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June 2002

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Abstract

We consider the Helmholtz equation with a variable index of refraction $n(x)$, which is not necessarily constant at infinity but can have an angular dependency like $n(x) \to n_\infty(x/|x|)$ as $|x| \to \infty$. We prove that the Sommerfeld condition at infinity still holds true under the weaker form

$$\frac{1}{R} \int_{|x| \leq R} \left| \nabla u - i n^{1/2}(x/|x|)u \frac{x}{|x|} \right|^2 \, dx \to 0, \quad \text{as} \quad R \to \infty.$$  

Our approach consists in proving this estimate in the framework of the limiting absorption principle. We use Morrey-Campanato type of estimates and a new inequality on the energy decay, namely

$$\int_{\mathbb{R}^d} \left| \frac{\partial}{\partial \omega} n_\infty(x/|x|) \right|^2 \frac{|u|^2}{|x|} \, dx \leq C.$$  

It is a striking feature that the index $n_\infty$ appears in this formula and not the phase gradient, in apparent contradiction with existing literature.

1 Introduction

We consider the Helmholtz equation with a variable index of refraction $n(x)$, with a slow, and only radial decay to a constant $n_\infty(x/|x|)$ at infinity

$$i \varepsilon u_\varepsilon + \Delta u_\varepsilon + n(x)u_\varepsilon = -f(x), \quad \varepsilon > 0. \quad (1.1)$$
Our main interest is the so called limiting absorption principle (i.e. to study the limit when $\varepsilon > 0$ approaches to 0 in (1.1)) and the validity of the Sommerfeld radiation condition at infinity. One of the main results in this paper is to prove that

$$\frac{1}{R} \int_{|x| \leq R} \left| \nabla u - in_{\infty}^{1/2}(\frac{x}{|x|}) u \frac{x}{|x|} \right|^2 dx \leq \omega(R) \to 0, \text{ as } R \to \infty. \quad (1.2)$$

A direct consequence of this condition is the more classical setting

$$\liminf_{|x| = r} \left| \nabla u - in_{\infty}^{1/2}(\frac{x}{|x|}) u \frac{x}{|x|} \right|^2 d\sigma(x) \to 0, \text{ as } r \to \infty, \quad (1.3)$$

where $d\sigma$ denotes the Lebesgue measure on the sphere. It is a striking feature that the index $n_{\infty}$ appears in this formula and not the phase gradient, in apparent contradiction with existing literature. This phenomenon, as well as the proof of (1.3), can be explained by a new and fundamental energy estimate that we state later on.

In order to state precisely our results we need some notation. Firstly we define, for $j \in Z$, the annulus $C(j)$ by

$$C(j) = \{ x \in \mathbb{R}^d \text{ s.t. } 2^j \leq |x| \leq 2^{j+1} \}.$$ 

Then we set

$$\|u\|_{R_0}^2 := \sup_{R > R_0} \frac{1}{R} \int_{|x| \leq R} |u|^2 dx, \quad (1.4)$$

$$N_{R_0}(f) := \sum_{j > J} [2^{j+1} \int_{C(j)} |f|^2 dx]^{1/2} + [R_0 \int_{B_{R_0}} |f|^2 dx]^{1/2}, \quad (1.5)$$

with $J$ defined by $2^J < R_0 < 2^{J+1}$, and we drop the index $R_0$ if $R_0 = 0$. We also denote the radial and tangential derivatives by

$$\frac{\partial}{\partial r} u := \frac{x}{|x|} \cdot \nabla u \quad (1.6)$$

$$\frac{\partial}{\partial \tau} u := \nabla u - \frac{x}{|x|} \frac{\partial}{\partial r} u, \quad (1.7)$$

so that for a function $n(\omega) \in C^1(S^{d-1})$, we have

$$\frac{\partial}{\partial \tau} n(\frac{x}{|x|}) = \frac{\partial}{\partial \omega} n(\frac{x}{|x|})/|x|. \quad \text{2}$$
We are now ready to state our main assumptions:

\[ n \in L^\infty, \quad n > 0 ; \quad \text{(1.8)} \]

\[ 2 \sum_{j \in \mathbb{Z}} \sup_{C(j)} \frac{(x \cdot \nabla n(x))_+}{n(x)} := 2 \sum_{j \in \mathbb{Z}} \beta_j = \beta < +\infty ; \quad \text{(1.9)} \]

Then, we assume that there are \( 0 < \delta < 1/2, \Gamma > 0 \) and a function \( n_\infty(\frac{x}{|x|}) \) such that

\[
\begin{aligned}
&\{ n_\infty(\frac{x}{|x|}) \in C^3(S^{d-1}), \quad n_\infty(\frac{x}{|x|}) \geq n_0 > 0, \\
&|n(x) - n_\infty(\frac{x}{|x|})| \leq n_0 \frac{\Gamma}{|x|} \quad \text{for } |x| \text{ large enough.} \quad \text{(1.10)}
\end{aligned}
\]

Finally we will sometimes use one of the following two conditions

\[
|n(x) - n_\infty(\frac{x}{|x|})| \leq n(x) \frac{\tilde{\Gamma}}{|x|}, \quad \tilde{\Gamma} > 0 ; \quad \text{(1.11)}
\]

\[
\begin{aligned}
&\left\{ \right. \\
&\quad \text{there exists } \tilde{\beta} < 1, \delta > 0 \text{ and } \tilde{\Gamma} > 0 \text{ such that} \\
&\quad \left( \frac{\partial n - n_\infty}{\partial \omega} \cdot \frac{\partial \omega}{\partial \omega} \right)_- \leq \tilde{\beta} \left( \frac{\partial n - n_\infty}{\partial \omega} \right)^2 + n(x) \frac{\tilde{\Gamma}}{|x|}. \\
&\end{aligned} \quad \text{(1.12)}
\]

Notice that from assumptions (1.8), (1.9) we get that there is a radial limit \( n_\infty \) as assumed in (1.10). More precisely, along rays of direction \( \omega \in S^{d-1} \), we have

\[
\frac{\partial}{\partial r} \ln n(r\omega) \in L^1(0, +\infty). 
\]

Indeed,

\[
\int_0^\infty \left( \frac{\partial}{\partial r} \ln n(r\omega) \right)_- \leq \beta,
\]
and thus

\[
\int_0^\infty \left( \frac{\partial}{\partial r} \ln n(r\omega) \right)_+ \leq \beta + 2\| \ln n \|_{L^\infty}.
\]

Our first result is a uniqueness theorem.

