E(\bar{\beta}(\mu), \mu)$ for any $\mu$, it yields

$$\Sigma_N(\mu) \leq C_N \frac{e^{-\mu}}{\bar{\beta}^4} \gamma\left(\frac{EP}{E}\right) \leq E \frac{C_N}{C_E} \bar{\beta}(\mu) \frac{e^{-mc}}{\bar{\beta}^4} \gamma\left(\frac{EP}{E}\right) \to 0$$

when $\mu \to +\infty$ (and thus $\bar{\beta}(\mu) \to 0$).

We now claim that, there is $\alpha \geq 0$ and a sequence $(\mu_n)$ such that $\mu_n \to -\infty$ and

$$(2.19) \quad M_n(p) := e^{-\bar{\beta}(\mu_n)|p|^2 - \mu_n} \xrightarrow{n \to +\infty} \alpha \delta_0$$

in the sense of measure, and in fact, most precisely $\ll M_n, \varphi \gg \to \alpha \varphi(0)$ for any $\varphi \in C(\mathbb{R}^3)$ such that $|1 + |p|^{-m}| \varphi(p) \to 0$ when $p \to \infty$ for some $m \geq 0$. Indeed, on one hand if for some $p \in \mathbb{R}^3 \setminus \{0\}$ we have $M_n(2\bar{p}) \geq \Theta > 0$ for any $n \geq 0$, then for any $p \in B(0, |p|)$ we have

$$M_n(p) = M(2\bar{p}) e^{\bar{\beta}(\mu_n)(|2\bar{p}|^2 - p^2)} \geq \Theta e^{\bar{\beta}(\mu_n)(|2\bar{p}|^2 - p^2)} \to +\infty$$

so that $E(\bar{\beta}(\mu_n), \mu_n) \to +\infty$ which is in contradiction with the bound $E(\bar{\beta}(\mu_n), \mu_n) \leq E$. Therefore, $M_n(p) \to 0$ when $n \to +\infty$ for any $p \in \mathbb{R}^3 \setminus \{0\}$. On the other hand, fix $\bar{p} \in \mathbb{R}^3 \setminus \{0\}$. We have already proved that there is some $\Theta$ such that $M_n(\bar{p}) \leq \Theta$ for any $n \geq 0$. Therefore, for any $p \in \mathbb{R}^3$ such that $|p| > |\bar{p}|$ we have

$$M_n(p) = M(\bar{p}) e^{-\bar{\beta}(\mu_n)(|p|^2 - \bar{p}^2)} \leq \Theta e^{-|p|^2 - \bar{p}^2}.$$

Combining this with the bound $E(\bar{\beta}(\mu_n), \mu_n) \leq E$, we get (2.17) and in particular

$$N(\bar{\beta}(\mu_n), \mu_n) \to \alpha, \quad H(\bar{\beta}(\mu_n), \mu_n) \to 0, \quad E(\bar{\beta}(\mu_n), \mu_n) \to mc\alpha.$$

Now, passing to the limit in (2.14)-(2.15) we get $E = mc\alpha + \frac{P^2}{E}$ so that

$$\lim_{n \to +\infty} \Sigma_N(\mu) = \alpha \gamma\left(\frac{P}{E}\right) = \frac{E^2 - P^2}{mc E} \frac{1}{\sqrt{1 - (P/E)^2}} = \frac{\sqrt{E^2 - P^2}}{mc E}.$$
II.2. Bose gas.

Consider now a gas of Bose particles. As we already said, Bose [B] and Einstein [E1], [E2], noticed that in this case, the set of steady distributions had to include solutions containing a Dirac mass. It is then necessary to extend the entropy function $H$ defined in (1.11) with $\tau = 1$ to such distributions. This may well understood with the following remark from [CL].

Let $a \in \mathbb{R}^3$ be any fixed vector and $(\varphi_n)_{n \in \mathbb{N}}$ an approximation of the identity:

$$(\varphi_n)_{n \in \mathbb{N}}; \quad \varphi_n \to \delta_0.$$ 

For any $f \in L^1$ the quantity $H(f + \varphi_n)$ is well defined by (1.11) for every $n \in \mathbb{N}$. Moreover,

$$H(f + \varphi_n) \to H(f) \quad \text{as} \quad n \to \infty.$$ 

Suppose indeed, for the sake of simplicity that: $\varphi_n \equiv 0$ if $|p - a| \geq 2/n$. Therefore:

$$H(f + \varphi_n) = \int_{|p - a| \geq 2/n} h(f(p,t),p)dp + \int_{|p - a| \leq 2/n} h((f(p,t) + \varphi_n(p),p)dp$$

Since $|(1 + z)\ln(1 + z) - z \ln z| \leq c\sqrt{z}$,

$$\int_{|p - a| \leq 2/n} |(f(p,t) + \varphi_n(p)|dp$$

$$\leq c\frac{2}{\sqrt{n^3}}(\int_{|p - a| \leq 2/n} (f(p,t) + \varphi_n(p)dp)^{1/2} \to 0.$$ 

Finally,

$$\int_{\mathbb{R}^3} h(f(p,t),p)dp - \int_{|p - a| \geq 2/n} h(f(p,t),p)dp$$

$$\leq \int_{|p - a| \leq 2/n} |h(f(p,t),p)|dp \to 0,$$

which ends the proof.

This indicates that the expression of $H$ given in (1.11) may be extended to nonnegative measures and that, moreover, the singular part of the measure does not contributes to the entropy. More precisely, for any nonnegative measure $F$ of the form $F = gdp + G$, where $g \geq 0$ is an integrable function and $G \geq 0$ is singular with respect to the Lebesgue measure $dp$, we define the Bose-Einstein entropy of $F$ by

$$(2.20) \quad H(F) := H(g) = \int_{\mathbb{R}^3} [(1 + g)\ln(1 + g) - g \ln g] dp$$

On the other hand, an heuristic argument has shown that the regular solution to the maximum entropy problem should be Bose relativistic distributions

$$(2.21) \quad b(p) = \frac{1}{e^{\nu(p)} - 1} \quad \text{with} \quad \nu(p) = \beta^0 p^0 - \beta \cdot p + \mu.$$ 

The following result explains simply where the Dirac masses have now to be placed.
Performing a spherical change of coordinates, i.e. writing \( p = e \rho \cos \theta + \ldots \), we obtain

\[
N(b) = \int_{\mathbb{R}^3} e^{\beta \rho \rho^0 (\rho^0 - e^2 \rho^0_p e^0 + \mu)} \frac{dp}{\rho^2} = 2\pi \int_0^\pi \int_0^{2\pi} e^{\beta \rho \rho^0 (\rho^0 - e^2 \rho^0_p e^0 + \mu)} \frac{1}{\rho^2} \rho d\theta dr.
\]

Choosing \( s_0 \) and \( r_0 \) small enough. Since the last expression is \(+\infty\), we see that (23) cannot hold.

Now assume, again by contradiction, that

\[
\beta = \beta^0 e, \quad e \in S^2 \quad \text{and} \quad \beta^0 mc + \mu > 0.
\]

Writing \( p = te + q \) with \( q \in \mathbb{R}^3, \, |q| \leq 1 \) and \( q \perp e \) we compute

\[
N(b) = \int_{\mathbb{R}^2} e^{\beta \rho \rho^0 (\rho^0 - e^2 \rho^0_p e^0 + \mu)} \frac{dp}{\rho^2} \geq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\beta \rho \rho^0 (\rho^0 - e^2 \rho^0_p e^0 + \mu)} \frac{dq}{\rho^2} = 2\pi \int_0^\pi \int_0^{2\pi} e^{\beta \rho \rho^0 (\rho^0 - e^2 \rho^0_p e^0 + \mu)} \frac{1}{\rho^2} \rho d\theta dr.
\]

Choosing \( t \) large enough. Once again the last expression is \(+\infty\), so that (24) does not hold.

Finally, since \( \beta^0 mc + \mu = \nu(0) \geq 0 \), we must have \( |\beta| < \beta^0 \).

From the preceding computations, we deduce that \( \nu(p) \to +\infty \) when \( |p| \to +\infty \) and therefore \( \nu(p) \) reaches its minimum at a point \( p_{MC} \) which satisfies

\[
\nabla_p \nu(p_{MC}) = \beta^0 \frac{p_{MC}}{mc^2} - \beta = 0.
\]

The unique solution of this equation is \( p_{MC} \) given in (22.2). As a conclusion, we get \( \nu(p) > \nu(p_{MC}) \) for any \( p \neq p_{MC} \) and the condition \( \nu(p_{MC}) \geq 0 \) implies \( \mu \geq -mc \beta^0 \).

We define now the generalized Bose-Einstein relativistic distribution \( B \) by

\[
B(p) = b + \alpha \delta_{p_{MC}} = \frac{1}{e^{\nu(p)} - 1} + \alpha \delta_{p_{MC}}.
\]
with
\begin{equation}
\forall p \neq p_{mc} \quad \nu(p) = \beta^0 p_0 - \beta \cdot p + \alpha \nu(p_{mc}) \geq 0
\end{equation}

and the condition $\alpha \nu(p_{mc}) = 0$.

**Theorem 2.5.**

1. Given $E, N > 0, P \in \mathbb{R}^3$, there exists at least one measure $F \geq 0$ which realizes the momentum equation (2.1) if, and only if,
\begin{equation}
m^2 c^2 N^2 + |P|^2 \leq E^2.
\end{equation}
When (2.27) holds we will say that $(N, P, E)$ is a admissible.

2. For any admissible $(N, P, E)$ there exists at least one relativistic Bose-Einstein distribution $B$ corresponding to these momentum, i.e. satisfying (2.1).

3. Let $B$ be a relativistic Bose-Einstein distribution. For any measure $F \geq 0$ with same momentum, i.e. satisfying
\begin{equation}
\int_{\mathbb{R}^3} \left( \frac{1}{p_0} - \frac{1}{P_0} \right) dF(p) = \int_{\mathbb{R}^3} \left( \frac{1}{p_0} - \frac{1}{P_0} \right) dB(p),
\end{equation}
one has
\begin{equation}
H(F) - H(B) = H_1(g|b) + H_2(G|b)
\end{equation}
where
\begin{equation}
H_1(g|b) := \int_{\mathbb{R}^3} \left( (1 + g) \ln \frac{1 + g}{1 + b} - g \ln \frac{g}{b} \right) dp \quad \text{and} \quad H_2(G|b) := -\int_{\mathbb{R}^3} \nu(p) dG(p).
\end{equation}
Moreover, $H(F|B) \leq 0$ and vanishes if, and only if, $F = B$. In particular, $H(F) < H(B)$ if $F \neq B$.

4. As a conclusion, for any admissible $(N, P, E)$, the entropy problem (2.1)-(2.3), (2.20) has an unique solution, and this one is the relativistic Bose-Einstein distribution constructed just before.

The proof of part 1 of Theorem 2.5 is the same as the proof of (i) in Theorem 2.1. except that the Jensen inequality may be not strict since we deal with measures which can be concentrated at a single point. We present the proof of parts 2 and 3 pointing out the main differences with respect to the proof of (i) and (ii) of Theorem 2.1. Then point 4 is an immediate consequence of the preceding steps.

**Proof of Theorem 2.5.-2.** We adopt the notations presented in the proof of part (ii) in Theorem 2.1. Given $(N, P, E)$ admissible, we look for $B = b + \alpha \delta_{p_{mc}}$ such that
\begin{equation}
\begin{cases}
N = N(B) = N(b) + \alpha \\
E = E(B) = E(b) + \alpha P_{m,c} \\
P = P(B) = P(b) + \alpha P_{m,c}.
\end{cases}
\end{equation}
We introduce again the variables $(\tilde{b}, u)$ and the Bose-Einstein distribution in the rest frame
\[ \tilde{b}(p) = \frac{1}{e^{\beta p^0 + \mu} - 1} \]
We easily verify that
\[ p_{mc} = m \gamma u, \quad p_{mc}^0 = m c \gamma. \]
By (2.11), with $\mathcal{M}$ replaced by $b$ and $\widetilde{\mathcal{M}}$ replaced by $\tilde{b}$, to find $B$ such that (2.31) holds is equivalent to find $(\tilde{\beta}, u, \mu)$ which satisfies

\begin{equation}
\begin{cases}
N = N(\tilde{\beta}, \mu) \gamma + \alpha \\
P \frac{c}{u} - E = H(\tilde{\beta}, \mu) \\
E - P \frac{u}{c} - \alpha m \frac{c}{\gamma} = E(\tilde{\beta}, \mu),
\end{cases}
\end{equation}

where now $L(\tilde{\beta}, \mu)$ stands for $L(\tilde{b})$. Setting

$$u = u(\tilde{\beta}, \mu) := \frac{cP}{E + H(\tilde{\beta}, \mu)}, \quad \gamma(\tilde{\beta}, \mu) = \gamma(u(\tilde{\beta}, \mu)),$$

the system (2.32) is equivalent to

\begin{equation}
\begin{cases}
E - \alpha \frac{mc}{\gamma(\tilde{\beta}, \mu)} = E(\tilde{\beta}, \mu) + \frac{P}{c} u(\tilde{\beta}, \mu) \\
N - \alpha = N(\tilde{\beta}, \mu) \gamma(\tilde{\beta}, \mu)
\end{cases}
\end{equation}

Let us define the functions

\begin{equation}
\begin{cases}
\Gamma_E(\tilde{\beta}, \alpha) := \alpha \frac{mc}{\gamma(\tilde{\beta})} + E(\tilde{\beta}) + \frac{P}{c} u(\tilde{\beta}), & \Gamma_N(\tilde{\beta}, \alpha) := N(\tilde{\beta}) \gamma(\tilde{\beta}) + \alpha \\
\Sigma_E(\tilde{\beta}, \mu) := E(\tilde{\beta}, \mu) + \frac{P}{c} u(\tilde{\beta}, \mu) & \Sigma_N(\tilde{\beta}, \mu) := N(\tilde{\beta}, \mu) \gamma(\tilde{\beta}, \mu),
\end{cases}
\end{equation}

with the notation $L(\tilde{\beta}) := L(\tilde{\beta}, \mu_2)$, $\mu_2 = -mc\tilde{\beta}$ as in Lemma 2.4, for any $L = N, E, G, H$. The properties of $\Gamma_E$, $\Sigma_E$ and $\Sigma_N$ are then summarized in the following Lemma.

**Lemma 2.6.**

1. There is a continuous function $\tilde{\beta}_* : [0, N] \to (0, +\infty)$, $\alpha \mapsto \tilde{\beta}_*(\alpha)$, such that for any $\alpha \in [0, N]$, $\Gamma_E(\tilde{\beta}_*(\alpha), \alpha) = E$, and $\tilde{\beta}_* = \tilde{\beta}(\alpha)$ is the unique solution of $\Gamma_E(\tilde{\beta}_*, \alpha) = E$.

2. We set $\mu_* := \mu_{\tilde{\beta}_*(0)} = -mc\tilde{\beta}_*(0)$. There is a continuous function $\tilde{\beta} : [\mu_*, +\infty) \to (0, +\infty)$, $\mu \mapsto \tilde{\beta}(\mu)$, such that $\Sigma_E(\tilde{\beta}(\mu), \mu) = E$ for any $\mu \geq \mu_*$, and $\tilde{\beta} = \tilde{\beta}(\mu)$ is the unique solution of the equation $\Sigma_E(\tilde{\beta}, \mu) = E$. Moreover, the application $\tilde{\beta}$ is not increasing and $\tilde{\beta}(\mu) \to 0$ when $\mu \to +\infty$. Last, the function $\mu \mapsto \Sigma_N(\mu) := \Sigma_N(\tilde{\beta}(\mu), \mu)$ is continuous on $[\mu_*, +\infty)$ and $\Sigma_N(\mu) \to 0$ when $\mu \to +\infty$.

Let us use Lemma 2.6 to conclude the proof of part 2 of Theorem 2.5. Its proof is postponed to the end of the proof of Theorem 2.5.

**End of the Proof of part 2 of Theorem 2.5** In order to obtain the solution $b, u, \mu$ to (2.32) we consider the two cases: $N > \Gamma_N(\tilde{\beta}_*(0), 0)$ and $N \leq \Gamma_N(\tilde{\beta}_*(0), 0)$.

- If $N > \Gamma_N(\tilde{\beta}_*(0), 0)$, we remark that

$$\Gamma_N(\tilde{\beta}_*(N), N) = N(\tilde{\beta}_*(N)) \gamma(\tilde{\beta}_*(N)) + N > N.$$

Therefore, by the intermediary value Theorem there exists $\alpha^* \in ]0, N[$ such that $\Gamma_N(\tilde{\beta}_*(\alpha^*), \alpha^*) = N$. Therefore, we define

$$\tilde{\beta}^* := \tilde{\beta}_*(\alpha^*), \quad u^* := u(\tilde{\beta}^*), \quad \mu^* := 0.$$

- If $N \leq \Gamma_N(\tilde{\beta}_*(0), 0)$, we remark that $\Sigma_N(\mu) = \Gamma_N(\tilde{\beta}_*(0), 0) \geq N$ and $\Sigma_N(\mu) \to 0$ when $\mu \to +\infty$, and by the intermediate value Theorem there exists $\mu^* \in [\mu_*, +\infty)$ such that $\Sigma_N(\mu^*) = N$. We define now

$$\tilde{\beta}^* := \tilde{\beta}_E(\mu^*), \quad u^* := u(\tilde{\beta}^*, \mu^*), \quad \alpha^* := 0.$$
In both cases ($\beta\ast, \mu\ast, u\ast, \alpha\ast$) is a solution of (2.32), and the associated relativistic Bose state is a solution of (2.31).

**Proof of part 3 of Theorem 2.5.** Using the momentum condition, we compute

$$H(b) = \int_{\mathbb{R}^3} \ln(1 + b) \, dp + \int_{\mathbb{R}^3} b \ln \frac{1 + b}{b} \, dp = \int_{\mathbb{R}^3} \ln(1 + b) \, dp + \int_{\mathbb{R}^3} b \nu(p) \, dp$$

$$\begin{align*}
&= \int_{\mathbb{R}^3} \ln(1 + b) \, dp + \int_{\mathbb{R}^3} (g + G - \alpha \delta_{\beta M C}) \nu(p) \, dp \\
&= \int_{\mathbb{R}^3} \ln(1 + b) \, dp + \int_{\mathbb{R}^3} g \ln \frac{1 + b}{b} \, dp + \int_{\mathbb{R}^3} (G - \alpha \delta_{\beta M C}) \nu(p) \, dp \\
&= \int_{\mathbb{R}^3} ((1 + g) \ln(1 + b) - g \ln b) \, dp + \int_{\mathbb{R}^3} \nu(p) \, dG(p),
\end{align*}$$

from where we deduce (2.29),(2.30). On the other hand, for any $y > 0$, the function $x \mapsto h(x|y)$ defined for any $x \geq 0$ by

$$h(x|y) := (x + 1) \ln \frac{x + 1}{y + 1} - x \ln \frac{x}{y},$$

satisfies $h'(x|y) = 0$ if, and only if, $x = y$, $1(y|y) = 0$ and $h(x|y) < 0$ for any $x \neq y$. Thus, $H(g | B) \leq 0$ and $H(g | B) = 0$ if and only if, $g = b$. Moreover, from (2.6), we obviously have $H(G | B) \leq 0$ and $H(G | B) = 0$ if, and only if, $G = \alpha \delta_{\mu}$ with $\alpha = 0$ if $\mu > 0$ and $\alpha = m - M(b)$ if $\mu \geq 0$.

**Proof of Lemma 2.6.**

Let us prove point 1 of Lemma 2.6. To this end, consider the function $\Gamma_E(\beta, \alpha)$. Notice first that for $L = N, E, G, H$ the function $L(\beta)$ is nothing but $L(P_{\beta})$ where $P_{\beta}$ is the distribution

$$P_{\beta} = \frac{1}{e^{\beta(c^2 - m c)} - 1}.$$  

Moreover, the applications $L = N, E, G, H$ are defined on the set

$$O = \{ (\beta, \mu); \beta > 0, \mu \geq \mu_{\beta} \} = \{ (\beta, \mu); \mu \in \mathbb{R}, \beta \geq \beta_{\mu} = \max(0, -\mu/m c) \}.$$  

Since,

$$L'(\beta) = \frac{d}{d\beta} L(\beta) = \frac{\partial L}{\partial \beta} - mc \frac{\partial L}{\partial \beta},$$

we immediately obtain that the applications $\beta \mapsto L(\beta)$ are $C^1$, decreasing, $L(\beta) \to +\infty$ when $\beta \to \beta_{\mu}$ and $L(\beta) \to 0$ when $\beta \to +\infty$. Moreover for any fixed $\alpha$

$$\frac{\partial \Gamma_E(\beta, \alpha)}{\partial \beta} = \frac{\alpha mc}{\gamma(\beta)} E'(\beta) + \frac{P}{c} u'(\beta),$$

A straightforward calculation gives:

$$E'(\beta) + \frac{P}{c} u'(\beta) = (E_\beta - mc E_\beta) \left( 1 - \frac{1}{3} \frac{u(\beta)}{c} \right)^2 + \left( \frac{u(\beta)}{c} \right)^2 \frac{G_\beta - mc G_{\mu}}{3} < 0,$$

and

$$\frac{\alpha mc}{\gamma(\beta)} E'(\beta) = \frac{\alpha m u^2(\beta)}{P c^2} \gamma^{-1}(\beta)(H_\beta - mc H_{\mu}) < 0.$$  

We deduce that, for any fixed $\alpha \in [0, N]$, the application $\beta \mapsto \Gamma_E(\beta, \alpha)$ is $C^1$, decreasing, $\Gamma_E(\beta, \alpha) \to +\infty$ when $\beta \to \beta_{\mu}$ and

$$\Gamma_E(\beta, \alpha) \to \alpha \frac{mc}{\gamma(\theta_{P/E})} + \frac{P^2}{E} \quad \text{when} \quad \beta \to +\infty.$$
On the other hand, for any $\alpha \in [0, N]$, the admissibility condition (2.27) on $(N, P, E)$ gives

$$
\left( \frac{\alpha m^c}{\gamma(c P/E)} \right)^2 \leq \left( N m^c \sqrt{1 - (P/E)^2} \right)^2
$$

$$
\leq (E^2 - P^2) (1 - (P/E)^2) = \left( E - \frac{P^2}{E} \right)^2.
$$

Therefore, we get $\lim_{\beta \rightarrow +\infty} \Gamma_E (\beta, \alpha) \leq E$. Using once more the intermediate value Theorem, there exists an unique $\beta_\star = \beta_\star (\alpha) \geq \beta_\mu$ such that $E = \Gamma_E (\beta_\star (\alpha), \alpha)$. Finally, the function $\alpha \mapsto \beta_\star (\alpha)$ is continuous (by the implicit function Theorem, for instance).

Let us prove now part 2. Here the proof is very the same that the proof of Lemma 2.1. The only difference is that the applications $L$ are only defined on the set

$$
O = \{ (\beta, \mu) ; \; \beta > 0, \; \mu \geq \mu_\beta \} = \{ (\beta, \mu) ; \; \mu \in \mathbb{R}, \; \beta \geq \beta_\mu = \max (0, -\mu / m^c) \}.
$$

In fact, we are only interested by

$$
O_\star = \{ (\beta, \mu) ; \; \mu \geq \mu_\star, \; \beta \geq \beta_\star \}.
$$

We prove without any difficulty that for any $\mu \geq \mu_\star$ and any function $L = E, N, G$ and $H$, the functions $L(., \mu) : (\beta, \mu) \mapsto (0, +\infty)$, $\beta \mapsto L(\beta, \mu)$ are $C^1$, decreasing, $\lim_{\beta \rightarrow +\infty} L(\beta, \mu) = 0$ when $\beta \rightarrow +\infty$ and $\lim_{\beta \rightarrow +\beta_\mu} L(\beta, \mu) = +\infty$ when $\beta \rightarrow +\beta_\mu$. We deduce $\Sigma_E (., \mu) : (\beta, \mu) \mapsto (0, +\infty)$ is $C^1$, decreasing, $\lim_{\beta \rightarrow +\beta_\mu} \Sigma_E (\beta, \mu) = +\infty$ when $\beta \rightarrow +\beta_\mu$ and $\lim_{\beta \rightarrow +\beta_\mu} \Sigma_E (\beta, \mu) = P^2 / E \leq E$ when $\beta \rightarrow +\infty$. As a consequence, for any $\mu \geq \mu_\star$, there exists an unique $\beta = \beta (\mu)$ such that $E = \Sigma_E \beta_\mu (\beta (\mu), \mu)$.

On the other hand, one can also verify that for any $\beta > 0$ and any $L = E, N, G$ and $H$, $L(\beta_\mu) : (\mu_\mu, +\infty) \mapsto (0, +\infty)$ is $C^1$, decreasing and $\lim_{\mu \rightarrow +\infty} L(\beta, \mu) = 0$ when $\mu \rightarrow +\infty$. Therefore, as in Lemma 2.3, (ii), on checks that $\Sigma_E (\beta, \mu)$ is $C^1$ and decreasing in its two variables. We deduce that the function $\mu \mapsto \beta (\mu)$ is decreasing and $\lim_{\mu \rightarrow +\infty} \beta (\mu) = +\infty$. Finally, $\Sigma_N (\mu)$ is continuous and satisfies $\Sigma_N (\mu) \rightarrow 0$ when $\mu \rightarrow +\infty$. \(\square\)

**Nonrelativistic Bose particles.** For non relativistic particles the energy is $\mathcal{E}(p) = |p|^2 / 2m$. By Galilean invariance the problem (2.1) is then equivalent to the following simpler one: given three quantities $N > 0$, $E > 0$, $P \in \mathbb{R}^3$ find $F(p)$ such that

$$
\int_{\mathbb{R}^3} \frac{1}{1 + \frac{|p - E |}{2m}} \cdot F(p) \, dp = \left( \frac{N}{0} \frac{1}{E - \frac{|P|^2}{2mN}} \right).
$$

It is rather simple, using elementary calculus, to prove that for any $E, N > 0$, $P \in \mathbb{R}^3$ there exists a distribution of the form

$$
F(p) = \frac{1}{e^{a|b - \frac{P}{N}|^2} - b^- \delta_b^+}
$$

with $a \in \mathbb{R}$, $b \in \mathbb{R}$, $\nu \in \mathbb{R}$, $b^+ = \max (b, 0)$, $b^- = -\max (-b, 0)$ which satisfies (2.35).

Once such a solution (2.36) of (2.35) is obtained, the following Bose-Einstein distribution

$$
B(p) = \frac{1}{e^{a|p|^2} - 1 + \alpha \delta_p, \quad \nu (p) = a|p|^2 - \frac{2a}{N} \cdot p + (b^+ + a|P|^2 / N^2)}
$$

solves (2.1).
This shows that for non relativistic particles Theorem 2.5 remains valid under the unique following change: the points 1 and 2 have to be replaced by

1'. For every $E, N > 0$, $P \in \mathbb{R}^3$, there exists one relativistic Bose-Einstein distribution defined by (2.36) corresponding to these momentum, i.e. satisfying (2.1).

Points 3 and 4 of Theorem 2.5 remain unchanged.

It is particularly simple to observe in this case that, for $E > 0$ and $P \in \mathbb{R}^3$ fixed, there is no regular Bose-Einstein state for particle numbers $N$ larger than a critical value $N^*$. This easily follows from the fact that by (2.37),

$$E \geq \int_{\mathbb{R}^3} \frac{|p|^2}{e^{a|p|^2} - 1} dp = \int_{\mathbb{R}^3} \frac{|p|^2}{e^{1|p|^2} - 1} dp + \frac{|P|^2}{N}$$

$$= \frac{1}{a^2} \int_{\mathbb{R}^3} \frac{|p|^2}{e^{1|p|^2} - 1} dp + \frac{|P|^2}{N}$$

$$= \frac{1}{a^{3/2}} \int_{\mathbb{R}^3} \frac{|p|^2}{e^{1|p|^2} - 1} dp + |P|^2 a^{3/2} \left[ \int_{\mathbb{R}^3} \frac{dp}{e^{1|p|^2} - 1} \right]^{-1}.$$  

In particular,

$$a \geq \frac{1}{E} \int_{\mathbb{R}^3} \frac{|p|^2}{e^{1|p|^2} - 1} dp |p|^2 / =: a_0,$$

and therefore

$$N = \int_{\mathbb{R}^3} \frac{dp}{e^{a|p|^2} - 1} \leq \int_{\mathbb{R}^3} \frac{dq}{e^{a_0|q|^2} - 1} = N^*.$$  

II.3. Fermi-Dirac gas.

In this subsection we are interested in a Fermi-Dirac gas, which means that we consider the maximum entropy problem for the Fermi-Dirac entropy

$$(2.38) \quad H_{FD}(f) := -\int_{\mathbb{R}^3} ((1 - f) \ln(1 - f) + f \ln f) \, dp$$

In particular, this implies the constraint $0 \leq f \leq 1$ on the density $f$ of the gas.

From the heuristics argument presented in the introduction, we know that the solution $\mathcal{F}$ of (2.1)-(2.3), (2.38) is the Fermi-Dirac distribution

$$(2.39) \quad \mathcal{F}(p) = \frac{1}{e^\nu(p) + 1} \quad \text{with} \quad \nu(p) = \beta^0 \cdot p - \beta \cdot p + \mu.$$  

We also introduce the “saturated” Fermi-Dirac (SFD) density

$$(2.40) \quad \chi(p) = \chi_{\beta^0, \beta}(p) = 1_{\{\beta^0 \cdot p - \beta \cdot p \leq 1\}} = 1_{\mathcal{E}} \quad \text{with} \quad \mathcal{E} = \{\beta^0 \cdot p - \beta \cdot p \leq 1\}.$$  

with $\beta \in \mathbb{R}^3$ and $\beta^0 > |\beta|$.

Our main result is the following.

**Theorem 2.7.**

1. For any $P$ and $E$ such that $|P| < E$ there exists an unique SFD state $\chi = \chi_{P,E}$ such that $P(\chi) = P$ and $E(\chi) = E$. This one realizes the maximum of particle number for given energy $E$ and mean impulsion $P$. More precisely, for any $f$ such that $0 \leq f \leq 1$ one has

$$(2.41) \quad P(f) = P, \quad E(f) = E \quad \text{implies} \quad N(f) \leq N(\chi_{P,E}).$$  

As a consequence, for given $(N, P, E)$ there exists $F$ satisfying the momentum equation (2.1) if, and only if, $E > |P|$ and $0 \leq N \leq N(\chi_{P,E})$. In this case, we say that $(N, P, E)$ is admissible.
2. For any \((N, P, E)\) admissible there exists a Fermi-Dirac state \(\mathcal{F}\) ("saturated" or not) which realizes the momentum equation (2.1).

