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DMA - 02 - 07
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April 2002

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Abstract

We establish new regularity estimates, in terms of Sobolev spaces, of the solution $f$ to a kinetic equation. The right-hand side can contain partial derivatives in time, space and velocity, as in classical averaging, and $f$ is assumed to have a certain amount of regularity in velocity. The result is that $f$ is also regular in time and space, and this is related to a commutator identity introduced by Hörmander for hypoelliptic operators. In contrast with averaging, the number of derivatives does not depend on the $L^p$ space considered. Three type of proofs are provided: one relies on the Fourier transform, another one uses Hörmander’s commutators, and the last uses a characteristics commutator. Regularity of averages in velocity are deduced. We apply our method to the linear Fokker-Planck operator and recover the known optimal regularity, by direct estimates using Hörmander’s commutator.

Key-words: kinetic equations – regularity – Sobolev spaces – hypoelliptic operator – Hörmander’s commutator – averaging

Mathematics Subject Classification: 35H10, 82C40, 35B65
1 Introduction and main results

The classical averaging theory, developed in [11], [10], [9], [1], [14], [3], [15], [4], [5], state that the solution $f(t, x, v)$ to a kinetic equation, say

$$\partial_t f + v \cdot \nabla_x f = g \quad \text{in } \mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}^N_v,$$  \hfill (1.1)

has some regularity when averaged with respect to the velocity $v$. More precisely, if $f, g \in L^2(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}^N_v)$, then the average

$$\rho_\psi(t, x) = \int_{\mathbb{R}^N_v} f(t, x, v) \psi(v) \, dv$$ \hfill (1.2)

satisfies $\rho_\psi \in H^{1/2}(\mathbb{R}_t \times \mathbb{R}^N_x)$ for any $\psi \in C_c^\infty$. The aim of this paper is to develop several methods for proving the regularity of $f$ itself, without averaging, but by assuming some extra regularity in $v$ of $f$,

$$D_\psi^\beta f \in L^2(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}^N_v).$$ \hfill (1.3)

Here and all throughout the paper, we denote

$$D_\psi^\beta = (-\Delta_\psi)^{\beta/2}, \quad D_x^s = (-\Delta_x)^{s/2}.$$ \hfill (1.4)

In the most simple situation as above, the result is the following.

**Proposition 1.1** Assume that $f, g \in L^2(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}^N_v)$ satisfy (1.1) and that (1.3) holds for some $\beta \geq 0$. Then $D_x^{\beta/(1+\beta)} f \in L^2(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}^N_v)$ and

$$\|D_x^{\beta/(1+\beta)} f\|_{L^2} \leq C_{N, \beta} \|D_\psi^\beta f\|_{L^2}^{1/(1+\beta)} \|g\|_{L^2}^{\beta/(1+\beta)},$$ \hfill (1.5)

where $C_{N, \beta}$ is a constant depending only on $N$ and $\beta$.

The optimal regularity of the averages (1.2) under assumption (1.3) has been obtained recently in [13], and it can be also deduced from (1.5).
Corollary 1.2 Under the same assumptions as in Proposition 1.1,
\[
\frac{f}{1 + |v|^{\beta/(1+\beta)}} \in H^{(1+\beta)}_{t,x}(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v^N), \tag{1.6}
\]
and for any \( \psi \in C_c^\infty \),
\[
\int f(t, x, v) \psi(v) \, dv \in H^{\frac{1}{2} + \frac{\beta}{1+\beta}}_{t,x}(\mathbb{R} \times \mathbb{R}_x^N). \tag{1.7}
\]

We can also consider \( L^p \) data and derivatives in the right-hand side, as is done usually in averaging. Our most general result in this direction is the following.

Theorem 1.3 Assume that \( f \in L^p(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v^N) \), \( 1 < p < \infty \), satisfies
\[
\partial_t f + v \cdot \nabla_x f = (\kappa^2 - \Delta_x - \Omega^2 \partial_t^2)^{r/2} \sum_{|\alpha| \leq m} \partial^\alpha_x g_\alpha, \tag{1.8}
\]
for some \( g_\alpha \) such that
\[
(1 + \Omega^2 |v|^2)^{|\alpha|+1/2} g_\alpha \in L^p(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v^N), \tag{1.9}
\]
and that
\[
D^\beta_v f \in L^p(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v^N), \tag{1.10}
\]
where
\[
0 \leq r \leq 1, \quad m \in \mathbb{N}, \quad \beta \geq 0, \tag{1.11}
\]
and \( \kappa > 0, \Omega > 0 \) are given constants. Then \( f \in W^{s,\beta}_{t,x}(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v^N) \),
\[
s = (1-r) \frac{\beta}{m+1+\beta}, \tag{1.12}
\]
and
\[
\|D^\beta_v f\|_p + \Omega^s \|D^\alpha_x f\|_p \\
\leq C \kappa^s \|f\|_p \\
+ C \sum_{|\alpha| \leq m} \|D^\beta_v f\|_p \left( (1 + \Omega^2 |v|^2)^{|\alpha|+1/2} g_\alpha \|_{L^p} \kappa^{-(1-r)\frac{m+1+\beta}{m+1+\beta}} \right) \tag{1.13}
\]
\[
+ C \sum_{|\alpha| \leq m} \|D^\beta_v f\|_p \Omega^{|\alpha|} \left( (1 + \Omega^2 |v|^2)^{1/2} g_\alpha \|_{L^p} \kappa^{-(1-r)\frac{m}{m+1+\beta}} \right)^{\frac{1}{\beta}} ,
\]
where \( C \) only depends on \( N, r, m, \beta, p, \) and not on \( \kappa, \Omega \).

It is noticeable that the Sobolev exponent \( s \) in (1.12) does not depend on \( p \). This is very different from the usual case of the regularity of averages. Here, as \( p \) tends to 1, the smoothing in time and space remains as efficient as in the \( L^2 \) case. However, the exponent \( p \) appears again if we write the regularity of averages that we deduce.
Corollary 1.4 Under the assumptions of Theorem 1.3 and if $1 < p \leq 2$, we have for any $\psi \in C^\infty_c(\mathbb{R}^N_v)$

$$
\int f(t, x, v)\psi(v) \, dv \in W^{(1-r)\frac{4m+1}{m+3}, p}(\mathbb{R}_t \times \mathbb{R}^N_x).
$$

(1.14)

The proofs of the results above are detailed in Section 2, and rely mainly on the tools developed in averaging techniques.

Another type of proof of the direct estimate (1.5) on $f$ is developed in Section 3. Indeed, Proposition 1.1 is reminiscent of the regularity of the solution to hypoelliptic equations like the Vlasov-Fokker-Planck equation

$$
\partial_t f + v \cdot \nabla_x f - \sigma \Delta_v f = g \quad \text{in } \mathbb{R}_t \times \mathbb{R}^N_v \times \mathbb{R}^N_v,
$$

(1.15)

with $\sigma > 0$. Indeed this equation gives obviously an estimate on $\nabla_v f$, and we are in the situation of Theorem 1.3. However, this approach does not give the best Sobolev exponent for the regularity in $(t, x)$, which is known to be $2/3$, as proved in [16]. It is possible to prove this regularity directly via an explicit formula for the fundamental solution, that was given in [12]. This approach was retained for example in [2] and [8]. However, when more complicate operators than (1.15) are involved, it is desirable to prove the regularity of $f$ without any use of the fundamental solution, nor on Fourier transform representations. That was done in [16], that improves the first work [12] of Hörmander, and the proof uses a condition introduced also in [12] on the brackets of the vector fields involved. Here this condition reduces to the simple commutator identity

$$
\partial_{x_j} = \partial_{v_j} (\partial_t + v \cdot \nabla_x) - (\partial_t + v \cdot \nabla_x) \partial_{v_j},
$$

(1.16)

which is the heart of the regularizing effect in Proposition 1.1. The drawback of [16] is that the estimates are very complicate. A simplified approach was proposed in [6], but however, the author does not get the optimal exponent. We propose here to improve this last approach in order to get the optimal regularity. The key point is to prove that we have indeed two derivatives in the variable $v$.