**Theorem 1.1** For dimensions \( d \geq 2 \), assume (1.8), (1.9) and that \( n(x) > n_0 > 0 \) for \( |x| \) large enough. If \( u \) is a solution of \( \Delta u + nu = 0 \) with \( u \) and \( \nabla u \) locally in \( L^2 \), and such that

\[
\liminf \int_{|x| = r} \left( |\nabla u|^2 + |u|^2 \right) d\sigma(x) \to 0, \quad \text{as } r \to \infty, \quad \text{(1.13)}
\]

then \( u = 0 \). Moreover a sufficient condition for (1.13) to hold is that, for some \( R_0, \|u\|_{R_0}^2 < +\infty \), and (1.2) holds.
The proof of this result follows well established ideas as can be found in [20], [17], and [23]. However it does not seem to appear in the literature when just the short range condition (1.9) is assumed. We sketch the proof in the appendix. We also want to remark that uniqueness is much more simple to establish if \( \beta < 1 \) as we will see from the proof of Theorem 3.1.

Our next result is a more general version of the bound derived in [19],

**Theorem 1.2** We assume one of the following three conditions

(i) \( d \geq 3 \), (1.8)–(1.9), \( \beta < 1 \) and \( R(n) = 0 \);

(ii) \( d = 2 \), (1.8)–(1.9), \( \beta < 1 \), \( n > n_0 > 0 \) and \( R(n) = n_0^{-1/2} \);

(iii) \( d \geq 2 \), (1.8)–(1.10) and either (1.11) or (1.12) and \( R(n) \) large enough (according especially to (3.6)).

Then the solution to the Helmholtz equation (1.1) satisfies,

\[
M^2 := \|\nabla u\|_{R(n)}^2 + \|\tilde{u}\|_{R(n)}^2 + \int_{|x| \geq R(n)} \frac{|\nabla_x u|^2}{|x|} \, dx \\
\leq C(\varepsilon + \|n\|_\infty) \, N_{R(n)} \left( \frac{f}{n^{1/2}} \right)^2. \tag{1.14}
\]

This theorem was obtained in [19] for the cases (i) and (ii) where \( \beta < 1 \), and the constant \( C \) only depends upon \( \beta \) and the dimension. Then, the homogeneity of the estimates makes it compatible with the high frequencies (replace \( n \) by \( \omega^2n \)). The case (iii) follows from a compactness argument and the uniqueness Theorem 1.1. In particular no information is available on the constant \( C \) which might depend on \( n \). Notice however that the above inequality is invariant under the rescaling \( n(x) \rightarrow \omega^2 n(\omega x), \quad \omega > 1 \). Then \( R(n) \) becomes \( R(n)/\omega \) and therefore the assumptions on \( f \) become weaker for larger \( \omega \).

Our main interest in [19] was to obtain estimates with the right scaling. In particular we were able to recover the well known estimate of Agmon and Hörmander in [3] for \( u \) in the constant coefficient case. Similar results but not scaling invariant were obtained in [11] and [24]. The scaling plays a fundamental role in the applications to nonlinear Schrödinger equations ([15]) and in the high frequency limit for Helmholtz equations ([5], [6]).

However in this paper our main interest is to get Sommerfeld radiation condition for solutions obtained from the limiting absorption principle, and at this level the smallness condition \( \beta < 1 \) turns out to be unnecessary.
We may now state our basic new estimate that leads to establish Sommerfeld condition. Its interest relies of course on the a priori bounds stated in Theorem 1.2.

**Theorem 1.3** For dimensions \( d \geq 2 \), we assume (1.8)-(1.10) and either (1.11) or (1.12), and we use the notations of Theorem 1.2. Then the solution to the Helmholtz equation (1.1) satisfies, for \( R \geq R(n) \) and \( R \) large enough,

\[
\tilde{M} := \int_{|x| \geq R} |\nabla_{\omega} n_{\infty}(\frac{x}{|x|})|^2 \frac{|u|^2}{|x|} \, dx \leq C \left[ (\varepsilon + \| n \|_{\infty}) N_{R(n)}(\frac{f}{n^{1/2}}) M + M^2 \right]
\]

for some constant \( C \) independent of \( \varepsilon \).

We would like to point out the sharpness of this inequality. It says that the points where \( |\nabla_{\omega} n_{\infty}(\frac{x}{|x|})|^2 \) vanishes on the sphere are in principle the directions where the energy \( |u|^2 \) irradiates. At this respect recall the uniqueness result given in Theorem 1.1. Indeed, we can derive from the Sommerfeld condition below the following proposition. The proof can be found at the end of section 3.

**Proposition 1.4** With the assumptions of Theorem 1.5 below, we have

\[
\lim_{R \to \infty} \frac{1}{R} \int_{|x| \leq R} n_{\infty}(\frac{x}{|x|}) |u|^2 \, dx = \mathcal{Im} \int_{\mathbb{R}^d} f(x) \, \bar{u}(x) \, dx.
\]

Therefore when, forgetting the factor \( |\nabla_{\omega} n_{\infty}(\frac{x}{|x|})|^2 \) in Theorem 1.3, we suppose the bound

\[
\int_{|x| \geq R} \frac{|u|^2}{|x|} \, dx \leq \infty,
\]

then we deduce that \( \mathcal{Im} \int_{\mathbb{R}^d} f(x) \, \bar{u}(x) \, dx = 0 \). In the constant coefficient case this leads ([12], p. 242) to a restrictive condition on the Fourier transform of \( f \), namely \( \hat{f}(\xi) = 0 \) on the sphere \( |\xi| = n^{1/2} \). A natural candidate to consider in our case is the generalized Fourier transform defined by S. Agmon, J. Cruz-Sampedro and I. Herbst in [2] which depends upon the construction of solutions to the associated eikonal equation. Notice however that, as we will explained later on, we do not have to solve the eikonal equation due to the existence of the inequality (1.15). The role played by the critical points of \( n_{\infty} \) was already pointed out by I. Herbst in [9].

We may now conclude with the statement of the main theorem of this paper which addresses the Helmholtz equation with \( \varepsilon = 0 \).
Theorem 1.5 For dimensions $d \geq 2$, assume (1.8)-(1.10) and either (1.11) or (1.12). Then, with $R(n)$ given in Theorem 1.2, there exists a unique solution to the Helmholtz equation with $\varepsilon = 0$, $M < \infty$, and which satisfies for $R \geq R(n)$,

$$
\frac{1}{R} \int_{|x| \leq R} |\nabla u - i \frac{x}{|x|} n^{1/2}_\infty u(x)|^2 \, dx \leq C \left( \int_{|x| \geq 1} \kappa_R(x) \frac{|
abla u|^2}{|x|} \, dx \right)^{1/2} \mathcal{N}(f) + C \left( \frac{\Gamma + \|n\|_{L^\infty}}{R^{\delta/2}} + \sum_{2^j > R(n)} \beta_j \kappa_R(2^j) \right) \mathcal{N}(f)^2. 
$$

(1.17)

Here $\kappa_R(x) = \min\{\frac{|x|}{R}, 1\}$, and $\mathcal{N}(f) := [N_{R(n)}(f)^2 + \|n\|_{L^\infty} N_{R(n)}(\frac{x}{|x|})^2]^{1/2}$.