3. Let \(\mathcal{F}\) be a Fermi-Dirac state. For any \(f\) such that \(0 \leq f \leq 1\) and

\[
\int_{\mathbb{R}^3} f(p) \left( \frac{1}{p^0} \right) dp = \int_{\mathbb{R}^3} \mathcal{F}(p) \left( \frac{1}{p^0} \right) dp.
\]

one has

\[
H_{FD}(f) - H_{FD}(\mathcal{F}) = H_{FD}(f|\mathcal{F}) := \int_{\mathbb{R}^3} ((1 - f) \ln \frac{1 - f}{1 - F} - f \ln \frac{f}{F}) dp.
\]

4. As a conclusion, for any admissible \((N, P, E)\) the entropy problem (2.1)-(2.3), (2.38) has a unique solution, and this one is the relativistic Fermi-Dirac distribution constructed just above.

The new difficulty with respect to the classic or Bose case is to manage with the constraint \(0 \leq f \leq 1\). Before proving Theorem 2.4 we present several auxiliary results.

**Proposition 2.8.**

1. For any \(P \in \mathbb{R}^3\) and \(E > 0\) such that \(|P| < E\) there exists a unique SFD state \(\chi = \chi_{P, E}\) such that \(P(\chi) = P\) and \(E(\chi) = E\). Moreover, for fixed \(P \in \mathbb{R}^3\), the application \(E \mapsto N(\chi_{P, E})\) is increasing.

2. For any \(N > 0\) and \(P \in \mathbb{R}^3\) there exists an unique SFD state \(\chi = \chi_{N, P}\) such that \(N(\chi) = N\) and \(P(\chi) = P\). Moreover, for fixed \(P \in \mathbb{R}^3\), the application \(N \mapsto E(\chi_{N, P})\) is increasing.

**Proof of Proposition 2.8.** We start with part 1. From (2.11) we know that for any SFD state \(\chi\) of the form (2.40) we have

\[
\begin{align*}
P(\chi) c/u - E(\chi) &= H(\beta) \\
E(\chi) - P(\chi) u/c &= E(\beta) \\
N(\chi) &= N(\beta) \gamma.
\end{align*}
\]

where \(L(\beta) = L(\chi), \chi(p) = 1_E, \beta = \{\beta p^0 \leq 1\}\) and \((\beta, u)\) is associated to \((\beta^0, \beta)\) thanks to (2.10).

On the other hand, from the two first equations in (2.43), to find \(\chi\) such that, for given \(P \in \mathbb{R}^3\) and \(E > |P|^2\), \(P(\chi) = P\) and \(E(\chi) = E\) is equivalent to find \((\beta, u)\) such that \(E\) satisfies

\[
E = \Xi(\beta, E) := E(\beta) + \frac{P^2}{E + H(\beta)}
\]

and we recover \(u\) by

\[
u = u(\beta) := \frac{P c}{E + H(\beta)}
\]

We have already done the analysis of such a function \(\Xi\) in Lemma 2.1. It is easy to verify that \(E(\beta), H(\beta)\) are smooth, decreasing for \(\beta \in (0, 1/mc)\), \(E(\beta), H(\beta) \to 0\) when \(\beta \to 1/mc\), \(E(\beta), H(\beta) \to \infty\) when \(\beta \to 0\) and

\[
\frac{\partial \Xi}{\partial \beta}(\beta, E) = E'(\beta) - \frac{P^2}{(E + H(\beta))^2} H'(\beta) < 0.
\]

Moreover, \(\Xi(\beta, E) \to \infty\) when \(\beta \to 0\) (since \(\Xi(\beta, E) \geq E(\beta)\)) and \(\Xi(\beta, E) \to P^2/E\) when \(\beta \to 1/mc\). This implies that there is a unique \(\beta \in (0, 1/mc)\) such that (2.44) holds (when \(|P|^2 < E\), we note \(\beta = \beta(E)\).

Finally,

\[
\frac{\partial}{\partial E} N(\chi_{P, E}) = \frac{d}{dE} N(\beta(E)) = N'(\beta(E)) \frac{d\beta}{dE} > 0,
\]

22
since $N'(\tilde{\beta}) < 0$ and $\frac{d\tilde{\beta}}{dE} < 0$, and therefore $E \mapsto N(\chi_{P,E})$ is increasing (for fixed $P \in \mathbb{R}^3$).

Let us prove now part 2. We eliminate the energy $E(\chi)$ in (2.43) to get

$$
\begin{align*}
&\left\{ \begin{array}{l}
\frac{1}{\gamma^2} P(\chi) \frac{c}{u} = E(\tilde{\beta}) + H(\tilde{\beta}) \\
N(\chi) = N(\tilde{\beta}) \gamma.
\end{array} \right.
\end{align*}
$$

Therefore, for given $N > 0$ and $P \in \mathbb{R}^d$, find $\chi$ such that $N(\chi) = N$ and $P(\chi) = P$ is equivalent to find $(\tilde{\beta}, u)$ such that $\tilde{\beta}$ satisfies

$$
N = N(\tilde{\beta}) \gamma(u(\tilde{\beta}))
$$

where $u$ is defined by

$$
u(\tilde{\beta}) = \frac{c P \ N^2(\tilde{\beta})}{N^2 E(\tilde{\beta}) + H(\tilde{\beta})}.
$$

In order to simplify the computations in what follow we introduce the variable $a = 1/\tilde{\beta}$ and we define $\tilde{L}(a) = L(1/\tilde{\beta})$. We make the elementary computations

$$
\begin{align*}
\tilde{N}(a) &= \frac{(a^2 - 1)^{3/2}}{3}, \quad \tilde{N}'(a) = 4a \pi (a^2 - m^2 c^2)^{1/2} \\
\tilde{E}(a) &= \frac{(a^2 - 1)^2}{4}, \quad \tilde{E}'(a) = 4\pi a^2 (a^2 - m^2 c^2)^{1/2} \\
\tilde{H}(a) &= 0, \quad \tilde{G}'(a) = 4\pi (a^2 - m^2 c^2)^{1/2}.
\end{align*}
$$

We look now for a solution $a \in (1, \infty)$ to

$$
N(a) = \Xi(a, N) := \tilde{N}(a) \gamma(\tilde{u}(a)),
$$

where the dependency of $\Xi$ with respect to $N$ comes from the dependence of $\tilde{u}(a)$ on $N$ in (2.46). For fixed $N > 0$, the function $a \mapsto \tilde{N}(a)$ is continuous and the same holds for the functions $a \mapsto \tilde{u}(a)$ and $a \mapsto \gamma(\tilde{u}(a))$. Moreover,

$$
\lim_{a \to 1} \tilde{N}(a) \gamma(\tilde{u}(a)) \leq \gamma(\tilde{u}(2)) \lim_{a \to 1} \tilde{N}(a) = 0
$$

and

$$
\lim_{a \to +\infty} \tilde{N}(a) \gamma(\tilde{u}(a)) \geq \gamma(\tilde{u}(2)) \lim_{a \to +\infty} \tilde{N}(a) = +\infty.
$$

Therefore, for any $N > 0$ the equation (2.47) has a solution $a \in (0, \infty)$. This provides a solution $(1/a, u(1/a))$ for (2.46) and thus a SFD distribution $\tilde{\chi}$ such that $N(\tilde{\chi}) = N$ and $P(\tilde{\chi}) = P$.

In order to show that the application $N \mapsto E(\tilde{\chi}, N)$ is increasing, we argue as follows. First, we notice that the application $N \mapsto a(N)$ with $a(N)$ such that $N = \Xi(a(N), N)$ is continuous and thus the application $E(\tilde{\chi}, N)$ is also continuous. Moreover, this application is injective. Indeed, if $N, N' > 0$ are such that $E(\tilde{\chi}, N) = E(\tilde{\chi}, N') =: E$, then using the uniqueness proved in part 1 (or also Lemma 2.9 below), we obtain $\tilde{\chi}_{N,P} = \tilde{\chi}_{N',P} = \chi_{P,E}$, and then $N = N'$. This implies that $E(\tilde{\chi}, N)$ is a monotone function of $\mathbb{R}_+$. Since

$$
E(\tilde{\chi}, N) \geq m c N \to \infty \quad \text{when} \quad N \to \infty,
$$

it follows that it may only be increasing.

\underline{Lemma 2.9.} For any SFD state $\chi = 1_E$, $E = \{ \beta^0 p^0 - \beta \cdot p \leq 1 \}$ and any Borel set $B$, we have

$$
N(1_B) = N(\chi), \quad P(1_B) = P(\chi) \quad \text{implies} \quad E(1_B) \geq E(\chi)
$$
where the inequality is strict if \( B \neq \mathcal{E} \) a.e., and

\[
P(1_B) = P(\chi), \quad E(1_B) = E(\chi) \quad \text{implies} \quad N(1_B) \leq N(\chi),
\]

and the inequality is strict if \( B \neq \mathcal{E} \) a.e.

**Remark 2.10.** As a consequence of Lemma 2.9, we recover that for any given \( N > 0 \) and \( P \in \mathbb{R}^3 \) there exists at most one SFD state \( \chi \) such that \( N(\chi) = N \) and \( P(\chi) = P \) and that for any given \( P \in \mathbb{R}^3 \) and \( E > 0 \) there exists at most one SFD state \( \chi \) such that \( P(\chi) = P \) and \( E(\chi) = E \). Indeed, for example for the first claim, if \( \chi = 1_\mathcal{E} \) and \( \chi' = 1_{\mathcal{E}'} \) are two SFD distributions such that \( N(\chi) = N(\chi') \) and \( P(\chi) = P(\chi') \), we have, using twice Lemma 2.9, \( E(\chi) = E(\chi') \) and therefore \( \mathcal{E} = \mathcal{E}' \) a.e., so that \( \chi = \chi' \).

**Proof of Lemma 2.9.** We only prove (2.48) since the proof of (2.49) is exactly the same. Given a SFD \( \chi = 1_\mathcal{E} \) and a Borel set \( B \) satisfying \( N(1_B) = N(\chi), \quad P(1_B) = P(\chi) \), we just compute

\[
E(1_B) - E(\chi) = \int_{B} p^0 \, dp - \int_{\mathcal{E}} p^0 \, dp = \int_{B \setminus \mathcal{E}} p^0 \, dp - \int_{\mathcal{E} \setminus B} p^0 \, dp \\
\geq \frac{1}{\beta_0} \left[ \int_{B \setminus \mathcal{E}} (1 + \beta \cdot p) \, dp - \int_{\mathcal{E} \setminus B} (1 + \beta \cdot p) \, dp \right] \\
\geq \frac{1}{\beta_0} \left[ \int_{B} (1 + \beta \cdot p) \, dp - \int_{\mathcal{E}} (1 + \beta \cdot p) \, dp \right] = 0,
\]

and the inequality is strict if \( B \neq \mathcal{E} \) a.e. \( \square \)

**Lemma 2.11.** Consider \( \varepsilon : \mathbb{R}^2 \rightarrow \mathbb{R} \) a radial and increasing (with respect to the radius) function. For any \( 0 \leq g \in L^1(\mathbb{R}^2) \), let define \( \rho \geq 0 \) by

\[
\pi \rho^2 = \int_{\mathbb{R}^2} g(p') \, dp' \quad \text{and} \quad g^* = 1_{\{p' \mid < \rho\}}.
\]

For any \( 0 \leq \phi \in L^1(\mathbb{R}^2) = L^1(\mathbb{R}^2; \mathbb{R}) \) let also define

\[
n(\phi) := \int_{\mathbb{R}^2} \phi(p') \, dp', \quad e(\phi) := \int_{\mathbb{R}^2} \phi(p') \varepsilon(p) \, dp'.
\]

Therefore, for any \( g \in L^1 \) we have

\[
n(g^*) = n(g) \quad \text{and} \quad e(g^*) \leq e(g).
\]

**Proof of Lemma 2.11.** It is done in several steps.

**Step 1.** Assume that \( g(p') = \lambda(t) 1_{\{p \mid \varepsilon(p) \leq t\}} \) with \( 0 < \lambda < 1 \), and define \( g_{\lambda(t)}(p') = \lambda(t) 1_{\{p \mid \varepsilon(p) \leq b_t\}} \) with \( b_t = b - t \) and \( \lambda(t) \) defined by the condition \( n(g_{\lambda(t)}) = n(g) \) \( \forall t \geq 0 \). This condition implies that

\[
\frac{d}{dt}[n(g_{\lambda(t)})] = \frac{d}{dt}[\lambda(t) \int_{a}^{b_t} \varepsilon(r) \, dr] = \lambda(t) \int_{a}^{b_t} \varepsilon(r) \, dr + \lambda(t) b_t b'_t = 0,
\]

and thus

\[
\lambda(t) \int_{a}^{b_t} \varepsilon(r) \, dr = \lambda(t) b_t.
\]

Then, we compute

\[
\frac{d}{dt}[e(g_{\lambda(t)})] = \frac{d}{dt}[\lambda(t) \int_{a}^{b_t} \varepsilon(r) \, dr] = \lambda(t) \int_{a}^{b_t} \varepsilon(r) \, dr + \lambda(t) \varepsilon(b_t) b_t b'_t \\
= \frac{\lambda(t) b_t b'_t}{\int_{a}^{b_t} \varepsilon(r) \, dr} \int_{a}^{b_t} (\varepsilon(r) - \varepsilon(b_t)) \, dr \leq 0,
\]

24
and therefore, in particular, the function \( g_1 = 1_{\{a \leq |\nu'| \leq c_1\}} \), uniquely defined by \( n(g_1) = n(g) \), satisfies \( e(g_1) \leq e(g) \).

**Step 2.** Assume now that \( g(p') = 1_{\{a \leq |\nu'| \leq b\}} \) and let define \( g^t(p') = 1_{\{a + t \leq |\nu'| \leq b_t\}} \) with \( a_k = a - t \) and \( b_t \) defined by the condition \( n(g^t) = n(g) \forall t \geq 0 \). This condition implies

\[
\frac{d}{dt}[n(g^t)] = \frac{d}{dt}\int_{a_t}^{b_t} \nu(r) \, dr = \frac{1}{2} (b_t b'_t - a_t a'_t) = 0,
\]

so that \( b_t b'_t = -a_t \). Then, we compute

\[
\frac{d}{dt}[e(g^t)] = \frac{d}{dt}\int_{a_t}^{b_t} \nu(r) \, dr = -a_t (\nu(b_t) - \nu(a_t)) \leq 0,
\]

and therefore, for any \( t \geq 0 \), we have \( n(g^t) = n(g) \), \( e(g^t) \leq e(g) \).

**Step 3.** Assume now that \( g \) is a step function, of the form

\[
g(p') = \sum_{k=1}^{K} \lambda_k 1_{\{a_k \leq |\nu'| \leq b_k\}}
\]

with \( 0 < \lambda_k \leq 1 \) and \( a_1 < b_1 \leq a_2 < b_2 \leq \ldots \leq a_K < b_K \). Using Step 1 we construct \( g_1 \) of the form

\[
g_1(p') = \sum_{k=1}^{K} 1_{\{a_k \leq |\nu'| \leq c_k\}},
\]

with \( a_1 < c_1 \leq a_2 < c_2 \leq \ldots \leq a_K < c_K \), such that \( n(g_1) = n(g) \) and \( e(g_1) \leq e(g) \). By induction on \( k = 1, \ldots, K \), using Step 2 on each function \( 1_{\{a_k \leq |\nu'| \leq c_k\}} \) we obtain that (2.52) holds for any step function \( g \).

**Step 4.** For a radial function \( g \) we argue by density. We consider a sequence \( \{g_n\} \) of step functions of the form (2.53) and such that \( g_n \rightarrow g \) in \( L^1_c(\mathbb{R}^2) \), in particular \( n(g_n) \rightarrow n(g) \) and \( e(g_n) \rightarrow e(g) \). This implies that \( \rho_n \rightarrow \rho \), where \( \rho \) and \( \rho_n \) are defined by (2.49), and therefore that \( g^*_n = 1_{\{|\nu'| \leq \rho_n\}} \) is a step function of the form (2.52) and we can pass to the limit: (2.50) is proved for any radial function.

**Step 5.** Finally, for a general function \( g \) we define the radial function \( \tilde{g} \) by

\[
\tilde{g}(p') := \frac{1}{2\pi} \int_{S^1} g(|p'| \omega) \, d\omega.
\]

Since \( \nu \) is radial we get \( n(\tilde{g}) = n(g) \) and \( e(\tilde{g}) = e(g) \). We apply Step 4 to conclude. 

\[ \square \]

**Proposition 2.12.** For any SFD state \( \gamma \) and any \( f \) such that \( 0 \leq f \leq 1 \), we have

\[
N(f) = N(\gamma), \quad P(f) = P(\gamma) \quad \text{implies} \quad E(f) \geq E(\gamma).
\]

**Proof of Proposition 2.12.** Let introduce \( e_1 \in S^2 \) such that \( P = \{p\} e_1 \) and let us write \( p = (p_1, p') := p_1 e_1 + p' \perp e_1 \). We have

\[
N(f) = \int_{\mathbb{R}} n(f(p_1,\cdot)) \, dp_1, \quad P(f) = \int_{\mathbb{R}} p_1 n(f(p_1,\cdot)) \, dp_1,
\]

(2.54)

\[
E(f) = \int_{\mathbb{R}} e(f(p_1,\cdot), p_1) \, dp_1.
\]
where for any \( \phi \geq 0 \) measurable, \( n(\phi) \) is defined by (2.49) and

\[
e(\phi, p_1) = \int_{\mathbb{R}^2} \phi(p') \sqrt{m^2 c^2 + |p_1|^2 + |p'|^2} \, dp'.
\]

For a.e. \( p_1 \in \mathbb{R} \) we define \( \rho(p_1) \) by

\[
\pi \rho(p_1)^2 = \int_{\mathbb{R}^2} f(p_1, p') \, dp', \quad f^*(p) = 1_{B(0, \rho(p_1))}(p') = 1_B,
\]

where \( B \) is nothing but \( B := \{ p \in \mathbb{R}^3, p_1 \in \mathbb{R}, p' \in B(0, \rho(p_1)) \} \}. Of course, \( N(f^*) = N(f) \) and \( P(f^*) = P(f) \). Therefore thanks to Lemma 2.9, we obtain \( B \) such that

\[
N(1_B) = N(f), \quad P(1_B) = P(f) \quad E(1_B) \leq E(f),
\]

and we conclude thanks to Lemma 2.9.

**Proof of part 1 of Theorem 2.7.** From Proposition 2.8 we already know that for any \( P \in \mathbb{R}^3 \) and \( E > |P| \) there exists a unique SFD state \( \chi_{P,E} \) such that \( P(\chi_{P,E}) = P \) and \( E(\chi_{P,E}) = E \). Consider now \( f \) such that \( 0 \leq f \leq 1 \) and define

\[
N := N(f), \quad P := P(f) \quad \text{and} \quad E := E(f),
\]

we have just to prove that

\[
N(f) \leq N(\chi_{P,E}).
\]

Thanks to Proposition 2.8 we know that there exists an unique SFD state \( \chi_{P,E} \) such that

\[
P(\chi_{P,E}) = P, \quad \text{and} \quad E(\chi_{P,E}) = E,
\]

and there also exists an unique SDF state \( \tilde{\chi}_{N,P} \) such that

\[
N(\tilde{\chi}_{N,P}) = N, \quad \text{and} \quad P(\tilde{\chi}_{N,P}) = P.
\]

Moreover, from Proposition 2.10

\[
E(\chi_{P,E}) = E(f) \geq E(\tilde{\chi}_{N,P}).
\]

But since from Proposition 2.8 the application \( N \mapsto E(\tilde{\chi}_{N,P}) \) is increasing this implies that

\[
N(\chi_{P,E}) \geq N(\tilde{\chi}_{N,P}) = N(f),
\]

and this conclude the proof.

**Proof of part 2 of Theorem 2.7.** We only need to prove that for any \( P \in \mathbb{R}^3 \) and any \( E > 0 \) such that \( |P| < E \) and for any \( N \in [0, N(\chi_{P,E})] \) there exists \( \mathcal{F} \) a Fermi-Dirac distribution such that

\[
N(\mathcal{F}) = N, \quad P(\mathcal{F}) = P, \quad E(\mathcal{F}) = E.
\]

It is straightforward to check that \( \mathcal{F} \) realizes the momentum equations (2.55) if, and only if

\[
\begin{cases}
N = N(\tilde{\beta}, \mu) \gamma(u) \\
E - \frac{P}{u} = -H(\tilde{\beta}, \mu) \\
E - Pu = E(\tilde{\beta}, \mu)
\end{cases}
\]
where
\[ L(\tilde{\beta}, \mu) = L(\tilde{\mathcal{F}}), \quad \mathcal{F} = \mathcal{F}_{\tilde{\beta}, 0, \mu} = \frac{1}{e^{\tilde{\beta} p + \mu} + 1}. \]

We then define,
\[ u = u(\tilde{\beta}, \mu) := \frac{P}{E + H(\tilde{\beta}, \mu)} \]
and reduce the system (2.56) to
\[
\begin{align*}
\Sigma_E(\tilde{\beta}, \mu) &:= E(\tilde{\beta}, \mu) + \frac{P^2}{E + H(\tilde{\beta}, \mu)} = E \\
\Sigma_N(\tilde{\beta}, \mu) &:= N(\tilde{\beta}, \mu) \gamma(u(\tilde{\beta}, \mu)) = N.
\end{align*}
\]

We conclude thanks to the following result.

**Lemma 2.13.**

1. For any fixed \( \mu \) the function \( \Sigma_E(\cdot, \mu) : \mathbb{R}_+^* \to \mathbb{R} \) is continuous, decreasing. \( \Sigma_E(\tilde{\beta}, \mu) \to +\infty \) when \( \tilde{\beta} \to 0 \) and \( \Sigma_E(\tilde{\beta}, \mu) \to 0 \) when \( \tilde{\beta} \to +\infty \). Therefore, for any \( \mu \) there exists a unique \( \tilde{\beta} = \tilde{\beta}(\mu) > 0 \) such that \( \Sigma_E(\tilde{\beta}, \mu) = E \).

2. The function \( \tilde{\beta} : \mathbb{R} \to \mathbb{R} \) is continuous and decreasing, \( \tilde{\beta}(\mu) \to +\infty \) and \( \tilde{\beta}(\mu) + \mu \to +\infty \) when \( \mu \to -\infty \) and \( \tilde{\beta}(\mu) \to 0 \) when \( \mu \to +\infty \).

3. The function \( \tilde{E} : \mathbb{R} \to \mathbb{R} \), \( \tilde{E}(\mu) := \Sigma_N(\tilde{\beta}(\mu), \mu) \) is continuous, decreasing, \( N(\mathcal{F}) \to N(\chi_{P, E}) \) when \( \mu \to -\infty \) and \( \tilde{E}(\mu) \to 0 \) when \( \mu \to +\infty \).

**Proof of Lemma 2.13.** The proofs of the two first points are very similar to those of the same points in Lemma 2.3 and Lemma 2.6. We therefore do not repeat them. The only new point is the third, and more precisely the behavior of \( \Sigma_N \) for \( \mu \to -\infty \). In order to prove it, let \( (\mu_n) \) be a decreasing sequence such that \( \mu_n \to -\infty \) and set \( \tilde{\beta}_n = \tilde{\beta}(\mu_n) \).

**Step 1.** We write
\[ 0 < E - P \leq E(\tilde{\beta}, \mu) = \int_{\mathbb{R}^3} \frac{p^3}{e^{\tilde{\beta} (p^2 - 1)} + \tilde{\beta} + \mu} \, dp. \]
When \( \mu \to -\infty \) we have \( \tilde{\beta} = \tilde{\beta}(\mu) \to +\infty \), and this implies that \( \tilde{\beta} + \mu \to -\infty \). In particular, we have \( \tilde{\beta} + \mu \leq 0 \) so that
\[ -\frac{\mu}{\tilde{\beta}(\mu)} \geq 1. \]

**Step 2.** Since
\[ \int_{\mathbb{R}^3} \frac{|z|}{e^{\tilde{\beta} mc + \mu} (|z| - 1)} + 1 \, dz \geq \int_{B_1} \frac{|z|}{e^{\tilde{\beta} mc + \mu} (|z| - 1)} + 1 \, dz \geq \int_{B_1} \frac{|z|}{|z| - 1} \, dz =: C_E > 0, \]
we deduce
\[ E \geq E(\tilde{\beta}, \mu) \geq \int_{\mathbb{R}^3} \frac{|p|}{e^{\tilde{\beta} (p^2 + mc^2)} + 1} \, dp = \left( \frac{\tilde{\beta} mc + \mu}{\tilde{\beta}} \right)^4 \int_{\mathbb{R}^3} \frac{|z|}{e^{\tilde{\beta} mc + \mu} (|z| - 1)} + 1 \, dz \geq \left( \frac{\tilde{\beta} mc + \mu}{\tilde{\beta}} \right)^4 C_E, \]
Therefore
\[ \left| \frac{\mu_n}{\tilde{\beta}_n} \right| \leq mc + \left| mc + \frac{\mu_n}{\tilde{\beta}_n} \right| \leq \left( \frac{E}{C_E} \right)^{1/4} + mc, \]

27
and there exists \( a \in \mathbb{R} \) and a subsequence \((n')\) such that
\[
\frac{\mu_{n'}}{\beta_{n'}} \to -a,
\]
and by (2.59) \( a \geq 1 \).

**Step 3.** We remark now that if \( n \) is large enough \( \beta_n \geq 1 \) and \( |\mu_n|/\beta_n \leq 2a \), from where we deduce that
\[
0 \leq \frac{1}{e^{\beta_n (p^0 - |\mu_n|/\beta_n)} + 1} \leq 1_{p^0 \leq 2a} + \frac{1}{e^{p^0 - 2a} + 1} 1_{p^0 \geq 2a} \in L^1(\mathbb{R}^3).
\]
Moreover, for a.e. \( p \in \mathbb{R}^3 \) we have
\[
\mathcal{F}_n = \frac{1}{e^{\beta_n (p^0 - |\mu_n|/\beta_n)} + 1} \to 1_{\{p^0 \leq a\}}.
\]
We conclude by dominated convergence Lebesgue Theorem that
\[
L(\beta_{n'}, \mu_{n'}) = L(\mathcal{F}_{n'}) \to \tilde{L}(a) := L(1_{\{p^0 \leq a\}}) \quad \text{for} \quad L = N, E, G, H.
\]

**Step 4.** Passing to the limit in (2.57) and (2.58) we obtain
\[
E = \tilde{E}(a) + P \frac{\tilde{u}(a)}{c}, \quad \tilde{u}(a) = \frac{P}{E + H(a)}
\]
and
\[
\lim_{\mu \to -\infty} \tilde{\Sigma}_N(\mu) = N(a) \gamma(\tilde{u}(a)).
\]
By (2.43) this means precisely that the SFD state associated to \((\tilde{\beta}, \tilde{u}) = (1/a, \tilde{u}(1/a))\) satisfies
\[
\lim_{\mu \to -\infty} \tilde{\Sigma}_N(\mu) = N(\chi), \quad P(\chi) = P, \quad E(\chi) = E.
\]
Therefore, \( \chi = \chi_{P,E} \) and \( \tilde{\Sigma}_N(\mu) = N(\mathcal{F}) \to N(\chi_{P,E}) \). \( \square \)

**Nonrelativistic Fermi-Dirac particles.** Here again, since the energy is \( \mathcal{E}(p) = |p^2|/2m \) the problem (2.1) is equivalent to (2.35): given three quantities \( N > 0, E > 0, P \in \mathbb{R}^3 \) find \( f(p) \) such that \( 0 \leq f \leq 1 \) and satisfying (2.35).

On the other hand, given \( N > 0 \) and \( P \in \mathbb{R}^3 \), we have
\[
\min_{\{0 \leq f \leq 1, \int f dp = N, \int f p dp = P\}} \frac{1}{2m} \int |p|^2 f(p) dp = \min_{\{0 \leq g \leq 1, \int g dp = N, \int g |p|^2 dp = 0\}} \frac{1}{2m} \int |p - \frac{P}{N}|^2 g(p) dp
\]
and this minimum is reached for a distribution of the form \( g(p) = 1_{\{|p - \frac{P}{N}| \leq c\}} \), for some \( c > 0 \) and for a distribution of the form \( f(p) = 1_{\{|p| \leq c\}} \). Since
\[
\int_{\mathbb{R}^3} 1_{\{|p| \leq c\}} dp = \omega_2 \frac{c^3}{3} \quad \text{and} \quad \int_{\mathbb{R}^3} |p|^2 1_{\{|p| \leq c\}} dp = \omega_2 \frac{c^5}{5}
\]

28
we deduce that the problem (2.1) has a solution \( f \) satisfying \( 0 \leq f \leq 1 \) only if
\[
(2.60) \quad E \geq \frac{3\sqrt{3}(4\pi)^{2/3}}{5} N^{5/3}.
\]
On the other hand, it is simple, using elementary calculus, to prove that for any \( E, N > 0, P \in \mathbb{R}^3 \) such that (2.58) holds, there exists a distribution of the from
\[
(2.61) \quad F(p) = \begin{cases} 
\frac{1}{e^{\phi(p)} + 1} & \text{if } E > \frac{3\sqrt{3}(4\pi)^{2/3}}{5} N^{5/3}, \\
1_{\{|p - \frac{E}{N}| \leq c\}} & \text{if } E = \frac{3\sqrt{3}(4\pi)^{2/3}}{5} N^{5/3}
\end{cases}
\]
with \( a \in \mathbb{R}, b \in \mathbb{R} \) and \( c \in \mathbb{R} \) such that \( \omega_2 c^3/3 = N \) satisfying (2.35).

Then, for every \( E, N > 0 \) and \( P \in \mathbb{R}^3 \) satisfying (2.60), the Fermi Dirac distribution:
\[
(2.62) \quad \mathcal{F}(p) = \begin{cases} 
\frac{1}{e^{\phi(p)} + 1} \cdot \nu(p) = a|P|^2 - \frac{2a}{N} P \cdot P + (b + \frac{2|P|^2}{N}) & \text{if } E > \frac{3\sqrt{3}(4\pi)^{2/3}}{5} N^{5/3}, \\
1_{\{|p - \frac{E}{N}| \leq c\}} & \text{if } E = \frac{3\sqrt{3}(4\pi)^{2/3}}{5} N^{5/3}
\end{cases}
\]
solves (2.1).

This shows that for non relativistic particles Theorem 2.7 remains valid under the unique following change: the points 1 and 2 have to be replaced by

1. For every \( E, N > 0, P \in \mathbb{R}^3 \), satisfying (2.60) there exists a non relativistic fermi Dirac state, saturated or not, defined by (2.62) corresponding to these momentum, i.e. satisfying (2.1).

Points 3 and 4 of Theorem 2.7 remain unchanged.

### III. THE BOLTZMANN EQUATION FOR ONE SINGLE SPECIE OF QUANTUM PARTICLES.

We consider now the homogeneous Boltzmann equation for Quantum non relativistic particles, and treat both Fermi-Dirac and Bose-Einstein particles. We begin with the Fermi-Dirac Boltzmann equation for which we may slightly improve the existence result of J. Dolbeault [JD] and P.-L. Lions [L]. We also state a very simple (and weak) result concerning the long time behavior of solutions. We finally consider the Bose-Einstein Boltzmann equation. We discuss the work of X. Lu [XL] and slightly extend some of its results to the natural framework of measure distributions.