**Theorem 1.5** Assume that (1.15) holds with $f, g \in L^2(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v^N), \nabla_v f \in L^2(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v^N)$. Then $\partial_t f + v \cdot \nabla_x f$ and $\sigma \Delta_v f$ both belong to $L^2(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v^N)$ and

$$
\|\partial_t f + v \cdot \nabla_x f\|_2 + \sigma \|\Delta_v f\|_2 \leq C_N \|g\|_2.
$$

(1.17)

Moreover, $D^{2/3}_x f \in L^2(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v^N)$ and

$$
\|D^{2/3}_x f\|_2 \leq \frac{C_N}{\sigma^{1/3}} \|g\|_2.
$$

(1.18)

The technique is more general and is also able to prove other generalizations of Proposition 1.1, when the right-hand side is itself regular.
**Theorem 1.6** Assume that $f, g \in L^2(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v^N)$ satisfy

$$\partial_t f + v \cdot \nabla_x f = g,$$  

and

$$D_v^\text{max(\beta, \gamma)} f, D_v^\gamma g \in L^2(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v^N),$$  

with

$$\gamma \geq 0, \quad 0 \leq 1 - \gamma \leq \beta.$$  

Then $D_x^s f \in L^2(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v^N)$ with

$$s = \frac{\beta}{1 - \gamma + \beta}, \quad s = 1 \text{ if } 1 - \gamma = \beta = 0,$$  

and

$$\|D_x^s f\|_2 \leq C_N \|D_v^\beta f\|_2^{1-s} \|D_v^\gamma g\|_2^s.$$  

We have to notice that it we make $\gamma = 0$ in Theorem 1.6, we recover Proposition 1.1, but with the extra assumption that $\beta \geq 1$. Another interesting choice is $\gamma = 1$, in this case $s = 1$ and we obtain a full derivative in $x$ on $f$.

The remainder of the paper is organized as follows. Section 2 is devoted to the proof of Proposition 1.1, Corollary 1.2, Theorem 1.3 and Corollary 1.4 via the classical tools of velocity averaging which are the Fourier transform and interpolation techniques. Then, we expose in Section 3 the proof of Theorems 1.5 and 1.6 by the commutator method of [6]. Finally, we show in Section 4 how it is possible, with very simple characteristics formulas, to get similar results, at least in the case where there is no derivative in the right-hand side.

## 2 The Fourier method

The heart of this section is the proof of Theorem 1.3. We first show in Subsection 2.1 how simple $L^2$ estimates via Fourier transform and smoothing in velocity can lead to Proposition 1.1. Then in Subsection 2.2, we prove Theorem 1.3 and several results of the same type.

### 2.1 $L^2$ estimates

Let us denote by $\hat{f}(\omega, k, v)$ the Fourier transform of $f$ in the $(t, x)$ variables. Then (1.1) gives

$$i(\omega + v \cdot k) \hat{f} = \hat{g}.$$  

**Proof of Corollary 1.2.** Let us denote $s = \beta/(1 + \beta)$. Then from (2.1),

$$|\omega|^{s} \hat{f} = \frac{|\omega|^s \omega}{\omega^2 + |v|^2|k|^2} (\hat{g}/i - v \cdot k \hat{f}) + \frac{|\omega|^s |v|^2|k|^{2-s}}{\omega^2 + |v|^2|k|^2} |k|^s \hat{f},$$  

(2.2)
and since by (1.5), \(|k|^s \hat{f} \in L^2_{\omega,k,v}\), we get obviously that \(\omega^s \hat{f}/(1+|v|^s) \in L^2_{\omega,k,v}\), which gives (1.6). Next, let \(\rho(t,x) = \int f \psi(v)dv\), and \(h = D_v^s f \in L^2\). Then
\[
\partial_t h + v \cdot \nabla x h = D_v^s g,
\]
and by classical averaging, \(\int h \psi(v)dv \in H^{(1-s)/2}\), which means that \(D_v^s \rho \in H^{(1-s)/2}\). We get similarly that \(D_v^s \rho \in H^{(1-s)/2}\), and therefore \(\rho \in H^{s+(1-s)/2}\). □

**Proof of Proposition 1.1.** We introduce a smoothing sequence in velocity
\[
\rho_\varepsilon(v) = \frac{1}{\varepsilon N} \rho_1\left(\frac{v}{\varepsilon}\right), \quad \rho_1 \in C^\infty, \quad \int \rho_1 = 1, \quad \int v^a \rho_1 = 0 \text{ for } 1 \leq |\alpha| < \beta.
\]
The idea is to decompose, at fixed \((\omega, k)\),
\[
\hat{f}(\omega, k, v) = (\rho_\varepsilon \ast \hat{f})(\omega, k, v) + \left(\hat{f}(\omega, k, v) - (\rho_\varepsilon \ast \hat{f})(\omega, k, v)\right).
\]
Since by (2.4) \(|1 - \hat{\rho}_\varepsilon(v)| \leq C_{N,\beta} \varepsilon v^\beta\), we can estimate the second term by
\[
\|\hat{f}(\omega, k, \cdot) - (\rho_\varepsilon \ast \hat{f})(\omega, k, \cdot)\|_{L^2(\mathbb{R}^N)} \leq C_{N,\beta} \varepsilon \|D_v^\beta \hat{f}(\omega, k, \cdot)\|_{L^2(\mathbb{R}^N)}.
\]
Then, we estimate the first term in (2.5) as usual in averaging lemma. We introduce an interpolation parameter \(\lambda > 0\), and from (2.1) we get
\[
(\lambda + i(\omega + v \cdot k)) \hat{f}(\omega, k, v) = \lambda \hat{f}(\omega, k, v) + \tilde{g}(\omega, k, v),
\]
which yields
\[
\hat{f}(\omega, k, v) = \frac{\lambda \hat{f}(\omega, k, v) + \tilde{g}(\omega, k, v)}{\lambda + i(\omega + v \cdot k)},
\]
and
\[
(\rho_\varepsilon \ast \hat{f})(\omega, k, v) = \int \frac{\lambda \hat{f}(\omega, k, \eta) + \tilde{g}(\omega, k, \eta)}{\lambda + i(\omega + \eta \cdot k)} \rho_\varepsilon(v - \eta) \, d\eta.
\]
We estimate this integral by the Cauchy-Schwarz inequality,
\[
\left| (\rho_\varepsilon \ast \hat{f})(\omega, k, v) \right| \leq \left( \|\hat{f}(\omega, k, \cdot)\|_{L^2(\mathbb{R}^N)} + \|\tilde{g}(\omega, k, \cdot)\|_{L^2(\mathbb{R}^N)} \right) \int_{\mathbb{R}^N} \frac{|\rho_\varepsilon(v - \eta)|}{1 + i(\omega + \eta \cdot k)/\lambda^2} \, d\eta^{1/2} \left( \int_{\mathbb{R}^N} \frac{|\rho_\varepsilon(v - \eta)|}{1 + i(\omega + \eta \cdot k)/\lambda^2} \, d\eta \right)^{1/2}.
\]
In order to estimate the last integral, we notice that \(|\rho_\varepsilon(v)| \leq C_{N,\beta} \varepsilon^{-N} \mathbb{1}_{|v| < \varepsilon}\), and writing the decomposition \(\eta = \tilde{\eta} + \eta'\) with \(\tilde{\eta} = \eta \cdot \frac{k}{|k|}\) and \(\eta' \cdot k = 0\), we obtain
\[
\int_{\mathbb{R}^N} \frac{|\rho_\varepsilon(v - \eta)|}{1 + i(\omega + \eta \cdot k)/\lambda^2} \, d\eta \leq C_{N,\beta} \int_{\mathbb{R}} \frac{\mathbb{1}_{\frac{1}{\varepsilon} \mathbb{1}_{|\tilde{\eta}| < \varepsilon}}}{1 + i(\omega + |\tilde{\eta}|)/\lambda^2} \, d\tilde{\eta} \leq C_{N,\beta} \frac{\lambda}{\varepsilon |k|}.
\]
Therefore, taking the $L^2$ norm in velocity in (2.10), we get

$$
\|(\rho_\varepsilon \ast \hat{f})(\omega, k, \cdot)\|_{L^2(\mathbb{R}^N)} \leq C_{N, \beta} \left( \frac{\lambda}{\varepsilon |k|} \right)^{1/2} \left( \||\hat{f}(\omega, k, \cdot)\|_{L^2(\mathbb{R}^N)} + \|\hat{g}(\omega, k, \cdot)\|_{L^2(\mathbb{R}^N)} / \lambda \right). \tag{2.12}
$$

Now, we choose $\lambda = \|\hat{g}(\omega, k, \cdot)\|_{L^2} / \|\hat{f}(\omega, k, \cdot)\|_{L^2}$, which depends on $\omega$ and $k$, but this is not a problem since the previous computations are valid at fixed $\omega$ and $k$, and this yields

$$
\|(\rho_\varepsilon \ast \hat{f})(\omega, k, \cdot)\|_{L^2} \leq \frac{C_{N, \beta}}{\sqrt{\varepsilon |k|}} \|\hat{f}(\omega, k, \cdot)\|_{L^2}^{1/2} \|\hat{g}(\omega, k, \cdot)\|_{L^2}^{1/2}. \tag{2.13}
$$

Together with (2.6), this enables to estimate (2.5),

$$
\|\hat{f}(\omega, k, \cdot)\|_{L^2} \leq \frac{C_{N, \beta}}{\sqrt{\varepsilon |k|}} \|\hat{f}(\omega, k, \cdot)\|_{L^2}^{1/2} \|\hat{g}(\omega, k, \cdot)\|_{L^2}^{1/2} + C_{N, \beta} \varepsilon^{\beta} \|D_v^\beta \hat{f}(\omega, k, \cdot)\|_{L^2}. \tag{2.14}
$$