Let us compare the above theorem with previously known results. There is a very extensive literature on the limiting absorption principle, see for example [7], [8], [1], [16], [4], [11], [23] and references there in. The situation for the Sommerfeld radiation condition is different. When $n = \lambda + V(x)$ and $V$ a short range potential the question was settled by Ikebe and Saito in [10]. Mochizuku and Uchiyama study in [18] large range potentials with mild radial oscillations at infinity like $V(x) \sim \sin(\ln |x|)$. Hörmander in [13], chapter XXX, characterizes the incoming/outgoing solutions obtained from the limiting absorption principle by some asymptotic behaviour, but in his case $n_\infty = \lambda = \text{constant}$. More general long range potentials were considered by Saito in [21]. Although in this latter work perturbations of first order terms (”magnetic potentials”) are also considered let us fix the attention in the conditions for $V$. Saito writes $V = p + Q$ where $Q$ is a short range perturbation, while $p$ is a bounded real function which belongs to $C^2(R^n - \{0\})$ and such that

$$
|D^\alpha p(x)| \leq c|x|^{-|\alpha|} \quad |\alpha| \leq 2.
$$

Then he proves a Sommerfeld radiation condition for $\lambda$ large enough given by $\nabla u \pm i \sqrt{\lambda}(\nabla R) u$, where $R$ is an appropriate solution of the associated eikonal equation

$$
|\nabla R|^2 = 1 - \frac{p(x)}{\lambda}.
$$

Therefore one can not expect that in general the vector $\nabla R$ points at the direction $x/|x|$. One example of this situation is to consider

$$
n(x) = 1 - \frac{x_1}{\lambda |x|}.
$$
In this case and for $\lambda$ large enough, see Remark 1.3 in [21], $R(x) = a(\lambda)|x| - b(\lambda)x_1$ with $a(\lambda) = 1/2[(1 + 1/\lambda)^{1/2} + (1 - 1/\lambda)^{1/2}]$ and $b(\lambda) = 1/2[(1 + 1/\lambda)^{1/2} - (1 - 1/\lambda)^{1/2}]$. This boundary condition differs from ours in all points except when $\nabla n = 0$. Then the apparent contradiction is clarified thanks to the estimate (1.15). As a conclusion of this estimate we can say that the solution $R$ of the eikonal equation is only relevant for the critical points of $n_\infty$. At the same time Saito’s assumptions are not comparable with ours. He imposes some smallness condition assuming $\lambda$ has to be large enough that we do not need. On the other hand he does not assume that $p(tx/|x|)$ has to have a limit for $t$ large and fixed $x$ as we do, but our condition (1.12) is satisfied by examples that he can not consider.

The fact that our Sommerfeld condition is fulfilled with $n_\infty$ and rather than the usual phase gradient ([21], [2]) seems to be compatible with a general feature of eikonal equation $|\nabla \varphi| = n_\infty^{1/2}(x)$. Its solution $\varphi(x)$ is given by the infimum of the energy functional $\int_0^T n_\infty^{1/2}(X(t)) \, dt$ for trajectories such that $X(0) = x$, $X(T) = 0$ and $|\dot{X}(t)| \leq 1$. Therefore, along the radial lines corresponding to a minimum of $n_\infty$, the minimum energy consists in staying on that line and one always has a radial gradient $\nabla \varphi = n_\infty^{1/2}(|x_1|) \frac{x_1}{|x_1|}$. This leads to conjecture that, coming back to our energy estimate (1.15), the energy in fact concentrates on the minima of $n_\infty$. This will be studied elsewhere.

We would like to make a few comments about our assumptions (1.8)-(1.10). The first one can be weaken a little bit around a ball centered at the origin. There we could write $n = n_1 + n_2$ with $n_1$ bounded and $n_2$ small and a good weight for Sobolev’s inequality and such that the unique continuation argument is true (i.e. $n_2 \in L^{d/2}$ if $d > 2$).

Assumptions (1.9) and (1.10) are rather natural if one keeps in mind the possibility of the existence of wave guides. In the appendix we give a simple example which implies that (1.9)-(1.10) can not be relaxed.

Acknowledgements We want to thank T. Hoffmann-Ostenhoff and G. Barles for enlightening conversations.

2 Proof of Theorem 1.3

The proof consists in using the basic inequality (4.3) (see the appendix) with a test function that carries information on the behavior of $n(x)$ at infinity.
We choose

\[ \Psi(x) = q\left(\frac{|x|}{R}\right) n_\infty\left(\frac{x}{|x|}\right) \]

for some nondecreasing smooth truncation function \( q(r) = 0 \) for \( r \leq 1 \) and \( q(r) = r \) for \( r \geq 2 \).

With this choice, we show that the only new information (compared to Theorem 1.2) in (4.3), is carried by the term

\[ \int_{\mathbb{R}^d} \nabla n(x) \cdot \nabla \Psi(x) \left| u(x) \right|^2 \, dx \]

the other terms being bounded thanks to (1.14).

As a first step, we consider that term and use the notation \( q = q\left(\frac{|x|}{R}\right) \),

\begin{align*}
\int_{\mathbb{R}^d} \nabla n(x) \cdot \nabla \Psi(x) \left| u(x) \right|^2 \, dx &= \int_{\mathbb{R}^d} q \left| \frac{\partial}{\partial \omega}\right. n_\infty\left(\frac{x}{|x|}\right) \left| \frac{u(x)}{|x|} \right|^2 \, dx \\
+ \int_{\mathbb{R}^d} q \left(\frac{\partial}{\partial r}\right) n(x) \left| u(x) \right|^2 \, dx + \int_{\mathbb{R}^d} q \left(\frac{\partial}{\partial \omega}\right) \left(r - n_\infty\left(\frac{x}{|x|}\right)\right) \frac{\partial}{\partial \omega} n_\infty \left| \frac{u(x)}{|x|} \right|^2 \, dx.
\end{align*}

The first term in the right hand side gives the control we look for. As for the second term, we lower bound it (we refer to [19] for this) by

\[ -\frac{C}{R} \beta \left\| n_\infty \right\|_{L^\infty} \left\| n^{1/2} u \right\|_{L^2(n)}. \]