Let us then consider the non relativistic quantum Boltzmann equation
\[
(3.1) \quad \begin{cases} 
\frac{\partial f}{\partial t} = Q(f) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} w \delta_c q(f) dv_* dv', \\
q(f) = (f' f_* (1 + \tau f) (1 + \tau f_*) - f f_* (1 + \tau f') (1 + \tau f'_*))
\end{cases}
\]
with \( \tau = \pm 1 \), where \( C \) is the non relativistic collision manifold of \( \mathbb{R}^{12} \), precisely
\[
(3.2) \quad (C) : \quad \begin{cases} 
|v_* + v'_*| = v + v_* \\
\frac{|v_*|^2}{2} + \frac{|v'_*|^2}{2} = \frac{|v|^2}{2} + \frac{|v_*|^2}{2},
\end{cases}
\]
where we have assumed, without any loss of generality in this Section that the mass \( m \) of the particles is one.
III.1. The Boltzmann equation for Fermi-Dirac particles.

We first want to give a mathematical sense to the collision operator $Q$ in (3.1) under the physical natural bounds on the distribution $f$. Of course, if $f$ is smooth (say $C_c(\mathbb{R}^3)$) and $w$ is smooth (for instance $w = 1$) the collision term $Q(f)$ is defined in the distributional sense as it has been mentioned in the Introduction. But, as we have already seen, the physical space of densities in the Fermi-Dirac particles framework is $L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ so that any assumption of regularity on $f$ is meaningless.

First, in order to give a pointwise sense of the formula (3.1) we recall the following elementary argument from [Gl]. After integration with respect to the variable $\nu'$ we have

\[ Q(f) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} w(q(f)) \delta(\frac{\|v\|^2}{2} - \frac{\|v_*\|^2}{2} - \frac{\|v - v_*\|^2}{2} - \frac{\|v - v_*\|^2}{2}) \, dv \, dv'. \]

By the change of variable $v' \rightarrow (r, \omega)$ with

\[ r \in \mathbb{R}, \quad \omega \in S^2 \quad \text{and} \quad v' = v + r \omega, \]

and using Lemma A1, we obtain

\[ Q(f) = \frac{1}{2} \int_{\mathbb{R}^3} \int_{S^2} \int_0^\infty w(q(f)) \delta(r - (v_* - v) \cdot \omega) \, r^2 \, dr \, d\sigma \, dv_*, \]

(3.3)

where now $v'$ and $v_*$ are defined by

\[ v' = v + (v_* - v) \omega, \quad v_* = v_* - (v_* - v) \omega. \]

Formula (3.3) gives a pointwise sense to $Q(f)$, say for $f \in C_c(\mathbb{R}^3)$. In order to extend the definition of $Q(f)$ to measurable functions we proceed as follow.

Next, we make the following assumption on the cross-section

\[ B = \frac{1}{2} w |(v_* - v, \omega)| \quad \text{is a function of } v_* - v \text{ and } \omega \]

and

\[ B \in L^1(\mathbb{R}^3 \times S^2). \]

(3.6)

Though (3.5) is a natural assumption in view of section 3.1, (3.6) is a strong restriction, in particular it does not hold when $w = 1$. With these assumptions, first introduced in [D], we explain how to give a sense to the collision term $Q(f)$ when $f \in L^1 \cap L^\infty$. In one hand, since $\Phi : (v, v_*, \omega) \mapsto (v', v'_*, \omega)$ (with $v'$, $v'_*$ given by (3.4)) is a $C^1$-diffeomorphism on $\mathbb{R}^3 \times \mathbb{R}^3 \times S^2$ with Jacobian $\text{Jac} \Phi = 1$, we can multiply that $\Phi(v, v_*, \omega) \mapsto f f'_* \text{ is a measurable function of } \mathbb{R}^3 \times \mathbb{R}^3 \times S^2$. On the other hand, performing a change of variable, we get

\[ \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} f f'_* \, dv \, d\omega = \int_{\mathbb{R}^3 \times \mathbb{R}^3} f f_* \left( \int_{S^2} B \, d\omega \right) (v_* - v) \, dv \, dv_*, \]

\[ \leq \|f\|_{L^1} \|f\|_{L^\infty} \|B\|_{L^1} < \infty, \]

and we deduce by the Fubini-Tonelli Theorem

\[ \int_{S^2} B f f' \, d\omega \in L^1(\mathbb{R}^3 \times \mathbb{R}^3). \]
Therefore, that gives a sense to the gain term

\[ Q^+(f) = \int_{\mathbb{R}^3} (1 - f) (1 - f_\ast) \left( \int_{S^2} B f' f' \, d\omega \right) \, dv_\ast \]

as a \( L^1 \) function. In the same way we give a sense to the loss term \( Q^-(f) \) as a \( L^1 \) function.

Notice that, under the assumption

\[ B \in L^1_{\text{loc}}(\mathbb{R}^3 \times S^2) \quad \text{and} \quad \frac{1}{1 + |z|^2} \int_{B_k \times S^2} B(z + v, \omega) \, d\omega \, dv \xrightarrow{|v|\to\infty} 0 \quad \forall R > 0, \]

we may give a sense to \( Q^\pm(f) \) as a function of \( L^1(B_R) \) (\( \forall R > 0 \)) for any \( f \in L^1_2 \cap L^\infty \), see [L3]. In particular, the cross-section \( B \) associated to \( w = 1 \) satisfies (3.7).

Finally, we can make a third assumption on the cross-section, namely

\[ 0 \leq B(z, \omega) \leq (1 + |z|^{\gamma}) \zeta(\theta), \quad \text{with} \quad \gamma \in (-5, 0), \quad \int_0^{\pi/2} \theta \zeta(\theta) \, d\theta < \infty. \]

This assumption allows singular cross-sections, both in the \( z \) variable and the \( \theta \) variable, near the origin. In that case, the following distributional sense may be given to the collision term

\[ \int_{\mathbb{R}^3} Q(f) \varphi \, dv = -\int_{\mathbb{R}^3} \int_{S^2} f \cdot \left( 1 - f' - f'_\ast \right) B K_\varphi \, dv_\ast \, d\omega \]

with the notation

\[ K_\varphi = \varphi' + \varphi'_\ast - \varphi - \varphi_\ast. \]

In order to see why the collision integral (3.9) is well defined we notice that

\[ |K_\varphi| \leq C_\varphi |v - v_\ast|^2 \theta. \]

see e.g. [V], so that

\[ \left| f \cdot \left( 1 - f' - f'_\ast \right) B K_\varphi \right| \leq C_\varphi \theta \zeta(\theta) \int f \cdot (|v - v_\ast|^2 + |v - v_\ast|^{\gamma + 2}) \in L^1(\mathbb{R}^3 \times \mathbb{R}^3 \times S^2). \]

To see this last claim, we just put \( g(z) = |z|^\alpha \); if \( \alpha \in (-3, 0) \) one has \( g \in L^1 + L^\infty \) and therefore \( f \ast (\ast g) \in L^1 \) when \( f \in L^1 \cap L^\infty \), and if \( \alpha \in [0, 2] \) then writing \( g(v - v_\ast) \leq (1 + |v|^2)(1 + |v_\ast|^2) \) we see that \( f \ast (\ast g) \in L^1 \) when \( f \in L^2_2 \).

**Theorem 3.1.** Assume that one of the conditions (3.6), (3.7) or (3.8) holds. For any \( f_{in} \in L^1_2(\mathbb{R}^3) \) such that \( 0 \leq f_{in} \leq 1 \) there exists a solution \( f \in C([0, +\infty); L^1(\mathbb{R}^3)) \) to equation (3.1). Furthermore,

\[ \int_{\mathbb{R}^3} f(t, v) \, dv = \int_{\mathbb{R}^3} f_{in} \, dv =: N, \]

\[ \int_{\mathbb{R}^3} f(t, v) v \, dv = \int_{\mathbb{R}^3} f_{in}(v) v \, dv =: P, \]

\[ \int_{\mathbb{R}^3} f(t, v) \frac{|v|^2}{2} \, dv \leq \int_{\mathbb{R}^3} f_{in}(v) \frac{|v|^2}{2} \, dv =: E, \]

and

\[ \int_0^\infty D(f) \, dt \leq C(f_{in}), \]
Villani. Finally, bounds on modified entropy dissipation term of the kind of \((\beta \partial f_0^*)\) has been introduced.

Assumption \((\beta \partial f_0^*)\) \((without \text{Grad's cut-off})\) has been established by L. Arkeryd, T. Goudon and C. Villani for the Boltzmann-Bose equation.

Concerning the behaviour of the solutions we prove the following result.

**Theorem 3.3.** For any sequence \((t_n)\) such that \(t_n \to +\infty\) there exists a subsequence \((t_{n'})\) and a stationary solution \(S\) such that

\[
 f(t_{n'} + \ldots) \xrightarrow{n' \to \infty} S \quad \text{in } C([0,T]; L^1 \cap L^\infty(\mathbb{R}^3) \text{ weak}) \quad \forall T > 0
\]

and

\[
 \int_{\mathbb{R}^3} S \, dv = N, \quad \int_{\mathbb{R}^3} S \, v \, dv = P, \quad \int_{\mathbb{R}^3} \frac{\lvert v \rvert^2}{2} \, dv \leq E.
\]

**Remark 3.4.** By stationary solution we mean a function \(S\) satisfying

\[
 S' \left(1 - S\right) \left(1 - S_*\right) = S \, S_* \left(1 - S'\right) \left(1 - S'_*\right).
\]

Of course, such a function is, formally at least, a Fermi-Dirac distribution. We refer to [D] for the proof of this claim in a particular case. Theorem 3.2 also holds for the non homogeneous Boltzmann-Fermi equation, when, for example, the position space is the torus, and under assumption (3.7) on \(B\) (in order to have existence).

**Open questions.**

1. Is Theorem 3.1 true under assumption (3.8) for all \(\gamma \in (-5, -2)\)?
2. Is it possible to prove the entropy identity (1.14) instead of the modified entropy dissipation bound (1.14)? Of course (1.14) implies the dissipation entropy bound (3.14) as it will be clear in the proof of Theorem 3.2.
3. Is any function satisfying (3.18) a Fermi-Dirac distribution?
4. Is it true that

\[
 \sup_{[0, \infty)} \int_{\mathbb{R}^3} f(t, v) \, \lvert v \rvert^{2+\varepsilon} \, dv \leq C(f_m, \varepsilon)
\]

for some \(\varepsilon > 0\)? Notice that with this estimate in hand we should be able to prove conservation of energy (instead of (3.17)) and convergence to the Fermi-Dirac distribution \(\mathcal{F}_{N, P, E}\), if we can also give a positive answer to 3.
5. Finally, is it possible to improve the convergence (3.16), and prove for instance strong \(L^1\) convergence?

**Proof of Theorem 3.1.** Let \(B\) satisfies (3.6), (3.7) or (3.8) and let define \(B_\varepsilon = B \mathbb{1}_{\theta > \varepsilon} \mathbb{1}_{\varepsilon < |z| < 1/\varepsilon}\). Notice that \(B_\varepsilon\) satisfies (3.6), \(0 \leq B_\varepsilon \leq B\) and \(B_\varepsilon \to B\) a.e. From [D] there exists a sequence of solutions \((f_\varepsilon)\) to (3.1) corresponding to \((B_\varepsilon)\). Moreover, for any \(\varepsilon > 0\), the solution \(f_\varepsilon\) satisfies (3.13) and (1.14). In [L], P.L. Lions has shown that

\[
 a_\varepsilon f_\varepsilon f_\varepsilon^* (1 - f_\varepsilon - f_\varepsilon^*) \xrightarrow{n' \to \infty} a f f_* (1 - f - f_*) \quad L^1 \text{ weak.}
\]

(3.19)

\[
 a_\varepsilon f_\varepsilon f_\varepsilon^* (1 - f_\varepsilon^* - f_*^*) \xrightarrow{n' \to \infty} a f f_* (1 - f^* - f_*^*) \quad L^1 \text{ weak.}
\]

where

\[
 \tilde{D}(f) := \int \int \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} B_2 \left(f' f_0^* (1 - f) (1 - f_*) - f f_* (1 - f') (1 - f_*)^2\right) \, d\omega \, dv, \, dv.
\]
for any sequences \((a_\varepsilon)\) such that \(f_\varepsilon \to f\) in \(L^1 \cap L^\infty\),
\[
\int_{\mathbb{R}^3} f_\varepsilon \psi \, dv \xrightarrow{\varepsilon \to 0} \int_{\mathbb{R}^3} f \, \psi \, dv \quad \forall \psi \in \mathcal{D}(\mathbb{R}^3),
\]
and any sequence \((a_\varepsilon)\) which satisfies (3.8) uniformly in \(\varepsilon\) and such that \(a_\varepsilon \to a\) a.e. In particular, under assumption (3.8) on \(B\) and taking \(a_\varepsilon = B_\varepsilon, a = B\), he was able to pass to the limit \(\varepsilon \to 0\) in equation (3.1) remarking that
\[
Q(f) = \int_{\mathbb{R}^3} B \left[ f' f'_s (1 - f - f_s) - f f_s (1 - f' - f'_s) \right] \, dw.
\]
That gives existence of a solution \(f\) to the Fermi-Boltzmann equation for \(B\) satisfying (3.7).

Now, for \(B\) satisfying (3.8), we just write
\[
Q_\varepsilon(f_\varepsilon) \varphi \, dv = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_\varepsilon f_* (1 - f'_\varepsilon - f'_s) B_\varepsilon K_\varphi \mathbf{1}_{\{\theta \geq \delta, |v - v_*| \geq \delta\}} \, dv \, dw
\]
\[
+ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_\varepsilon f_* (1 - f'_\varepsilon - f'_s) B_\varepsilon K_\varphi \mathbf{1}_{\{\theta \leq \delta, |v - v_*| \leq \delta\}} \mathbf{1}_{\{|v - v_*| \leq \delta\}} \, dv \, dw
\]
\[
= Q_{\delta, \varepsilon} + r_{\delta, \varepsilon}.
\]
For the first term \(Q_{\delta, \varepsilon}\) we easily pass to the limit \(\varepsilon \to 0\) using Lions' result (3.19). For the second term \(r_{\delta, \varepsilon}\), using (3.11), we have
\[
\lim_{\varepsilon \to 0} r_{\delta, \varepsilon} \leq \lim_{\varepsilon \to 0} C_\varphi \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_\varepsilon f_* (|v - v_*|^2 + |v - v_*|^{\gamma + 2}) \, dv \, dw
\]
\[
+ \lim_{\varepsilon \to 0} C_\varphi \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_\varepsilon f_* (|v - v_*|^2 + |v - v_*|^{\gamma + 2}) \mathbf{1}_{\{|v - v_*| \leq \delta\}} \, dv \, dw
\]
\[
\leq C_{\varphi, f} \int_{\mathbb{R}^3} \theta \zeta(\theta) \mathbf{1}_{\{\theta \leq \delta\}} \, dw + C_{\varphi, \zeta} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_\varepsilon f_* (|v - v_*|^2 + |v - v_*|^{\gamma + 2}) \mathbf{1}_{\{|v - v_*| \leq \delta\}} \, dv \, dw.
\]
Therefore,
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} Q_\varepsilon(f_\varepsilon) \varphi \, dv = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f f_* (1 - f' - f'_s) B K_\varphi \mathbf{1}_{\{\theta \geq \delta, |v - v_*| \geq \delta\}} \, dv \, dw + r_\delta,
\]
with \(r_\delta \to 0\) when \(\delta \to 0\). Since we also have
\[
\int_{\mathbb{R}^3} Q(f) \varphi \, dv = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f f_* (1 - f' - f'_s) B K_\varphi \mathbf{1}_{\{\theta \geq \delta, |v - v_*| \geq \delta\}} \, dv \, dw + \tilde{r}_\delta,
\]
with \(\tilde{r}_\delta \to 0\) when \(\delta \to 0\), we conclude that
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} Q_\varepsilon(f_\varepsilon) \varphi \, dv = \int_{\mathbb{R}^3} Q(f) \varphi \, dv
\]
and \(f\) is a solution to (3.1). We refer to [L] and [V] for more details.

In order to establish (3.14), we take \(a_\varepsilon = a = \sqrt{B_\delta}\) in (3.19) and we then have
\[
\sqrt{B_\delta} \left[ f'_\varepsilon f'_s (1 - f_\varepsilon - f_* - f) - f f_* (1 - f'_\varepsilon - f'_s) \right] \xrightarrow{\varepsilon \to 0}
\]
\[
\sqrt{B_\delta} \left[ f' f'_s (1 - f - f_s) - f f_s (1 - f' - f'_s) \right]
\]
(3.20)
Using the elementary inequality

\[(b - a)^2 \leq j(a, b) \quad \forall a, b \in [0, 1],\]

we have for \(\varepsilon \in (0, \delta]\)

\[
\int_0^\infty \dot{D}_\delta(f_\varepsilon) \, dt \leq \int_0^\infty \dot{D}_\varepsilon(f_\varepsilon) \, dt \leq \int_0^\infty \dot{D}_\varepsilon(f) \, dt \leq C(f_{in}).
\]

Gathering (3.20) and (3.21) we obtain, using the convexity of \(s \mapsto s^2\),

\[
\int_0^\infty \dot{D}_\delta(f) \, dt \leq \liminf_{\varepsilon \to 0} \int_0^\infty \dot{D}_\varepsilon(f_\varepsilon) \, dt \leq C(f_{in}).
\]

and we recover (3.15) letting \(\delta \to 0\). \(\square\)

**Proof of Theorem 3.3.** Let consider \(f_n = f(t + t_n, \cdot)\) as in the statement of the Theorem 3.3. We know that there exists \(n^*\) and \(S\) such that \(f_n^* \to_{n^* \to \infty} S\) weakly in \(L^1 \cap L^\infty((0, T) \times \mathbb{R}^3) \quad \forall T > 0\), and we only have to identify the limit \(S\). On one hand, \(S\) satisfies the moment equation (3.17). On the other hand by (3.21) recalled above and s.c.i. we get

\[
\int_0^T \dot{D}_\delta(S) \, ds \leq \liminf_{\varepsilon \to 0} \int_0^T \dot{D}(f_{n^*}) \, ds \leq \liminf_{t \to t_n^*} \int_0^{T+t_n^*} \dot{D}(f) \, ds = 0
\]

for any \(\delta > 0\), so that

\[
S' S_{s_\nu} (1 - S) (1 - S_{s_\nu}) - S S_{s} (1 - S') (1 - S'_s) = S' S_{s_\nu} (1 - S - S_{s}) - S S_{s} (1 - S' - S_s) = 0
\]

for a.e. \((v, v_*, \omega) \in \mathbb{R}^3 \times \mathbb{R}^3 \times S^2\). In particular,

\[
\frac{\partial S}{\partial t} = Q(S) = 0
\]

and \(S\) is a constant function in time. We last improve the convergence of \(f_{n^*}\) to \(S\) and establish (3.16).

Fix now \(\psi \in C_c(\mathbb{R}^3)\). One has

\[
\frac{d}{dt} \int_{\mathbb{R}^3} f_n \psi \, dv = \int_{\mathbb{R}^3} Q(f_n) \psi \, dv
\]

bounded in \(L^1(0, T)\), so that \(\int_{\mathbb{R}^3} f_n \psi \, dv\) is bounded in \(BV(0, T)\). This implies that we can extract a second subsequence (not relabeled) such that the sequence \(\left(\int_{\mathbb{R}^3} f_n \psi \, dv\right)\) converges strongly \(L^1(0, T)\) and a.e. on \((0, T)\) and the limit is obviously \(\int_{\mathbb{R}^3} S \psi \, dv\). Let fix \(\tau \in (0, T)\) a time such that the above convergence holds, we deduce that

\[
\int_{\mathbb{R}^3} f_n (t, \cdot) \psi \, dv = \int_{\mathbb{R}^3} f_n (\tau) \psi \, dv - \int_{\tau}^t \int_{\mathbb{R}^3} Q(f_n) \psi \, dvds
\]

and using Lions’ result we get

\[
\int_0^T \int_{\mathbb{R}^3} Q(f_n) \psi \, dvds \to \int_0^T \int_{\mathbb{R}^3} Q(S) \psi \, dvds = 0.
\]
and therefore
\[
\sup_{[0,T]} \left| \int_{\mathbb{R}^3} f(t, \nu(\w)) \psi \, d\nu - \int_{\mathbb{R}^3} S \psi \, d\nu \right| \to 0.
\]

\[\square\]

III.2. Bose-Einstein Collision operator for Isotropic density.

We come now to the analysis of the Boltzmann equation for Bose-Einstein particles and take \(\tau = 1\) in (3.1). We start with an elementary remark. Assume that \(w = 1\) and consider a sequence \((f_\varepsilon)\) defined by \(f_\varepsilon(\nu) := \varepsilon^{-3} f(\nu / \varepsilon)\) for a given \(0 \leq f \in C_c(\mathbb{R}^3)\), \(f \neq 0\). An elementary change of variables lead to
\[
\|Q^\pm(f_\varepsilon)\|_{L^1} = \frac{1}{\varepsilon^3} \|Q^\pm(f)\|_{L^1} \to +\infty.
\]

Therefore, no a priori estimate of the form
\[
\|Q(f)\| \leq \Phi(\|f\|_{L^1}), \text{ with } \Phi \in C(\mathbb{R}_+),
\]
can be expected. In particular we will not be able to give a sense to the kernel \(Q(f)\) under the only physical bound \(f \in M^1(\mathbb{R}^3)\) for such a \(w\).

This first remark motivates the two following simplifications, originally performed by X. Lu [Lu], in order to give a sense to \(Q(f)\): we assume that the density is isotropic and we make a strong (and unphysical) truncature assumption on \(w\). We use them here, in a slightly different way that we believe to be simpler.

Assume then until the end of this Section that the density \(f\) only depends of the quantity \(|v|\), and denote \(f(v) = f(|v|) = f(r)\) with \(r = |v|\). For a given function \(q = q(r, r_*; r', r'_*)\) we define
\[
(3.22) \quad Q[q](v) := \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}^3} w \, q \, \delta_C \, dv \, dv' \, dv'_*.
\]

Introducing the new function
\[
(3.23) \quad \hat{w}(r, r_*; r', r'_*) := \int_{\mathbb{R}^3} w(v, v_*, v', v'_*) \delta_{v + v_* = v' + v'_*} \, dv_* \, dv' \, dv'_*.
\]

and the notations \(v_* = r_* \, \sigma_*\), \(v_* = r' \, \sigma'\), \(v'_* = r'_* \, \sigma'_*\), \(d\hat{r}_* = r_*^2 \, dr_*\), \(d\hat{r}'_* = r'_*^2 \, dr'_*\), \(d\hat{r}''_* = r''_*^2 \, dr''_*\)

\[
(3.24) \quad \hat{C} = \{ (r, r_*; r', r'_*) \in \mathbb{R}^3_+, \ r^2 + r_*^2 = r'^2 + r'_*^2 \}.
\]

we may write \(Q[q]\) in the following simpler way
\[
(3.25) \quad Q[q](r) = \int_{\mathbb{R}^3_+} \hat{w} \, \delta_{\hat{C}} \, q \, d\hat{r}_* \, d\hat{r}'_* \, d\hat{r}_**_*.
\]

Let us emphasize that \(\hat{w}\) is indeed a function of \(r = |v|\) (and not on the whole variable \(v\)) since, for any \(R \in SO(3)\), we have
\[
\int_{\mathbb{R}^3_+} \hat{w}(r, r_*; r', r'_*) \delta_{v + v_* = v' + v'_*} \, dv_* \, dv' \, dv'_* = \int_{\mathbb{R}^3} w(Rv, Rv_*, Rv', Rv'_*) \delta_{v + v_* = v' + v'_*} \, dv_* \, dv' \, dv'_*.
\]

35
by change of variables \((\sigma_\ast, \sigma', \sigma'_\ast) \rightarrow (R\sigma_\ast, R\sigma', R\sigma'_\ast)\) and rotation invariance of \(w\). Moreover, \(\dot{w}\) clearly satisfies the following property

\[
(3.26) \quad \dot{w}(r, r_\ast, r', r'_\ast) = \dot{w}(r, r, r', r'_\ast) = \dot{w}(r', r'_\ast, r, r_\ast).
\]

**Lemma 3.5** Assume that \(w\) is such that \(B\) defined by (3.5) satisfies

\[
(3.27) \quad \sup_{z \in \mathbb{R}^3} \frac{1}{1 + |z|^s} \int_{S^2} B(z, \omega) d\omega < \infty
\]

with \(s = 2\) and that \(\dot{w}\) defined by (3.26) satisfies

\[
(3.28) \quad \dot{w}(r + r_\ast + r' + r'_\ast) \in L^\infty(\mathbb{R}^4).
\]

Then for any isotropic function \(f \in L^2_2(\mathbb{R}^3)\) the kernel \(Q^\pm(f)\) is well defined and

\[
(3.29) \quad \|Q^\pm(f)\|_{L^1} \leq C_B \|f\|_{L^2_2(\mathbb{R}^3)}^2 + C_w \|f\|_{L^1(\mathbb{R}^3)}^3.
\]

A refined version of bound (3.29) has been used, by X. Lu [L], in order to prove a global existence result when \(s = 0\) in (3.27). For \(s = 1\), X. Lu also prove an existence result making the additional assumption that \(B\) has the particular shape: \(B(z, \omega) = |z|^\gamma \zeta(\theta)\) with \(\gamma \in [0, 1], \zeta \in L^1\). Here condition (3.28) has to be understand as a truncature assumption (near the origine)

\[
(3.30) \quad \exists B_0 \in (0, \infty), \quad B(z, \omega) \leq B_0 (\cos \theta)^2 \sin \theta |z|^3,
\]

introduced in [Lu]. In order to clarify assumption (3.28) we state the following result.

**Lemma 3.6.** 1. For \(w = 1\),

\[
(3.31) \quad \dot{w} = \frac{4\pi^2}{rr_\ast r'r'_\ast} \min(r, r_\ast, r', r'_\ast).
\]

2. As a consequence, any cross-section \(w\) such that

\[
(3.32) \quad \exists w_0 \in (0, \infty), \quad w \leq w_0 ((|r' - v| \mid v - v| \mid 1)
\]

satisfies (3.27)-(3.28).

Notice that condition (3.32) is exactly the X. Lu’s assumption (3.30) near the origine. This condition kills interaction between particles with low energy. We emphasize that this assumption is never satisfied by physically relevant cross-section.

**Proof of Lemma 3.5** Since we may write

\[
(3.33) \quad f' f'_\ast (1 + f) (1 + f_\ast) - \frac{f}{1 + f_\ast} f_\ast (1 + f') (1 + f'_\ast) = f' f'_\ast (1 + f + f_\ast) - \frac{f}{1 + f_\ast} f_\ast (1 + f' + f'_\ast)
\]

we have to define \(Q[q]\) for two kinds of terms \(q\): for the quadratic terms \(q = f f_\ast, \frac{f}{1 + f_\ast} f_\ast\) and \(q = f' f'_\ast, \frac{f}{1 + f_\ast} f_\ast\) and for cubic terms \(q = f' f f_\ast, \frac{f}{1 + f_\ast} f f_\ast, q = f f_\ast f', \frac{f}{1 + f_\ast} f f_\ast, q = f f_\ast f'\) and \(q = f' f f_\ast, \frac{f}{1 + f_\ast} f f_\ast\). The quadratic terms may be defined in the same way that for the Fermi-Dirac Boltzmann equation in section 3.1 thanks to assumption (3.28) and they are bounded by the first term in the right hand side of estimate (3.29).

We then focus on the cubic terms. We define them making one more integration in the representation formula (3.25) (on one of the variables \(r_\ast, r', r'_\ast\)) but we will continue to use the formula (3.25) in order to conserve the symmetries of \(Q\).

Let us first assume that moreover, \(f \in C_c(\mathbb{R}^3)\). Performing in (3.25) the integration in the \(r_\ast\) variable we get, thanks to Lemma A.1,

\[
(3.34) \quad Q[q](r) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} w \frac{r_\ast}{2} 1_{\{r^2 + r'_\ast^2 \geq r^2\}} q \, dr' dr'_\ast.
\]
where now $r_*$ stands for
\[ r_* = \sqrt{r^2 + r_*^2 - r^2}. \]

It is clear that (3.33) defined $Q[q]$ as measurable function.

Moreover, writing
\[ \int_{\mathbb{R}^n} Q[q](r) \, d\tilde{r} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{w} \, \delta_{Q} q \, d\tilde{r} \, d\tilde{r}' \, d\tilde{r}^* \]
we first perform an integration in the lacking variable of $q$ (we mean in $r_*$ if $q = f^t f_*^t f$, in $r$ if $q = f^t f_*^t f$, and so on) and we get (thanks to Lemma A.1 again)
\[ \int_{\mathbb{R}^n} Q[q](r) \, d\tilde{r} \leq \|\tilde{w} (r + r_* + r' + r'_*)\|_{L^1(\mathbb{R}^n)} \|f\|_{L^1(\mathbb{R}^n)}, \]
where in the first inequality we have used the notation $r_4 = \sqrt{r_1^2 + r_2^2 - r_3^2}$. We then clearly get (3.29) using a density-continuity argument.

**Proof of Lemma 3.6.** We start proving (3.31). Using that
\[ \delta(v + v_* - v' - v'_*) = \int_{\mathbb{R}^3} e^{i(z,v,v,v',v')} \, dz \left( \frac{2\pi}{2\pi} \right)^3 = \int_{S^2} e^{i(z,v,v,v')} \, d\sigma \left( \frac{2\pi}{2\pi} \right)^3 \]
where $z = |z| \sigma$ and $\varepsilon = |z|$ in polar coordinates, we obtain
\[ \int_{S^2} \int_{S^2} \int_{S^2} \delta(v + v_* - v' - v'_*) \, d\sigma \, d\sigma' \, d\sigma^* \]
\[ = \int_{S^2} \int_{S^2} \int_{S^2} \int_{S^2} e^{i(z,v,v,v')} \, d\sigma \, d\sigma' \, d\sigma^* \, d\sigma^* \]
\[ = \int_{S^2} \int_{S^2} e^{i(z,v,v')} \, d\sigma \int_{S^2} e^{i(z,v,v)} \, d\sigma^* \int_{S^2} e^{i(z,v,v)} \, d\sigma' \int_{S^2} e^{i(z,v,v)} \, d\sigma^* \]
\[ = \frac{16\pi}{rr_* r'_* r'_*} \int_{0}^{\infty} \frac{\varepsilon^2 \, d\varepsilon}{(2\pi)^3} \sin(\varepsilon r) \sin(\varepsilon r_* \sin(\varepsilon r')) \sin(\varepsilon r'_*). \]

Then (3.31) follows by an elementary (but a bit tedious) trigonometric computation.