Next, we choose $\varepsilon$ in order to optimize the right-hand side (thus $\varepsilon$ also depends on $\omega$ and $k$), and we obtain

$$
\|\hat{f}(\omega, k, \cdot)\|_{L^2} \leq C_{N, \beta} \|D_v^\beta \hat{f}(\omega, k, \cdot)\|_{L^2}^{1/2} \|\hat{g}(\omega, k, \cdot)\|_{L^2} / |k| \right)^{\beta/(1+2\beta)}. \tag{2.15}
$$

Finally, we simplify this inequality to get

$$
\|\hat{f}(\omega, k, \cdot)\|_{L^2} \leq \left[ C_{N, \beta} \|D_v^\beta \hat{f}(\omega, k, \cdot)\|_{L^2}^{1/2} \|\hat{g}(\omega, k, \cdot)\|_{L^2} / |k| \right]^{\beta/(1+2\beta)} + C_{N, \beta} \|D_v^\beta \hat{f}(\omega, k, \cdot)\|_{L^2}^{1/2} \|\hat{g}(\omega, k, \cdot)\|_{L^2} / |k| \right)^{\beta/(1+\beta)}, \tag{2.16}
$$

which yields (1.5) by integration in $(\omega, k)$. \hfill \Box

**Remark 2.1** The same type of interpolation argument by convolution in velocity that uses the regularity in $v$ of $\hat{f}$ is used in [7].

### 2.2 $L^p$ estimates

We shall now consider a simplified version of Theorem 1.3, that only takes into account the regularity in $x$.

**Theorem 2.1** Assume that $f \in L^p(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v^N)$, $1 < p < \infty$, satisfies

$$
\partial_t f + v \cdot \nabla_x f = (\kappa^2 - \Delta_x)^{r/2} \sum_{|\alpha| \leq m} \partial_{v \alpha}^\beta g_\alpha, \tag{2.17}
$$

for some $g_\alpha \in L^p(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v^N)$, and that

$$
D_v^\beta f \in L^p(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v^N), \tag{2.18}
$$


where
\[ 0 \leq r \leq 1, \quad m \in \mathbb{N}, \quad \beta \geq 0, \] (2.19)
and \( \kappa > 0 \) is a given constant. Then \( f \in W_{x}^{s,p}(\mathbb{R}_{t} \times \mathbb{R}_{x}^{N} \times \mathbb{R}_{v}^{N}) \),

\[ s = (1 - r) \frac{\beta}{m + 1 + \beta}, \] (2.20)

and
\[
\| D_{x}^{s} f \|_{p} \\
\leq C \kappa \| f \|_{p} + C \sum_{|\alpha| \leq m} \| D_{x}^{\alpha} f \|_{p}^{\frac{1}{|\alpha|+1+\beta}} \left( \| g_{\alpha} \|_{p} \kappa^{-s} \right)^{\frac{m-|\alpha|}{m+1+\beta}}. \] (2.21)

where \( C \) only depends on \( N, r, m, \beta, p \), and not on \( \kappa \).
If moreover \( |v|^{|\alpha|+1} g_{\alpha} \in L^{p}(\mathbb{R}_{t} \times \mathbb{R}_{x}^{N} \times \mathbb{R}_{v}^{N}) \), then \( f \in W_{x}^{s,p}(\mathbb{R}_{t} \times \mathbb{R}_{x}^{N} \times \mathbb{R}_{v}^{N}) \).

**Corollary 2.2** If in Theorem 2.1 we have \( D_{x}^{r} \) in the right-hand side of (2.17) instead of \( (\kappa^{2} - \Delta_{x})^{r/2} \), then the same result holds, but with the estimate
\[
\| D_{x}^{s} f \|_{p} \leq C \sum_{|\alpha| \leq m} \| f \|_{p}^{\frac{m-|\alpha|}{m+1+\beta}} \| D_{x}^{\alpha} f \|_{p}^{\frac{1}{|\alpha|+1+\beta}} \| g_{\alpha} \|_{p}^{\frac{\beta}{m+1+\beta}}. \] (2.22)

where \( C \) only depends on \( N, r, m, \beta, p \).

Before going into proofs, let us state the main interpolation tool that we use for getting \( L^{p} \) bounds. It can be found in [1].

**Lemma 2.3** Let \( m(k_{1}, k_{2}) \) be a function of \( (k_{1}, k_{2}) \in \mathbb{R}_{N_{1}} \times \mathbb{R}_{N_{2}} \) which is \( C_{\infty} \)
out of the set \( [k_{1} = 0 \text{ or } k_{2} = 0] \), verifying for all \( \gamma_{1}, \gamma_{2} \)
\[
\left| \partial_{k_{1}}^{\gamma_{1}} \partial_{k_{2}}^{\gamma_{2}} m(k_{1}, k_{2}) \right| \leq \frac{C_{\gamma_{1}, \gamma_{2}}}{|k_{1}|^{\gamma_{1}}|k_{2}|^{\gamma_{2}}}. \] (2.23)

Then \( m \) defines a bounded Fourier multiplier on \( L^{p}(\mathbb{R}_{N_{1}} \times \mathbb{R}_{N_{2}}) \) for any \( 1 < p < \infty \), with a bound depending linearly on a finite number of constants \( C_{\gamma_{1}, \gamma_{2}} \).

**Proof of Theorem 2.1.** Consider as in the proof of Proposition 1.1 a smoothing sequence in velocity
\[
\rho_{\varepsilon}(v) = \frac{1}{\varepsilon^{N}} \rho_{1}(v / \varepsilon), \quad \rho_{1} \in C_{\infty}, \quad \int \rho_{1} = 1, \quad \int v^{\alpha} \rho_{1} = 0 \text{ for } 1 \leq |\alpha| < \beta. \] (2.24)

We choose \( \varepsilon = \varepsilon(k) \) with \( k \) the dual variable of \( x \), as follows,
\[
\varepsilon(k) = \frac{\varepsilon_{0}}{(\kappa^{2} + |k|^{2})^{\frac{1}{2}(1-r)/(m+1+\beta)}}, \] (2.25)
where $\varepsilon_0 > 0$ will be chosen later on. As in (2.5), we decompose $f$ in several parts,

$$f = \chi_\kappa \ast f + (\delta - \chi_\kappa) \ast (f - Pf) + (\delta - \chi_\kappa) \ast (Pf),$$

(2.26)

where $\delta$ stands for the Dirac distribution in $x$ at the origin,

$$\hat{\chi}_\kappa(k) = \hat{\chi}_1(k/\kappa), \quad \hat{\chi}_1 \in C^\infty, \quad \hat{\chi}_1(k) = 1 \text{ if } |k| < 1/2,$$

(2.27)

and the operator $P$ is defined through

$$\hat{P} f = \rho_\varepsilon(k) \ast \hat{f}.$$

(2.28)

Here $\hat{f}(\omega, k, \nu)$ denotes again the Fourier transform of $f$ in $(t, x)$. The operator $P$ is well-defined because if we perform the Fourier transform $\mathcal{F}$ in all variables $(t, x, v)$, it becomes a multiplication by a bounded $C^\infty$ function,

$$\mathcal{F}(Pf)(\omega, k, \nu) = \hat{\rho}_1(\varepsilon(k)\nu)\mathcal{F}f(\omega, k, \nu).$$

(2.29)

We observe that the multiplier

$$\Upsilon(k, \nu) = \frac{1 - \hat{\rho}_1(\varepsilon(k)\nu)}{\varepsilon(k)^3|\nu|^3}$$

(2.30)

verifies the estimates

$$|\partial_k^{\gamma_1} \partial_\nu^{\gamma_2} \Upsilon(k, \nu)| \leq \frac{C_{\gamma_1, \gamma_2}}{|k|^{\gamma_1}|\nu|^{\gamma_2}},$$

(2.31)

for some constants $C_{\gamma_1, \gamma_2}$ independent of $\varepsilon_0$ and $\kappa$. According to Lemma 2.3, $\Upsilon$ is therefore a bounded multiplier on $L^p(\mathbb{R}^N \times \mathbb{R}^N)$ for any $1 < p < \infty$. Thus,

$$\|\varepsilon_0^\beta (\kappa^2 - \Delta_x)^{r/2} (f - Pf)\|_{L^p_{txv}} \leq C\|D_\nu^\beta f\|_{L^p_{txv}},$$

(2.32)

where $C$ depends neither on $\varepsilon_0$, nor on $\kappa$. This enables to estimate (2.26) as follows,

$$\|D_\nu^\beta f\|_p \leq C\kappa^\beta \|f\|_p + C\varepsilon_0^\beta \|D_\nu^\beta f\|_p + \|D_\nu^\beta (\delta - \chi_\kappa) \ast (Pf)\|_p.$$ 

(2.33)