Let us consider first condition (1.11). Then the last term (2.1) is bounded below as follows. After integration by parts, it is also given by

\begin{align*}
-\Re \int_{\mathbb{R}^d} q \left( n(x) - n_\infty\left(\frac{x}{|x|}\right)\right) \left(D^2 n_\infty \left| u(x) \right|^2 + 2 \frac{\partial}{\partial \omega} n_\infty \frac{\partial}{\partial \omega} u(x) \, \bar{u}\right) \, dx \\
\geq -\tilde{\Gamma} \left[ \left\| n_\infty \right\|_{C^2} \right] \int_{\mathbb{R}^d} q \left| n\left(\frac{u(x)}{|x|}\right) \right|^2 \, dx + \left\| n \right\|_{L^\infty} \int_{\mathbb{R}^d} \frac{q \left| \frac{\partial}{\partial \omega} n_\infty \left| \nabla u(x) \right| \right| u \right| \, dx \\
\geq -\frac{\tilde{\Gamma}}{R^2} \left\| n_\infty \right\|_{C^2} \left\| n^{1/2} u \right\|^2_{L^2} - \frac{1}{2} \int_{\mathbb{R}^d} q \left| \frac{\partial}{\partial \omega} n_\infty \right|^2 \left| \frac{u(x)}{|x|} \right|^2 \, dx \\
\geq -\frac{\tilde{\Gamma}^2}{R} \left\| n \right\|^2_{L^\infty} \int_{|x| \geq R} \frac{\left| \nabla u(x) \right|^2}{|x|} \, dx.
\end{align*}
Let us assume now (1.12), then the last term of (2.1) is bounded below by
\[-\bar{\lambda} \int_{\mathbb{R}^d} q \left| \frac{\partial}{\partial \omega} n_\infty \right|^2 \frac{|u(x)|^2}{|x|^2} \, dx - C n_0 \frac{R}{R^8} \|n^{1/2}u\|^2_{R(n)}.
\]

As a conclusion of this first step we have obtained, choosing \(R\) larger than 1 and than \(R(n)\),
\[
\int_{\mathbb{R}^d} q \left( \frac{|x|}{R} \right) \left| \frac{\partial}{\partial \omega} n_\infty \left( \frac{x}{|x|} \right) \right|^2 \frac{|u(x)|^2}{|x|^2} \, dx
\leq C_1 \int_{\mathbb{R}^d} \nabla n(x) \cdot \nabla \Psi(x) \, |u(x)|^2 \, dx + C_2 \, M^2.
\] (2.2)

The second step consists in estimating the first term in the right hand side of this inequality thanks to the basic equality (4.3). We now provide a control on all the remaining terms under the assumption \(R > 1\).

We begin with computing
\[
\nabla \tilde{u}(x) \cdot D^2 \Psi(x) \cdot \nabla u(x) = \frac{q'}{R \, |x|} n_\infty |\nabla \tau u|^2 + \frac{q''}{R^2} n_\infty \left| \frac{\partial}{\partial \tau} u \right|^2
\]
\[+ 2 \left( \frac{q'}{R \, |x|} - \frac{q}{|x|^2} \right) \Re \left[ \frac{\partial}{\partial \tau} \tilde{u} \cdot \frac{\partial}{\partial \omega} n_\infty \cdot \nabla \tau u \right] + \frac{q}{|x|^2} \nabla \tau \tilde{u} \cdot D_\omega n_\infty \cdot \nabla \tau u. \] (2.3)

And because the terms \(\frac{q'}{R \, |x|} - \frac{q}{|x|^2}\) and \(q''\) are supported in the ball \(|x| \leq R\), we see that all the terms in the corresponding integral are exactly controlled as we did it in the first step by \(C \|n_\infty\|_{C^2} M/R\).

Next, we consider the term
\[
\int_{\mathbb{R}^d} \Delta^2 \Psi(x) |u(x)|^2 = -2 \Re \int_{\mathbb{R}^d} \nabla \Delta \Psi \cdot \nabla u \tilde{u}
\leq 2 \int_{\mathbb{R}^d} \frac{|\nabla \Delta \Psi|}{n^{1/2}} |\nabla u| |n^{1/2} u|.
\]

But we have
\[
|\nabla \Delta \Psi(x)| \leq C \|n_\infty\|_{C^3} \left[ \frac{q}{|x|^3} + \frac{q'}{R \, |x|^2} \right].
\]

Henceforth this term is again under the same control as in the first step. Since \(n\) does not appear here, we only have to assume \(R\) large enough and obtain a control by \(C \|n^{1/2}_n\|_{C^3} M/R\).
For the right hand side terms containing \( f \), they can be treated exactly as above and are respectively upper bounded by 
\[
C \| n_{\infty} \|_{C^1} N_{R(n)}(f) \| \nabla u \|_{R(n)},
\]
and by 
\[
C \frac{n_{\infty}}{n_0^{1/2}} \| C^1 N_{R(n)}(f) \| n^{1/2} \| u \|_{R(n)}.
\]

The last term in \( \varepsilon \), following [19], can be upper bounded by 
\[
C \frac{n_{\infty}}{n_0^{1/2}} \| C^1 (\varepsilon + \| n \|_{L^\infty}) N_{R/n}(f/n^{1/2}) \| n^{1/2} \| u \|_{R}.
\]

Adding together the different upper bounds this completes the proof of Theorem 1.3.

3 Proof of Theorems 1.5 and 1.2, and of Proposition 1.4

A first consequence of Theorem 1.3 is the derivation of Sommerfeld condition at infinity in a weak form and at the \( \varepsilon \) level. It is the following result.

**Theorem 3.1** For dimensions \( d \geq 2 \), assume (1.8)-(1.10). Then the solution to the Helmholtz equation (1.1) satisfies, for \( R \geq R(n) \) and \( R(n) \) large enough according to (1.10),

\[
\frac{1}{R} \int_{R(n) \leq |x| \leq R} |\nabla u - i \frac{x}{|x|} n_{\infty}^{1/2} u|^2 \, dx + \varepsilon \int_{R(n) \leq |x| \leq R} \frac{\kappa_R}{n_{\infty}^{1/2}} |\nabla u - i \frac{x}{|x|} n_{\infty}^{1/2} u|^2 \, dx
\]

\[
\leq C \left[ M [N_{R(n)}(\kappa_R f) + \frac{1}{R} n_0^{1/2} N_{R(n)}(f) + \frac{1}{R} N_{R(n)}(f/n^{1/2})] \right. \\
+ M^2 \left( \sum_{2^j > R(n)} \beta_j \kappa_R(2^j) + C \varepsilon \right) \\
+ C \frac{1}{R^{d/2}} [M^2 + M N_{R(n)}(f/n^{1/2})] + C \left( \int_{|x| \geq R(n)} \frac{\kappa_R}{|x|} |\nabla x u|^2 \, dx \right)^{1/2} \bar{M},
\]

with \( \kappa_R(x) = \min\{|x|/R, 1\} \), a function that vanishes as \( R \to \infty \), and \( M, \bar{M} \) defined in Theorems 1.2 and 1.3.