It is clear, thanks to (3.25), that (3.32) implies (3.27). We then have to prove that (3.32) implies (3.28). For any $r, r_*, r', r'_*$ given, we set $m_1 = \min(r, r_*, r', r'_*)$, $m_2 = \min\{r, r_*, r', r'_*\} \setminus \{m_1, m_2\}$ and finally, $m_4 = \max(r, r_*, r', r'_*)$. Since $|v - v'| = |v_* - v'_*|$, $|v - v'_*| = |v_* - v'|$ and $\{r, r_*\} \neq \{m_3, m_4\}$, the assumption (3.32) leads to
\[ w \leq w_0 \left\{ \left[ 4 \min(\max(r, r'), \max(r_*, r'_*)) \right] \min(\max(r, r_*), \max(r_*, r')) \right\} \wedge 1 \]
\[ \leq w_0 \left\{ \left[ 4 m_2 m_3 \right] \wedge 1 \right\}. \]

Gathering (3.31) and this last inequality we deduce
\[ \tilde{w} (r + r_* + r' + r'_*) \leq \frac{4 \pi^2 m_1}{rr_* r'_* r'_*} (4w_0 m_2 m_3) (r + r_* + r' + r'_*) \leq 64 \pi^2 w_0, \]
and that concludes the proof. \( \Box \)
We want now to extend the previous arguments and give a sense to \(Q(F)\) for \(F \in M^1(\mathbb{R}^3)\). For the quadratic term that problem has been solved by Povzner in [P]. For the cubic term we write for a radial measure \(dF = f r^2 \, dr\)

\[
< Q[q], \phi > = < F \otimes F \otimes F, B_w[\psi] >
\]

with

\[
B_w[\psi](r_1, r_2, r_3) := \hat{w}(r_1, r_2, r_3, r_4) \frac{r_4}{2} \mathbf{1}_{\{r_1^2 + r_2^2 \geq r_4^2\}} \psi
\]

and \(r_4 = \sqrt{r_1^2 + r_2^2 - r_3^2}\) and \(\psi(r_1, r_2, r_3) = \phi(r_i)\) for \(i = 1, 2, 3\) or \(4\) depending of \(q\).

Under the condition

\[
B_w[1] \in C(\mathbb{R}^3)
\]

we clearly have \(B_w[\psi] \in C_c(\mathbb{R}^3)\) for any \(\phi \in C_c(\mathbb{R}^3)\) and then the right hand side term of (3.36) is well defined. The assumption

\[
B \in C(\mathbb{R}^3 \times S^2),
\]

which is satisfied for hard potential, guaranties that the quadratic term is well defined and that \(\hat{w} \in C(E \setminus \{0\})\) where \(E = \{(r_1, r_2, r_3, r_4) \in \mathbb{R}_+^4, r_1^2 + r_2^2 - r_3^2 - r_4^2 = 0\}\). The condition (3.37) is then satisfied if moreover

\[
\lim_{(r_1, r_2, r_3, r_4) \to 0} \hat{w}(r_1, r_2, r_3, r_4) r_4 = 0.
\]

From the computations performed in the proof of Lemma 3.4. it is clear that the last condition holds if we assume, for instance

\[
w \leq w_0 |v' - v| \wedge 1, \quad \gamma > 1.
\]

As a conclusion, under assumption (3.27)-(3.38)-(3.39) we may define the collision kernel for general non negative bounded and isotropic measures.

From all the above one easily deduce the following existence result for (3.1) with \(\tau = 1\) in the framework of non negative bounded measures.

**Theorem 3.7.** Assume \(w\) satisfies (3.39). Let \(0 \leq F_{in} \in M^1_{rad}(\mathbb{R}^3)\). Then, there exists an unique global solution \(F = g + G \in C([0, \infty), M^1_{rad}(\mathbb{R}^3))\) to (1.1).

**Remark 3.8.** It is straightforward that in the radially symmetric case Theorem 2.5 gives:

**Theorem.** Let \(0 \leq F \in M^1_{rad}(\mathbb{R}^3)\) an isotropic measure on \(\mathbb{R}^3\) such that

\[
\int_{\mathbb{R}^3} \left( \frac{1}{2} F \right) dF(v) = \left( \frac{N}{E} \right) \quad \text{and of course} \quad \int_{\mathbb{R}^3} v \, dF(v) = 0,
\]

with \(N > 0, E > 0\). Therefore the two following assertions are equivalent:

(i) \(F\) is the Bose-Einstein distribution \(B[N, 0, E]\),

(ii) \(F\) is the solution of the maximisation problem:

\[
\mathcal{H}(F) = \max \{\mathcal{H}(F'), \quad F' \text{ satisfying (3.41)}\}.
\]

**Open questions.** In the \(L^1\) setting X. Lu has proved in [Lu]:
(i) For any \((t_n)\) such that \(t_n \to \infty\) there exists \(m' \leq m\), \(E' \leq E\) and a subsequence \((t_{n'})\) such that 
\[
g(t_{n'}) \to b_{m',0,E'} \quad \text{biting } L^1_{\text{rad}} \hat{\gamma}_{\text{weak}}.
\]

(ii) For a given \(E\) there exists \(N_c = N_c(E)\) such that if \(N(f_{in}) < N_c\) and \(E(f_{in}) = E\) then 
\[
g(t) \to B_{N,0,E} = b_{N,0,E} \quad L^1_{\text{rad}} \hat{\gamma}_{\text{weak}}.
\]

Where the distributions \(b\) and \(B\) are defined in (2.22) and (2.25). The two following questions are then natural.

1. Is it possible to construct (global?) solution to (3.1) for \(\tau = 1\) without the strong truncation condition (3.39); for instance for \(w = 1\)? What is the qualitative behavior a such a solution?

2. With the strong truncation condition (3.39): is it possible to prove that 
\[
F(t) \to B_{N,0,E} \quad \text{weakly } \sigma(M^1,C_c) \quad \text{and} \quad g \to b_{N,0,E} \quad \text{strongly } \hat{\gamma} L^1(\mathbb{R}^3 \setminus \{0\})
\]
as it may be expected from the stationary analysis? When \(N(g_{in}) < N_c\) it is possible to prove strong convergence instead of result (ii) in Theorem 3.5?

**Remark 3.9.** The Boltzmann–Compton equation, introduced in Section 5 below, is a particular case of (3.1) with \(\tau = 1\). It has been proved in [EM1], [EM2] that it also has global solutions \(F = g + G \in C([0, \infty), M^1_{\text{rad}}(\mathbb{R}^3))\), where \(g\) is the regular and \(G\) the singular part of the measure \(F\) with respect to the Lebesgue measure. Moreover it was proved that the Boltzmann–Compton may be splitted as a system of two coupled equations for the pair \((g, G)\). This allows in particular for a detailed study of the asymptotic behaviour of the solutions. Let us briefly show that this is not true for the general isotropic solutions \(F = g + G\) of the equation (3.1) unless \(G\) is one single Dirac mass.

Let us write \(F = g + G\) with \(g\) regular with respect to the Lebesgue measure \((g \ll dv)\) and \(G\) singular with respect to the Lebesgue measure \((G \perp dv)\). We have then

\[
F' F' (1 + F') (1 + F) - F' F_\ast (1 + F') (1 + F') =
\]
\[
= (1 + g) (1 + g_\ast + G_\ast) (g' + G') (g_\ast' + G_\ast') + G (1 + F_\ast) F' F_\ast'
\]
\[
- \quad g (g_\ast + G_\ast) (1 + g' + G') (1 + g_\ast' + G_\ast') - G F_\ast (1 + F') (1 + F')
\]
\[
= (1 + g) \left \{ (1 + g_\ast) g' g_\ast' + (1 + g_\ast) g' G_\ast' + (1 + g_\ast) G' g_\ast' + (1 + g_\ast) G' G_\ast' \right \}
\]
\[
+ G_\ast g' g_\ast' + G_\ast g' G_\ast' + G_\ast G' g_\ast' + G_\ast G' G_\ast'
\]
\[
- g \left \{ g_\ast (1 + g') (1 + g_\ast') + g_\ast (1 + g') G_\ast' + g_\ast G' (1 + g_\ast') + g_\ast G' G_\ast' \right \}
\]
\[
+ G_\ast (1 + g') (1 + g_\ast') + G_\ast (1 + g') G_\ast' + G_\ast G' (1 + g_\ast') + G_\ast G' G_\ast'
\]
\[
= q(g) + q_1(g,G) + q_2(g,G) + q_3(G,g).
\]

with

\[
q(g) := (1 + g) (1 + g_\ast) g' g_\ast' - g g_\ast (1 + g') (1 + g_\ast'),
\]
\[
q_1(g,G) := G_\ast [(1 + g) g' g_\ast' - g (1 + g') (1 + g_\ast')]
\]
\[
+ G' [(1 + g) (1 + g_\ast) g_\ast' - g g_\ast (1 + g_\ast')]
\]
\[
+ G' [(1 + g) (1 + g_\ast) g_\ast' - g g_\ast (1 + g')].
\]
\[
q_2(g,G) := G_\ast G' [(1 + g) g' g_\ast' - g (1 + g_\ast')]
\]
\[
+ G_\ast G' [(1 + g) g' - g (1 + g')]
\]
\[
+ G' G_\ast [(1 + g) (1 + g_\ast) - g g_\ast],
\]
\[
q_3(G) := (1 + g) G_\ast G_\ast G_\ast' - g G_\ast G_\ast' G_\ast',
\]
\[
q_4(G,g) := G (1 + F_\ast) F' F_\ast' - G F_\ast (1 + F') (1 + F_\ast').
\]
Defining
\[ Q_4(g, G) = \int\int_{\mathbb{R}^3} w \delta_{C} q_4(g, G) \, dv \, dv', \]
we may write
\[ Q(F) = Q(g) + Q_1(g, G) + Q_2(g, G) + Q_3(G) + Q_4(g, G). \]
The key result in all our analysis is the following result

**Lemma 3.10.** Assume (3.27), (3.38) and (3.39). For any \( 0 \leq F \in \mathcal{M}_{rad}^1(\mathbb{R}^3) \) we have
\[ Q(g), \ Q_1(g, G), \ Q_2(g, G) \in L_{rad}^1(\mathbb{R}^3), \]
\[ Q_4(g, G) \in \mathcal{M}_{rad}^1(\mathbb{R}^3) \text{ and } Q_4(g, G) \perp dv. \]

For any finite sum of Dirac masses \( G \) the kernel \( Q_2(G) \) is also finite sum of Dirac masses and, moreover, if \( G \) is not single Dirac mass then \( \text{supp} G \setminus \{0\} \) is strictly contained in \( \text{supp} Q_2(G) \). Last, there exists \( G \) singular such that \( Q_2(G) \) is regular.

**Proof of Lemma 3.5.** From the above, we already know that \( Q(g) \in L_{rad}^1(\mathbb{R}^3) \) and \( Q_1(g, G), \ Q_2(g, G), \ Q_4(g, G), Q_3(G) \in M_{rad}^1(\mathbb{R}^3) \). Let denote by \( q \) the first term in \( q_1(g, G) \) and write for a given \( \phi \)
\[ < Q[q], \phi > = \int\int_{\mathbb{R}^3} \hat{w} \, \hat{\delta}_C \, g \delta \phi \, dG \, dr \, dr', \]
with
\[ \psi = \int_{\mathbb{R}^3} \hat{w} \, \hat{r} \, 1_{\{r^2 + r'^2 \geq r_0^2\}} \, \phi(r) \, g \, dr', \]
where, of course, in the expression of \( \psi \) we have used the notation \( r = \sqrt{r^2 + r'^2 - r_0^2} \). Observe that, by assumption, \( (r, r', r_0') \mapsto \hat{w} \, \hat{r} \, 1_{\{r^2 + r'^2 \geq r_0^2\}} \) is continuous so that \( (r, r') \mapsto \psi \) is continuous for any \( \phi \in L_{rad}^1(\mathbb{R}^3) \). Moreover, taking \( \phi = 1_A \) for any set \( A \) with Lebesgue measure equal zero we get \( \psi = 0 \), so that the Radon-Nykodim Theorem implies that \( Q[q] \in L^1 \). By the same way we prove that all the terms in \( Q_1(g, G) \) and \( Q_2(g, G) \) belongs to \( L_{rad}^1(\mathbb{R}^3) \).

Next, it is clear that taking \( q = F^1 F^1 - F_3 F^3 - F^3 F^3 - F^3 F^3 \) we have \( Q[q] \in C_b(\mathbb{R}^3) \) so that \( Q_4(g, G) \perp dv. \)

For given \( R, R', R'_* \geq 0 \) we set \( q = \delta_{R_0} \), \( (r, r', r'_*) \mapsto \hat{w} \, \hat{r} \, 1_{\{r^2 + r'^2 \geq R_0^2\}} \) and we verify
\[ < Q[q], \phi > = \phi(R) \int \hat{w} \, (R, R_0, R', R'_*) \, 1_{\{r^2 + r'^2 \geq R_0^2\}} \]
where \( R \) is defined by \( R = \sqrt{R^2 + R'^2 - R_0^2} \) if \( R^2 + R'^2 \geq R_0^2 \). In particular, if \( G = \alpha \delta_0 \) then \( Q_3(G) = 0 \) and if \( G = \alpha \delta_a \) with \( \alpha, \alpha \neq 0 \) then \( Q_3(G) = \beta \delta_a \) with \( \beta > 0 \).

If
\[ G = \sum_{a \in E} \alpha_a \delta_a \]
then
\[ Q_3(G) = \sum_{b \in E'} \beta_b \delta_b, \]
with \( E' = \{ b \geq 0, \exists a, a', a'_* \in E \text{ s.t. } (b, a, a', a'_*) \in \hat{C} \} \). That proves the claim of the Lemma since \( E' \) strictly contain \( E \) if \( E \) is not a single point.

Finally, let \( G \) be a measure supported by \( \sqrt{C} \), where \( C \) is the Cantor set. For instance, define \( C_n := \{ x \in [0, 1], \exists k \in \mathbb{N}, 2k \leq 3^n x \leq 2k + 1 \} \), so that \( C_n \searrow C \). Then \( g_n := |C_n|^{-1} \, 1_{C_n} \rightarrow H \) a singular non
negative measure with support $C$ and take $G := H \circ s$ with $s : \mathbb{R} \to \mathbb{R}$, $s(r) = \sqrt{r}$. For any $\phi \in C_{rad, \varepsilon}(\mathbb{R}^3)$ we have
\[
< Q_3(G) = \int \int_{\mathbb{R}_+^3} dG(r) dG(r') dG(r'_*) [\phi(r) r \dot{\varphi} 1_{\{r^2 + r'^2 + r'^2_* \geq r^2\}}].
\]
Since
\[
\{z \in \mathbb{R}_+ : \exists \varepsilon, \varepsilon', \varepsilon_* \in C z = \varepsilon + \varepsilon' - \varepsilon_*\} = \mathbb{R}_+
\]
we see that $< Q_3(G); \phi > > 0$ for any non negative and not vanishing $\phi$, and thus $Q_3(G)$ has a regular part which is not equal to 0.

\[\square\]

**Remark 3.11.** If we assume that for every time $t > 0$, $G(t) \equiv \alpha(t) \delta_0$, then the equation (3.1) with $\tau = 1$ may be split into a coupled system of equations for the pair $(g, \alpha)$. Nevertheless, due to our truncation hypothesis (3.39) on the function $w$ to give a sense to the equation we do not go further in this way.

**IV. BOLTZMANN EQUATION FOR TWO SPECIES**

We consider in this Section a system of two coupled homogeneous equations describing a Bose gas interacting with a heat bath chosen to be a Fermi gas. This is a particular case of a gas composed of two species and has already been considered in the physical literature (see references below). One of the species are Bose-Einstein particles and the other are either Fermi-Dirac particles or non quantum particles. We only deal with non relativistic particles, and the next section will be concerned with relativistic particles, in particular with photons. In fact, in order to avoid lengthy repetitions, we do not specify the energy (relativistic or not relativistic) unless necessary.

From a mathematical point of view, the study of Boltzmann equation for Bose-Einstein particles alone is rather difficult, as we have already seen it in the preceding section. But it is possible to derive, from the Boson-Fermion interaction system, a physically relevant model which turns out to be a sort of "linearisation" of equation (3.1). This model is then simpler and gives some insight on the behaviour of the Boltzmann equations for quantum particles. It describes the interaction of Bose particles with isotropic distribution and non quantum Fermi particles at isotropic equilibrium with non truncated cross-section. The equation is now quadratic instead of cubic and its mathematical analysis is easier. We prove that Bose-Einstein condensation takes place in infinite time, in contrast with the finite time condensation which is expected for the Bose-Bose interaction equation. Similar results had previously been obtained in the physical literature, using formal and numerical methods for similar situations (cf. [LY1], [LY2], [ST1] and [ST2]).

We thus consider in what follows a gas composed of two species of particles. The first one are Bose particles. The second are called Fermi particles, even when they are considered in the non quantum approximation. We assume that when one Bose particle of impulsion $p$ encounter one Fermi particle with impulsion $p_*$ they perform an elastic collision, so that the total impulsion and the total energy of the system constituted by these pairs of particles are conserved. More precisely, denoting by $E_1(p)$ the energy of Bose particles with impulsion $p$ and by $E_2(p_*)$ the energy of Fermi particles with impulsion $p_*$ we assume that after collision the particles have impulsion $p'$ (for Bose particles) and $p'_*$ (for Fermi particles) which satisfy

\[\begin{align*}
E_1(p') + E_2(p'_*) &= E_1(p) + E_2(p_*), \\
\end{align*}\]

\[4.1\]

The gas is described by the density $F(t, p) \geq 0$ of Bose particles and the density $f(t, p) \geq 0$ of Fermi particles. We assume that the evolution of the gas is given by the following Boltzmann equation (see for instance [CC])

\[\begin{align*}
\frac{\partial F}{\partial t} &= Q_{1,1}(F, F) + Q_{1,2}(F, f) \\
\frac{\partial f}{\partial t} &= Q_{2,1}(f, F) + Q_{2,2}(f, f)
\end{align*}\]

\[4.2\]
The collision terms \( Q_{1,1}(F,F) \) and \( Q_{2,2}(f,f) \) stand for collisions between particles of the same specie and therefore are given by \( (1.5) \). The collision terms \( Q_{1,2}(F,f) \) and \( Q_{2,1}(f,F) \) stand for collisions between particles of the two different species, they are given by

\[
\begin{align*}
Q_{1,2}(F,f) &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} w_{1,2} \delta_{c_{1,2}} q_{1,2} dp dp' dp'_s, \\
q_{1,2} &= F'_s f'_s (1 + F) (1 + \tau f_s) - F f_s (1 + F'_s) (1 + \tau f'_s)
\end{align*}
\]  

with \( \tau = -1 \) when the second specie is composed of true Fermi particles and \( \tau = 0 \) when the second specie is constituted of non quantum particles. Collision kernel \( Q_{2,1}(f,F) \) is given by an obvious similar expression. The cross-section \( w_{1,2} = w_{1,2}(p,p_s,p'_s,p'_s) \) corresponds to the probability transition that in a collision the pair of particles changes the pair of impulsions \((p, p_s)\) to \((p', p'_s)\). It satisfies the micro-reversibility hypothesis

\[
w_{1,2}(p', p'_s, p, p_s) = w_{1,2}(p, p_s, p', p'_s),
\]

but not the indistinguishability \( w_{1,2}(p, p_s, p', p'_s) = w_{1,2}(p, p_s, p'_s, p'_s) \) as in \((1.7)\) since the two species are now distinguishable one from the other. When both energies are non relativistic \( w_{1,2} \) is invariant by Galilean transformation, see \((1.7)\) and when both energies are relativistic it is invariant by Lorentz transformation. In the mixed case of one non relativistic specie and one relativistic specie, the situation is a little more complicated and we postpone the analysis to the next Section.

We begin by some simple formal properties of the solutions of equation \((4.2)\). Thanks to the symmetry \((4.4)\) performing a change of variables \((p', p'_s, p, p_s) \rightarrow (p, p_s, p', p'_s)\) we get the fundamental formula: for any \( \psi = \psi(p) \)

\[
\int_{\mathbb{R}^3} Q_{1,2}(F,f) \psi dp = \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} w_{1,2} \delta_{c_{1,2}} \left( F'_s f'_s (1 + F) (1 + \tau f_s) - F f_s (1 + F'_s) (1 + \tau f'_s) \right) \psi dp dp'_s dp'_s,
\]

and a similar formula holds for \( Q_{2,1}(f,F) \).

After integration of each equation of \((4.2)\) separately, and using \((1.7)\) and \((4.5)\) we formally get the particle number conservation of each specie:

\[
\int_{\mathbb{R}^3} F(t, p) dp = \int_{\mathbb{R}^3} F_{in}(p) dp, \quad \int_{\mathbb{R}^3} f(t, p) dp = \int_{\mathbb{R}^3} f_{in}(p) dp
\]

Next, multiplying both equations in \((4.2)\) by \( \psi(p) = p \), summing and using \((1.7)\), \((4.5)\) and \( w_{1,2} = w_{2,1} \) we obtain the global impulsion conservation

\[
\int_{\mathbb{R}^3} (F(t, p) + f(t, p)) p dp = \int_{\mathbb{R}^3} (F_{in}(p) + f_{in}(p)) p dp.
\]

Then, multiplying the first equation of \((4.2)\) by \( E_1 \), the second equation of \((4.2)\) by \( E_2 \) and using again symmetries \((1.7)\), \((4.5)\) and collision invariance \((4.1)\) we get the global energy conservation

\[
\int_{\mathbb{R}^3} (F(t, p) E_1(p) + f(t, p) E_2(p)) dp = \int_{\mathbb{R}^3} (F_{in}(p) E_1(p) + f_{in}(p) E_2(p)) dp.
\]

Finally, we define the entropy of the system by

\[
H_S(F, f) := H_1(F) + H_\tau(f)
\]

with \( H_\tau \) given by \((1.1)\). Multiplying the first equation by \( h_1^1(F) \) and the second equation by \( h_\tau^1(f) \) we obtain thanks to \((4.5)\)

\[
\frac{d}{dt} H_S(F, f) = D_S(F, f) := -\frac{1}{4} D_1(F) + \frac{1}{2} D_{1,2}(F, f) + \frac{1}{4} D_\tau(f) \geq 0,
\]

42
where $D_1(F)$ and $D_2(f)$ are the usual dissipation entropy production of one specie (1.13) and $D_{1,2}(F, f)$ is the mixed dissipation entropy production given by

$$
D_{1,2}(F, f) := \iiint_{\mathbb{R}^3} w_{1,2} \delta_{c_{1,2}} j(F'; f') (1 + F') (1 + \tau f_*),
$$

$$
F' (1 + F') (1 + \tau f'_*) \, dp dp' dp'_* .
$$

(4.11)

We list now several questions that one may naturally ask about the system (4.2).

1. As for the single Quantum equation, one may consider the maximisation entropy problem under particle number, momentum and energy restriction i.e.: for any given $N_B, N_F, E > 0, P \in \mathbb{R}^3$, to find a pair of functions $(F, f)$ such that

$$
\int_{\mathbb{R}^3} dF(p) = N_B, \quad \int_{\mathbb{R}^3} f(p) \, dp = N_F
$$

$$
\int_{\mathbb{R}^3} P (F(p) + f(p)) \, dp = P, \quad \int_{\mathbb{R}^3} (F(p) \mathcal{E}_1(p) + f(p) \mathcal{E}_2(p)) \, dp = E.
$$

(4.12)

and satisfying to

$$
H_S(F, f) = \max_{G, g \in \mathcal{I}(4.12)} H_S(G, g), \quad \text{with } H_S(G, g) = H_{BE}(G) + H_{T}(g).
$$

(4.13)

We believe that a complete analysis of this problem can be done using the same ideas used in the second section.

2. One can also address the well posedness of the Cauchy problem for the system (4.2). Of course the situation here is the same as for the Bose-Einstein equation. The analysis performed in the section 3.2 may be resdily extended to prove existence under the assumption of isotropy of the distribution and with Lu’s truncature on the cross sections. A simpler question would be to consider the case when $Q_{1,1}(F, F)$ and $Q_{2,2}(f, f)$ vanish and to address the well posedness of the Cauchy problem in this case. Even when $\tau > 0$ (which gives an $L^\infty$ a priori bound on the Fermi density $f$) we do not know if it is possible to give a sense to the collision terms $Q_{1,2}(F, f)$ and $Q_{2,1}(f, F)$ without Lu’s truncature on the cross sections.

Nevertheless, we do not try to go further in these two directions and consider instead the following question. In the study of gases formed by Bose and Fermi particles, it is particularly relevant to consider the case where the Fermi particles are at equilibrium and where the collisions between Bose particles can not distort significantly their distribution function [cf. [LY1], [LY2]]. This moreover constitutes a first important simplification from a mathematical point of vue. The system (4.2) reduces then to a unique equation on the Bose distribution $F$. Moreover this equation is quadratic and not cubic (c.f. sub Section IV.1)

A second simplification arises if we consider non relativistic, isotropic densities and we assume on physical grounds, that $w_{1,2}$ is constant, (c.f below). In that case, we keep the same quadratic structure for the equation on the Bose distribution $F$, but we obtain an explicit and quite simple cross-section (c.f. sub Section IV.2).

In both situations, our main concern is to understand if it is possible to obtain a global existence result without the Lu’s truncature on the cross sections and then to describe the long time asymptotic behaviour of the solutions.

**IV.1. Second specie at thermodynamical equilibrium.**

Let us then assume that $f = F$ is at thermodynamical equilibrium, which means that it is a Fermi or a Maxwellian distribution defined by

$$
F(p) = \frac{1}{e^{\nu(p)} - \tau}, \quad \nu(p) = \beta \mathcal{E}_2(p) - \beta \cdot p + \mu.
$$

(4.14)
This greatly simplifies the situation since the system (4.2) reduces now to a single equation for the Bose distribution $F$ which reads

\begin{equation}
\frac{\partial F}{\partial t} = Q_{BQ}(F) := Q_{1,2}(F,F)
\end{equation}

with

\begin{equation}
Q_{BQ}(F) = \int_{\mathbb{R}^3} S(p, p') (F' (1 + F) e^{-\beta \varepsilon_1(p)} - F (1 + F') e^{-\beta \varepsilon_1(p')}) dp',
\end{equation}

\begin{equation}
S(p, p') = \int_{\mathbb{R}^3} \left( \frac{w_{1,2}}{2} \, \delta_{C_{12}} e^{\beta \varepsilon_1(p)} \right) F' (1 + \tau F_*) dp_* dp_*'.
\end{equation}

In order to establish (4.16)-(4.17) we have used the following elementary identity

\begin{equation}
e^{\beta_0 \varepsilon_1(p)} F' (1 + \tau F_*) = e^{\beta_0 \varepsilon_1(p')} F' (1 + \tau F_*')
\end{equation}

which holds on $C_{12}$. Just remark that using the micro-reversibility symmetry (4.4) and the identity (4.18) we have

\begin{equation}
S(p', p) = \int_{\mathbb{R}^3} \left( \frac{w_{1,2}}{2} \, \delta_{C_{12}} e^{\beta \varepsilon_1(p)} \right) F' (1 + \tau F_*) dp_* dp_*'
\end{equation}

so that $S$ is symmetric.

Using the symmetry above one can observe that, at least formally, a solution $F$ of equation (4.15) still satisfies the qualitative properties

\begin{equation}
\int_{\mathbb{R}^3} F dp = \int_{\mathbb{R}^3} F_{in} dp
\end{equation}

and

\begin{equation}
\frac{d}{dt} H_{BQ}(F) = D_{BQ}(F).
\end{equation}

with

\begin{equation}
H_{BQ}(F) = \int_{\mathbb{R}^3} \left( (1 + F) \ln(1 + F) - F \ln F - F \beta_0 \varepsilon_1(p) \right) dp
\end{equation}

and

\begin{equation}
D_{BQ}(F) = \int_{\mathbb{R}^3} S(p, p') j (F' (1 + F) e^{-\beta \varepsilon_1(p)} - F (1 + F') e^{-\beta \varepsilon_1(p')} ) dp dp'.
\end{equation}

In other words, the particle number is preserved along the trajectories and $H_{BQ}$ is a Lyapunov function (the relative entropy $H_{BQ}$ is a decreasing function along the trajectories).

**Non relativistic particles, fermions at isotropic Fermi Dirac equilibrium.**

It is possible, under further simplifications of the model, to obtain more explicit expressions of the cross section $S$. As a first step in that direction we consider nonrelativistic particles. We also assume, without
any loss of generality, that the two particles have the same mass \( m = 1 \) from where their energies are \( E_i(p) = \frac{p^2}{2}, i = 1, 2 \). We assume moreover that fermions are at isotropic equilibrium (see (2.39)):

\[
\mathcal{F}(p) = \frac{1}{e^{\beta p \cdot \frac{p^2}{2}} - 1}
\]

for some \( \theta > 0, b \geq 0 \).

We introduce now the Carleman parametrisation of the collision manifold (4.1). Starting by performing the \( dp_\ast \) integration one finds

\[
S(p, p') = \int_{\mathbb{R}^3} w_{1,2}(p, p', p', p_\ast) \delta_{|p'|^2 + |p'| - |p'| - |p|} e^{\beta p \cdot \frac{p^2}{2}} \mathcal{F}_0' (1 + \tau \mathcal{F}) dp_\ast,
\]

where now \( p_\ast \) is defined by

\[
\]

An elementary computation shows that

\[
|p'|^2 + |p'|^2 - |p|^2 = -2 (p'_2 - p)(p'_2 - p),
\]

so that in (4.23) \( p'_2 \) describes all the plane \( E_{p,p'} \) orthogonal to \( p' - p \) and containing \( p \). Then, using the distributional Lemma A.1, we get

\[
S(p, p') = \int_{E_{p,p'}} \frac{w_{1,2}(p, p', p', p_\ast)}{|p' - p|} e^{\beta p \cdot \frac{p^2}{2}} \mathcal{F}_0' (1 + \tau \mathcal{F}) dp_\ast dE(p'_2),
\]

where \( dE(p'_2) \) stands for the Lebesgue measure on \( E_{p,p'} \) and \( p_\ast \) is again defined thanks to (4.24).