It remains mainly to estimate the last term. In order to do so, we write that from (2.17),

$$i(\omega + v \cdot k) \hat{f} = (\kappa^2 + |k|^2)^{r/2} \sum_{|\alpha| \leq m} \partial_\nu^\alpha \hat{g}_\alpha.$$ 

(2.34)

We introduce in interpolation parameter $\lambda(k) > 0$, that is indeed chosen as

$$\lambda(k) = \lambda_0 (\kappa^2 + |k|^2)^{\frac{1}{2} + \frac{r}{2\gamma_1}}$$

(2.35)

for some $\lambda_0 > 0$, and we write

$$\hat{f} = \frac{\lambda(k)}{\lambda(k) + i(\omega + v \cdot k)} \hat{f} + \frac{(\kappa^2 + |k|^2)^{r/2}}{\lambda(k) + i(\omega + v \cdot k)} \sum_{|\alpha| \leq m} \partial_\nu^\alpha \hat{g}_\alpha.$$ 

(2.36)
By applying the operator $P$, this yields
\[
\tilde{P}f(\omega, k, v) = \int \frac{\lambda(k)}{\lambda(k) + i(\omega + \eta \cdot k)} \rho_{\varepsilon(k)}(v - \eta) \hat{f}(\omega, k, \eta) \, d\eta \\
+ \int \frac{(k^2 + |k|^2)^{r/2}}{\lambda(k) + i(\omega + \eta \cdot k)} \rho_{\varepsilon(k)}(v - \eta) \sum_{|s| \leq m} (\partial_s^3 \hat{g}_s)(\omega, k, \eta) \, d\eta.
\] (2.37)

Let us first study the first term by defining the operator $Q$ by
\[
\tilde{Q}f(\omega, k, v) = \int \frac{\lambda(k)}{\lambda(k) + i(\omega + \eta \cdot k)} \rho_{\varepsilon(k)}(v - \eta) \hat{f}(\omega, k, \eta) \, d\eta,
\] (2.38)
which can be written as $Q = PM$, with $M$ defined by
\[
\tilde{M}f(\omega, k, v) = \frac{\lambda(k)}{\lambda(k) + i(\omega + v \cdot k)} \hat{f}(\omega, k, v).
\] (2.39)

In order to prove the boundedness of $Q$ on $L^p$, we introduce the transformation
\[
f_*(t, x, v) = f(t, x + vt, v),
\] (2.40)
and define an operator $M_*$ by
\[
\tilde{M_*}f_*(\omega, k, v) = \frac{\lambda(k)}{\lambda(k) + i\omega} \hat{f}_*(\omega, k, v).
\] (2.41)

Since \( \hat{f}_*(\omega, k, v) = \hat{f}(\omega - v \cdot k, k, v) \), we have that
\[
(Mf)_* = M_*f_*. \tag{2.42}
\]

But the multiplier $\varphi(\omega, k) = 1/(1 + i\omega/\lambda(k))$ satisfies the estimates
\[
|\partial_\omega^m \partial_k^2 \varphi(\omega, k)| \leq \frac{C_{\gamma_1,\gamma_2}}{\lambda(\omega)\|k\|^2\|x\|^2},
\] (2.43)

for some constants $C_{\gamma_1,\gamma_2}$ independent of $\lambda_0$ and $\kappa$, thus we deduce by Lemma 2.3 that it defines a bounded operator on $L^p(\mathbb{R} \times \mathbb{R}^N)$ for any $1 < p < \infty$. Therefore, by the similarity formula (2.42) and since the transformation (2.40) is an isometry of $L^p$, we conclude that $M$ is bounded on $L^p(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N)$ with a constant independent of $\lambda_0$ and $\kappa$. But by a direct estimate similar to (2.30)-(2.31), $P$ is bounded on $L^p(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N)$, hence $Q$ also by composition,
\[
\|Qf\|_{L^p(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N)} \leq C\|f\|_{L^p(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N)}, \tag{2.44}
\]
where $C$ is independent of $\lambda_0$, $\varepsilon_0$ and $\kappa$. Next, we notice that the estimates (2.9)-(2.12) in the proof of Proposition 1.1 give
\[
\|\tilde{Q}f(\omega, k, \cdot)\|_{L^2(\mathbb{R}^N)} \leq C_{\varepsilon,\beta} \left( \frac{\lambda(k)}{|k|\varepsilon(k)} \right)^{1/2} \|\hat{f}(\omega, k, \cdot)\|_{L^2(\mathbb{R}^N)}, \tag{2.45}
\]

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thus with the cutoff function $1 - \chi_\kappa(k)$ from (2.27), and taking into account the definitions (2.25), (2.35) of $\varepsilon(k)$ and $\lambda(k)$,

$$\|Qf - \chi_\kappa \ast (Qf)\|_{L^2(\mathbb{R}_x \times \mathbb{R}^N_\xi \times \mathbb{R}^N_\xi)} \leq C_{N, \beta} \left( \frac{\lambda_0}{\varepsilon_0} \right)^{1/2} \|f\|_{L^2(\mathbb{R}_x \times \mathbb{R}^N_\xi \times \mathbb{R}^N_\xi)}. \quad (2.46)$$

Now we interpolate (2.44) and (2.46). The exponent $p$ being given, we choose either $1 < p_1 < p$ if $p < 2$, or $p < p_1 < \infty$ if $p > 2$. Since from (2.44) $\|Qf - \chi_\kappa \ast (Qf)\|_{p_1} \leq C \|f\|_{p_1}$, we conclude by classical $L^p$ interpolation that

$$\|Qf - \chi_\kappa \ast (Qf)\|_p \leq C \left( \frac{\lambda_0}{\varepsilon_0} \right)^t \|f\|_p, \quad t = \frac{1/2 - 1/p_1}{1/2 - 1/p_1} > 0. \quad (2.47)$$

Therefore we get

$$\|D_x^e(Qf - \chi_\kappa \ast (Qf))\|_p \leq C \left( \frac{\lambda_0}{\varepsilon_0} \right)^t \|D_x^e f\|_p, \quad (2.48)$$

noticing that we can assume that $D_x^e f \in L^p$ even if we may need further smoothing of $f$ in $x$. Next, let us study the second term in (2.37). After integration by parts, we are led to operators of the form

$$\tilde{W}\tilde{g}(\omega, k, \nu) = \int \frac{(\kappa^2 + |k|^2)^{s/2}k_{\alpha_1}}{[\lambda(k) + i(\omega + \eta \cdot k)]^{1/2}} \partial^{\alpha_2} \rho_{\varepsilon(k)}(v - \eta) \tilde{g}(\omega, k, \eta) d\eta, \quad (2.49)$$

with $|\alpha_1| + |\alpha_2| = |\alpha| \leq m$. We can write

$$\tilde{W}\tilde{g} = \left( \frac{\kappa^2 + |k|^2)^{s/2}k_{\alpha_1}}{\lambda(k)^{1/2} \varepsilon(k)^{1/2}} \right) \tilde{W} \tilde{g}, \quad (2.50)$$

with

$$\tilde{W}\tilde{g}(\omega, k, \nu) = \int \frac{1}{1 + i(\omega + \eta \cdot k)^{1/2}} \frac{1}{\varepsilon(k)^{1/2}} \partial^{\alpha_2} \rho_{\varepsilon(k)}(v - \eta) \tilde{g}(\omega, k, \eta) d\eta. \quad (2.51)$$

By the same analysis as that on $Q$ above, $(\delta - \chi_\kappa) \ast \tilde{W}$ is bounded on $L^p(\mathbb{R}_x \times \mathbb{R}^N_\xi \times \mathbb{R}^N_\xi)$ in $C(\lambda_0/\varepsilon_0)^t$, with $C$ a constant independent of $\lambda_0$, $\varepsilon_0$ and $\kappa$. Since the multiplier

$$\frac{(\kappa^2 + |k|^2)^{s/2}k_{\alpha_1}}{\lambda(k)^{1/2} \varepsilon(k)^{1/2}} \left( \kappa^2 + |k|^2 \right)^{s/2}$$

$$= \frac{1}{\lambda_0^{1/2} \varepsilon_0^{1/2}} \left( \kappa^2 + |k|^2 \right)^{s/2} \left( \frac{\kappa^2 + |k|^2)^{s/2}k_{\alpha_1}}{\lambda(k)^{1/2} \varepsilon(k)^{1/2}} \right) \quad (2.52)$$

is bounded on $L^p(\mathbb{R}^N)$ in $C/(\kappa^{(1-r)(m-|\alpha|)/(m+1+\beta)} \lambda_0^{1/2} \varepsilon_0^{1/2})$, we deduce that

$$\| (\kappa^2 - \Delta_x)^{s/2}(\delta - \chi_\kappa) \ast (Wg)\|_p \leq C \left( \frac{\lambda_0}{\varepsilon_0} \right)^t \frac{C}{\kappa^{(1-r)(m-|\alpha|)/(m+1+\beta)} \lambda_0^{1/2} \varepsilon_0^{1/2}} \|g\|_p. \quad (2.53)$$
Together with (2.48) and (2.37), we are now able to estimate the last term in (2.33), and we obtain
\[
\|D_x^s f\|_p \leq C \kappa^s \|f\|_p + C \varepsilon_0^\beta \|D^\beta f\|_p + C \|\lambda_0/\varepsilon_0\|^\beta \|D_x^s f\|_p
\]
\[
+ C \sum_{|\alpha| \leq m} \sum_{\alpha_1 + \alpha_2 = \alpha} \frac{\lambda_0/\varepsilon_0}{\kappa^{(1-r)/m+1+\beta} \varepsilon_0^{|\alpha|+1}} \|g_\alpha\|_p.
\]
(2.54)