**Proof of Theorem 3.1.** We shall do the proof for \( d > 2 \). Recall that if \( d = 2 \) the difficulty comes from the estimates close to the origin. In the proof that follows the multipliers vanish in a sufficiently big ball of radius \( R(n) \).
Thanks to assumption (1.10), we can always assume that outside of this ball 
\[ n(x) > \frac{2n}{\pi}. \] This observation and the arguments in [19] section 5 are enough
to extend the proof to the case \( d = 2 \).

We will get the Sommerfeld condition using the three estimates in the
appendix. We derive the calculations in three steps and build successively
the full form of Sommerfeld’s term at \( \varepsilon \) level.

As a first step we follow the procedure given in [19], but we need a minor
modification due to the weaker condition (1.9). We choose a cut off function
\( \theta \) with \( \theta(r) = 0 \) if \( r < 1/2 \), \( \theta(r) = 1 \) if \( |x| > 1 \), and \( \theta' \geq 0 \) for all \( r \). Then
define \( \theta_{R(n)}(x) = \theta(\frac{x}{R(n)}) \). Now for \( R > R(n) \), add up (4.1) and (4.3) with

\[
\nabla \Psi_R = \begin{cases} \frac{2}{R} \theta_{R(n)}(x) & \text{for } |x| \leq R, \\ \frac{2}{R} & \text{for } |x| \geq R. \end{cases} \tag{3.2}
\]

\[
\psi_R = \begin{cases} \frac{2}{R} \theta_{R(n)}(x) & \text{for } |x| \leq R, \\ 0 & \text{for } |x| \geq R. \end{cases} \tag{3.3}
\]

This combination is chosen to provide, using explicitly that \( \theta' \) is nonnegative, the
inequality

\[
\frac{1}{R} \int_{R(n)<|x|<R} [\nabla u]^2 + n|u|^2 - 2\varepsilon \mathcal{W} \int_{\mathbb{R}^d} \nabla \Psi_R \cdot \nabla u \, \bar{u} - 2 \int_{\mathbb{R}^d} \nabla \Psi_R \cdot \nabla n \, |u|^2 + \left[ \int_{\mathbb{R}^d} 2f(x) \nabla \Psi_R \cdot \nabla \bar{u} \right] + \int_{|x|\leq R} \frac{C}{R} |f(x)\bar{u}| + \int_{|x|> R} \frac{d-1}{|x|} |f(x)\bar{u}| \bigg] \\
+ C_0 \frac{1}{R (R(n))^2} \int_{R(n)/2<|x|<R(n)} |u|^2. \tag{3.4}
\]

The term involving \( \nabla n \) gives

\[
2 \int_{\mathbb{R}^d} (\nabla \Psi_R \cdot \nabla n) \, |u|^2 \leq \left( \sum_{2^j > R(n)} \beta_j \min \left( \frac{2^j}{R}, 1 \right) \right) \|u\|^2_{R(n)} \leq M^2 \sum_{2^j > R(n)} \beta_j \min \left( \frac{2^j}{R}, 1 \right)
\]

The terms containing \( f \) are not dangerous and can be upper bounded by

\[
C_{R(n)} \|u\|_{R(n)} \frac{N(f)}{R} + C_{R(n)} \|\nabla u\|_{R(n)} N(\min \{ \frac{|x|}{R}, 1 \} f).
\]

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As a second step, the appearance of the $\varepsilon$ term motivates to add to the above inequality the identity (4.2) with the choice of a test function

$$\psi_R(x) = \begin{cases} 2n^{1/2}\frac{1}{R} \theta_R(x) & \text{for } |x| \leq R, \\ 2n^{1/2} & \text{for } |x| \geq R, \end{cases}$$

Using the first step, (1.15) and the lower bound for $n_\infty$, we get the inequality

$$\varepsilon \int_{\mathbb{R}^d} \psi_R(x) |u|^2 + \frac{1}{R} \int_{|x|<|x'|<R} \nabla u - i \frac{x}{|x|} n_\infty^{1/2} u^2 \leq$$

$$C_{R(n)} M [N(\kappa R f) + \frac{N(f)}{Rn_0^{1/2}}] + \frac{1}{R} \int_{|x|<|x'|<R} (n_\infty - n) |u|^2$$

$$-2\varepsilon \text{Im} \int_{|x| \leq R} \nabla \psi_R \cdot \nabla u \bar{u} + \frac{C}{R} \int_{|x|<|x'|<R(n)} (|\nabla u|^2 + |u|^2)$$

$$+ \left( \sum_{j \in \mathbb{Z}} \beta_j \min \left( \frac{2j}{R}, 1 \right) \right) \|u\|_{L^2(R(n))}^2$$

$$+ \frac{C}{n_0^{1/2}} \left( \int_{|x| \geq R(n)} \kappa_R \frac{\nabla u \nabla u}{|x|} dx \right)^{1/2} \left( \int_{|x| \geq R(n)} 1_{n_\infty} \frac{\nabla u \nabla u}{|x|} dx \right)^{1/2}. \quad (3.5)$$

As a third step, we subtract to the above inequality, the identity (4.1) multiplied by $\varepsilon$ with the choice of a test function

$$\varphi_R(x) = \begin{cases} \frac{1}{R} \theta_R(x) & \text{for } |x| \leq R, \\ \frac{1}{n_\infty^{1/2}} & \text{for } |x| \geq R. \end{cases}$$

We obtain, after using the two $\varepsilon$ terms in (3.5), a new square

$$\frac{1}{R} \int_{|x|<|x'|<R} \nabla u - i \frac{x}{|x|} n_\infty^{1/2} u^2 + \varepsilon \int_{\mathbb{R}^d} \varphi_R |\nabla u - i \frac{x}{|x|} n_\infty^{1/2} u|^2 \leq$$

$$C_{R(n)} M [N(\kappa R f) + \frac{N(f)}{Rn_0^{1/2}}] + \frac{1}{R} \int_{|x|<|x'|<R} (n_\infty - n) |u|^2$$

$$+ \varepsilon \int_{\mathbb{R}^d} \varphi_R (n_\infty - n) |u|^2 + \frac{\varepsilon}{2} \int_{\mathbb{R}^d} \Delta \varphi_R |u|^2 + \frac{\varepsilon}{n_0} N(f) \|u\|_{L^2(R(n))}^2$$

$$+ \left( \sum_{j \in \mathbb{Z}} \beta_j \min \left( \frac{2j}{R}, 1 \right) \right) \|u\|_{L^2(R(n))}^2 + \frac{C}{R} \int_{|x|<|x'|<R(n)} (|\nabla u|^2 + |u|^2) \quad (3.5)$$
\[ + \frac{C}{n_0^{1/2}} \left( \int_{|x| \geq 1} \kappa_R \frac{|
abla u|^2}{|x|} \, dx \right)^{1/2} \bar{M}. \]