A further simplification may be performed, noticing that, on physical ground (cf. Appendix 3), the Boson-Fermion interaction is a short range interaction and that the velocities of the particles undergoing scattering are small. This allows to consider that \( w_{1,2} \) is constant, assuming without loss of generality that \( w_{1,2} = 1 \). We then obtain a more explicit expression of \( S \) in (4.25). Namely:

\[
S(p, p') = \int_{E_{p,p'}} \frac{e^{\beta p \cdot \frac{p^2}{2}}}{|p' - p|} \frac{1}{e^{\beta p \cdot \frac{p^2}{2}} - 1} \frac{e^{\beta p_\ast \cdot \frac{p^2}{2} + b}}{e^{\beta p_\ast \cdot \frac{p^2}{2} + b} - 1} dE(p'_2).
\]

Notice that

\[
\frac{e^b}{1 + e^b} \leq \frac{e^{\beta p_\ast \cdot \frac{p^2}{2} + b}}{e^{\beta p_\ast \cdot \frac{p^2}{2} + b} - 1} < 1
\]

from where,

\[
\frac{e^b}{1 + e^b} \int_{E_{p,p'}} \frac{e^{\beta p \cdot \frac{p^2}{2}}}{|p' - p|} e^{-\beta p \cdot \frac{p^2}{2}} dE(p'_2)
\]

\[
\leq S(p, p') \int_{E_{p,p'}} \frac{e^{\beta p \cdot \frac{p^2}{2}}}{|p' - p|} e^{-\beta p \cdot \frac{p^2}{2}} dE(p'_2).
\]

Let us consider then the function

\[
S_0(p, p') \equiv \int_{E_{p,p'}} \frac{e^{\beta p \cdot \frac{p^2}{2}}}{|p' - p|} e^{-\beta p \cdot \frac{p^2}{2}} dE(p'_2)
\]

In the orthonormal basis \( \{ \hat{k}, \hat{i}, \hat{j} \} \) where \( \hat{k} := (p' - p)/|p' - p| \), \( p'_2 \) may be written as follows \( p'_2 = p + s \hat{i} + t \hat{j} \) with \( s, t \in \mathbb{R} \)

\[
|p'_2|^2 = |p|^2 + (s + p \cdot i)^2 - (p \cdot i)^2 + (t + p \cdot j)^2 - (p \cdot j)^2
\]

\[
= \left( p \cdot \frac{p' - p}{|p' - p|} \right)^2 + (s + p \cdot i)^2 + (t + p \cdot j)^2.
\]
and, integrating (4.25), we find

\[ S_0(p, p') = \frac{e^{\beta p \cdot \mathbf{p}'}}{|\mathbf{p}' - \mathbf{p}|} \int \int_{\mathbb{R}^3} e^{-\beta \mathbf{p}' \cdot \mathbf{p}'/2} \, ds \, dt \]
\[ = \frac{2 \pi}{\beta^3} \frac{\mathbf{p}^2}{|\mathbf{p}' - \mathbf{p}|} \exp \left[ -\beta \left( \frac{\mathbf{p}^2}{2} - (p, \frac{\mathbf{p}' - \mathbf{p}}{|\mathbf{p}' - \mathbf{p}|})^2 \right) \right]. \]

This can also be written in the symmetric form

\[ S_0(p, p') = \frac{2 \pi}{\beta^3} \frac{\mathbf{p}^2}{|\mathbf{p}' - \mathbf{p}|} \exp \left[ -\beta \left( \frac{\mathbf{p}^2}{4} + |p'|^2 - (p, \frac{\mathbf{p}' - \mathbf{p}}{|\mathbf{p}' - \mathbf{p}|})^2 - (p', \frac{\mathbf{p}' - \mathbf{p}}{|\mathbf{p}' - \mathbf{p}|})^2 \right) \right]. \]

**Remark 4.1.** If one considers the system (4.2) when collisions between Boson particles are very weak (so that they may be neglected) and collisions between Fermi particles are very strong (in such a way that the distribution of Fermi particles goes through thermodynamical equilibrium very rapidly) we may consider as relevant the following the scaling

\[
\begin{aligned}
\frac{\partial F_\varepsilon}{\partial t} &= \varepsilon Q_{1,1}(F_\varepsilon, F_\varepsilon) + Q_{1,2}(F_\varepsilon, f_\varepsilon) \\
\frac{\partial f_\varepsilon}{\partial t} &= Q_{2,1}(f_\varepsilon, F_\varepsilon) + \frac{1}{\varepsilon} Q_{2,2}(f_\varepsilon, f_\varepsilon),
\end{aligned}
\]

in the limit \( \varepsilon \to 0 \). Observe that particle number conservation (4.6) and energy conservation (4.8) provide the a priori bounds

\[ \sup_{t > 0} \sup_{p \in [0, \infty)} \int_{\mathbb{R}^3} F_\varepsilon(t, p)(1 + \mathcal{E}_1(p)) \, dp, \quad \int_{\mathbb{R}^3} f_\varepsilon(t, p)(1 + \mathcal{E}_2(p)) \, dp \leq C. \]

Moreover, since

\[ \frac{d}{dt} H_S(F_\varepsilon, f_\varepsilon) = \frac{\varepsilon}{4} D_1(F_\varepsilon) + \frac{1}{2} D_{1,2}(F_\varepsilon, f_\varepsilon) + \frac{1}{4 \varepsilon} D_{2}(f_\varepsilon), \]

and all the terms at the right hand side are positive, we obtain

\[ \int_0^\infty D_2(f_\varepsilon) \leq \varepsilon C(F_{in}, f_{in}). \]

Formally, these bounds imply that, up to the extraction of a subsequence, we have

\[ f_\varepsilon \to \mathcal{F}, \quad F_\varepsilon \to F, \]

where \( \mathcal{F} \) has same momentum that \( f_{in} \) and \( D_2(\mathcal{F}) = 0 \) so that \( \mathcal{F} \) is the Fermi or Maxwellian distribution associated to \( f_{in} \) given by (4.4) and \( F \) solves the Quadratic Bose equation (4.15)-(4.17).

**Remark 4.2.** It is important to notice that there is no conservation of the energy for equation (4.15)-(4.16). The conserved quantity in the system (4.2) is the total energy of the bosons-fermions gas but not of any of the two species as it is shown by (4.8).

**Open questions.**

1. To establish rigorously (4.30).
2. To solve the Cauchy problem (4.15)-(4.17) for physically relevant \( S \).

**IV.2. Isotropic distribution and second specie at the thermodynamical equilibrium.**
We consider now that the Fermi particles are at non-relativistic isotropic equilibrium and the Bose distributions are also isotropic. In other words, we suppose that

\[(4.31) \quad f(p, t) \equiv f(|p|) = \frac{1}{e^{\beta p^2/2} + b} \quad \text{and} \quad F(p, t) = F(|p|, t).\]

The interaction of an isotropic gas of Bose particles and Fermi particles at equilibrium has been considered by E. Levich and V. Yakhot in [LY1] and [LY2], by means of formal arguments, to study the occurrence of Bose Einstein condensation. Numerical simulations showing Dirac mass formation in infinite time for a related equation have been obtained by D.V. Semikoz and I.I. Tkachev in [ST2] and [ST2]. The Fermi distributions considered in that case are at a quantum, isotropic, saturated Fermi Dirac distribution (SFD in Section II). This corresponds to the choice

\[f(p) \equiv f(|p|) = 1_{\{0 \leq |p| \leq \mu}\} \]

for some \(\mu > 0\), and gives rise to a slightly different equation than ours.

The Cauchy problem for the resulting quadratic Bose equation is a “linearised model” of the the Bose-Bose interaction equation considered in Section III, where moreover, the function \(S(p, p')\) may be calculated explicitly.

Similar equations have been considered in [EM1],[EM2]. Nevertheless, the global existence results obtained in these references do not apply to our case, because the collision kernel does not fulfill the required hypothesis. Therefore, we end this Section proving global existence of solutions, with integrable initial data, to our problem. Finally, the long time behaviour of these global solutions may be adressed exactly as in [EM2] and then, we only state the result for the sake of completeness.

We start with the following:

**Proposition 4.3.** If the function \(F\) is radially symmetric then so is \(Q_{BQ}(F)\). In that case we write \(Q_{BQ}(F)p = Q_{BQ}(F)(\frac{|p|^2}{2\theta})\) by abuse of notation. Moreover,

\[(4.32) \quad Q_{BQ}(F)(\varepsilon) = \int_0^\infty S[F' (1 + F) e^{-\varepsilon} - F (1 + F') e^{-\varepsilon'}] \, d\varepsilon',\]

with \(S(\varepsilon, \varepsilon') = \Sigma(\varepsilon, \varepsilon')/\sqrt{\varepsilon}\) and

\[(4.33) \quad \Sigma(\varepsilon, \varepsilon') = \int_{\max(0, \varepsilon - \varepsilon')}^{\infty} e^{\varepsilon' - \varepsilon} \frac{e^{\varepsilon' + \varepsilon - \varepsilon} - b}{e^{\varepsilon' + \varepsilon - \varepsilon} + b - \tau} \min(\sqrt{\varepsilon}, \sqrt{\varepsilon'}, \sqrt{\varepsilon'}, \sqrt{\varepsilon}) \, d\varepsilon'.\]

**Proof of Proposition 4.3.** It follows from (4.17) and (4.31) that

\[S(p, p') = e^{\beta \frac{|p|^2}{2}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \delta[p^\prime + p'] \delta[p^\prime - |p|] \delta[p'] \, dp_\ast \, dp_\ast' \]

\[= \int_0^\infty \int_0^\infty \frac{e^{\beta \frac{|p'|^2}{2}}}{e^{\beta \frac{|p|^2}{2}} + b - \tau} \frac{dp_\ast}{e^{\beta \frac{|p'|^2}{2}} + b - \tau} \]

\[\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \delta[p' + p' - p - p'] \, dp_\ast \, dp_\ast' \, dp_\ast \, dp_\ast'.\]

therefore,
By a change of variables in the integral definition of \( Q_{BQ} \) see that it is a symmetric function. We may then write equation \((4.15)\) as

\[
\int_{S^2} \int_{S^2} \delta_{p' + p' - p - p' = 0} \, d\omega \, d\omega' = \frac{4\pi^2}{|p'|} \min [|p|, |p'|, |p' + |p'|] 
\]

from where,

\[
Q_{BQ}(F) = \frac{4\pi^2}{|p|} \int_0^\infty \int_0^\infty \delta_{p^2 + p^2 - p^2 - p^2 = 0} \frac{e^{\beta \frac{|p|^2}{2}}}{e^{\beta \frac{|p|^2}{2} + \tau}} \frac{e^{\beta \frac{|p|^2}{2} + \tau}}{e^{\beta \frac{|p'|^2}{2} + \tau}}.
\]

Finally we have already seen in Section III,

\[
\int_{S^2} \int_{S^2} \delta_{p' + p' - p - p' = 0} \, d\omega \, d\omega' = \frac{4\pi^2}{|p'|} \min [|p|, |p'|, |p' + |p'|],
\]

If we define now:

\[
\varepsilon = \beta \frac{|p|^2}{2}, \quad \varepsilon' = \beta \frac{|p|^2}{2}, \quad \varepsilon* = \beta \frac{|p|^2}{2}, \quad \varepsilon* = \beta \frac{|p|^2}{2},
\]

\[
\mathcal{S}(\sqrt{2\varepsilon/\beta^0}, \sqrt{2\varepsilon'/\beta^0}) = 4\pi^2 \sqrt{2(\beta^0)^{-3/2}} e^{-b} \int_0^\infty e^{-\varepsilon' - \varepsilon* - \varepsilon - b} \frac{e^{\varepsilon' + \varepsilon* - \varepsilon - b}}{e^{\varepsilon' + \varepsilon* - \varepsilon - b} - \tau} \left(\varepsilon' + \varepsilon* \geq \varepsilon\right) \cdot 
\]

\[
\min (\sqrt{\varepsilon}, \sqrt{\varepsilon'}, \sqrt{\varepsilon*}, \sqrt{\varepsilon*}, \sqrt{\varepsilon}, \sqrt{\varepsilon'}, \sqrt{\varepsilon*}, \sqrt{\varepsilon*}) \, d\varepsilon',
\]

\[
\equiv S(\varepsilon, \varepsilon')
\]

Finally

\[
Q_{BQ}(F) = \int_0^\infty \left[ F'(1 + F) e^{-\tau} - F(1 + F') e^{-\tau} \right] S d\varepsilon'
\]

with \( S(\varepsilon, \varepsilon') = \Sigma(\varepsilon, \varepsilon')/\sqrt{\varepsilon} \)

and

\[
\Sigma(\varepsilon, \varepsilon') = \int_0^\infty e^{-\varepsilon' - \varepsilon - b} \frac{e^{\varepsilon' + \varepsilon* - \varepsilon - b}}{e^{\varepsilon' + \varepsilon* - \varepsilon - b} - \tau} \, \min (\sqrt{\varepsilon}, \sqrt{\varepsilon'}, \sqrt{\varepsilon*}, \sqrt{\varepsilon*}, \sqrt{\varepsilon}, \sqrt{\varepsilon'}, \sqrt{\varepsilon*}, \sqrt{\varepsilon*}) \, d\varepsilon'.
\]

By a change of variables in the integral definition of \( \Sigma \) see that it is a symmetric function. We may then write equation \((4.15)\) as

\[
\sqrt{\varepsilon} \frac{\partial F}{\partial t} = \int_0^\infty \Sigma \left[ F'(1 + F) e^{-\tau} - F(1 + F') e^{-\tau} \right] d\varepsilon'.
\]

48
and perform the usual change of variable
\[
\sqrt{\varepsilon} F \to F, \quad b(\varepsilon, \varepsilon') = \frac{\Sigma(\varepsilon, \varepsilon')}{\sqrt{\varepsilon}}
\]
to obtain the more suitable form
\[
\frac{\partial F}{\partial t} = \int_0^\infty b(\varepsilon, \varepsilon') [F' (\sqrt{\varepsilon} + F') e^{-\varepsilon} - F (\sqrt{\varepsilon} + F') e^{-\varepsilon'}] \, d\varepsilon',
\]
(4.34)

Notice that \(b\) is singular near the origin and as a consequence we can not apply the results obtained in [EM 1], [EM2].

We first want to give a precise mathematical sense to the collision term in (4.34). Notice that it can be written in the following way
\[
Q(F) = \int_0^\infty b(\varepsilon, \varepsilon') \sqrt{\varepsilon} e^{-\varepsilon} F' \, d\varepsilon' - F \int_0^\infty b(\varepsilon, \varepsilon') \sqrt{\varepsilon} e^{-\varepsilon'} \, d\varepsilon'
\]
\[
+ \int_0^\infty b(\varepsilon, \varepsilon') (e^{-\varepsilon} - e^{-\varepsilon'}) F F' \, d\varepsilon'.
\]

**Lemma 4.4.** The function
\[
\ell(\varepsilon) := \int_0^\infty b(\varepsilon, \varepsilon') \sqrt{\varepsilon} e^{-\varepsilon'} \, d\varepsilon'.
\]
(4.37)
satisfies \(\ell \in C([0, \infty))\), \(\ell \leq C_1 (1 + \sqrt{\varepsilon})\) for some positive constant \(C_2\), \(\ell \to C_2 (b, \tau)\) when \(\varepsilon \to 0\) and \(\ell \sim \gamma \sqrt{\varepsilon}\) when \(\varepsilon \to \infty\), with \(\gamma > 0\). Moreover, we have
\[
\chi(\varepsilon, \varepsilon') := (e^{-\varepsilon} - e^{-\varepsilon'}) b(\varepsilon, b') \in C_b([0, \infty) \times [0, \infty)).
\]
(4.38)

As a conclusion, for any \(F\) such that \((1 + \sqrt{\varepsilon}) F \in L^1\) then \(Q(F)\) belongs to \(L^1\) and the application \(F \mapsto Q(F)\) is continuous from \(L^1_{1/2}\) to \(L^1\).

**Proof of Lemma 4.4.** In order to prove the properties of \(\ell\) we write
\[
\ell(\varepsilon) = \frac{1}{\sqrt{\varepsilon}} \int_0^\varepsilon \zeta_b(\varepsilon') e^{-\varepsilon'} \, d\varepsilon' + \frac{1}{\sqrt{\varepsilon}} \int_0^\varepsilon \xi_b(\varepsilon, \varepsilon') e^{-\varepsilon'} \, d\varepsilon'
\]
where,
\[
\zeta_b(\varepsilon) = \int_0^\infty e^{z-k} \frac{e^{k+b}}{e^{k+b} - \tau} \min(\sqrt{\varepsilon}, \sqrt{k}) \, dk
\]
\[
\xi_b(\varepsilon, y) = \int_0^\infty e^{-z-k} \frac{e^{z+y+b}}{e^{z+y+b} - \tau} \min(\sqrt{\varepsilon}, \sqrt{k}) \, dk.
\]

Notice that,
\[
\zeta_b(\varepsilon) \leq \int_0^\infty e^{-z} \min(\sqrt{\varepsilon}, \sqrt{k}) \, dk = \gamma(\varepsilon) \varepsilon^2 + \sqrt{\varepsilon} \equiv \zeta(\varepsilon) \quad \text{with} \quad \gamma(\varepsilon) = \int_0^\varepsilon e^{-y} \sqrt{y} \, dy
\]
and also \(\xi_b(\varepsilon, y) \leq \zeta(\varepsilon)\). Assume first that \(\varepsilon \to 0\). Then,
\[
\frac{1}{\sqrt{\varepsilon}} \int_0^\varepsilon \zeta_b(\varepsilon') e^{-\varepsilon'} \, d\varepsilon' \leq \frac{1}{\sqrt{\varepsilon}} \int_0^\varepsilon \zeta(\varepsilon') e^{-\varepsilon'} \, d\varepsilon' \to 0 \quad \text{as} \ \varepsilon \to 0.
\]

49
On the other hand,

\[
\frac{1}{\sqrt{\varepsilon}} \int_0^\infty \xi_b(\varepsilon, \varepsilon') e^{-\varepsilon^{\prime}} d\varepsilon' = \frac{1}{\sqrt{\varepsilon}} \int_0^\infty e^{-\varepsilon^{\prime}} \int_0^\infty e^{\varepsilon^{\prime} - \varepsilon^{\prime\prime}} \frac{e^{\varepsilon^{\prime\prime} - \varepsilon^{\prime} + b}}{e^{\varepsilon^{\prime\prime} - \varepsilon^{\prime} + b} - \tau} \sqrt{\varepsilon^{\prime\prime}} d\varepsilon^{\prime\prime} d\varepsilon^{\prime} + \\
\int_0^\infty e^{-\varepsilon'} \int_0^\infty e^{\varepsilon^{\prime} - \varepsilon^{\prime\prime}} \frac{e^{\varepsilon^{\prime\prime} - \varepsilon^{\prime} + b}}{e^{\varepsilon^{\prime\prime} - \varepsilon^{\prime} + b} - \tau} \sqrt{\varepsilon^{\prime\prime}} d\varepsilon^{\prime\prime} d\varepsilon^{\prime} \\
\to \int_0^\infty e^{-\varepsilon} \int_0^\infty e^{\varepsilon^{\prime} - \varepsilon^{\prime\prime}} \frac{e^{\varepsilon^{\prime\prime} - \varepsilon^{\prime} + b}}{e^{\varepsilon^{\prime\prime} - \varepsilon^{\prime} + b} - \tau} \sqrt{\varepsilon^{\prime\prime}} d\varepsilon^{\prime\prime} d\varepsilon^{\prime} = C_2(b, \tau) \quad \text{as } \varepsilon \to 0.
\]

Suppose now that \( \varepsilon \to \infty \). We first notice that,

\[
\frac{1}{\sqrt{\varepsilon}} \int_0^\varepsilon \xi_b(\varepsilon, \varepsilon') e^{-\varepsilon^{\prime}} d\varepsilon' \leq \frac{1}{\sqrt{\varepsilon}} \int_0^\varepsilon \xi(\varepsilon') e^{-\varepsilon^{\prime}} d\varepsilon' \to 0 \quad \text{as } \varepsilon \to \infty.
\]

Finally,

\[
\frac{1}{\sqrt{\varepsilon}} \int_0^\varepsilon e^{-\varepsilon'} \xi_0(\varepsilon') d\varepsilon' = \frac{1}{\sqrt{\varepsilon}} \int_0^\varepsilon \int_0^\varepsilon e^{-\varepsilon^{\prime}} \frac{e^b}{e^{b + \tau} - \tau} \sqrt{k} dk d\varepsilon' \\
+ \frac{1}{\sqrt{\varepsilon}} \int_0^\varepsilon \sqrt{\varepsilon'} \int_0^\varepsilon k e^{-k} \frac{e^{b + \tau}}{e^{b + \tau} - \tau} dk d\varepsilon' \equiv I_1 + I_2
\]

and,

\[
I_2 \leq \frac{1}{\sqrt{\varepsilon}} \int_0^\varepsilon \sqrt{\varepsilon'} e^{-\varepsilon'} d\varepsilon' = O\left(\frac{1}{\sqrt{\varepsilon}}\right)
\]

\[
\lim_{\varepsilon \to \infty} \frac{1}{\sqrt{\varepsilon}} \int_0^\varepsilon \int_0^\varepsilon e^{-k} \sqrt{k} dk d\varepsilon' = \int_0^\infty e^{-k} \sqrt{k} dk \equiv \gamma
\]

In order to prove (4.38) we just have to consider the two cases: If \( \min(\varepsilon, \varepsilon') \to 0 \) then

\[
\chi \sim \frac{e^{-\max} - e^{-\min}}{\sqrt{\max}} \quad \text{max} \to 0, \infty \quad \text{uniformly in min.}
\]

If \( \min(\varepsilon, \varepsilon') \to \infty \) then \( \varepsilon, \varepsilon' \to \infty \) and

\[
|\chi| \leq \frac{e^{-\min} - e^{-\max}}{\sqrt{\max} \sqrt{\min}} e^{-\min} \leq 1.
\]

Since the particle number of a solution to (4.33) is conserved along time (at least formally) and this gives a first a priori bound in \( L^1 \), we have to control momentum of \( F \). We perform now a formal computation which show that one can bound, uniformly in time polynomial momentum. Note also that exponential momentum of a solution \( F \) may be controlled by the same way.

**Lemma 4.5.** A solution \( F \) to equation (4.34) satisfies (formally at least)

\[
\frac{d}{dt} Y_1(F) + \frac{\gamma}{4} \int_0^\infty \int_0^\infty F(1 + \sqrt{\varepsilon}) \varepsilon d\varepsilon \leq C_{in},
\]

with

\[
Y_0(F) := \int_0^\infty \varepsilon^0 d|F|(\varepsilon).
\]
Proof of Lemma 4.5. Let consider a solution $F$ to (4.34) and write

$$
\frac{d}{dt} \gamma_1(F) = \int_0^\infty \int_0^\infty b F' F' [e^{-\varepsilon} - e^{-\varepsilon'}] \varepsilon d\varepsilon' d\varepsilon
+ \int_0^\infty \int_0^\infty b \sqrt{\varepsilon} F' \varepsilon e^{-\varepsilon} - \sqrt{\varepsilon'} F e^{-\varepsilon'} \varepsilon d\varepsilon' d\varepsilon
= \int_0^\infty \int_0^\infty \frac{b}{2} F F' [e^{-\varepsilon} - e^{-\varepsilon'}] (\varepsilon - \varepsilon') d\varepsilon' d\varepsilon
+ \int_0^\infty \int_0^\infty b \sqrt{\varepsilon} F e^{-\varepsilon'} [\varepsilon' - \varepsilon] d\varepsilon' d\varepsilon.
$$

Since the first term is non positive, we just have to manage with the second term. In order to estimate it we compute

$$
I(\varepsilon) := \int_0^\infty b \sqrt{\varepsilon} e^{-\varepsilon'} [\varepsilon' - \varepsilon] d\varepsilon'
= \int_0^\varepsilon \frac{\zeta(\varepsilon')}{\varepsilon} e^{-\varepsilon'} [\varepsilon' - \varepsilon] d\varepsilon' + \frac{\zeta(\varepsilon)}{\sqrt{\varepsilon}} \int_\varepsilon^\infty e^{-\varepsilon'} [\varepsilon' - \varepsilon] d\varepsilon' = I_1 + I_2.
$$

On one hand

$$
I_2 = \frac{\zeta(\varepsilon)}{\sqrt{\varepsilon}} e^{-\varepsilon} = \frac{\gamma(\varepsilon)}{\sqrt{\varepsilon}} \leq C.
$$

On the other hand

$$
I_1 \leq \frac{1}{\sqrt{\varepsilon}} \int_0^\varepsilon \frac{\gamma(\varepsilon')}{\sqrt{\varepsilon'}} (\varepsilon' - \varepsilon) d\varepsilon' \leq \frac{\gamma(m)}{\sqrt{\varepsilon}} \int_0^\varepsilon (\varepsilon' - \varepsilon) d\varepsilon' = \frac{\gamma(m)}{\sqrt{\varepsilon}} \left( -\frac{\varepsilon^2}{2} + m \varepsilon - \frac{m^2}{2} \right).
$$

Since $\gamma(m) \to \gamma$ when $m \to \infty$ and the term of greater order have a minus sign, we obtain

$$
I \leq C - \frac{\gamma}{4} \left( 1 + \sqrt{\varepsilon} \right) \varepsilon,
$$

and (4.39) follows. \[\square\]

We can now state the global existence result for the equation associated to the collision kernel (4.32).

Theorem 4.6. For any initial datum $F_{in} \in L_1^1(\mathbb{R}^+)$, $F \geq 0$, there exists a solution $F \in C([0, \infty), L_1^1(\mathbb{R}^+))$ to the equation

$$
\frac{\partial F}{\partial t} = Q_{BQ}(F), \quad \text{for} \quad \varepsilon > 0, \quad t > 0
$$

with $Q_{BQ}$ defined in (4.32), and such that

$$
\lim_{t \to 0} ||F(t) - F_{in}||_{L_1^1(\mathbb{R}^+)} = 0.
$$

Proof of Theorem 4.6. We first introduce a regularized problem. Let consider a solution $F_n$ to the equation

$$
\frac{\partial F_n}{\partial t} = \int_0^\infty b_n(\varepsilon, \varepsilon') \left[ F_n' (\sqrt{\varepsilon_n + F_n}) e^{-\varepsilon} - F_n (\sqrt{\varepsilon_n' + F_n}) e^{-\varepsilon'} \right] d\varepsilon', 
$$

(4.41)
with
\[ b_n(\epsilon, \epsilon') = \frac{\zeta(\epsilon \wedge \epsilon' \wedge n)}{\sqrt{\epsilon \wedge \epsilon' \wedge n} \sqrt{\epsilon \vee \epsilon' \vee 1/n}}, \quad \epsilon_n = \epsilon \vee \frac{1}{n} \]

Existence of a solution \( F_n \) to this equation is given by the existence result of [EM] since \( b_n \in L^\infty \). We obtain a priori estimates on the sequence \( (F_n) \) in the two following Lemmas and then pass to the limit using Lemma 4.4

**Lemma 4.7.** The sequence of solution \( (F_n) \) satisfies

\[ \frac{d}{dt} Y_1(F_n) + \frac{\gamma}{3} \int_0^\infty F_n \sqrt{\epsilon}(\epsilon \wedge n) d\epsilon \leq C_{in} + Y_1(F_n). \]

for \( n \) large enough. This implies that \( Y_1(F_n) \) is bounded in \( L^\infty(0, T) \) and \( \int_0^\infty F_n \sqrt{\epsilon}(\epsilon \wedge n) d\epsilon \) is bounded in \( L^1(0, T) \) for any \( T > 0 \).

**Proof of Lemma 4.7.** We perform the same computation as in Lemma 4.3 and obtain

\[ \frac{d}{dt} Y_1(F_n) = \int_0^\infty \int_0^\infty b_n \frac{2}{3} F_n F'_n [e^{-\epsilon'} - e^{-\epsilon}] (\epsilon - \epsilon') d\epsilon' d\epsilon + \int_0^\infty F_n I_n d\epsilon \]

with the first term non positive and with

\[ I_n(\epsilon) := \int_0^\infty b_n \sqrt{\epsilon' \wedge n} e^{-\epsilon'} [\epsilon' - \epsilon] d\epsilon' \leq \int_0^\epsilon \frac{\gamma (\epsilon' \wedge n)}{\sqrt{\epsilon' \wedge n}} \sqrt{\epsilon' \wedge n} e^{\epsilon' \wedge n} - \epsilon' [\epsilon' - \epsilon] d\epsilon' + \frac{\zeta(\epsilon \wedge n)}{\sqrt{\epsilon \wedge n}} e^{-\epsilon}.
\]

The last term is bounded. Let fix \( \ell \geq 1 \) such that \( \gamma(\ell) > 2 \gamma/3 \). For \( n \) and \( \epsilon \) such that \( \epsilon \wedge n \geq \ell \) we write

\[ \int_0^\epsilon \frac{\gamma (\epsilon' \wedge n)}{\sqrt{\epsilon' \wedge n}} \sqrt{\epsilon' \wedge n} e^{\epsilon' \wedge n} - \epsilon' [\epsilon' - \epsilon] d\epsilon' \leq \frac{2 \gamma}{3 \sqrt{\epsilon}} \int_0^\epsilon \frac{\sqrt{\epsilon'}}{\sqrt{\epsilon' \wedge n}} e^{\epsilon' \wedge n} - \epsilon' [\epsilon' - \epsilon] d\epsilon' + \gamma \int_0^\ell \frac{\epsilon}{\sqrt{\epsilon' \wedge n}} d\epsilon' \leq \frac{2 \gamma}{3 \sqrt{\epsilon}} \int_0^\ell (\epsilon' - \epsilon) d\epsilon' + \gamma 2 \sqrt{\ell \epsilon} \leq 2 \sqrt{\ell \epsilon} - \frac{\gamma}{3} \sqrt{\epsilon (\epsilon \wedge n)}.
\]

We then conclude as in the proof of Lemma 4.5. \( \Box \)

**Lemma 4.8.** The set \( (F_n) \) is a Cauchy sequence in \( C([0, T]; L^1_{1/2}(\mathbb{R}_+)) \) for any \( T > 0 \).

**Proof of Lemma 4.8.** We just compute

\[ \frac{\partial}{\partial t} (F_m - F_n) = Q_n(F_m) - Q_n(F_n) + Q_m(F_m) - Q_n(F_m) \]

and

\[ \frac{d}{dt} \|F_m - F_n\|_{L^1_{1/2}} \leq \int_0^\infty |F_m - F_n| b_n \sqrt{\epsilon' \wedge n} e^{-\epsilon'} [\sqrt{\epsilon} - \sqrt{\epsilon'}] d\epsilon' d\epsilon \]

\[ + \int_0^\infty \int_0^\infty |F_m F'_m - F_n F'_n| e^{-\epsilon'} - e^{-\epsilon} b_n (1 + \sqrt{\epsilon}) d\epsilon' d\epsilon \]

\[ + \int_0^\infty \int_0^\infty (b_m - b_n) F_m F'_m e^{-\epsilon'} - e^{-\epsilon} |(1 + \sqrt{\epsilon}) d\epsilon' d\epsilon \]

\[ + \int_0^\infty \int_0^\infty |b_m \sqrt{\epsilon' \wedge n} - b_n \sqrt{\epsilon' \wedge n}| F_m e^{-\epsilon'} [2 + \sqrt{\epsilon'} + \sqrt{\epsilon}] d\epsilon' d\epsilon.
\]

\[ \equiv J_1 + J_2 + J_3 + J_4 \]
with $b_{m,n} = b_m - b_n$.