Since \( \varepsilon > 0 \), we can choose now \( \lambda_0/\varepsilon_0 \) small enough so that the third term on the right-hand side has a coefficient which is less than \( 1/2 \), and this yields
\[
\|D_x^s f\|_p \leq C \kappa^s \|f\|_p + C \varepsilon_0^\beta \|D^\beta f\|_p + C \sum_{|\alpha| \leq m} \frac{\|g_\alpha\|_p}{\kappa^{(1-r)/m+1+\beta} \varepsilon_0^{|\alpha|+1}},
\]
(2.55)

for any \( \varepsilon_0 > 0 \). By choosing
\[
\varepsilon_0 = \sum_{|\alpha| \leq m} \left[ \frac{\|g_\alpha\|_p}{\kappa^{(1-r)/m+1+\beta} \|D^\beta f\|_p} \right]^{1/\kappa^{(1-r)/m+1+\beta}},
\]
(2.56)

we finally obtain (2.21). The last statement of Theorem 2.1 concerning regularity in time is indeed a consequence of Theorem 1.3 that is proved below, because (2.17) can be put in the form (1.8) since \((\kappa^2 - \Delta_x)^{r/2} / (\kappa^2 - \Delta_x - \Omega^2 \partial_t^2)^{r/2}\) is bounded on \( L^p \) independently of \( \kappa \) and \( \Omega \), and this ensures that (1.13) is valid for any \( \Omega > 0 \).

**Proof of Corollary 2.2.** We can write for any \( \kappa > 0 \)
\[
D_x^s g_\alpha = (\kappa^2 - \Delta_x)^{r/2} \left[ \frac{D_x^s}{(\kappa^2 - \Delta_x)^{r/2}} g_\alpha \right].
\]
(2.57)

Since the term between brackets belongs to \( L^p \) with a norm less than \( C \|g_\alpha\|_p \) independent of \( \kappa \), we can apply Theorem 2.1 and we get the estimate (2.21) for any \( \kappa > 0 \). In the case \( s > 0 \), we just choose
\[
\kappa^s = \sum_{|\alpha| \leq m} \frac{\|D^\beta f\|_p \|g_\alpha\|_p}{\|f\|_p \|g_\alpha\|_p},
\]
(2.58)

which gives the result. In the case \( s = 0 \), we notice that the term \( C \kappa^s \|f\|_p \) in (2.21) comes in fact from \( \|D_x^s \chi_k \ast f\|_p \) in the proof. Thus we have for any \( \kappa > 0 \)
\[
\|f\|_p \leq C \|\chi_k \ast f\|_p + C \sum_{|\alpha| \leq m} \|D^\beta f\|_p \|g_\alpha\|_p.
\]
(2.59)

But as \( \kappa \to 0 \), \( \|\chi_k \ast f\|_p \to 0 \) (argue by density of \( C^\infty \) in \( L^p \)), therefore
\[
\|f\|_p \leq C \sum_{|\alpha| \leq m} \|D^\beta f\|_p \|g_\alpha\|_p,
\]
(2.60)
and thus for any \( \varepsilon > 0 \), by distinguishing if \( \varepsilon \leq 1 \) or \( \varepsilon \geq 1 \),
\[
\|f\|_p \leq \varepsilon \|f\|_p + C \sum_{|\alpha| \leq m} \|D^2 f\|_p \frac{\|k^{\alpha+\beta}\|_0 \|g\|_p}{\varepsilon^{\alpha+\beta}}.
\] (2.61)

Finally, by taking
\[
\varepsilon = \sum_{|\alpha| \leq m} \|D^2 f\|_p \frac{\|k^{\alpha+\beta}\|_0 \|g\|_p}{\varepsilon^{\alpha+\beta}} \|f\|_p^{-1},
\] (2.62)
we get the result. \( \Box \)

We wish now to establish estimates for derivatives with respect to time in order to obtain Theorem 1.3. This is done by first proving a generalization of the estimate (2.11).

**Lemma 2.4** Whenever \( \Omega > 0 \) and \((\omega, k) \in \mathbb{R} \times \mathbb{R}^N \setminus \{(0, 0)\}\),
\[
\int_{\mathbb{R}} \frac{d\tilde{\eta}}{1 + (\omega + |k|\tilde{\eta})^2(1 + \Omega^2 \tilde{\eta}^2)} \leq \frac{C}{(\tilde{\eta}^2 + \Omega^2 \omega^2)^{1/2}}.
\] (2.63)

**Proof.** Denote by \( I \) the integral above. Then obviously
\[
I \leq \int \frac{d\tilde{\eta}}{1 + \omega + |k|\tilde{\eta}^2} = \frac{\pi}{|k|}.
\] (2.64)

Then,
\[
\int_{|\tilde{\eta}| \leq \frac{|k|}{\sqrt{2}}} \frac{d\tilde{\eta}}{1 + (\omega + |k|\tilde{\eta})^2\Omega^2 \tilde{\eta}^2} \leq \int_{|\tilde{\eta}| \leq \frac{|k|}{\sqrt{2}}} \frac{d\tilde{\eta}}{1 + \tilde{\eta}^2\Omega^2 \tilde{\eta}^2} \leq \frac{2\pi}{\Omega|\omega|},
\] (2.65)
and
\[
\int_{|\tilde{\eta}| > \frac{|k|}{\sqrt{2}}} \frac{d\tilde{\eta}}{1 + (\omega + |k|\tilde{\eta})^2\Omega^2 \tilde{\eta}^2} \leq \int_{|\tilde{\eta}| > \frac{|k|}{\sqrt{2}}} \frac{d\tilde{\eta}}{1 + (\omega + |k|\tilde{\eta})^2\Omega^2 \tilde{\eta}^2} \frac{\Omega^2 \omega^2}{4|k|^2} \leq \frac{2\pi}{\Omega|\omega|},
\] (2.66)
which yields that \( I \leq 4\pi/\Omega|\omega|. \) \( \Box \)

**Proof of Theorem 1.3.** It is very close to that of Theorem 2.1, but we need to treat further growth of \( \psi \) at infinity. We take now \( \varepsilon = \varepsilon(\omega, k) \) as follows,
\[
\varepsilon(\omega, k) = \frac{\varepsilon_0}{(k^2 + |k|^2 + \Omega^2 \omega^2)^{\frac{1}{2}}(1-r)/(m+1)},
\] (2.67)
and we decompose \( f \) as
\[
f = \chi_k * f + (\delta - \chi_k) * (f - Pf) + (\delta - \chi_k) * (Pf),
\] (2.68)
where $\delta$ stands for the Dirac distribution in $(t, x)$ at the origin,

$$\hat{\chi}_c(\omega, k) = \hat{\chi}_1 \left( \frac{\Omega \omega}{k}, \frac{k}{\kappa} \right), \quad \hat{\chi}_1 \in C_0^\infty, \quad \hat{\chi}_1(\omega, k) = 1 \text{ if } (|k|^2 + \omega^2)^{1/2} < 1/2,$$

and

$$\hat{P} f = \rho_{\varepsilon(\omega, k)} \ast \hat{f}.$$  

The multiplier

$$\Upsilon(\omega, k, \nu) = \frac{1 - \hat{\rho}_1(\varepsilon(\omega, k)\nu)}{\varepsilon(\omega, k)\beta|\nu|^\beta}$$

verifies the estimates

$$\left| \partial_\omega^j \partial_k^j \partial_\nu^p \Upsilon(\omega, k, \nu) \right| \leq \frac{C_{\varepsilon, \gamma, \gamma_2} \Omega^j}{(|k|^2 + \Omega^2 \omega^2)^{j/p} + |\nu|^p},$$  

for some constants $C_{\varepsilon, \gamma, \gamma_2}$ independent of $\varepsilon_0$ and $\kappa$, $\Omega$. According to Lemma 2.3, $\Upsilon$ is therefore a bounded multiplier on $L^p(\mathbb{R}_x \times \mathbb{R}_x^N \times \mathbb{R}_x^N)$ for any $1 < p < \infty$,

$$\|\varepsilon_0^{-\beta}(k^2 - \Delta_x - \Omega^2 \partial_t)^{s/2}(f - Pf)\|_{L^p} \leq C\|D_\varepsilon^{\beta} f\|_{L^p},$$

where $C$ depends neither on $\varepsilon_0$, nor on $\kappa$, $\Omega$. Thus we can estimate (2.68) as

$$\|D_x^k f\|_p + \Omega^s\|D_x^k f\|_p \leq C\kappa\|f\|_p + C\varepsilon_0^{-\beta}\|D_\varepsilon^{\beta} f\|_p + \Omega^s\|D_\varepsilon^{\beta} f\|_p (P f)\|_p + \Omega^s\|D_\varepsilon^{\beta} f\|_p (P f)\|_p.$$