To complete the estimate, we define \( \tilde{\varphi}_R = \varphi_R / \theta_R(n) \) and after computing \( \nabla \tilde{\varphi}_R \),

\[
\frac{\varepsilon}{2} \int_{\mathbb{R}^d} \Delta \varphi_R |u|^2 \leq -\frac{\varepsilon}{2} \int_{\mathbb{R}^d} \theta_R(n) \nabla \varphi_R \nabla |u|^2 + \frac{C\varepsilon}{n_0} \frac{R(n)}{R(n)} \int_{R(n)/2 < |x| < R(n)} (|n| |u|^2 + |\nabla u|^2)
\]

\[
\leq \frac{\varepsilon}{n_0} \left\| \nabla u \right\|_{R(n)} \left\| n \right\|^{1/2} \left\| u \right\|_{R(n)} + \varepsilon \frac{n_\infty \| C \right\|_{R} \frac{n_0}{n_0^{3/2}} \int_{|x| > R(n)/2} \left\| \nabla u \right\| \left\| u \right\| + \frac{C\varepsilon}{n_0} M^2
\]

\[
\leq \frac{C\varepsilon}{n_0} M^2 + C \frac{n_\infty \| C \right\|_{R} \frac{n_0}{n_0^{2}} M \left( \varepsilon + \| n \|_{L\infty} \right) N(f \frac{1}{n_1/2}),
\]

(see the treatment of the term \( 2\varepsilon \int_{R(n)} \nabla \Psi_{R(n)} \cdot \nabla u \bar{u} \) in [19]).

Similarly, we have

\[
\varepsilon \int_{\mathbb{R}^d} \varphi_R \left| n - n_\infty \right| |u|^2 \leq \left\| \frac{n}{n_0^{1/2}} \right\|_{L\infty} \frac{\varepsilon}{R(d)} \int_{\mathbb{R}^d} |u|^2
\]

\[
\leq \left\| \frac{n}{n_0^{1/2}} \right\|_{L\infty} \frac{\Gamma \left( \frac{f}{n_1/2} \right)}{R(d)} N_R(n_0)
\]

It remains now, using (1.10), to upper bound the term

\[
\frac{1}{R} \int_{R(n)/2 < |x| < R} \left| n - n_\infty \right| |u|^2 \leq \frac{C}{R} \int_{R(n)/2 < |x| < R} |u|^2 + \frac{\Gamma}{R^{1+\delta/2}} \int_{R^{1-\delta/2} < |x| < R} n |u|^2
\]

\[
\leq C \frac{1 + \Gamma}{R^{\delta/2}} M^2.
\]

Therefore the proof of Theorem 3.1 is over.

**Proof of Theorem 1.5.** Theorem 1.5 follows, passing to the limit \( \varepsilon \to 0 \) by a combination of Theorems 1.2, 1.3 and 3.1.

**Proof of Theorem 1.2.** It remains to prove the case (iii) of Theorem 1.2. We now choose \( R(n) \) again large enough such that

\[
2 \sum_{j: \varphi > R(n)} \sup_{c(j)} \left( \frac{x \cdot \nabla n(x)}{n(x)} - \frac{1}{4} \right) < 1/4.
\]

(3.6)
As a conclusion of the above proof, we have obtained the desired bound as long as
\[
\sup_{\varepsilon} \int_{|x|<R(n)} (|u|^2 + |\nabla u|^2) \leq C N^2_{R(n)}(f). \tag{3.7}
\]
Assume on the contrary that (3.7) does not hold. Then there are \(\varepsilon_m, u_m,\) and \(f_m\) with \(\lim_{m} N(f_m) = 0, \Delta u_m + (n + i\varepsilon_m)u_m = f_m\) and
\[
\int_{|x|<R(n)} |u_m|^2 + |\nabla u_m|^2 = 1. \tag{3.8}
\]
Taking a subsequence we get \(u\) and \(\varepsilon_0\) such that
\[
\Delta u + (n + i\varepsilon_0)u = 0, \quad \int_{|x|<R(n)} |u|^2 > 0,
\]
otherwise (4.1) contradicts (3.8). If \(\varepsilon_0 > 0\) this is false because \(\varepsilon_0 \int |u|^2 = 0.\) If \(\varepsilon_0 = 0\) then we can use (3.8) and the three steps already obtained in the proof of Theorem 3.1. We first deduce \(L^2_{pol}\) bounds using [19] and hiding the term \(\nabla \Psi_{R(n)} \cdot \nabla n|u|^2\) in the left hand side thanks to (3.6). Next, we may pass to the limit first in \(\varepsilon_m\) and then in \(R\) we get that \(u\) satisfies Sommerfeld radiation condition. Therefore Theorem 1.1 applies and we conclude that \(u = 0\) which is a contradiction. Therefore (3.7) holds and Theorem 1.2 is proved.

**Proof of Proposition 1.4.** We multiply in the equation \(\Delta u + nu = f\) by \(\bar{u}\) and integrate by parts in the ball of radius \(R\). Then we use Sommerfeld radiation condition to get
\[
\lim_{R} \int_{S_R} n_{\infty} |u|^2 = \mathcal{I}m \int_{\mathbb{R}^d} f(x)\bar{u}(x),
\]
which is in fact a stronger statement than that in Proposition 1.4.