We now estimate each of the terms $J_i$, $i=1,2,3,4$ separately.

1. **Estimate of $J_1$.** The first term is nothing but

$$J_1(n) = \int_0^\infty |F_m - F_n| I_n \, dz$$

with

$$I_n(\varepsilon) := \int_0^\infty b_n \sqrt{b^2_\varepsilon - \varepsilon^2} \left[ \sqrt{b^2_\varepsilon - \varepsilon^2} \right] \, dz \leq \frac{\zeta(\varepsilon \wedge n)}{\sqrt{\varepsilon \wedge n}} \int_\varepsilon^\infty \varepsilon^{-\varepsilon} \left[ \sqrt{b^2_\varepsilon - \varepsilon^2} \right] \, dz =: I'_n.$$

For $\varepsilon \geq 1$ and integrating by parts we get

$$I'_n = \frac{\zeta(\varepsilon \wedge n)}{\sqrt{\varepsilon \wedge n}} \int_{\varepsilon}^\infty \frac{e^{-\varepsilon'}}{\sqrt{\varepsilon \wedge n}} \, dz \leq \frac{\zeta(\varepsilon \wedge n)}{\sqrt{\varepsilon \wedge n}} e^{-\varepsilon} \in L^\infty(\mathbb{R}_+).$$

Since $I'_n$ is bounded for $\varepsilon \leq 1$, there exists a constant $K_1 > 0$ such that

(4.44) $J_1 \leq K_1 \|F_m - F_n\|_{L^1}.$

2. **Estimate of $J_2$.** Noticing that

$$\chi_n := b_n \left( e^{-\varepsilon} - e^{-\varepsilon'} \right) \leq \frac{\zeta(\varepsilon \wedge \varepsilon')}{\sqrt{\varepsilon \wedge \varepsilon'}} \frac{e^{-\varepsilon} - e^{-\varepsilon'}}{\sqrt{\varepsilon \wedge \varepsilon'}} 1_{\varepsilon \wedge \varepsilon' \leq 1} + \frac{\zeta(\varepsilon \wedge \varepsilon' \wedge n)}{\sqrt{\varepsilon \wedge \varepsilon' \wedge n}} \max(e^{-\varepsilon}, e^{-\varepsilon'}) 1_{\varepsilon \wedge \varepsilon' \geq 1}$$

is uniformly bounded in $\mathbb{R}_+^2$ as we have already shown it in the proof of Lemma 4.4, we deduce

\begin{align*}
J_2 &\leq \|\chi_n\|_{L^\infty(\varepsilon' \wedge 1)} \|\|F_m - F_n\|(1 + \sqrt{\varepsilon})\|_{L^1} \|F_m\|_{L^1} + \|F_m - F_n\|_{L^1} \|F_m(1 + \sqrt{\varepsilon})\|_{L^1} \\
&\leq K_2 \|F_m\|_{L^1} \|F_m - F_n\|_{L^1},
\end{align*}

for some constant $K_2 > 0$.

3. **Estimate of $J_3$.** Since that $x \mapsto \zeta(x)/\sqrt{x}$ is increasing (at least for the large values of $x$) we have $0 \leq b_n \leq b_m$. We then remark that

$$0 \leq b_m - b_n = \frac{\zeta(\varepsilon \wedge \varepsilon')}{\sqrt{\varepsilon \wedge \varepsilon'}} \left( \frac{1}{\sqrt{\varepsilon \wedge \varepsilon' \wedge 1/m}} - \frac{1}{\sqrt{\varepsilon'}} \right) 1_{\varepsilon \wedge \varepsilon' < 1/m}$$

$$+ \frac{1}{\sqrt{\varepsilon'}} \left( \frac{\zeta(\varepsilon \wedge \varepsilon' \wedge m)}{\sqrt{\varepsilon \wedge \varepsilon' \wedge m}} - \frac{\zeta(\varepsilon \wedge \varepsilon')}{\sqrt{\varepsilon'}} \right) 1_{\varepsilon \wedge \varepsilon' > 1/m}.$$

On one hand we have

$$\left( b_m - b_n \right) |e^{-\varepsilon} - e^{-\varepsilon'}| 1_{\varepsilon \wedge \varepsilon' < 1} \leq \frac{\zeta(\varepsilon \wedge \varepsilon')}{\sqrt{\varepsilon \wedge \varepsilon'}} \frac{1}{\sqrt{\varepsilon \wedge \varepsilon' \wedge 1/m}} |e^{-\varepsilon} - e^{-\varepsilon'}| 1_{\varepsilon \wedge \varepsilon' < 1/m}$$

$$\leq \|\zeta(x)\|_{L^{\infty}(0,1)} \|\sqrt{\varepsilon \wedge \varepsilon'} 1_{\varepsilon \wedge \varepsilon' < 1/m} \leq \frac{K_2}{\sqrt{\varepsilon}} 1_{\varepsilon \wedge \varepsilon' < 1/m},$$

and on the other hand

$$\left( b_m - b_n \right) |e^{-\varepsilon} - e^{-\varepsilon'}| 1_{\varepsilon \wedge \varepsilon' \geq 1} \leq \frac{\zeta(\varepsilon \wedge \varepsilon' \wedge m)}{\sqrt{\varepsilon \wedge \varepsilon' \wedge m}} \frac{1}{\sqrt{\varepsilon \wedge \varepsilon'}} |e^{-\varepsilon} - e^{-\varepsilon'}| 1_{\varepsilon \wedge \varepsilon' \geq 1/m}$$

$$\leq \frac{2}{\sqrt{\varepsilon \wedge \varepsilon'}} \|\zeta(x)\|_{L^{\infty}(0,1)} 1_{\varepsilon \wedge \varepsilon' \geq 1/m} \leq \frac{K_2}{\sqrt{\varepsilon}} 1_{\varepsilon \wedge \varepsilon' > 1/m},$$
for some constant $K_3 > 0$. We deduce that

\begin{equation}
J_3 \leq \frac{K_3}{\sqrt{n}} \gamma_0(F_m) \|F_m\|_{L_1}.
\end{equation}

4. Estimate of $J_4$. Since $0 \leq \sqrt{e_m^}\ e_n \leq \sqrt{e_m^}\ b_m$ for $n$ large enough, we may write

\[
\sqrt{e_m^}\ b_m - \sqrt{e_n^}\ b_n \leq \frac{\sqrt{e_m^} + 1/m}{\sqrt{e_n^} + 1/m} \zeta(e \wedge e^') 1_{e < 1/n} + \frac{\sqrt{e_m^}}{\sqrt{e_n^}} \zeta(e \wedge e^' \wedge m) 1_{e > 1/n}.
\]

On one hand we compute

\begin{equation}
\int_{0}^{\infty} (b_m \sqrt{e_m^} - b_n \sqrt{e_n^}) e^{-e^'} [2 + \sqrt{e} + \sqrt{e^'}] 1_{e \leq 1/n} \, de^' \leq 4 \int_{0}^{1/n} (b_m \sqrt{e_m^} - b_n \sqrt{e_n^}) \, de^' \leq 4 \|\zeta(\x)\|_{L_{\infty}(0, 1)} \frac{1}{n}.
\end{equation}

On the other hand,

\[
\int_{0}^{\infty} (b_m \sqrt{e_m^} - b_n \sqrt{e_n^}) e^{-e^'} [2 + \sqrt{e} + \sqrt{e^'}] 1_{e \leq 1/n} \, de^' \\
\leq \int_{0}^{e} \frac{\sqrt{e^'} \zeta(e^' \wedge m)}{\sqrt{e} \zeta(e^' \wedge m)} e^{-e^'} [2 + \sqrt{e} + \sqrt{e^'}] \, de^' 1_{e \geq n} + \zeta(e \wedge m) \int_{e}^{\infty} e^{-e^'} [3 + \sqrt{e} + \sqrt{e^'}] \, de^' 1_{e \geq n} \\
\leq \int_{0}^{e} \frac{\zeta(e^' \wedge m)}{\sqrt{e^' \wedge m}} e^{-e^'} [3 + \sqrt{e} + \sqrt{e^'}] \, de^' \leq 4 \|\zeta(\x)\|_{L_{\infty}(0, 1)} (1 + \sqrt{m}) \frac{1}{n}.
\]

In order to estimate the first term in the last right hand side term, we consider successively the cases $e \leq m$ and $e \geq m$ and make the use of the following elementary estimate

\[
\int_{m}^{e} (e')^\gamma e^{-e^'} \, de^' \leq (1 + m^\gamma) e^{-m},
\]

with $\gamma = 1/2$ and $\gamma = 1$. When $e \leq m$ we have

\begin{equation}
\int_{0}^{e} \frac{\zeta(e^' \wedge m)}{\sqrt{e^' \wedge m}} e^{-e^'} [3 + \sqrt{e} + \sqrt{e^'}] \, de^' = \int_{0}^{e} \zeta(e^') e^{-e^'} [3 + \sqrt{e} + \sqrt{e^'}] \, de^' \leq 4 \|\zeta(\x)\|_{L_{\infty}(0, 1)} (1 + \sqrt{m}).
\end{equation}

Now, when $e \geq m$, we get

\[
\int_{0}^{e} \frac{\zeta(e^' \wedge m)}{\sqrt{e^' \wedge m}} e^{-e^'} [3 + \sqrt{e} + \sqrt{e^'}] \, de^' = \int_{m}^{e} \zeta(e^') e^{-e^'} [3 + \sqrt{e} + \sqrt{e^'}] \, de^' + \frac{\zeta(m)}{\sqrt{m}} \int_{m}^{e} e^{-e^'} [3 + \sqrt{e} + \sqrt{e^'}] \, de^' \\
\leq 4 \|\zeta(\x)\|_{L_{\infty}(0, 1)} (1 + \sqrt{m}) + \|\zeta(\x)\|_{L_{\infty}(0, 1)} (1 + \sqrt{m}),
\]

using (4.48). As a conclusion, we have proved that

\begin{equation}
\int_{0}^{\infty} (b_m \sqrt{e_m^} - b_n \sqrt{e_n^}) e^{-e^'} [2 + \sqrt{e} + \sqrt{e^'}] 1_{e \leq 1/n} \, de^' \leq K_4 ((e \wedge m) + \sqrt{e} + 1) 1_{e \geq m}.
\end{equation}
Therefore, combining (4.47) and (4.49), we get

$$J_4 \leq \frac{K_4}{\sqrt{n}} \int_0^\infty F_m \left( 1 + \sqrt{\varepsilon (\varepsilon + m)} \right) \, dz.$$  

From (4.42)-(4.46), (4.50) and Lemma 4.7 we obtain

$$\frac{d}{dt} \|F_m - F_n\|_{L_{1/2}^1} \leq A(t) \|F_m - F_n\|_{L_{1/2}^1} + \frac{B(t)}{\sqrt{n}}$$

with $A \in L^\infty(0, T)$ and $B \in L^1(0, T)$ for any $T > 0$. We end the proof using the Gronwall Lemma. \[\square\]

The equation (4.15)-(4.32) is of the form given by equation (1.1) in [EM1], [EM2]:

$$\varepsilon \frac{\partial F}{\partial t}(\varepsilon) = \int_0^\infty b(\varepsilon, \varepsilon')(F' (1 + F)e^{-\varepsilon} - F(1 + F)e^{-\varepsilon'}) \, d\varepsilon'$$

with $\varepsilon^2 b(\varepsilon, \varepsilon') = S(\varepsilon, \varepsilon')$. We then refer to Theorem 2 in [EM2] for the proof of the following proposition.

**Proposition 4.9.** Let $0 \leq F \in M^1_{rad}([0, \infty))$ such that

$$M(F) := \int_0^\infty v^2 \, dF(v) = N \geq 0$$

Then the two following assertions are equivalent:

(i) $F = B_N$ with

$$\varepsilon^2 B_N(\varepsilon) = \begin{cases} \frac{\varepsilon^2}{\varepsilon + \mu - 1}, & \text{if } N \leq N_0 \equiv \int_0^\infty \frac{\varepsilon^2 \, d\varepsilon}{\varepsilon - 1}, \\ \frac{\varepsilon^2}{\varepsilon - 1} + (N - N_0)\delta_0 & \text{otherwise} \end{cases}$$

(ii) $F$ is the solution of the maximisation problem:

$$\mathcal{H}(F) = \max\{\mathcal{H}(F'), \; F' \text{ satisfying (4.51)}\},$$

where the entropy $\mathcal{H}$ given by (4.21) reads:

$$H(F) = \int_0^\infty [(1 + F) \ln(1 + F) - F \ln F - \varepsilon F] \varepsilon^2 \, d\varepsilon.$$

**Theorem 4.10.** Asymptotic behaviour. For every $F_{in} \in L^1_1(\mathbb{R}^+)$, $F_{in} \geq 0$, let $N = M(F_{in})$, and $F \in C([0, \infty), L^1_{1/2}(\mathbb{R}^+))$ the corresponding solution to (4.33) with initial data $F_{in}$. Then:

$$\begin{cases} F(t, .) \rightharpoonup B_N \text{ weakly } \ast \text{ in } (C_r(\mathbb{R}_+))' \\
\lim_{t \to \infty} \|g(t, .) - B_N\|_{L^1(0, \infty)} = 0 \quad \forall k_0 > 0. \end{cases}$$

**Proof.** The proof is the same as that of Theorem 6 in [EM2] and is thus skipped for the sake of brevity.

V. THE COLLISION INTEGRAL FOR RELATIVISTIC QUANTUM PARTICLES.
We particularly follow in this Section the books by S. R. de Groot, W. A. van Leeuwen and Ch. G. van Weert [GLW] and R. T. Glassey [GI]

**V.1. Parametrisations**

In this section we introduce the Boltzmann equation for relativistic particles. A particle is now determined by the pair \((X, P)\) of position and impulsion in the time-position and impulsion space \(\mathbb{R}^4 \times \mathbb{R}^4\) where we write \(X = (X^0, x)\), with \(X^0 = t, (X^1, X^2, X^3) = x\), \(P = (P^\mu)\) with \((P^1, P^2, P^3) = p\). Moreover, since the particle has to be on its mass shell, we have

\[
P^0 \equiv p^0 := \sqrt{|p|^2 + m^2c^2}.
\]

where \(c\) is the speed of the light. The gas is described by its density \(F = F(X, P)\). In this context, the Boltzmann equation as it may be found in [GLW] and [GI] is

\[
\langle P, \nabla_X F \rangle = Q(F),
\]

where the collision kernel reads

\[
Q(F)(P) = \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} W q(F) \delta_{P+P_*-P} \chi(P) \chi(P') dP' dP' dP_*.
\]

Here, \(\nabla_X = (c^{-1} \partial_t, \nabla_x)\) and \(\langle \cdot, \cdot \rangle\) represents the Lorentz inner product in \(\mathbb{R}^4\) and we use the abbreviation \(|P|^2 = P \cdot P\) for any \(P \in \mathbb{R}^4\). We refer to the Appendix 2 for more details on the Lorentz space. Notice nevertheless that

\[
\langle P, \nabla_X F \rangle = \sum_{\mu, \nu = 1}^{4} \eta_{\mu \nu} P^\mu (\nabla_X F)^\nu, \quad \text{and} \quad (\nabla_X F)^\nu = (c^{-1} \partial_t, -\nabla_x)\]

Moreover, \(W = W(P, P_*, P', P'_*)\) is a given non negative function related to the differential cross-section \(\sigma\), see (5.10) and (5.11) below, and as before,

\[
q(F) = F' F'_* (1 + \tau F) (1 + \tau F_*) - F F_* (1 + \tau F') (1 + \tau F'_*)
\]

with \(\tau \in \{-1, 0, 1\}\), \(F = F(X, P)\), \(F_* = F(X, P_*)\), \(F' = F(X, P')\), \(F'_* = F(X, P'_*)\).

Finally, we have defined

\[
\chi(R) = \delta_{R^2 - m^2c^2} H(|R|^2)
\]

where \(H\) stands for the Heaviside function. With these notations, if we denote \(f(t, x, p) := F(X, P)\), we have

\[
\langle P, \nabla_X f \rangle = \frac{p^0}{c} \frac{\partial}{\partial t} + p \cdot \nabla_x f
\]

and therefore equation (5.2) reads

\[
\frac{\partial}{\partial t} f + \frac{cp}{p^0} \cdot \nabla_x f = Q(F(t, x, \cdot))(p^0, p) := \frac{c}{p^0} Q(F(t, x, \cdot))(p^0, p).
\]

Since from Lemma A.1

\[
\chi(P) = \frac{1}{2p^0} \delta_{P^0 - p^0}.
\]

we obtain, performing the integration in variables \(P^0, \; P^0, \; P_*^0, \; P'_*^0\),

\[
Q(F)(p^0, p) = \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \frac{cW}{8p^0 p^0 p_*^0 p'_*^0} q(f) \delta_c dp' dp'_* dp_*.
\]

56
where \( C \) is defined by (1.1) with of course \( E(p) = p^0/c \). Gathering (5.7) and (5.9) we then recover (1.5), (1.6) with

\[
(5.10) \quad w = \frac{c \mathcal{V}}{8p^0p^0'p'^0p'^0}.
\]

The function \( w \) is determined by the differential cross section as follows

\[
(5.11) \quad c \frac{s \sigma(s, \theta)}{2p^0p^0'p'^0p'^0} = w
\]

where

\[
(5.12) \quad s = (P + P_*)^2, \quad \cos \theta = \frac{(P_* - P) \cdot (P'_* - P')}{(P_* - P)^2}.
\]

The parameter \( s \) times \( c^2 \) is the square of the energy in the center of momentum system and \( \theta \) is the scattering angle (see Remark 5.1). The differential cross section \( \sigma = \sigma(s, \theta) \) is a function of the energy and the scattering angle. Since we consider identical particles it satisfies the symmetry relation:

\[
\sigma(s, \theta) = \sigma(s, \pi - \theta).
\]

Thus, the differential cross section determines the structure of the collision integral. See Appendix 3 for more details about this function and its physical meaning.

The 12-fold integral in (5.3) can be reduced to a 5-fold integral by carrying out the delta function integrations. This can be done in two different ways. They correspond to the two different parametrisations of the collision manifold which are well known for the classical Boltzmann equations.

**The center of mass parametrisation.**

We deduce this parametrisation starting from the formulation (5.3) of the collision kernel. To this end we first perform a change of variables in the \( P' \) and \( P'_* \) integrals. So, given \( P \) and \( P_* \) fixed, consider the Lorentz transformation \( \Lambda \) from \( \mathbb{R}^4 \) into itself defined by:

\[
(5.13) \quad \Lambda \left( \begin{array}{c} \sqrt{s} \\ 0 \end{array} \right) = (P + P_*).
\]

It is given by the following \( 4 \times 4 \) matrix:

\[
(5.14) \quad \Lambda = \begin{pmatrix} \gamma(v) & \gamma(v) v^\top \\ \gamma(v) v & I + \frac{\gamma(v) - 1}{|v|^2} v v^\top \end{pmatrix}
\]

with

\[
(5.15) \quad v = \frac{p + p_*}{\sqrt{|p + p_*|^2 + s}} \in \mathbb{R}^3, \quad \gamma(v) = \frac{1}{\sqrt{1 - |v|^2}} \in \mathbb{R}
\]

and

\[
(5.16) \quad (v v^\top)_{ik} = v_i v_k, \quad v v^\top \in \mathcal{M}_{3 \times 3}.
\]

Define now,

\[
(5.17) \quad P' = \Lambda Q', \quad P'_* = \Lambda Q'_*.
\]
For the sake of brevity we shall denote:

$$P' = P'(Q'), \quad P_* = P'_*(Q'_*), \quad p' = p'(Q'), \quad p'_* = p'_*(Q'_*).$$

By the definition of Lorentz transform we have that $s$ and $\theta$ are invariant (see Appendix 2) by the change of variable. Then:

$$Q(f)(p) = \frac{4c}{p^0} \int_{\mathbb{R}^3} \int_{\mathbb{R}} s \sigma(s, \theta) q(f) \delta_{\Lambda^{-1}(p+P_*)-Q'-Q'_*} \chi(T^0_p \chi(Q'^0_*) \chi(Q^0_*) dQ' dQ'_* dp_*,$$

with now

$$q(f) = f(p'_*(Q'_*)) f(p'(Q')) (1 + \tau f(p)) (1 + \tau f(p_*)),$$

where we have used

if $(u_0, u_*, u_2, u_3) = \Lambda(v_0, v_*, v_2, v_3)$, then $\text{sign } u_0 = \text{sign } v_0$.

and

$$\delta(P + P_* - Q' - Q'_*) = \delta(\Lambda^{-1}(P + P_*) - Q' - Q'_*).$$

We perform now the integration with respect to $Q'^0_*$, $Q'^0_*$ and $Q'^0_*$. Thanks to (5.8) we have

$$Q(f)(p) = \frac{c}{2p^0} \int_{\mathbb{R}^3} \int_{\mathbb{R}} s \sigma(s, \theta) q(f) \delta_{\Lambda^{-1}(\sqrt{s}, 0)-Q'-Q'_*} \frac{dq'}{q'^0} \frac{dq'_*}{p'_*} dp_*,$$

with now

$$P = (p^0_0, p), \quad P_* = (p^0_*, p_*), \quad Q' = (q'^0, q'), \quad Q'_* = (q'^0_*, q'_*).$$

Observing that

$$(q'^0, q') + (q'^0_*, q'_*) = (\sqrt{s}, 0) \quad \text{iff} \quad q' = -q'_*, \quad q'^0 + q'^0_* = \sqrt{s},$$

we get

$$q'^0 = q'^0_* = \frac{\sqrt{s}}{2}$$

Therefore, performing the integration in $q'_*$ we get

$$Q(f)(p) = \frac{c}{p^0} \int_{\mathbb{R}^3} \int_{\mathbb{R}} \sigma q(f) \delta_{q'^0} - \sqrt{q'^0 + m^2 c^2} dq \frac{dp_*}{p_*}$$

We finally, change to spherical coordinates in the $q'$ integral:

$$dq'^0 = |q'|^2 dq' d\Omega$$

and use, again by Lemma A3.

$$\delta_{\frac{\sqrt{s}}{2}} - \sqrt{\frac{|q'|^2 + m^2 c^2}{s - 4m^2 c^2} min}$$

to obtain

$$Q(f)(p) = \frac{c}{p^0} \int_{\mathbb{R}^3} \int_{\mathbb{R}} \sigma q(f) \delta_{\sqrt{s} - \frac{3}{2} \sqrt{\frac{|q'|^2}{s - 4m^2 c^2}} dq' |q'|^2 d\Omega dq_*^0 dp_* / p_*$$

58
(5.26) \[ Q(f)(p) = \frac{c}{4p^9} \int \int_{S^2} \sqrt{s(s-4m^2c^2)} \sigma q(f) d\Omega \frac{dp}{p^9}. \]

Let us come back to the expression of \( q(f) \) in order to give an explicit formula in terms of \( p, p_*, \) and \( \Omega \). Remember that:

\[
q(f) = f(p_*(Q_*')) f(p'(Q')) (1 + \tau f(p)) (1 + \tau f(p_*)) - f(p) f(p_*) (1 + \tau f(p'(Q')) (1 + \tau f(p_*)(Q_*'))
\]

where

\[ P' = \Lambda Q', \quad P_*' = \Lambda Q_*' \quad \text{and} \quad \Lambda(\sqrt{s},0) = (P + P_*) = (p^0 + p_*^0, p + p_*), \]

so that we just have to express \( p' = p'(Q') \) and \( p_*' = p_*'(Q_*') \). As we have said just before

\[ Q' = \left( \frac{\sqrt{s}}{2}, |q'|\Omega \right), \quad Q_*' = \left( \frac{\sqrt{s}}{2}, -|q'|\Omega \right). \]

Therefore

\[
p' = \Lambda(\frac{\sqrt{s}}{2}, q') = \left( \gamma(v) \frac{\sqrt{s}}{2} + \gamma(v) |q'| v^T \Omega, \quad \gamma(v) v \frac{\sqrt{s}}{2} + |q'| \Omega \right) = \frac{\gamma(v) - 1}{v^2} |q'| v^T \Omega
\]

\[
p_*' = \Lambda(\frac{\sqrt{s}}{2}, -q') = \left( \gamma(v) \frac{\sqrt{s}}{2} - \gamma(v) |q'| v^T \Omega, \quad \gamma(v) v \frac{\sqrt{s}}{2} - |q'| \Omega \right) = \frac{\gamma(v) - 1}{v^2} |q'| v^T \Omega
\]

with

\[
v = \frac{p + p_*}{\sqrt{|p + p_*|^2 + s}} = \frac{p + p_*}{p^0 + p_*^0}, \quad \gamma(v) = \frac{1}{\sqrt{1 - |v|^2}} = \frac{p^0 + p_*^0}{\sqrt{(p^0 + p_*^0)^2 - |p + p_*|^2}}
\]

and

\[
|q'| = \frac{1}{2} \sqrt{s - 4m^2c^2} = \frac{1}{2} \sqrt{(p^0 + p_*^0)^2 - |p + p_*|^2 - 4m^2c^2}.
\]

We finally find:

\[
\begin{align*}
p'(p) &= \frac{p + p_*}{2} + \frac{1}{2} \sqrt{(p^0 + p_*^0)^2 - |p + p_*|^2 - 4m^2c^2} (\Omega + \frac{\gamma(v) - 1}{v^2} v v^T \Omega), \\
p_*(p) &= \frac{p + p_*}{2} - \frac{1}{2} \sqrt{(p^0 + p_*^0)^2 - |p + p_*|^2 - 4m^2c^2} (\Omega + \frac{\gamma(v) - 1}{v^2} v v^T \Omega).
\end{align*}
\]

Finally, the variable \( \Omega \) describes the whole sphere \( S^2 \). It can then be parametrised in spherical coordinates of polar axis \( q \):

\[
\Omega = \frac{q}{|q|} \cos \theta + (\cos \phi \begin{pmatrix} I \end{pmatrix} + \sin \phi \begin{pmatrix} J \end{pmatrix}) \sin \theta
\]

in such a way that:

\[
d\Omega = \sin \theta d\theta d\phi.
\]

59
In order to see this we observe that:

\[
\cos \theta = \frac{(P - P') \cdot (P - P')}{(P - P')^2} = \frac{(Q - Q') \cdot (Q - Q')}{(Q - Q')^2}
\]

\[
= \frac{(q_0^2 - q^2)(q_0^2 - q^2) - (q - q) \cdot (q - q)}{(q_0^2 - q^2)^2 - |q - q|^2} = \frac{q \cdot q'}{|q|^2},
\]

where use has been made of the fact that \(q_0 = q_0 = q_0 = q_0 = \sqrt{s}/2\) and \(q = -q, q' = -q'\). Finally since

\[
|q| = \sqrt{q^2 - m^2c^2} = \sqrt{q'^2 - m^2c^2} = |q|,
\]

we get

\[
\cos \theta = \frac{qq'}{|q||q'|}.
\]

**Remark 5.1.** The reference frame of the variables \((Q, Q, q, q')\) in the previous calculation is called the center of momentum system. The geometry of the collision is particularly simple in this frame since:

(5.32) \(Q + Q = Q' + q' = \left(\sqrt{s} \right)
\]

and the scattering angle \(\theta\) is

(5.33) \(\cos \theta = \frac{qq'}{|q||q'|}
\]

It is easily seen that this is the angle formed by the trajectories before and after the collision for each of the particles. Finally the Møller velocity is defined by:

(5.34) \(v = \frac{c}{2p^0p^0} \sqrt{s(4m^2c^2)}
\]

and the collision integral in (5.26) may be written:

(5.35) \(Q(f)(p) = \frac{1}{2} \int_{\mathbb{R}^3} \int_{S^2} v \sigma(s, \theta)q(f) \, d\Omega dp_\ast.
\]

**Another expression for the collision integral.** (We closely follow in this part the Appendix II of R. T. Glassey and W. A. Strauss [GS], see also [Gl]). There is another way to reduce the 12-fold original integral to a 5-fold by using a slightly different parametrisation. Let us start again from the original equation, perform directly the integration in the \(p^0, p_\ast^0\) and \(p_\ast^0\) variables to obtain:

(5.36) \(Q(f)(p) = \frac{c}{2p^0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \delta((P + P_\ast) - \sqrt{s}|p|^2 + m^2c^2, p') - \delta(|p'|^2 + m^2c^2, p_\ast) \sigma \frac{dp'}{p^0} \frac{dp_\ast}{p_\ast^0} \frac{dp_\ast}{p_\ast^0}
\]

We integrate now in \(p_\ast\) to obtain

(5.37) \(Q(f)(p) = \frac{c}{16p^0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} \delta(p^0 + p_\ast^0 - p^0 - p_\ast^0) \sigma q(f) \frac{1}{p_\ast^0} \frac{dp'}{p^0} \frac{dp_\ast}{p_\ast^0}
\]

60
We perform now the change of variables:

\[
\begin{align*}
p' &= p + r \cdot \omega \\
p_*' &= p_* - r \cdot \omega
\end{align*}
\]

with \( \omega \in S^2 \) and \( r \in \mathbb{R} \) in such a way that

\[ p + p_* = p' + p_*'. \]

for every \( p \) and \( p_* \) in \( \mathbb{R}^3 \). We use now Lemma A3 (ii') to obtain:

\[
\delta(p^0 + p_*^0 - p^0 - p_*^0) = 2(p^0 + p_*^0)\delta((p^0 + p_*^0)^2 - (p^0 + p_*^0)^2) = 4(p^0 + p_*^0)p^0 p_*^0 \delta(4p^0 p_*^0 - [(p^0 + p_*^0)^2 - p^0 - p_*^0]^2).
\]

Observe now that:

\[
P(r) \equiv 4p^0 p_*^0 - [(p^0 + p_*^0)^2 - p^0 - p_*^0]^2 = 4p^0 p_*^0 - [(p^0 + p_*^0)^2 - (p^0 + p_*^0)^2]^2 = -(p^0 + p_*^0)^4 + 2(p^0 + p_*^0)^2(p^0 + p_*^0) - (p^0 - p_*^0)^2,
\]

where by the change of variables (5.38):

\[ p^0 - p_*^0 = |p|^2 - |p_*|^2 + 2r(p + p_*) \cdot \omega. \]

It is now a simple matter to check that \( P \) is a polynomial of degree two whose roots are \( r = 0 \) and

\[
\begin{align*}
a(p, p_*, \omega) &= \frac{2(p^0 + p_*^0 \omega \cdot (p - p_*)) p^0 p_*^0}{(p^0 + p_*^0)^2 - (\omega \cdot (p + p_*))^2} \equiv \frac{2N}{D}, \\
\hat{p} &= \frac{p}{p^0}, \quad \hat{p}_* = \frac{p_*}{p_*^0}.
\end{align*}
\]

Therefore,

\[ P(r) = Dr(r - a(p, p_*, \omega)). \]

We may then write:

\[
\delta(p^0 + p_*^0 - p^0 - p_*^0) \left( \frac{1}{p^0} \right) \frac{1}{p_*^0} dp' = \int \delta(p^0 + p_*^0 - p^0 - p_*^0) \left( \frac{1}{p^0} \right) \frac{1}{p_*^0} dp' = 4(p^0 + p_*^0)\delta(Dr - a(p, p_*, \omega)) r^2 dr d\omega.
\]

Using Lemma A.3 we get

\[
\delta(p^0 + p_*^0 - p^0 - p_*^0) \left( \frac{1}{p^0} \right) \frac{1}{p_*^0} dp' = 4(p^0 + p_*^0)\frac{1}{|D|} r^2 dr d\omega.
\]

The first delta function drops because of the factor \( r^2 \) and we are led them to:

\[
\delta(p^0 + p_*^0 - p^0 - p_*^0) \left( \frac{1}{p^0} \right) \frac{1}{p_*^0} dp' = 4(p^0 + p_*^0)2|D|^{-1} \delta(r - a) r^2 dr d\omega
\]

(5.44)
Finally,

\[
Q(f)(p) = \int_{\mathbb{R}^3} \int_{S^2} \Gamma(p, p*, \omega) q(f) \, d\omega dp*,
\]

(5.45)

\[
\Gamma(p, p*, \omega) = 4 cs \sigma \frac{(p^0 + p^0*)^2 |\omega \cdot (\hat{p} - \hat{p}*)|}{[(p^0 + p^0*)^2 - (\omega \cdot (p + p*))^2]^2}.
\]

Observe that

\[
\Gamma(p, p*, \omega) = \Gamma(p*, p, \omega)
\]

and since \( D \geq 2 \),

\[
0 \leq \Gamma(p, p*, \omega) \leq cs \sigma (p^0 + p^0*)^2 |\omega \cdot (\hat{p} - \hat{p}*)|.
\]

V.2 Particles with different masses.