We need to estimate the last two terms, and in order to do so we introduce $\lambda(\omega, k, \nu)$ depending now on all variables,

$$\lambda(\omega, k, \nu) = \frac{\lambda_0}{(1 + \omega^2)^{1/2} (k^2 + |\nu|^2 + \Omega^2 \omega^2)^{1/2}} (k^2 + |\nu|^2 + \Omega^2 \omega^2)^{1/2}\left(1 - \frac{1}{m+1/2}\right),$$

and with (1.8) we decompose $f$ as

$$\hat{f} = \frac{\lambda(\omega, k, \nu)}{\lambda(\omega, k, \nu) + i(\omega + v \cdot k)} f + \frac{\lambda(\omega, k, \nu) + i(\omega + v \cdot k)}{\lambda(\omega, k, \nu) + i(\omega + v \cdot k)} \sum |n| \leq m \partial_\nu^m g_n.$$

By applying the operator $P$, this yields

$$\hat{P} f(\omega, k, \nu) = \int \frac{\lambda(\omega, k, \eta)}{\lambda(\omega, k, \eta) + i(\omega + \nu \cdot k)} \rho_{\varepsilon(\omega, k)}(v - \eta) \hat{f}(\omega, k, \eta) d\eta$$

and

$$\hat{M} f(\omega, k, \nu) = \frac{\lambda(\omega, k, v)}{\lambda(\omega, k, v) + i(\omega + v \cdot k)} \hat{f}(\omega, k, v).$$

We denote as before by $\hat{Q} f$ the first term in the right-hand side, that is $Q = PM$, with $M$ defined by

$$\hat{M} f(\omega, k, \nu) = \frac{\lambda(\omega, k, v)}{\lambda(\omega, k, v) + i(\omega + v \cdot k)} \hat{f}(\omega, k, v).$$
Following the idea of [1], we introduce the transformation
\[ f_*(t, x, v) = f(R_{xv}^{-1}(t, x, v)), \]  
(2.79)
where \( R_v \) is a linear transformation on \( \mathbb{R} \times \mathbb{R}^N \) such that
\[ |\det R_v| = 1, \quad \| R_v(t, k) \| = \| (\omega, k) \|, \quad (1, v) \cdot R_v(t, k) = \omega \sqrt{1 + \Omega^2 |v|^2}, \]
where we denote \( \| (\omega, k) \| = \sqrt{|k|^2 + \Omega^2 \omega^2} \). Indeed we can take for \( R_v \)
\[ R_v(t, k) = \left( \frac{\omega - v \cdot k}{\sqrt{1 + \Omega^2 |v|^2}}, \frac{(v \cdot k) \omega + \Omega \omega v}{\sqrt{1 + \Omega^2 |v|^2}} + k - \frac{(v \cdot k) v}{|v|^2} \right). \]
(2.81)
We define an operator \( M_* \) by
\[ M_* f_*(\omega, k, v) = \frac{\lambda(\omega, k, v)}{\lambda(\omega, k, v) + i \omega \sqrt{1 + \Omega^2 |v|^2}} \tilde{f}_*(\omega, k, v), \]
(2.82)
and since \( \tilde{f}_*(\omega, k, v) = \tilde{f}(R_v^1(\omega, k), v) \), we have with (2.80)
\[ (Mf)_* = M_* f_. \]
(2.83)
But the multiplier \( \varphi_v(\omega, k) = 1/(1 + i \omega \sqrt{1 + \Omega^2 |v|^2} \lambda(\omega, k, v)) \) satisfies the estimates
\[ |\partial_\omega^j \partial_k^\alpha \varphi_v(\omega, k)| \leq \frac{C_{\gamma_1, \gamma_2}}{|\omega|^{\gamma_1} |k|^{\gamma_2}} \]
(2.84)
for some constants \( C_{\gamma_1, \gamma_2} \) independent of \( \lambda_0, \kappa, \Omega \) and \( v \), thus we deduce by Lemma 2.3 that it defines a bounded operator on \( L^p(\mathbb{R} \times \mathbb{R}^N) \) for any \( 1 < p < \infty \). Therefore, by the similarity formula (2.83) and since the transformation (2.79) is again an isometry of \( L^p \), we conclude that \( M \) is bounded on \( L^p(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N) \) with a constant independent of \( \lambda_0, \kappa, \Omega \), hence \( Q \) also by composition. Next, we can perform again the estimates (2.9)-(2.12) with the help of Lemma 2.4 that uses the \( v \) dependence of \( \lambda \), and it gives
\[ \|Qf(\omega, k, \cdot)\|_{L^2(\mathbb{R}^N)} \leq C_{N, \beta} \left( \frac{\lambda_0 \sqrt{k^2 + |k|^2 + \Omega^2 \omega^2}}{\varepsilon_0 \sqrt{|k|^2 + \Omega^2 \omega^2}} \right)^{1/2} \|\tilde{f}(\omega, k, \cdot)\|_{L^2(\mathbb{R}^N)}, \]
(2.85)
hence with the cutoff function (2.69),
\[ \|Qf - \chi_{(\varepsilon_0)}(Qf)\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R})} \leq C_{N, \beta} \left( \frac{\lambda_0}{\varepsilon_0} \right)^{1/2} \|f\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R})}. \]
(2.86)
Similarly as in the proof of Theorem 2.1 we conclude by interpolation that
\[ \|Qf - \chi_{(\varepsilon_0)}(Qf)\|_p \leq C \left( \frac{\lambda_0}{\varepsilon_0} \right)^t \|f\|_p, \quad t = \frac{1}{2} \left[ \frac{1}{p} - \frac{1}{p_1} \right] > 0. \]
(2.87)
Therefore we get
\[
\| D_x^s (Q f - x_\kappa, t_x (Q f)) \|_p \leq C \left( \frac{\lambda_0}{\varepsilon_0} \right)^\ell \| D_x f \|_p^n
\]
\[
\| D_t^s (Q f - x_\kappa, t_x (Q f)) \|_p \leq C \left( \frac{\lambda_0}{\varepsilon_0} \right)^\ell \| D_t f \|_p^n
\]  \tag{2.88}

For the second term in (2.77), after integration by parts, we are led to operators of the form
\[
\overrightarrow{W} g(\omega, k, v) = \int \frac{(\kappa^2 + |k|^2 + \Omega^2 \omega^2)^{s/2} \kappa^{a_1} \lambda(\omega, k, \eta) |\alpha_{31}|^{a_3} \varphi(\Omega k) \Omega^{a_2-1} |\alpha_4|}{\lambda(\omega, k, \eta) \sqrt{1 + \Omega^2 |\eta|^2} |\alpha_1|^{a_1} |\alpha_3|^{a_3} \varepsilon(\omega, k) |\alpha_2|}
\times \partial^a \rho_{e(\omega, k)} (v - \eta) \tilde{g}(\omega, k, \eta) \, d\eta,
\]
with \(|\alpha_1| + |\alpha_2| + |\alpha_3| + |\alpha_4| = |\alpha| \leq m\), and \(\varphi\) satisfies
\[
|\partial^a \varphi(v)| \leq \frac{C_{\gamma}}{(1 + |v|^2)^{1/2} |\alpha_{31}|^{a_3} |\alpha_4|^{a_4}}. \tag{2.89}
\]