4 Appendix

**A.1.- Basic identities.** Our proof combines three basic inequalities that have been used throughout this paper and that we state here without proof (see [19] for a proof). For real valued functions \(\Psi, \varphi, \psi \in \mathcal{S}(\mathbb{R}^d),\) we have
\[
-\int_{\mathbb{R}^d} \varphi(x)|\nabla u(x)|^2 + \frac{1}{2} \int_{\mathbb{R}^d} \Delta \varphi(x)|u(x)|^2 + \int_{\mathbb{R}^d} \varphi(x)n(x)|u(x)|^2
\]
We choose $A/2$ of the proof of Theorem 1.1. Passing to the limit in $\mathbb{R}^d$ of the proof of Theorem 1.3, (3.2) and (3.3), but instead of integrating in $\mathbb{R}^d$, we obtain

$$
\varepsilon \int_{\mathbb{R}^d} \psi(x)|u(x)|^2 - \mathcal{I}m \int_{\mathbb{R}^d} \nabla \psi(x) \cdot \nabla u(x) \bar{u}(x) = \mathcal{I}m \int_{\mathbb{R}^d} f(x) \bar{u}(x) \psi(x),
$$

(4.2)

$$
\int_{\mathbb{R}^d} \left[ \nabla \bar{u}(x) \cdot D^2 \Psi(x) \cdot \nabla u(x) - \frac{1}{4} \Delta^2 \Psi(x) |u(x)|^2 + \frac{1}{2} \nabla n(x) \cdot \nabla \Psi(x) |u(x)|^2 \right] = -\mathcal{R}e \int_{\mathbb{R}^d} f(x) \left( \nabla \Psi(x) \cdot \nabla \bar{u}(x) + \frac{1}{2} \Delta \Psi(x) \bar{u}(x) \right) - \varepsilon \mathcal{I}m \int_{\mathbb{R}^d} \nabla \Psi(x) \cdot \nabla \bar{u}(x) u(x).
$$

(4.3)

**A.2.- Proof of Theorem 1.1.** Again we shall reduce ourselves to $d > 2$. We choose $R(n)$ as in (3.6) and use the same multipliers as in the beginning of the proof of Theorem 1.3, (3.2) and (3.3), but instead of integrating in the full space we just do it on $|x| < R_1$ for some $R_1 > R$. We do this for all $R$ with $R(n) < R < R_1$ and then take the supremum in $R$ in both sides to hide the term on $(x \cdot \nabla n)_-$ as done in [19]. Then after computing the corresponding boundary terms we obtain ($\varepsilon = 0$):

$$
\sup_{R(n) < R < R_1} \frac{1}{R} \int_{|x| < R} |\nabla u|^2 + n|u|^2 
\leq C\|n\|_\infty \int_{S_{R_1}} |\nabla u|^2 + |u|^2 + C \sup_{R(n) < R < R_1} \frac{1}{R(R(n))^2} \int_{|x| < R(n)} |u|^2.
$$

Passing to the limit in $R_1$ and using (1.13) we get that,

$$
\sup_{R(n) < R} \frac{1}{R} \int_{|x| < R} |\nabla u|^2 + n|u|^2 \leq C \frac{1}{R(n)^3} \int_{|x| < R(n)} |u|^2.
$$

Notice that the above argument can be repeated choosing as $R(n)$ any $R > R(n)$ to obtain

$$
\int_{|x| < 2R} |\nabla u|^2 + n|u|^2 \leq C \frac{1}{R^2} \int_{|x| < R} |u|^2.
$$

Then

$$
\int_{|x| < R} |\nabla u|^2 + n|u|^2 < \sum_j \int_{2^j |x| < 2^{j+1}} |\nabla u|^2 + \frac{n}{2} |u|^2.
$$
\[ < C \sum_{j} 2^{-j} \sup_{R(n) < R} \frac{1}{R} \int_{R(n) < |x| < R} |u|^2 \leq C^2 \frac{1}{(R(n))^{2}} \int_{R(n)/2 < |x| < R(n)} |u|^2. \]

Hence
\[ \int_{R(n) < |x|} \left[ |\nabla u|^2 + n|u|^2 \right] < C \frac{1}{(R(n))^{2}} \int_{R(n)/2 < |x| < R(n)} |u|^2. \quad (4.4) \]

In fact in the above inequality we can replace \( R(n) \) by any \( R \) such that \( (3.6) \) holds.

Our next step is to prove that if \( j = 0, 1, 2, ... \), and \( R > R(n) \), \( \tau = 2^j \), then
\[ \int_{2^j R < |x|} \left[ |\nabla u|^2 + |u|^2 \right] |x|^\tau < \frac{(4C)^{j+1}}{R^2} \int_{R/2 < |x| < R} |u|^2. \quad (4.5) \]

We do it by induction. We have using \((4.1)\) and \((4.3)\) again,
\[ \int_{2^j R < |x|} \left[ |\nabla u|^2 + |u|^2 \right] |x|^\tau < \frac{(4C)^{j+1}}{R^2} \int_{R/2 < |x| < R} |u|^2. \]

Therefore we have for all \( \tau = 2^j \), that
\[ \int_{R < |x|} \left[ |\nabla u|^2 + |u|^2 \right] |x|^\tau < C_\tau. \]

By assuming a stronger condition on the decay of \( n - n_\infty \) like \((1.11)\), we could use the well known Carleman estimate proved in \([12]\) p. 264. Notice that in the proof of that result the eigenvalue \( \lambda \) can be changed into the function \( \lambda(x/|x|) \) and the argument works the same. In order to avoid this extra assumption we have to work a little bit more. We follow the arguments in \([20]\) p. 226, \([17]\), and \([23]\).

We use again for \( R > R(n) \) the identities \((4.1)\) and \((4.3)\), this time with the multipliers \( \nabla \Psi(x) = \theta_R |x|^\tau x \) and \( \varphi(x) = (1/2)\theta_R |x|^\tau \). Adding both identities and after some manipulation we get
\[ \int [ |\nabla u|^2 + |u|^2 ] |x|^\tau \theta_R \]
\[ < C \tau^3 \int [ |\nabla u|^2 + |u|^2 ] |x|^{-2} \theta_R + \frac{C \tau^2}{R^2} \int_{R/2 < |x| < R} |u|^2 |x|^{-2}. \]
Then we make $\tau = \lambda m$, with $0 < \lambda < 2/3$, multiply both sides of the above inequality by $t^m_m$, $t > 0$. Then adding up we get
\[
\int \sum_{j=3}^{M} |x|^\lambda \frac{t^m_m}{m!} \left(|\nabla u|^2 + |u|^2 \right) \theta_R < C \int \sum_{j=3}^{M} |x|^{(\lambda m - 2)} \frac{t^m_m}{m!} \left(|\nabla u|^2 + |u|^2 \right) \theta_R + C \int_{R/2 < |x| < R} \sum_{j=3}^{M} |x|^{(\lambda m - 2)} \frac{t^m_m}{m!} |u|^2.
\]
Hence using that $0 < \lambda < 2/3$, taking $R$ large enough depending on $t$ and passing to the limit in $N$, we get
\[
\int_{|x| < R} e^{t|x|^\lambda} \left(|\nabla u|^2 + |u|^2 \right) < C(t).
\]
In order to conclude we define $v = u \exp \left(\frac{\tau x}{2} \right)$. Then $v$ solves
\[
\Delta v + \left(n + h(|x|)\right)v - t\lambda |x|^\lambda x \cdot \nabla v = 0, \quad (4.6)
\]
with $h(r) = c_1 t^2 r^{2(\lambda - 1)} - c_2 \lambda^2 r^{\lambda - 2}$, $c_1 = \lambda^2 / 4 > 0$, and $|h'(r)| < 2(\lambda - 1)h(r)/r + c_3 t r^{\lambda - 3}$. Hence if $t > 1$ and $r$ is sufficiently large (independent of $t$) we have $h(r) > 0$ and if $1/2 < \lambda < 2/3$ we also have that $|h'(r)| < h(r)/r$. Therefore we can use identities (4.1)-(4.3) to equation (4.6) with $\nabla \Psi(x) = \theta_R x$ and $\varphi(x) = (1/2)\theta_R$. Adding both identities and after some manipulation we obtain with $1/2 < \lambda < 2/3$ and for $t > 1$ (see [23] where all the details are included)
\[
\int_{R < |x|} \left(|\nabla v|^2 + c_4 \left(n + h(|x|)\right)|v|^2 + 2\lambda t |x|^\lambda |\partial_v|^2 \right) < C \int_{R/2 < |x| < R} |v|^2,
\]
with $c_4 > 0$. Writing the above inequality in terms of $u$ and passing to the limit in $t$ we conclude $u = 0$ if $|x| > 2R$. Then a unique continuation argument can be used, see for example [14], to conclude that $u = 0$.