Similar calculations can be performed if we consider collisions of two quantum relativistic particles of two different masses \( m_* \) and \( m_2 \) with momenta \( P = (p^0, p) \) and \( P_2 = (p^0_2, p_2) \) such that

\[
p^0 = \sqrt{|p|^2 + m_*^2 c^2}, \quad p^0_2 = \sqrt{|p_2|^2 + m_2^2 c^2}.
\]

(5.46)

Let us denote by \( f \) the density distribution of the particles \( P \) of mass \( m_* \) and \( g \) that of the particles \( P_2 \) of mass \( m_2 \). These functions satisfy the coupled system (4.2). Let us consider here only the integral \( Q_{1,2}(f, g) \equiv Q(f, g) \) since \( Q_{2,1}(g, f) \) is completely similar:

\[
Q(f, g) = \frac{8c}{p^0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} s \sigma q(f, g) \delta_{P+P_2-P_1-P_2} = 0 \chi_2(P_0^0).
\]

(5.47)

\[
\chi_1(P_0^0) \chi_2(P_0^0) \, dP^0 dP_2^0 dP_1^0
\]

\[
q(f, g) = g(p_2^0) f(p^0)(1 + \tau f(p))(1 + \tau^* g(p_2^0)) - f(p) g(p_2^0)(1 + \tau f(p^0))(1 + \tau^* g(p_2^0)),
\]

with \( \chi_i(P) = \delta_{P^0-i m_i^2 c^2} H(P^0) \) for \( i = 1, 2 \), and \( \tau, \tau^* \in \{-1, 0, 1\} \).

As before the integral collision may be written as

\[
Q(f, g) = \int_{\mathbb{R}^3} \int_{S^2} \nu \sigma q(f, g) d\Omega dp_*
\]

(5.48)

where the Møller velocity is in this case: (where \( \Omega \) and \( \theta \) are defined as above)

\[
\nu = \frac{c}{2p^0 p_*^0} \sqrt{(s - (m_* - m_2)^2 c^2)(s - (m_* + m_2)^2 c^2)}.
\]

(5.49)

and the dependence of \( p' \) and \( p_2^0 \) in function of \( p, p_* \) and \( \Omega \) in the expression (5.47) of \( q(f, g) \) is

\[
p' = \frac{p + p_*}{2} + \frac{\sqrt{(s - (m_* - m_2)^2 c^2)(s - (m_* + m_2)^2 c^2)}}{2\sqrt{s}} \left( \Omega + \frac{\gamma(v) - 1}{|v|^2} vv^\top \Omega \right),
\]

\[
p_2^0 = \frac{p + p_*}{2} - \frac{\sqrt{(s - (m_* - m_2)^2 c^2)(s - (m_* + m_2)^2 c^2)}}{2\sqrt{s}} \left( \Omega + \frac{\gamma(v) - 1}{|v|^2} vv^\top \Omega \right).
\]

An expression similar to (5.45) may also be obtained for the collision integral \( Q(f, g) \). From the expression in (5.47) we perform an integration in the \( p^0, p^0_2 \) and \( p^0_* \) variables to obtain:

\[
Q(f, g)(p) = \frac{c}{p^0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \delta((P + P_2) - (\sqrt{|p'|^2 + m_*^2 c^2}, p') - (\sqrt{|p_2^0|^2 + m_2^2 c^2}, p_2^0))
\]

\[
\frac{s \sigma q(f, g) \, dp' dp_2^0 dp_*^0}{p_2^0 p_*^0 p^0}
\]

62
We integrate now in \( p'_* \) to obtain

\[
Q(f, g)(p) = \frac{c}{p^0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \delta(p^0 + p'_* - p^0 - p^0_*) s \sigma q(f, g) \frac{1}{p^0_2} \frac{dp^1 dp'_*}{p^0_*}. \tag{5.50}
\]

We may then apply the calculations from (5.38) to (5.44) and obtain finally

\[
\begin{cases}
Q(f, g)(p) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Gamma(p, p_*, \omega) q(f, g) d\omega dp_*, \\
\Gamma(p, p_*, \omega) = 8 c s \sigma \frac{(p^0 + p^0_*)^2 |\omega \cdot (\hat{p} - \hat{p}_*)|}{[(p^0 + p^0_*)^2 - (\omega \cdot (p + p_*)^2)]^2}.
\end{cases} \tag{5.51}
\]

\textbf{Remark 5.2.} The expressions (5.34)-(5.35) and (5.48)-(5.49) correspond to the center of mass angular parametrisations of the collisions in the classical and non quantum Boltzmann equation.

\textbf{V.3. Boltzmann-Compton equation for photon-electron scattering.}

We consider in this Section the system of Boltzmann equations for a dilute gas of low energy electrons and weakly dense photons. We assume that particles \( P \) are photons and so are massless, \( m_* = 0 \) and particles \( P_* \) are electrons of mass \( m_2 > 0 \) that we take to be \( m_2 = m \). From the preceding sub Section V.2, if \( f \) describes the density of photons and \( g \) the density of electrons the collision integral writes

\[
Q(f, g)(p) = \frac{c}{2|p|} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (s - m^2 c^2) \sigma(s, \theta) q(f, g) d\Omega \frac{dp_0}{p^0_*}. \tag{5.52}
\]

where,

\[
s = (P + P_*)^2 = (|p| + \sqrt{|p_*|^2 + m^2 c^2})^2 - |p + p_*|^2 = c^2 + 2|p|\sqrt{|p_*|^2 + m^2 c^2} - 2(p, p_*) \tag{5.53}
\]

and

\[
p' = \frac{p + p_*}{2} + \frac{s - m^2 c^2}{2\sqrt{s}}(\Omega + \frac{\gamma(v) - 1}{v^2} vv^\top \Omega), \tag{5.54}
\]

\[
p'_* = \frac{p + p_*}{2} - \frac{s - m^2 c^2}{2\sqrt{s}}(\Omega + \frac{\gamma(v) - 1}{v^2} vv^\top \Omega). \tag{5.55}
\]

In this context, the differential cross section \( \sigma \) is given by the Klein Nishina formula given in Appendix 3. Consider now the so called classical limit when \( c \to \infty \). We first have

\[
s - m^2 c^2 \sim 2|p|m c, \quad \text{and} \quad v \sim 1 \quad \text{as} \quad c \to \infty \tag{5.56}
\]

and, using that the photons have low energy, \( p^0 \sim p^0_0 \) from where we deduce

\[
\sigma(s, \theta) \sim \frac{1}{2} \frac{\gamma^2}{\gamma^2_0} \{1 + \cos^2 \theta\}, \quad \text{as} \quad c \to \infty, \tag{5.57}
\]

(c.f. [PS], p.163). Finally

\[
\lim_{c \to \infty} \frac{s - m^2 c^2}{2\sqrt{s}} = |p|. \tag{5.58}
\]
Then, in the classical limit which amounts to consider the higher order term in (5.52) we obtain:

\[ Q(f, g)(p) = \frac{c r_0^2}{2} \int_{\mathbb{R}^3} \int_{S^2} \{1 + \cos^2 \theta \} q(f, g) d\Omega dp \]

with

\[ p = \frac{p + p^*}{2} + |p| \, \Omega, \quad p^* = \frac{p + p^*}{2} - |p| \, \Omega \]

**Dilute and low energy electron gas at equilibrium.** We assume moreover that the dilute and low energy electron gas is at equilibrium. Then, the density of electrons is given by a Maxwell Boltzmann distribution. It is therefore possible to write the corresponding collision integral in a more explicit way as it was already done in Section 4. To this end it is convenient to express this collision integral in a Carleman’s type parametrisation. It should be possible to obtain it from (5.5) but nevertheless we proceed in a simpler way. We consider then the classical limit, and the integral collision reads:

\[ Q(f)(p) = \frac{c r_0^2}{2|p||p'|} \int_{\mathbb{R}^3} \int_{S^2} \int_{\mathbb{R}^3} (1 + \cos^2 \theta) q(f) \delta_{\Sigma} dp dp dp' \]

where \( \Sigma \) is now the manifold of 4-uplets \((p, p^*, p', p_\star')\) such that,

\[ p + p^* = p' + p_\star', \quad |p| + |p| \, |p'| = |p'| + |p_\star'| \]

By the hypothesis on the electron gas, we may take \( q(p_\star') = e^{-\beta^0 |p_\star'|^2} e^\mu \) for some \( \beta^0 > 0 \) and \( \mu \in \mathbb{R} \) constants. By (5.61) we deduce

\[ q(p_\star') = e^{-\beta^0 |p_\star'|^2} e^\mu, \]

and the integral collision reads,

\[ Q(f)(p) = \frac{r_0^2}{2} e^\mu \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(1 + \cos^2 \theta)}{|p||p'|} e^{-\beta^0 |p_\star'|^2 + \beta^0 |p'|} q(f) \delta_{\Sigma} dp dp dp'. \]

with

\[ q(f) = e^{-\beta^0 |p| f(p')} (1 + f(p)) - e^{-\beta^0 |p'} f(p) (1 + f(p')). \]

Notice that the integration in the \( p_\star' \) variable is straightforward. Let us state the following auxiliary Lemma.

**Lemma 5.3.**

\[ S(p, p') = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \delta_{\Sigma} e^{-\beta^0 \frac{|p_\star'|^2}{2mc}} dp dp' = \frac{2\pi m^2 c^2}{\beta^0 \omega} e^{-\beta^0 \frac{|w|^2}{2mc}} \]

where

\[ A = |p| - |p'| + \frac{|p - p'|^2}{2mc} \quad \text{and} \quad w = p' - p. \]

**Proof.** Since

\[ |p| + \frac{|p_\star'|^2}{2mc} - |p'| - \frac{|p + p^* - p'|^2}{2mc} = A - \frac{1}{mc} (p_\star, w), \]

\[ \frac{r_0^2}{2} e^\mu \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(1 + \cos^2 \theta)}{|p||p'|} e^{-\beta^0 |p_\star'|^2 + \beta^0 |p'|} q(f) \delta_{\Sigma} dp dp dp'. \]

with

\[ q(f) = e^{-\beta^0 |p| f(p')} (1 + f(p)) - e^{-\beta^0 |p'} f(p) (1 + f(p')). \]

Notice that the integration in the \( p_\star' \) variable is straightforward. Let us state the following auxiliary Lemma.

**Lemma 5.3.**

\[ S(p, p') = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \delta_{\Sigma} e^{-\beta^0 \frac{|p_\star'|^2}{2mc}} dp dp' = \frac{2\pi m^2 c^2}{\beta^0 \omega} e^{-\beta^0 \frac{|w|^2}{2mc}} \]

where

\[ A = |p| - |p'| + \frac{|p - p'|^2}{2mc} \quad \text{and} \quad w = p' - p. \]

**Proof.** Since

\[ |p| + \frac{|p_\star'|^2}{2mc} - |p'| - \frac{|p + p^* - p'|^2}{2mc} = A - \frac{1}{mc} (p_\star, w), \]
we have
\[
S(p, p') = \int_0^\infty \int_{S^2} \delta_{A - \frac{1}{m\omega} (\Omega, w)} d\Omega_{\ast} e^{-\beta \frac{\omega^2}{2m} |p_{\ast}|^2} |p_{\ast}|^2 d|p_{\ast}|.
\]
From
\[
\int_{S^2} \delta_{A - \frac{1}{m\omega} (\Omega, w)} d\Omega_{\ast} = \frac{2\pi mc}{|p_{\ast}| |w|} H(1 - \frac{A^2 m^2 c^2}{|p_{\ast}|^2 |w|^2})
\]
(where \(H\) stands for the Heaviside's function) we obtain:
\[
S(p, p') = \frac{2\pi mc}{|w|} \int_0^\infty |p_{\ast}| e^{-\beta \frac{w^2}{2m} |p_{\ast}|^2} d|p_{\ast}|
\]
and then conclude.

The collision kernel reads now
\[
Q(f)(p) = \frac{c r_0^2}{2} e^{\mu} \int_{\mathbb{R}^3} \frac{S(p, p')}{|p| |p'|} \left(1 + \cos^2 \theta\right) e^{\beta |w'|^2} q(f) d|p'|
\]

If we now assume that the photon distribution is radial, we may write in polar coordinates, denoting \(\varepsilon = |p|, \varepsilon' = |p'|\), and, with a slight abuse of notation \(S(p, p') = S(\varepsilon, \varepsilon', \theta)\),
\[
Q(f)(\varepsilon) = \frac{c r_0^2}{2 \varepsilon^2} e^{\mu} \int_0^\infty B(\varepsilon, \varepsilon') q(f) e^{\beta \varepsilon' \varepsilon} d\varepsilon',
\]
with
\[
B(\varepsilon, \varepsilon') = 2\pi \varepsilon \varepsilon' \int_0^\pi (1 + \cos^2 \theta) S(\varepsilon, \varepsilon', \theta) \sin \theta d\theta.
\]
If we call now \(F(\varepsilon, t) = \varepsilon^2 f(\varepsilon, t)\), the equation reads:
\[
\begin{cases}
\partial_t F = Q(F) = \int_0^\infty b(\varepsilon, \varepsilon') (F'(\varepsilon^2 + F) e^{-\beta \varepsilon^2} - F(\varepsilon' + F') e^{-\beta \varepsilon'^2}), \\
\text{where } b(\varepsilon, \varepsilon') = \frac{c r_0^2}{2} e^{\mu} \frac{B(\varepsilon, \varepsilon')}{\varepsilon^2 \varepsilon'^2} e^{\beta \varepsilon' \varepsilon} = \frac{2 m^2 c^3 r_0^2 \pi^2}{\beta^2 \varepsilon^2} e^{\mu} \varepsilon e^{\beta \varepsilon^2} \int_0^\pi (1 + \cos^2 \theta) \frac{1}{|w|} e^{-\beta \frac{w^2}{2m} |w|^2} \sin \theta \theta d\theta,
\end{cases}
\]
with \(|w|^2 = \varepsilon^2 + \varepsilon'^2 - 2 \varepsilon \varepsilon' \cos \theta\).

**Remark 5.3.** Up to the change of variable \(F = \varepsilon^2 f\), the equation (5.6) is formally the same as (4.15)-(4.32) and (1.1) of [EM2]. In particular, these equations have the same entropy function, given by (4.52). Nevertheless, in each of these cases, the functions \(b(\varepsilon, \varepsilon')\) which appear in the collision have different, although similar, behaviours in the domain \(\varepsilon > 0, \varepsilon' > 0\). It is in particular easy to check that the function \(b\) in (5.65) does not satisfy any of the conditions (2.1), (2.2) (2.3) imposed in the existence theorems obtained in [EM2]. The global existence of solutions in \(L^1\) is then still an open question for this equation.

## Appendix 1. A distributional Lemma.

For any function \(\Psi : \mathbb{R}^m \rightarrow \mathbb{R}\) one defines \(\delta_{\Psi} = \delta_{\Psi - 0}\) as the following measure of \(\mathbb{R}^m\)
\[
(\text{A1.1}) \quad \forall \varphi \in C_c(\mathbb{R}^m) \quad <\delta_{\Psi = 0}, \varphi> = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^m} \rho_\varepsilon(\Psi(y)) \varphi(y) dy
\]
where \(\rho_\varepsilon\) is any approximation of the 1-dimension Dirac measure in 0.
Lemma A1.1  For any $a, b \in \mathbb{R}$, $a \neq 0$

(A1.2) \[ \delta_a x - b = \frac{1}{a} \delta_{x/b/a}. \]

For $a \neq b$,

(A1.3) \[ \delta_{(x-a)}(x-b) = \frac{1}{|b - a|} (\delta_{x-a} + \delta_{x-a}). \]

Remark A1.2  Two particular cases are

(A1.4) \[ \forall a > 0 \quad \delta_{x-a^2}1_{x>0} = \frac{1}{2a} \delta_{x-a} \]

and

(A1.5) \[ \forall a > 0 \quad \delta_{x-(x-a)}1_{x>0} = \frac{1}{a} \delta_{x-a}. \]

Proof of Lemma A1.1.  Since (A1.2) is evident and (A1.4), (A1.5) are immediate consequences of (A1.3), we just prove (A1.3).  Let $(\rho_x)$ be a sequence of $L^1(\mathbb{R})$ such that $\rho_x \rightarrow \delta$ in $\mathcal{D}'(\mathbb{R})$.  Then

\[
< \delta_{(x-a)}(x-b), \phi > := \lim_{\epsilon \rightarrow 0} < \rho_x((x-a) (x-b)), \phi > = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} \rho_x((x-a) (x-b)) \phi \, dx = \lim_{\epsilon \rightarrow 0} I_x + J_x,
\]

with

\[
I_x = \int_{-\infty}^{a+b} \rho_x((x-a) (x-b)) \phi \, dx, \quad J_x = \int_{a+b}^{\infty} \rho_x((x-a) (x-b)) \phi \, dx.
\]

Without loss of generality we may assume that $a < b$.  Set $y = (x-a) (x-b) = x^2 - (a + b)x + ab$.  The function $x \mapsto y(x)$ is monotone for any $x \leq (a+b)/2$ so that it is an allowed change of variable.  We compute

\[
2x = a + b \pm \sqrt{4y + (a-b)^2} \quad \text{and} \quad dy = [2x - (a+b)] \, dx = \pm \sqrt{4y + (a-b)^2} \, dx,
\]

and hence

\[
I_x \rightarrow \frac{1}{|b - a|} \phi \left( \frac{a + b}{2} \pm \sqrt{y + \frac{(a-b)^2}{4}} \right) \frac{dy}{\sqrt{4y + (a-b)^2}} = \frac{\phi(a)}{|b - a|}
\]

Similarly, we prove $J_x \rightarrow \frac{\phi(a)}{|b - a|}$.

Appendix 2. Minkowsky space and Lorentz transform.

We note $P = (P^0, p) \in \mathbb{R}^4$ with $P^0 \in \mathbb{R}$, $p \in \mathbb{R}^3$ or indifferently $P = (P^\mu)$.  Let define the Lorentz metric:

(A2.1) \[ < P, Q > = P^0 Q^0 - p \cdot q \quad \forall P, Q \in \mathbb{R}^4. \]

We also write

\[
< P, Q > = P^\mu Q_\mu = P^\top \eta Q = \sum_{\mu, \nu=0}^4 \eta_{\mu \nu} P^\mu Q^\nu, \quad \text{with} \quad Q_\mu = \eta_{\mu \nu} Q^\nu,
\]

66
where
\[
\eta = (\eta_{\mu\nu}) = \begin{pmatrix}
1 & 0^T \\
0 & -I_3
\end{pmatrix}
\]
is the Minkowsky matrix. The inner product \(<,>\) on \(\mathbb{R}^4\) is symmetric, non degenerated but not positive.

**Definition A2.1** A Lorentz transform is a linear operator \(\Lambda : \mathbb{R}^4 \to \mathbb{R}^4\) such that
\[
< \Lambda P, \Lambda Q > = < P, Q > \quad \forall P, Q \in \mathbb{R}^4.
\]

**Example 1: Rotations.** For any rotation \(R\) of \(\mathbb{R}^3\) \((R \in SO(3))\),
\[
\Lambda = \begin{pmatrix}
1 & 0^T \\
0 & R
\end{pmatrix}
\]
is a Lorentz transform.

**Example 2: Boosts.** For any \(v \in \mathbb{R}^3\) such that \(v := |v| < 1\),
\[
\Lambda = \begin{pmatrix}
\gamma & \gamma v^T/\sqrt{1-v^2} \\
\gamma v & I + \gamma v^T v v^T
\end{pmatrix}, \quad \gamma = \frac{1}{\sqrt{1-v^2}}
\]
is a Lorentz transform.

**Remark A2.2.** Any Lorentz transform is the composition of a boost and a rotation, see \([\mathbf{1}]\). For \((\beta^0, \beta) \in \mathbb{R}^4\) with \(\beta^0 > |\beta|\) we define \(\beta > 0\) and \(u \in \mathbb{R}^3\) by
\[
\beta^2 := (\beta^0)^2 - |\beta|^2, \quad \frac{u}{c} := \frac{\beta}{\beta^0}.
\]
Then, setting
\[
\gamma := \frac{1}{\sqrt{1-(u/c)^2}}, \quad \text{we have} \quad (\beta^0, \beta) = (\gamma \beta^0, \gamma \beta \frac{u}{c}).
\]
For such a 4-vector \((\beta^0, \beta)\) we define the boost transform \(\Lambda = \Lambda_{(\beta^0)}\) associated to \(v := u/c\). It satisfies
\[
\Lambda \begin{pmatrix}
\beta \\
0
\end{pmatrix} = \begin{pmatrix}
\beta^0 \\
\beta
\end{pmatrix}.
\]

**Lemma A2.3** Any Lorentz transform \(\Lambda\) satisfies \(\det \Lambda = \pm 1\).

**Proof of Lemma A2.3.** Define
\[
\Lambda^* = \begin{pmatrix}
a & -c^T \\
b & d^T
\end{pmatrix} \quad \text{for} \quad \Lambda = \begin{pmatrix}
a & b^T \\
c & d
\end{pmatrix}
\]
with \(a \in \mathbb{R}, \ b, c \in \mathbb{R}^3\) and \(d \in M(\mathbb{R}^3)\). We easily verify that
\[
< \Lambda P, Q > = < P, \Lambda^* Q > \quad \text{for any} \ P, Q \in \mathbb{R}^4.
\]
In particular, by definition of a Lorentz transform, one has
\[
< \Lambda^* \Lambda P, Q > = < \Lambda P, \Lambda Q > = < P, Q > \quad \text{for any} \ P, Q \in \mathbb{R}^4,
\]
so that \(\Lambda^* \Lambda = Id_4\). Since \(\det \Lambda^* = \det \Lambda\), we get \((\det \Lambda)^2 = 1\).\(\square\)

67
Definition A2.4 For $s \in \mathbb{R}$ we define the hyperboloid

$$M^s_+ := \{ P \in \mathbb{R}^4, < P, P > = s, P^0 > 0 \} = \{(\sqrt{s + |p|^2}, p), p \in \mathbb{R}^3 \}.$$  

We write $\Lambda \in \mathcal{L}^1_+$, if $\Lambda$ is a Lorentz transform such that $\det \Lambda = +1$ and $(\Lambda P)^0 > 0$ for any $P \in M^s_+$, with $s > 0$.

The boost $\Lambda$ associated to $(\beta^0, \beta)$ belongs to $\mathcal{L}^1_+$ since $\det \Lambda = 1$ and

$$(\Lambda P)^0 = \gamma P^0 + \gamma u \cdot p = \frac{1}{\beta} [\beta^0 P^0 + \beta \cdot p] \geq \frac{1}{\beta} [\beta^0 |p| - |\beta| |p|] \geq 0$$

for any $P \in M^s_+$ with $s > 0$.

Lemma A2.5 Let $f : \mathbb{R} \to \mathbb{R}$ and $(\beta^\mu) = (\beta^0, \beta) \in \mathbb{R}^4$ such that $\beta^2 := \beta^0 - |\beta|^2 > 0$. We define $F : \mathbb{R}^3 \to \mathbb{R}$ by $F(p) = f(\beta^\mu p^\mu) = f(\beta^0 P^0 - \beta \cdot p)$ with $p^0 := \sqrt{s + |p|^2}$, $s > 0$. There is some constants $A_i = A_i(f, \beta)$ such that

$$\int_{\mathbb{R}^3} p^\mu F dp = A_1 \beta^\mu$$

$$\int_{\mathbb{R}^3} p^\mu p^\nu F dp = A_2 \eta^{\mu \nu} + A_3 \beta^\mu \beta^\nu.$$  

In particular, one has

$$N(F) := \int_{\mathbb{R}^3} F dp = A_1 \beta^0$$

$$P(F) := \int_{\mathbb{R}^3} F dp \, dp = A_3 \beta^0 \beta$$

$$E(F) := \int_{\mathbb{R}^3} F dp \, dp = A_2 + A_3 (\beta^0)^2$$

$$G(F) := s \int_{\mathbb{R}^3} F \frac{dp}{p^0} = 4 A_2 + A_3 \beta^2.$$  

Proof of Lemma A2.5. Using Lemma A2.3 we have, denoting by $H$ the Heaviside function,

$$\delta_{p^0-s} H(p^0) = \left( \delta_{p^0-\sqrt{s + |p|^2}} (p^0 + \sqrt{s + |p|^2}) \right) H(p^0)$$

$$= \frac{1}{2 \sqrt{s + |p|^2}} \left( \delta_{p^0-\sqrt{s + |p|^2}} + \delta_{p^0+\sqrt{s + |p|^2}} \right) H(p^0)$$

Therefore, we get the fundamental identity

$$\int_{\mathbb{R}^4} F(P) \delta_{p^0-s} H(p^0) \, dP = \int_{\mathbb{R}^3} \left\{ \int_{\mathbb{R}} F(P) \frac{1}{2 \sqrt{s + |p|^2}} \delta_{p^0-\sqrt{s + |p|^2}} \, dp \right\} \, dp$$

$$= \int_{\mathbb{R}^3} F(p^0, P) \frac{dp}{p^0}$$

For $\Lambda \in \mathcal{L}^1_+$ we get

$$\int_{\mathbb{R}^3} F(\Lambda(p^0, P)) \frac{dp}{p^0} = \int_{\mathbb{R}^4} F(\Lambda P) \delta_{p^0-s} H(p^0) \, dP$$

$$= \int_{\mathbb{R}^4} F(Q) \delta_{Q^0=s} H(q^0) \, dQ = \int_{\mathbb{R}^3} F(p^0, P) \frac{dp}{p^0}$$

68
Now we choose as $\Lambda$ the boost associated to $(\beta^\mu)$ and using (A2.8) we have, setting $P = \Lambda Q$, we get

$$\int_{\mathbb{R}^3} p^\mu f(\beta^\nu p^\nu) \frac{dp}{p^0} = \int_{\mathbb{R}^3} p^\mu f(\vec{\beta} (\Lambda^{-1} P)^0) \frac{dp}{p^0} = \Lambda^\mu \int_{\mathbb{R}^3} q f(\vec{\beta} q^0) \frac{dq}{q^0} = \Lambda^\nu \left( \vec{\beta} A_1 \right),$$

with

$$A_1 := \int_{\mathbb{R}^3} q^0 f(\vec{\beta} q^0) \frac{dq}{q^0},$$

since

$$\int_{\mathbb{R}^3} q^i f(\vec{\beta} q^0) \frac{dq}{q^0} = 0$$

for $i = 1, 2, 3$ by rotation symmetry. This proves (A2.10).

Similarly,

$$\int_{\mathbb{R}^3} p^\mu p^\nu f(\beta^\sigma p^\sigma) \frac{dp}{p^0} = \Lambda^\mu_\nu \Lambda^\nu_\mu \int_{\mathbb{R}^3} q^\mu q^\nu f(\vec{\beta} q^0) \frac{dq}{q^0} = \Lambda^\mu_\nu \Lambda^\nu_\mu \alpha_\mu \delta_{\mu \nu},$$

with

$$\alpha_0 = \int_{\mathbb{R}^3} (q^0)^2 f(\vec{\beta} q^0) \frac{dq}{q^0}$$

$$\alpha_1 = \alpha_2 = \alpha_3 = \int_{\mathbb{R}^3} (q^1)^2 f(\vec{\beta} q^0) \frac{dq}{q^0}$$

Just compute

$$\Lambda^2 = \left( \begin{array}{cc} 2 \gamma^2 - 1 & \gamma^2 \vec{\gamma}^T \vec{v} \\ \gamma^2 \vec{v} \vec{v}^T & 2 \gamma^2 v^2 I + 2 \gamma^2 \vec{v} \vec{v}^T \end{array} \right) = - (\eta)^\mu_\nu + 2 \left( \begin{array}{cc} \gamma^2 & \gamma (\gamma u/c)^T \\ \gamma (\gamma u/c)^T & (\gamma u/c)(\gamma u/c)^T \end{array} \right)$$

$$= - (\eta)^\mu_\nu + \frac{2}{\beta^2} \beta^\mu \beta^\nu,$$

and

$$\Lambda^\mu_\nu = \left( \begin{array}{c} \gamma \\ \gamma u \end{array} \right) = \frac{\beta^\mu}{\beta}.$$

Therefore

$$\int_{\mathbb{R}^3} p^\mu p^\nu f(\beta^\sigma p^\sigma) \frac{dp}{p^0} = \alpha_1 \Lambda^\mu_\nu \Lambda^\nu_\mu + (\alpha_0 - \alpha_1) \Lambda^\mu_\nu \Lambda^\nu_\mu$$

$$= \alpha_1 (\Lambda^2)^{\mu \nu} + (\alpha_0 - \alpha_1) \Lambda^\mu_\nu \Lambda^\nu_\mu$$

$$= -\alpha_1 (\eta)^{\mu \nu} + \frac{\alpha_0 + \alpha_1}{\beta^2} \beta^\mu \beta^\nu,$$

and (A2.11) is proved. The points (A2.12), (A2.13) and (A2.14) follow. Finally, we compute, thanks to (A2.11),

$$\int_{\mathbb{R}^3} \eta^\mu_\nu p^\mu p^\nu f \frac{dp}{p^0} = A_1 \eta^\mu_\nu \eta^\nu_\mu + A_2 \beta^\mu \beta^\nu \eta^\mu_\nu,$$

so that

$$\int_{\mathbb{R}^3} [(p^0)^2 - |p|^2) f \frac{dp}{p^0} = 4 A_1 + A_2 [(\beta^0)^2 - |\beta|^2] = 4 A_1 + A_2 \beta^2,$$

and (A2.15) follows, remarking that $(p^0)^2 - |p|^2 = s. \Box$

Appendix 3: Differential cross section.
We present in this Appendix some short notes, mainly taken from the volumes 3 and 10 of the Landau and Lifshitz. We only wish to present some basic but important informations about what the differential cross section is and where does it come from.