We can write
\[
\overrightarrow{W} g = \frac{(\kappa^2 + |k|^2 + \Omega^2 \omega^2)^{s/2} \kappa^{a_1} \lambda^{a_3} \varphi(\Omega k) \Omega^{a_2-1} |\alpha_4|}{\lambda(\omega, k, \eta) \sqrt{1 + \Omega^2 |\eta|^2} |\alpha_1|^{a_1} |\alpha_3|^{a_3} \varepsilon(\omega, k) |\alpha_2|}
\overrightarrow{W} g
\]  \tag{2.91}

with
\[
\overrightarrow{W} g(\omega, k, v) = \int \frac{\varphi(\Omega k)}{1 + |v|^2} |\alpha_{31}|^{a_3} |\alpha_4|^{a_4}
\times \frac{1}{\varepsilon(\omega, k)} \partial^a \rho_{e(\omega, k)} \left( \frac{v - \eta}{\varepsilon(\omega, k)} \right) \left( 1 + \Omega^2 |\eta|^2 \right)^{a_1-1} \tilde{g}(\omega, k, \eta) \, d\eta,
\]  \tag{2.92}

where we use an incorrect notation in (2.91), but in fact the term \(\lambda(\omega, k, \eta) \times \sqrt{1 + \Omega^2 |\eta|^2}\) does not depend on \(\eta\). By the same analysis as that on \(Q\) above,
\[
\| \overrightarrow{W} g - x_\kappa, t_x \overrightarrow{W} g \|_p \leq C (\lambda_0/\varepsilon_0)^\ell (1 + \Omega^2 |v|^2)^{a_1+1} \| g \|_p^n \tag{2.93} \]

with \(C\) a constant independent of \(\lambda_0, \varepsilon_0\) and \(\kappa, \Omega\). Since the multiplier
\[
\frac{(\kappa^2 + |k|^2 + \Omega^2 \omega^2)^{s/2} \kappa^{a_1} \lambda^{a_3} \varphi(\Omega k) \Omega^{a_2-1} |\alpha_4|}{\lambda(\omega, k, \eta) \sqrt{1 + \Omega^2 |\eta|^2} |\alpha_1|^{a_1} |\alpha_3|^{a_3} \varepsilon(\omega, k) |\alpha_2|}
\]
\[
= \frac{\lambda_0^{a_1+1} \varepsilon_0^{a_3} (\kappa^2 + |k|^2 + \Omega^2 \omega^2)^{s/2} (1-s)}{m+1} \left( \frac{\kappa^2 + |k|^2 + \Omega^2 \omega^2}{\lambda_0} \right)^{s/2}
\]  \tag{2.94}
is bounded on \(L^p(\mathbb{R} \times \mathbb{R}^N)\) in \(C \Omega^{\lambda_{1}+1+\lambda_{4}}/(\kappa(1-r)(m-|\alpha_{2}|)(m+1+\beta)\lambda_{0}^{\lambda_{1}+1+\lambda_{4}}\), we deduce that

\[
\| (\kappa^2 - \Delta x - \Omega^2 \partial_t^s)^{s/2} (\delta - \chi_{\kappa}) \ast (Wg) \|_p \\
\leq \frac{C (\lambda_0/\varepsilon_0)^t \Omega^{\lambda_{3}+1+\lambda_{4}}}{\kappa(1-r)\frac{m-|\alpha_{2}|}{m+1+\beta}\lambda_{0}^{\lambda_{1}+1+\lambda_{4}} \varepsilon_0} \| (1 + \Omega^2 |v|^2)^{\frac{\lambda_{1}+1}{2}} g \|_p.
\] (2.95)

Together with (2.88) and (2.77), we are now able to estimate the last two terms in (2.74), and we obtain

\[
\| D_x^s f \|_p + \Omega^s \| D_x^s f \|_p \\
\leq C \kappa^s \| f \|_p + C \varepsilon_0^\beta \| D^S_x f \|_p + C (\lambda_0/\varepsilon_0)^t (\| D_x^s f \|_p + \Omega^s \| D_x^s f \|_p) \\
+ C \sum_{|\alpha| \leq m} \frac{(\lambda_0/\varepsilon_0)^{-|\alpha|-1}}{\kappa(1-r)\frac{m-|\alpha_{2}|}{m+1+\beta}\varepsilon_0^{-1}} \left( \frac{\Omega\varepsilon_0}{\lambda_{0}^{1+\lambda_{4}} k^{1+\lambda_{4}}} \right)^{|\alpha|+1} \| (1 + \Omega^2 |v|^2)^{\frac{\lambda_{1}+1}{2}} g_\alpha \|_p.
\] (2.96)

Since \(t > 0\), we can choose now \(\lambda_0/\varepsilon_0\) small enough so that the third term on the right-hand side has a coefficient which is less than 1/2, and this yields

\[
\| D_x^s f \|_p + \Omega^s \| D_x^s f \|_p \\
\leq C \kappa^s \| f \|_p + C \varepsilon_0^\beta \| D^S_x f \|_p \\
+ C \sum_{|\alpha| \leq m} \frac{1}{\kappa(1-r)\frac{m-|\alpha_{2}|}{m+1+\beta}\varepsilon_0^{-1}} \left[ \| (1 + \Omega^2 |v|^2)^{\frac{\lambda_{1}+1}{2}} g_\alpha \|_p \\
+ \left( \frac{\Omega\varepsilon_0}{\lambda_{0}^{1+\lambda_{4}} k^{1+\lambda_{4}}} \right)^{|\alpha|} \| (1 + \Omega^2 |v|^2)^{\frac{1}{2}} g_\alpha \|_p \right],
\] (2.97)

for any \(\varepsilon_0 > 0\). By choosing

\[
\varepsilon_0 = \sum_{|\alpha| \leq m} \left[ \| (1 + \Omega^2 |v|^2)^{\frac{\lambda_{1}+1}{2}} g_\alpha \|_p \frac{1}{\| D^S_x f \|_p} \right]^\frac{1}{\lambda_{1}+\lambda_{4}} \\
+ \sum_{|\alpha| \leq m} \left[ \frac{\Omega^{|\alpha|}}{\kappa(1-r)\frac{m-|\alpha_{2}|}{m+1+\beta}} \| (1 + \Omega^2 |v|^2)^{1/2} g_\alpha \|_p \right]^\frac{1}{\lambda_{1}+\lambda_{4}},
\] (2.98)

we finally obtain (1.13). \(\square\)

**Proof of Corollary 1.4.** We have \(\psi(v)f \in W_{t,x}^{s,p}(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_x^N)\), and \(h = (\kappa^2 - \Delta x - \Omega^2 \partial_t^s)^{s/2} \psi f \in L^p\) satisfies

\[
\partial_t h + v \cdot \nabla_x h = (\kappa^2 - \Delta x - \Omega^2 \partial_t^s)^{(r+s)/2} \sum_{|\alpha| \leq m} \psi(v) \partial_x^\alpha g_\alpha.
\] (2.99)

By the result of [1], we thus have

\[
\int h(t,x,v) \, dv \in W_{t,x}^{d,p}(\mathbb{R}_t \times \mathbb{R}_x^N),
\] (2.100)
with \( \theta = \frac{1-(r+\beta)}{m+1}(1 - \frac{1}{p}) \), which gives that
\[
\int f(t, x, v) \psi(v) \, dv \in W^{s+\theta, p}(\mathbb{R}_t \times \mathbb{R}^N).
\] (2.101)

We just notice that
\[
r + s = \frac{r(m + 1) + \beta}{m + 1 + \beta}, \quad \theta = \frac{1 - r}{m + 1 + \beta}(1 - \frac{1}{p}),
\] (2.102)
which gives the desired result. \( \square \)

3 The Hörmander commutator

We prove here Theorems 1.5 and 1.6, with the method of [6], which is based on writing the commutator identity
\[
\partial_{x_j} f = \partial_{x_j} (\partial_t f + v \cdot \nabla x f) - (\partial_t + v \cdot \nabla x) \partial_{x_j} f,
\] (3.1)
and taking the \( L^2 \) bracket of it against an \( x \)-derivative of \( f \). We shall denote
\[
\langle f, g \rangle = \iint_{\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N} f g \, dtdx dv.
\] (3.2)

The main drawback of this method is however that it cannot handle \( L^p \) functions.

Proof of Theorem 1.6. By suitable smoothing and cutoff, we can reduce to the case where \( f, g \in C^\infty_c \) (this is the reason why we need that \( D^\text{max}(\beta, \gamma) f \in L^2 \), and not only \( D^\beta f \in L^2 \)). Therefore, let us assume that \( f, g \in C^\infty_c(\mathbb{R}_t \times \mathbb{R}^N \times \mathbb{R}^N) \). Noticing that \( 1/2 \leq s \leq 1 \), we compute that
\[
\begin{align*}
\|D_x^{-(1-s)} \partial_{x_j} f\|_2^2 &= \langle D_x^{-(1-s)} \partial_{x_j} f, \partial_{x_j} f \rangle \\
&= \langle D_x^{-(1-s)} \partial_{x_j} f, \partial_{x_j} g - (\partial_t + v \cdot \nabla x) \partial_{x_j} f \rangle \\
&= -\langle \partial_{x_j} D_x^{-(1-s)} \partial_{x_j} f, \partial_{x_j} g \rangle + \langle (\partial_t + v \cdot \nabla x) D_x^{-(1-s)} \partial_{x_j} f, \partial_{x_j} f \rangle \\
&= -\langle D_x^{-(1-s)} \partial_{x_j} \partial_{x_j} f, \partial_{x_j} g \rangle + \langle D_x^{-(1-s)} \partial_{x_j} f, \partial_{x_j} f \rangle \\
&= -2 \Re \langle D_x^{-(1-s)} \partial_{x_j} \partial_{x_j} f, \partial_{x_j} f \rangle \\
&\leq 2 \|D_x^{-(1-s)} D_v^{1-\gamma} f\|_2 \|D_v \gamma g\|_2.
\end{align*}
\] (3.3)