Finally assume that Sommerfeld boundary condition (1.2) holds. Solutions of $\Delta u + nu = 0$ satisfy (just multiply by $\bar{u}$ and integrate in a ball of radius $R$) $\mathcal{I}m \int_{|x|=R} \bar{u} \partial_n u \, d\sigma = 0$. Therefore
\[
\mathcal{I}m \int_{|x|=R} \bar{u} |\partial_n u - in^{1/2} u|^2 \, d\sigma = \int_{|x|=R} n^{1/2} |u|^2 \, d\sigma,
\]
and from (1.2) and Cauchy-Schwarz inequality we get (1.13). The proof of Theorem 1.1 is complete.

**A.3.- Some Examples.** In this section we shall give examples of indices $n$ which satisfy

$$
\frac{(x \cdot \nabla n(x)) -}{n(x)} := \tilde{\beta} < +\infty
$$

(4.7)

with $\tilde{\beta}$ as small as wanted and such that Theorem 1.5 does not hold true. Condition (4.7) is weaker than (1.9) and appears naturally in the study of the absence of embedded eigenvalues in the continuous spectrum for the Schrödinger operator $\Delta + n$. Recall at this respect the well known example due to Von Neumann and Wigner of a potential which satisfies (4.7) for $\tilde{\beta}$ large enough and has an embedded eigenvalue, see [20] p. 233.

We give examples of wave guides which satisfy (4.7) but with a scaling which does not leave invariant that condition, and therefore there is no possible $\tilde{\beta}$ good for all of them. For these examples condition (1.9) is not fulfilled either.

Define $Q$ as the unique positive solution with $Q(\pm \infty) = 0$ of

$$
Q'' + \left( Q^2 - \frac{1}{2} \right) Q = 0, \quad y \in \mathcal{R}.
$$

That is to say $Q(y) = sech \left( y/\sqrt{2} \right)$. Also for $\lambda > 0$ take $Q_{\lambda}(y) = Q(\lambda y)$, which solves

$$
Q_{\lambda}'' + \lambda^2 \left( Q_{\lambda}^2 - \frac{1}{2} \right) Q_{\lambda} = 0.
$$

Set $\theta \in \mathcal{C}^\infty$ a bump function around the origin with $\theta(x) = 0$ if $|x| < 1$ and $\theta(x) = 1$ if $|x| > 2$. Then call

$$
u_{\lambda}^\varepsilon(x, y) = Q_{\lambda}(y) \theta(x)e^{i\sqrt{1+i\varepsilon}|x|},$$

with $0 < \varepsilon < 1$ and $\text{Im} \sqrt{1+i\varepsilon} > 0$. Then $\nu_{\lambda}^\varepsilon$ solves

$$
\Delta \nu_{\lambda}^\varepsilon + \left( n_{\lambda} + i\varepsilon \right) \nu_{\lambda}^\varepsilon = f_\varepsilon \quad (x, y) \in \mathcal{R}^2,
$$

with

$$
n_{\lambda} := \tilde{n}_{\lambda} + 1 - \frac{1}{2} \lambda^2; \quad \tilde{n}_{\lambda} := -\frac{Q_{\lambda}''}{Q_{\lambda}} = \lambda^2 Q_{\lambda}^2(\lambda y),
$$

and

$$
f_\varepsilon(x, y) = \left( 2i \sqrt{1+i\varepsilon} \text{sgn}(x) \theta'(x) + \theta''(x) \right) Q_{\lambda}(y)e^{i\sqrt{1+i\varepsilon}|x|}.
$$
Now it is straightforward to check \( N(f_\varepsilon) < \infty \). Passing to the limit in \( \varepsilon \) we get that if \( u_\lambda = \lim_{\varepsilon \to 0^+} u_\lambda^\varepsilon = Q_\lambda(y) \theta(x)e^{i|x|} \), then
\[
\Delta u_\lambda + n_\lambda u_\lambda = f(x, y) := \left( \text{sign}(x) \theta'(x) + \theta''(x) \right) Q_\lambda(y)e^{i|x|}.
\]

On the other hand if \( 0 < \lambda < 1/2 \) and
\[
n_\lambda^\lambda = 1 - \frac{\lambda^2}{2},
\]
we get
\[
n_\lambda - n_\lambda^\lambda = \tilde{n}_\lambda = \lambda^2 Q^2(\lambda y).
\]

Therefore
\[
\left( \frac{\partial}{\partial r} \tilde{n}_\lambda \right)_r \leq C \frac{\lambda^2}{r} \sup \left| (Q^2)' \right|.
\]

Hence
\[
\frac{\left( \frac{\partial}{\partial r} \tilde{n}_\lambda \right)_r}{n_\lambda} < 2C\lambda^2 \quad r > 1,
\]
and can be made as small as wanted. On the other hand straightforward computations prove that \( 1/R \int_{|x| \leq R} |u_\lambda|^2 < \infty \) and that there is \( c_0 > 0 \) such that
\[
\frac{1}{R} \int_{|x| \leq R} \left| \frac{\partial}{\partial r} u_\lambda - i \sqrt{1 - \frac{\lambda^2}{2}} |u_\lambda| \right|^2 \geq c_0.
\]

References