Following Landau and Lifshitz vol 10, §2 let us consider collisions between two molecules of a monatomic gas, one of which has momentum \( p \) in a given range \( dp \) and the other in a range \( dp' \), and which acquire in the collision values in the ranges \( dp' \) and \( dp' \). For brevity we refer simply to a collision of molecules with \( p \) and \( p' \), resulting in \( p' \) and \( p' \). The total number of such collisions per unit time and unit volume of the gas may be written as a product of the number of molecules per unit volume, \( f(t, p)dp \), and the probability that any of them has a collision of the type concerned. This probability is always proportional to the number of molecules \( p \) per unit volume, \( f(t, p)dp \), and to the ranges \( dp' \) and \( dp' \) of the values of \( p \) for the two molecules after collision. Thus the number of collisions \( p, p \rightarrow p', p' \) per unit time and volume may be written as

\[
W(p', p'; p, p) f f, (1 + \tau f') (1 + \tau f') dp dp dp dp
\]

where as usual, \( \tau = 1 \) for Bose particles, \( \tau = -1 \) for Fermi particles and \( \tau = 0 \) for non quantum particles. The coefficient \( W \) is a function of all its arguments. The ratio of \( W dp dp' \) to the absolute value of the relative velocity \( v - v' \) of the colliding molecules has the dimensions of area, and is the effective collision cross section:

\[
d\sigma = W(p', p'; p, p) |p - p| dp dp
\]

The function \( W \) can in principle be determined only by solving the mechanical problem of collision of particles interacting according to some given law. However, certain properties of this function can be elucidated from general arguments.

The first property is called detail balancing and reads:

\[
W(p', p'; p, p) = W(p, p'; p', p')
\]

I.e. each microscopic collision process is balanced by the reverse process. It is a direct consequence of two symmetries.

1.- The symmetry of the laws of the mechanic (classical or quantum) under time reversal: according to this, the number of collisions \( p, p \rightarrow p', p' \) is equal; in equilibrium, to the number \( -p', -p' \rightarrow -p, -p \) from where one obtains:

\[
W(p', p'; p, p) = W(-p, -p'; -p', -p').
\]

2.- The symmetry of the molecules under spatial inversion i.e. change of the signs of all coordinates. This symmetry implies

\[
W(-p, -p'; -p', -p') = W(p, p'; p', p')
\]

and the detail balancing follows.

Moreover, we may also use the fact that, we do not integrate in the whole momentum space but only along the manifold determined by the conservation of energy and conservation of the momentum. Let us assume for the sake of brevity that the two particles have the same mass equal to one. Then, the two conservation properties read:

\[
\begin{cases}
  p' + p' = p + p \\
  \frac{|p|^2}{2} + \frac{|p|^2}{2} = \frac{|p'|^2}{2} + \frac{|p'|^2}{2}.
\end{cases}
\]

The expression of \( W \) may therefore be written like:

\[
W(p', p'; p, p) dp dp = H(p', p'; p, p) \delta(p' + p' - p - p) dp dp
\]

\[
= H(p + p - p', p'; p, p) \delta(\frac{|p'|^2 + |p'|^2 - |p|^2 - |p|^2}{2}) dp' dp'
\]

70
On this manifold we can moreover write:

\[
\begin{cases}
    p' = q(p, p_*, \omega) = p - (p - p_*)\omega \\
    p'_* = q_*(p, p_*, \omega) = p_* + (p - p_*)\omega, \quad \omega \in S^2
\end{cases}
\]

from where,

\[ W(p', p'_*; p, p_*) dp' dp'_* = B(p, p_*, \omega) d\omega. \]

Finally, since the two interacting particles constitute a closed physical system, \( W \) has to be Galilean invariant (or Lorentz invariant in the relativistic case). Consider for the sake of simplicity the classical case. This means:

\[ W(Tp', Tp'_*; Tp, Tp_*) = W(p', p'_*; p, p_*) \]

for any rotation \( T = R \in SO(3) \) and any translation \( T(p) = a + p \) with \( a \in \mathbb{R}^3 \). We have:

\[ (q(p + a, p_* + a, \omega), q_*(p + a, p_* + a, \omega)) = (q(p, p_*, \omega) + a, q_*(p, p_*, \omega) + a) \]

in such a way that,

\[ B(p + a, p_* + a, \omega) = W(p + a, p_* + a, p' + a, p'_* + a) = W(p, p_*, p', p'_*) = B(p, p_*, \omega) \]

and therefore,

\[ B(p, p_*, \omega) = B(0, p_* - p_*, \omega) \]

that we denote

\[ B(0, p_* - p_*, \omega) \equiv B(p_* - p_*, \omega). \]

On the other hand, we also have

\[ (q(Rp, Rp_*, R\omega), q_*(Rp, Rp_*, R\omega)) = (Rq(p, p_*, \omega), Rq_*(p, p_*, \omega)) \]

in such a way that:

\[ B(R(p_* - p_*, \omega) = B(Rp, Rp_*, R\omega) \]

\[ = W(Rp, Rp_*, Rp', Rp'_*) = W(p, p_*, p', p'_*) = B(p_* - p_*, \omega) \]

for every rotation \( R \in SO(3) \). Therefore we have

\[ B(p_* - p_*, \omega) = B(|p_* - p_|, \frac{p_* - p}{|p_* - p|}, \omega). \]

And, \( B(p_* - p_*, \omega) f(p_*) d\omega dp_* \) is the probability per unit time and unit volume that any of the colliding particles \( p \) has a collision of the type considered. The effective collision cross section is then defined by

\[ d\sigma = \frac{B(p_* - p_*, \omega)}{|p - p_*|} d\omega. \]

**Scattering theory.**

The differential cross section depends on a crucial way from the kind of interaction between the two colliding particles one considers. If the interaction between particles only depends on the distance between the two particles, we may assume without any loss of generality that

\[ B(|p_* - p|, \frac{p_* - p}{|p_* - p|}, \omega) = B(|p_* - p|, \frac{p_* - p}{|p_* - p|}, \omega), \]

71
i.e. the function $B$ only depends on the modulus of the difference of momentum of the two incident particles and of the angle $\alpha$ formed by the two directions $p_\ast - p$ and $\ell_\ast - \ell'$.

On the other hand, like any problem of two bodies, the problem of elastic collision amounts to a problem of the scattering of a single particle, with the reduced mass, in the field $U$ of a fixed centre force. This simplification is effected by changing to a system of coordinates in which the centre of mass of the two particles is at rest. We pose:

$$\Omega = \frac{p - p_\ast}{|p - p_\ast|} - 2\left(\frac{p - p_\ast}{|p - p_\ast|}, \omega\right)\omega$$

in such a way that with this new variable:

$$\begin{cases}
\ell' = \frac{p + p_\ast}{2} + \frac{|p - p_\ast|}{2} \Omega,

\ell'_\ast = \frac{p + p_\ast}{2} - \frac{|p - p_\ast|}{2} \Omega.
\end{cases}$$

We have then:

$$d\omega = \frac{1}{4 \cos \alpha} d\Omega,$$

$\alpha = \text{angle entre } p - p_\ast \text{ et } \omega$,

Then, we use a spherical coordinate system with axis $p - p_\ast$:

$$\Omega = \frac{p - p_\ast}{|p - p_\ast|} \cos \theta + (\cos \phi h + \sin \phi i) \sin \theta.$$ 

So in these new variables,

$$|(p - p_\ast, \omega)| = |p - p_\ast| \sin \frac{\theta}{2}$$

from where,

$$\cos \alpha = \sin \frac{\theta}{2}.$$ 

Therefore,

$$d\omega = \frac{d\Omega}{4 \cos \alpha} = \frac{1}{4 \cos \alpha} \frac{\sin \theta d\theta d\phi}{4 \cos \alpha} = \frac{\sin \theta}{4 \sin(\theta/2)} d\theta d\phi = \frac{1}{2} \cos(\theta/2) d\theta d\phi$$

from where,

$$B(z, \omega) d\omega = \frac{B(z, \omega)}{4 \cos \alpha} d\Omega = \frac{\cos(\theta/2)}{2} B(z, \omega) d\theta d\phi.$$ 

The differential cross section, $\sigma(z, \theta)$ is then such that

$$B(z, \omega) d\omega = \sigma(z, \theta) d\Omega \equiv \sigma(z, \theta) \sin \theta d\theta d\phi.$$ 

And we may then write:

$$d\sigma(p, p_\ast, \theta) = \sigma \left(\frac{|p - p_\ast| \theta}{|p - p_\ast|}\right) d\Omega.$$ 

The angle $\theta$ is the angle formed by the incident and scattered trajectory of the particle interacting with the central potential. In classical mechanics, collisions of two particles are entirely determined by their velocities and impact parameter. For a detailed study of the different differential cross section depending on the potential $U$ considered in the classical case the reader may consult the detailed work by Cercignani [C]. It is shown in particular that:

1.- If $U$ is a power law potential:

$$U(\rho) = |\rho|^{1-n}, \quad n \neq 2, 3.$$
then :

\[ B(|p_s - p|, \frac{p_s - p}{|p_s - p|}, \omega) = |p_s - p|^7 \beta(\frac{p_s - p}{|p_s - p|} \cdot \omega), \]

where \( p \) and \( p_s \) are the velocities of the two particles.

**Coulomb potential.** If \( U(p) = \alpha|p|^{-1} \) then

\[
\sigma(|p - p_s|, \theta) = \frac{\alpha^2}{16|p_s - p|^4 \sin^4 \frac{\theta}{2}}.
\]

This is known as the Rutherford’s formula (see Landau & Lifshitz vol. 1 §19).

**Hard sphere potential.** For the so called Hard sphere potential \( U \) defined as:

\[
U(p) = \lim_{n \to \infty} U_n(p), \quad U_n(r) = \begin{cases} 
  n & \text{if } |p| < a \\
  0 & \text{if } |p| > a
\end{cases}
\]

we have

\[ \sigma = a^2. \]

**Remark 1.** In the general case, where the two interacting particles \( p \) and \( p_s \) have masses \( m_1 \) and \( m_2 \) respectively, similar arguments and calculations can be performed. In particular, the center of mass parametrisation may be written:

\[
\begin{align*}
  p' &= \frac{m_1 m_2}{m_1 + m_2} (p + p_s) + \frac{m_1}{m_1 + m_2} |p - p_s| \Omega, \\
p'_s &= \frac{m_1 m_2}{m_1 + m_2} (p + p_s) - \frac{m_2}{m_1 + m_2} |p - p_s| \Omega.
\end{align*}
\]

One defines again the angle \( \theta \) by,

\[ \Omega = \frac{p - p_s}{|p - p_s|} \cos \theta + (\cos \phi h + \sin \phi i) \sin \theta, \]

from where, as before

\[ W(p', p'_s; p, p_s) dp' dp'_s = \sigma(z, \theta) d\theta d\phi, \]

and

\[ d\sigma(p, p_s, \theta) = \sigma(|p - p_s|, \theta) \frac{1}{|p - p_s|} d\theta d\phi. \]

**Remark 2.** ([LL] Vol.10, §2.) Although the free motion of particles is assumed classical, this does not at all mean that their collision cross section need not be determined quantum mechanically; in fact, it usually must be so determined.

In quantum mechanics the very wording of the scattering problem must be changed, since in motion with definite velocities the concept of path is meaningless, and so is the impact parameter. The purpose of the theory is then only to calculate the probability that, as a result of the collision, the particles will scattered through any given angle. It is not our purpose to present in detail the derivation and the properties of the differential cross section from the scattering theory in detail. We only want to present the general idea, the relevant results for our study and some precise references for the interested readers.

We shall only present here a brief description of what M. Reed and B. Simon present as “naive” scattering theory, or a stationary picture of it ([RS], Vol.3, Notes on §X1.6). It is nevertheless the usual in the texbooks of quantum mechanics as for instance vol.3, §123 of Landau Lifschitz.

73
For a fixed $\mathbb{R}^3$-vector $p$ and a positive real number $E$ let us consider the function of $\rho = (x, y, z)$ and $t$ called plane wave:

$$e^{-iE t} e^{i p \cdot \rho}.$$ 

This plane wave describes a state in which the particle has a definite energy $E$ and momentum $p$. The angular frequency of this wave is $E/\hbar$ and its wave vector $k = p/\hbar$; the corresponding wavelength $2\pi \hbar/|\beta p|$ is the de Broglie wavelength of the particle. The mass is $m = |p|^2/2E$ and the velocity is $v = p/m$.

Consider now a free particle, with mass $m$, total energy $E$, moving in the direction of the $z$-axis and by abuse of notation let us denote its plane wave as

$$\psi_1(\rho) = e^{i k z}, \quad \forall \rho = (x, y, z) \in \mathbb{R}^3.$$ 

Assume that it is scattered by a radially symmetric potential $U(r)$ ($\rho = (r, \theta, \varphi)$ in polar coordinates).

The basic ansatz of naïve scattering theory is that the scattering state is the solution of the linear Schrödinger equation:

$$\Delta \psi(\rho) + k^2 \psi(\rho) - \frac{2m}{\hbar^2} U(r) \psi(\rho) = 0$$

such that

$$\psi(\rho) \sim e^{i k z} + f(k, \theta) e^{i k r} \quad \text{as} \quad r \to \infty.$$ 

I.e., at large $r$ we see the incident plane wave moving in the positive sense of the $z$-axis and a spherical divergent wave, modulated by the function $f \equiv f(k, \theta)$, called the scattering amplitude. This extra term describes the “scattered particle”.

The scattered particle is described far from the center, as a spherical divergent wave, i.e. a wave moving in the “increasing” sense of the radial direction

$$f(k, \theta) e^{i k r}.$$ 

Remember that the square of the modulus of the wave $\psi$ is the density of probability to find the particle at the point $\rho$). The presence of the factor $1/r$ is just to preserve that property since we are in $\mathbb{R}^3$. The density of probability is not necessarily the same in all the points but has to be independent of the $\varphi$ and $r$ variables due to the spherical symmetry of the potential. This is taken into account by the coefficient $f(k, \theta)$ which is the amplitude diffusion and depends only on the angle $\theta$ between the direction of the incoming particle, which is $e_3 = (0, 0, 1)$, and the direction where we are looking for the scattered particle, i.e. $\rho/r$.

As it is pointed out in [RS], at first sight, this ansatz looks absurd, for, if $\psi \sim e^{i k z} + f(\theta) r^{-1} e^{i k r}$ for $r \to \infty$, then, for all the time $\psi$ has both a plane wave coming in and an outgoing spherical wave. The point of the argument is to consider an initial state which is more localized i.e. given by the function:

$$\psi(\rho) = \int g(k) \left( e^{i k z} + f(\theta) r^{-1} e^{i k r} \right) dk$$

and $g$ peaked around $k = k_0$. Then following the same idea, for $r$ and $t$ large, the wave would be given by

$$\psi(\rho) \sim \int g(k) e^{i k (z - k t)} dk + r^{-1} f(\theta) \int g(k) e^{i k (r - k t)} dk.$$ 

Then, essentially by the Riemann-Lebesgue lemma, the first integral for $z$ and $t$ large has appreciable size only for $z \sim k_0 t$ and the second integral if $r \sim k_0 t$. Therefore, if $t \to -\infty$, the second term is negligible for all $r \geq 0$ and we recover, asymptotically only the incident wave.

Therefore, the probability per unit time that the scattered particle will pass through the surface element $dS = r^2 d\Omega$ is $(v/|v|)|v|^2 d\Omega = |f(k, \theta)|^2 d\Omega$, where $v$ is the velocity of the particle. We have then:

$$\sigma(k, \theta) d\Omega = |f(k, \theta)|^2 d\Omega$$
and we recover the well known formula $d\sigma = |f(\theta)|^2 d\Omega$.

We are then lead to see how do we get information about the function $f(k, \theta)$. It is well known that the solutions of the linear Schrödinger equation may be written as

$$\psi(r) = \sum_{l=0}^{\infty} A_l P_l(\cos \theta) R_{k,l}(r)$$

where $A_l$ are constants and the $R_{k,l}(r)$ are radial functions satisfying the equation

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \left[ k^2 - \frac{l(l+1)}{r^2} - \frac{2m}{\hbar} U(r) \right] R = 0$$

and the $P_l$ are the Legendre polynomials. The coefficients $A_l$ are constants which have to be chosen so that the condition at $r \to \infty$ is fulfilled. This implies that:

$$A_l = \frac{1}{2k} (2l+1)^l \exp(i\delta_l)$$

where for every $l$, $\delta_l$ is a constant called phase shift.

To see that observe that, the asymptotic form of each of the functions $R_{k,l}$ is

$$R_{k,l}(r) \sim \frac{2}{r} \sin(kr - \frac{l\pi}{2} + \delta_l) = \frac{1}{ir} \{(-i)^l \exp[i(kr + \delta_l)] - i^l \exp[-i(kr + \delta_l)]\}$$

Therefore, it is formally deduced that:

$$\psi(r) \sim \sum_{l=0}^{\infty} A_l P_l(\cos \theta) \frac{i}{r} \{\exp[-i(kr - \frac{l\pi}{2} + \delta_l)] - \exp[i(kr - \frac{l\pi}{2} + \delta_l)]\}.$$ 

On the other hand, the plane wave’s expansion in spherical harmonics gives:

$$e^{ikz} = \sum_{l=0}^{\infty} (-i)^l (2l+1) P_l(\cos \theta) \frac{r}{kr} \left( \frac{1}{r} \frac{d}{dr} \right)^l \sin kr$$

and for $r \to \infty$ we have:

$$e^{ikz} \sim \frac{1}{kr} \sum_{l=0}^{\infty} i^l (2l+1) P_l(\cos \theta) \sin(kr - \frac{l\pi}{2})$$

$$= \sum_{l=0}^{\infty} i^l (2l+1) P_l(\cos \theta) \frac{i}{2kr} \{\exp[-i(kr - \frac{l\pi}{2})] - \exp[i(kr - \frac{l\pi}{2})]\}$$

If we choose now $A_l$ as indicated above:

$$A_l = \frac{1}{2k} (2l+1)^l \exp(i\delta_l)$$

then

$$\psi - e^{ikz} \sim \frac{i}{2kr} \sum_{l=0}^{\infty} (2l+1) P_l(\cos \theta)[(-1)^l e^{-ikr} - S_l e^{ikr}]$$

with

$$S_l = e^{2i\delta_l}$$
and finally,

\[ f(k, \theta) = \frac{1}{2i k} \sum_{l=0}^{\infty} (2l + 1) [S_l - 1] P_l(\cos \theta). \]

Integrating \( d\sigma \) over all the values of the angles we obtain the total cross section

\[ \sigma = 2\pi \int_0^\pi |f(k, \theta)|^2 \sin \theta d\theta = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l + 1) \sin^2 \delta_l, \]

since the Legendre polynomials are orthonormals and

\[ \int_0^\pi P_l^2(\cos \theta) \sin \theta d\theta = \frac{2}{2l + 1}. \]

The coefficients

\[ f_l(k, \theta) = \frac{1}{2i k} (S_l - 1) \]

are called partial amplitude diffusion.

**Study of the general formula of \( f(k, \theta) \).** This formula is valid for all radial potential \( U(r) \) vanishing at infinity. Its study reduces to that of the phases \( \delta_l \). (We quote [LL] Vol. 10, §124.)

Assume first that \( U(r) \sim r^{-n} \) as \( r \to \infty \).

1.- If \( n > 2 \) then the total cross section is finite and the differential cross section is integrable.

2.- If \( n \leq 2 \) the total cross section is infinite and the differential cross section is not integrable. From a physical point of view this is due to the fact that, since the field is slowly decreasing with the distance, the probability of diffusion of very small angles becomes very large. Remember that in classical mechanics, for every positive potential \( U(r) \) vanishing only at infinity, (any particle with large but finite impact parameter is deviated by a small but non zero angle) and so the total cross section is infinite, whatever is the decay of \( U(r) \). (In that sense, it may be considered that the quantum scattering is more regular, or less singular, than the classical one).

On the other hand, concerning the differential cross section itself:

a.- If \( n \leq 3 \) the differential cross section becomes infinite as \( \theta \to 0 \)

b.- If \( n > 3 \) the differential cross section is finite as \( \theta \to 0 \)

Finally, if \( n \leq 1 \), then the total cross section is infinite, i.e. the differential cross section is not integrable; the differential cross section is singular as \( \theta \to 0 \) but it is well defined for \( \theta \neq 0 \) and is given by

\[ f(k, \theta) = \frac{1}{2i k} \sum_{l=0}^{\infty} (2l + 1) P_l(\cos \theta) (e^{2i\delta_l} - 1). \]

**Non radial interaction.**

Let us consider again the incident plane wave \( e^{ikz} \) in the radial potential considered above, moving in the \((0, 0, 1)\) direction, which is reflected and which at a large distance point \( \rho = (r, \theta, \phi) \) is seen as \( e^{ikz} + f(\theta)e^{ikr}/r \).

Observe that, given the vector \( \rho = (x, y, z) = (r, \theta, \phi) \), we have:

\[ z = r e_3 \cdot \frac{\rho}{r}, \quad e_3 = (0, 0, 1), \]

i.e. \( z \) may be seen as \( r \) times the scalar product of the two unitary vectors giving the directions of the incident and scattered particle’s velocities. Consider now a general potential \( U(\rho) \), and \( n \) a unitary vector.
of \( \mathbb{R}^3 \). Consider then an incident particle in the direction \( n \), scattered by \( U(\rho) \). The wave describing this particle would then be the solution of the Schrödinger equation

\[-\Delta \psi(\rho) - k^2 \psi(\rho) + U(\rho)\psi(\rho) = 0\]

such that at the point \( \rho = (r, \theta, \psi) \) with \(|\rho| \to \infty\),

\[\psi(\rho) \sim e^{ikrnn'} + \frac{1}{r} f(n, n')e^{ikr} \]

where \( n' = r/\rho \). The amplitude diffusion depends on the two directions of the incident and scattered particles and not only on the angle they form.

**Born’s formula.** [LL] Vol. 3, §126. The Born’s formula gives an explicit relation between the differential cross section and the potential \( U(\rho) \), non necessarily radial. As we have seen, in that case the amplitude diffusion depends on the incident and scattered directions and not only on the angle they form. The explicit expression is:

\[f(q, q') = -\frac{m}{2\pi\hbar^2} \int_{\mathbb{R}^3} U(r)e^{-i(q' - q) \cdot \rho} dV(\rho)\]

\[\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4} \left| \int_{\mathbb{R}^3} U(\rho)e^{-i(q' - q) \cdot \rho} dV(\rho) \right|^2, \quad |q' - q| = 2k \sin \frac{\theta}{2},\]

where \( \theta \) is the angle between the two vectors \( q \) and \( q' \).

One may approximate the differential cross section by the Born’s formula whenever the perturbation field \( U(\rho) \), not necessarily spherically symmetric, may be considered as a perturbation. [This corresponds to the case where all the phases \( \delta_l \) are small]. This is possible when one of the following conditions are fulfilled:

\[|U| \ll \frac{\hbar^2}{ma^2}\]

\[|U| \ll \frac{\hbar v}{a} = \frac{\hbar^2}{ma^2}ka\]

where \( a \) is the rayon d’action of \( U(\rho) \) and \( U \) its order of magnitude in the main region of its existence. In the first case, the Born’s approximation may be applied for all the velocities. In the second case, it may be applied for particles with sufficiently large velocities.

If moreover the potential is spherically symmetric, \( U = U(r) \) then we obtain:

\[f(k, \theta) = -\frac{m}{\hbar^2} \int_0^\infty U(r) \frac{\sin[2rk \sin \frac{\theta}{2}]}{k \sin \frac{\theta}{2}} r dr.\]

When \( \theta = 0 \), this integral diverges whenever \( U(r) \) decreases like \( r^{-3} \) or slower as \( r \to \infty \).

**Scattering of slow particles.** ([LL] Vol. 3, §132.)

We consider here the limiting case where

1.- The potential \( U(r) \) is radial and decreases at large distances more rapidly than \( 1/r^3 \).

2.- The velocities of the particles undergoing scattering are so small that their wavelength is large compared with the radius of action \( a \) of the field \( U(r) \), i.e. \( ka << 1 \), and their energy is small compared with the field within that radius,

Under these conditions, the total amplitude may be approximated by the formula:

\[f(\theta) \sim f_0 : \text{the first partial amplitude},\]
and so

\[ d\sigma(k, \theta) = |f_0|^2 d\Omega \]

At low velocities, the scattering is isotropic, and the differential cross section is independent of the particle energy.

If the potential decreases as \( U(r) \sim r^{-n} \) with \( n < 3 \) the above approximation is not valid.

**Some important examples**

(i) **Coulomb interaction:** Rutherford formula ([LL] Vol. 3 §135.)

The scattering by central coulomb potential is particularly important with respect to the applications in physics. Moreover, the quantomic problem of the collisions may be solved explicitly until the end.

So, if we assume the potential to be \( U(\rho) = |\rho|^{-1} \), the differential cross section is:

\[ \sigma(k, \theta) = \frac{1}{4k^4 \sin^4 \frac{\theta}{2}} \]

Observe that it is the same as the differential cross section obtained for the classical Coulomb interaction.

(ii) **Hard sphere potential.** (Slow particles; [LL] Vol. 3 §132, Problem 2.)

For the Hard sphere potential \( U(r) = \lim_{n \to \infty} U_n(r) \), with

\[ U_n(r) = \begin{cases} n & \text{if } r < a \\ 0 & \text{if } r > a, \end{cases} \]

the differential cross section, under the conditions \( ka << 1 \) and for small values of \( k \) is

\[ \sigma = a^2. \]

Observe that this implies that the total cross section is \( 4\pi a^2 \), which is four times the result following classical mechanics.

(iii) **Yukawa potential.** (Born approximation; [LL] Vol. 3 §126, Problem 3.)

Assume that

\[ U(r) = \alpha e^{-r/a} \]

The Born approximation is:

\[ \sigma = \left( \frac{\alpha m a}{\hbar^2} \right)^2 \frac{4a^2}{(q^2 a^2 + 1)^2}, \]

with \( q = 2k \sin(\theta/2) \) This approximation is valid whenever \( \alpha m a / \hbar^2 << 1 \) or \( \alpha / \hbar \nu << 1 \). In the first case it is valid for all the velocities. In the second, only for velocities sufficiently large.

One may find in [LL], Vol. 3, §132, problem 1 and problem 2 the Born approximations of the differential cross sections corresponding to the spherical well and uniform potential barrier and in §126 problem 2 to the Gaussian potential.

**Collisions of identical particles.** ([LL], Vol. 3 §137.)

If the two particles are identical, then they are indiscernibles and reads:

\[ W(p'; p, p_*) = W(p'; p, p_*) \]

78
The wave function of a system of two particles has to be symmetric or antisymmetric with respect to the particles depending whether their total spin is odd or even. Therefore, the wave function, describing the scattering, obtained by solving the Schrödinger equation has to be symmetrised or antisymmetrised. Their asymptotic expansion has then to be written as:

$$\psi \sim e^{ikz} \pm e^{-ikz} + \frac{1}{r} e^{ikr} [f(\theta) \pm f(\pi - \theta)].$$

In that way, if the total spin of the particles in the collision is even, the differential section is:

$$d\sigma_s = [f(\theta) + f(\pi - \theta)]^2 d\Omega,$$

if it is odd:

$$d\sigma_a = [f(\theta) - f(\pi - \theta)]^2 d\Omega.$$

We have assumed in these two formulas that the total spin of the particles in the collision has a fixed value. But in general, we deal with collisions in which the particles do not have their spins in a determined state. In order to find the differential cross section one has then to take the mean over all the possible states of the spin where we consider all of them equiprobables. For an half integer $s$, the probability for a system of two particles of spin $s$ to have a spin $S$ even is $s/(2s + 1)$. The probability to have a spin $S$ odd is $(s + 1)/(2s + 1)$. Then the differential cross section for interacting identical particles of half integer spin $s$ is:

$$d\sigma = \frac{s}{2s + 1} d\sigma_s + \frac{s + 1}{2s + 1} d\sigma_a.$$

Similarly, for interacting identical particles of integer spin $s$:

$$d\sigma = \frac{s + 1}{2s + 1} d\sigma_s + \frac{s}{2s + 1} d\sigma_a.$$

**Relativistic case.** The differential cross sections in the relativistic case are calculated in a completely different way. Let us first mention here that for short range interaction, one still has $s\sigma(s, \theta) \equiv$ constant (cf. [PS]). We finally end with the following example.

**Scattering photon-electron.** Let $P = (p^0, p)$ and $P_\epsilon = (p_\epsilon^0, p_\epsilon)$ be the 4-momenta of the photon and electron before collision, and $P' = (p'\epsilon, p')$ and $P'_\epsilon = (p'_\epsilon, p'_\epsilon)$ their 4-momenta after the collision. Define the center of mass coordinates:

$$s = (P + P_\epsilon)^2, \quad t = (P - P')^2.$$

The differential Compton cross section is given by the Klein-Nishina formula:

$$\sigma(s, \theta) = \frac{1}{2} r_0^2 (1 - \xi) \left\{ 1 + \frac{\xi^2 (1 - x)^2}{1 - \frac{\xi}{2} (1 - x)} + \frac{1 - (1 - \frac{\xi}{2} (1 - x))}{1 - \frac{\xi}{2} (1 - x)} \right\}$$

where $m$ is the mass of the electron,

$$x = 1 + \frac{2st}{(s - m^2)^2}, \quad \xi = \frac{(P + P_\epsilon)^2 - m^2 c^2}{(P + P_\epsilon)^2}, \quad r_0 = \frac{e^2}{4\pi mc^2}.$$

See e.g. [GLW]. If the energy of the photons is low, the non relativistic limit of the Klein Nishina differential cross section is

$$\left( \xi \sim \frac{2|p|mc}{m^2 c^2} = \frac{2|m|}{mc} \right).$$
and so it gives
\[
\sigma(s, \theta) \sim \frac{1}{2} r_0^2 \{1 + \cos^2 \theta\}, \quad \text{as} \quad c \to \infty.
\]
This is the Thomson formula for the efficient cross section of the diffusion of an incident electromagnetic wave diffused by a single free charge at rest, (in that case, \(\theta\) is the angle formed by the direction of the diffusion and the direction of the electric field of the incident wave (see [LL], Vol. 2, §78.7).

REFERENCES