Therefore,
\[
\|D_v^{\theta} f\|_2^2 \leq C_N \|D_x^{-(1-s)} D_v^{1-\gamma} f\|_2 \|D_v \gamma g\|_2.
\] (3.4)

Now, let \( \theta = 2 - 1/s \in [0, 1] \). We have \( 1 - 2(1-s) = \theta s \) and \( 1 - \gamma = (1 - \theta) \beta \). Noticing that
\[
\|D_x^{\theta} D_v^{(1-\theta)\beta} f\|_2 \leq \|D_v^{\theta} f\|_2 \|D_v^{\beta} f\|_2^{1-\theta},
\] (3.5)
which is nothing else than a Hölder inequality in the Fourier variables \((k, \nu)\) of \((x, v)\),

\[
\int \frac{1}{\|D^\beta f\|_2^\theta} \|D^\gamma g\|_2^\theta \leq \left( \int \frac{1}{\|D^\beta f\|_2^\theta} \|D^\gamma g\|_2^\theta \right)^{1-\theta},
\]

we deduce that

\[
\|D^\beta f\|_2^\theta \leq C_N \|D^\gamma g\|_2^\theta.
\]

Therefore, after simplification,

\[
\|D^\beta f\|_2^\theta \leq C_N \|D^\gamma g\|_2^\theta,
\]

which yields (1.23). \(\square\)

**Proof of Theorem 1.5.** Once we have proved (1.17), (1.18) follows obviously by applying Theorem 1.6 (or Proposition 1.1) with \(\beta = 2\) and \(\gamma = 0\). Therefore, let us prove that (1.17) holds, and again it is enough to do it when \(f, g \in C^\infty_c(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N)\). We first write that

\[
(\partial_t + v \cdot \nabla_x - \sigma \Delta_v)(D^{1/3}_x f) = D^{1/3}_x g,
\]

and we take the \(L^2\) bracket against \(D^{1/3}_x f\). We get

\[
\sigma \iint |\nabla_v(D^{1/3}_x f)|^2 = \text{Re}\langle D^{1/3}_x f, D^{1/3}_x g \rangle = \text{Re}\langle D^{2/3}_x f, g \rangle,
\]

thus

\[
\sigma \|\nabla_v(D^{1/3}_x f)\|_2^2 \leq \|D^{2/3}_x f\|_2 \|g\|_2.
\]

Next, we take the bracket of (1.15) with \(-\Delta_v f\), which yields

\[
\text{Re}\langle -\Delta_v f, v \cdot \nabla_x f \rangle + \sigma \|\Delta_v f\|_2^2 = \text{Re}\langle -\Delta_v f, g \rangle.
\]

But

\[
\text{Re}\langle -\Delta_v f, v \cdot \nabla_x f \rangle = -\sum_j \text{Re}\langle \partial_{v_j} f, v \cdot \nabla_x f \rangle
\]

\[
= \sum_j \text{Re}\langle \partial_{v_j} f, (v \cdot \nabla_x) \partial_{v_j} f \rangle
\]

\[
= \text{Re}\langle \nabla_v f, \nabla_x f \rangle,
\]

thus (3.12) gives with (3.11)

\[
\sigma \|\Delta_v f\|_2^2 = \text{Re}\langle -\Delta_v f, g \rangle - \text{Re}\langle \nabla_v f, \nabla_x f \rangle
\]

\[
\leq \|\Delta_v f\|_2 \|g\|_2 + \|\nabla_v(D^{1/3}_x f)\|_2 \|D^{2/3}_x f\|_2
\]

\[
\leq \|\Delta_v f\|_2 \|g\|_2 + \frac{1}{\sqrt{\sigma}} \|D^{2/3}_x f\|_2^{3/2} \|g\|_2^{1/2}.
\]
By applying Theorem 1.6 (or Proposition 1.1) with $\beta = 2$ and $\gamma = 0$, we have also that
\[ \| D_x^{2/3}f \|_2 \leq C_N \| \Delta_v f \|_2^{1/3} \| \partial_t f + v \cdot \nabla_x f \|_2^{2/3}, \tag{3.15} \]
and since from (1.15) $\| \partial_t f + v \cdot \nabla_x f \| \leq \| g \| + \sigma \| \Delta_v f \|$, we obtain
\[ \| D_x^{2/3}f \|_2^{3/2} \leq C_N \| \Delta_v f \|_2^{3/2}(\| g \|_2 + \sigma \| \Delta_v f \|_2), \tag{3.16} \]
and with (3.14)
\[ \sigma \| \Delta_v f \|_2^2 \leq \| \Delta_v f \|_2 \| g \|_2 + \frac{C_N}{\sqrt{\sigma}} \| \Delta_v f \|_2^{1/2} \| g \|_2^{1/2}(\| g \|_2 + \sigma \| \Delta_v f \|_2). \tag{3.17} \]
Finally, by simplifying, we obtain
\[ \sigma^{3/2} \| \Delta_v f \|_2^{3/2} \leq \sqrt{\sigma} \| \Delta_v f \|_2^{1/2} \| g \|_2 + C_N \| g \|_2^{1/2}(\| g \|_2 + \sigma \| \Delta_v f \|_2), \tag{3.18} \]
and this yields obviously that $\sigma \| \Delta_v f \| \leq C_N \| g \|$, which concludes the proof. \( \square \)

4 The characteristics commutator

We prove here a result that is similar to Proposition 1.1, but with a very simple and direct approach that is based on the characteristics of the transport operator. It enables to consider data in $L^p$ for any $1 \leq p \leq \infty$, but however we get estimates in spaces that slightly differ from the usual Sobolev spaces. At the present time, we are not able to treat derivatives in the right-hand side.

Proposition 4.1 Assume that $f, g \in L^p(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v^N), 1 \leq p \leq \infty$ satisfy
\[ \partial_t f + v \cdot \nabla_x f = g, \tag{4.1} \]
and that
\[ \forall \eta \in \mathbb{R} \quad \| f(t, x, v + \eta) - f(t, x, v) \|_{L^p_{tv}} \leq A|\eta|^\beta, \tag{4.2} \]
for some $0 \leq \beta \leq 1$ and some constant $A \geq 0$. Then
\[ \forall h \in \mathbb{R} \quad \| f(t, x + h, v) - f(t, x, v) \|_{L^p_{tv}} \leq C A^\frac{1}{1+\beta} \| g \|_{p+x} \| h \|_{v}^{\frac{\beta}{1+\beta}}. \tag{4.3} \]

Proof. For any $\tau \neq 0$ we write the following decomposition, which involves somehow a commutator of the characteristics of $\partial_t + v \cdot \nabla_x$ and $\nabla_v$,
\[ f(t, x + h, v) - f(t, x, v) = \delta f^1 + \delta f^2 + \delta f^3 + \delta f^4, \tag{4.4} \]
with
\[ \delta f^1 = f(t - \tau, x - \tau v, v) - f(t, x, v), \]
\[ \delta f^2 = f(t - \tau, x - \tau v, v + h/\tau) - f(t, x - \tau v, v), \]
\[ \delta f^3 = f(t, x + h, v + h/\tau) - f(t - \tau, x - \tau v, v + h/\tau), \]
\[ \delta f^4 = f(t, x + h, v) - f(t, x + h, v + h/\tau). \tag{4.5} \]
We have
\[ \delta f^1 = -\int_0^T g(t - \sigma, x - \sigma v, v) \, d\sigma, \]  
(4.6)

thus
\[ \|\delta f^1\|_{L^p_{t,x,v}} \leq |\tau| \|g\|_{L^p_{t,x,v}}. \]  
(4.7)

Next, we have obviously
\[ \|\delta f^2\|_{L^p_{t,x,v}} \leq A \left| \frac{h}{\tau} \right|^\beta. \]  
(4.8)

Similarly,
\[ \|\delta f^3\|_{L^p_{t,x,v}} = \left\| \int_0^T g(t - \sigma, x + h - \sigma(v + h/\tau), v + h/\tau) \, d\sigma \right\|_{L^p_{t,x,v}} \]  
\[ \leq |\tau| \|g\|_{L^p_{t,x,v}}, \]  
(4.9)

and
\[ \|\delta f^4\|_{L^p_{t,x,v}} \leq A \left| \frac{h}{\tau} \right|^\beta. \]  
(4.10)

Therefore, we conclude that
\[ \|f(t, x + h, v) - f(t, x, v)\|_{L^p_{t,x,v}} \leq 2(|\tau| \|g\|_{L^p} + A|h|^\beta \|g\|_{L^p}), \]  
(4.11)

which gives the result by choosing \(|\tau|^{1+\beta} = A|h|^\beta \|g\|_{L^p} \). \( \square \)

**Acknowledgements**

The author is indebted to Cédric Villani and Laurent Desvillettes for fruitful discussions in the genesis of this work.

**References**


