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Département de mathématiques et applications - École normale supérieure
45 rue d’Ulm 75230 PARIS Cedex 05
Tel : (33)(1) 01 44 32 30 00
E-mail : preprints@dma.ens.fr

* DMA, Ecole normale supérieure et Laboratoire de Probabilités et Modèles aléatoires, Université Paris VI.
E-mail : Gregory.Miermont@ens.fr
Self-similar fragmentations derived from the stable tree II: splitting at nodes

Grégory Miermont
DMA, ENS,
et Laboratoire de Probabilités et Modèles aléatoires,
Université Paris VI.
45, rue d’Ulm,
75230 Paris Cedex 05

Abstract

We study a fragmentation process of the so-called stable tree introduced by Duquesne and Le Gall, which consists in removing the nodes of the tree according to a certain procedure that makes the fragmentation self-similar. We give formulas for the semigroup and provide asymptotic results. It turns out that the dislocations involved in this fragmentation follow the same mechanism as when fragmenting the stable tree by removing the vertices located below a certain height, which has been studied in a companion paper by the author. Though these two fragmentations may look very different, the only difference is thus the speed at which fragments decay. We also give another representation of this fragmentation, using paths of Lévy processes, that echoes the two alternative constructions of the standard additive coalescent by fragmenting the Brownian continuum random tree or using Brownian paths, respectively due to Aldous-Pitman and Bertoin.

Key Words. Self-similar fragmentation, stable tree, stable processes.

A.M.S. Classification. 60J25, 60G52.

1 Introduction

The goal of this paper is to investigate a Markovian fragmentation of the so-called stable tree, which is a model of continuum random tree (CRT) depending on a parameter $\alpha \in (1, 2]$, that has been introduced by Duquesne and Le Gall [16]. When $\alpha = 2$ this is the Brownian CRT of Aldous [2]. In a companion paper [22], we already studied such a fragmentation process called $(F^-(t), t \geq 0)$ with values in

$$ S := \left\{ s = (s_1, s_2, \ldots) : s_1 \geq s_2 \geq \cdots \geq 0, \sum_{i=1}^{\infty} s_i \leq 1 \right\}, $$
The motivation of the present paper is to look for possible generalizations of such a result to the other stable trees: does there exist another way as $F^-$ to log the stable tree, that would induce a self-similar fragmentation with the same dislocation measure but positive index? The naïve approach of this problem would be to mimic the description of the Aldous-Pitman fragmentation. This approach fails because the Aldous-Pitman fragmentation, which uses a Poisson cutting along the skeleton of the Brownian CRT, is binary (when a fragment dislocates, it gives birth to exactly two fragments), and it is not difficult to see that it is also the case when trying to generalize it to other stable trees, because the cutpoints of the Poisson processes have zero chance to fall on branchpoints of the tree. When $\alpha = 2$ this is not a problem since the associated $F^-$ is also binary, a property it inherits from the fact that the Brownian CRT is a binary tree. But when $\alpha < 2$, the situation is completely different and the branchpoints of the stable tree all have infinite degree, which implies that for $F^-$, every dislocation involves infinitely many fragments.

### Proposition 1.

Let $(F(t), t \geq 0)$ be a self-similar fragmentation with index $\beta \geq 0$ and erosion coefficient $c = 0$. Then for every function $G$ that is continuous and null on a neighborhood of $(1, 0, \ldots)$ in $S$,

$$ t^{-1}E[G(F(t))] \underset{t \to 0}{\to} \nu(G). $$

See [22] (or Theorem 1 below, but let us not anticipate) for an explicit formula for the dislocation measure of $F^-$ when $\alpha \in (1, 2)$. When $\alpha = 2$, the fragmentation $F^-$ corresponds to a fragmentation of the Brownian CRT, or equivalently of the normalized Brownian excursion, that has been studied in [10]. It was shown that the self-similarity index of $F^-$ is $-1/2$, which agrees with the above statement, and the dislocation measure was given explicitly. It turned out that this measure also arose in another self-similar fragmentation of the Brownian CRT introduced by Aldous and Pitman [3], which is related to the so-called standard additive coalescent. This fragmentation has index $1/2$ and the same dislocation measure, up to a multiplicative constant (which can be set equal to 1 up to a linear time-change of the fragmentation).
With these heuristics, it is natural to look for a fragmentation of the stable tree that would cut only at the branchpoints of the tree, which we call “hubs” in the sequel, because each branchpoint of the stable tree has an infinite degree. It is not difficult to see that one should cut these hubs at different rates according to their “magnitude” to obtain a self-similarity property, because some of the hubs are somehow more “surrounded” by leaves than others. The correct notion is the following.

We denote by $\mathcal{T}$ the stable CRT, which is a random metric space with respect to a certain distance $d$, whose elements $v$ are called vertices. One of these vertices is distinguished and called the root. This space is a tree in the sense that for $v, w$ two vertices there is a unique non-self-crossing path $[[v, w]]$ from $v$ to $w$ in $\mathcal{T}$, whose length equals $d(v, w)$. For every $v \in \mathcal{T}$, call height of $v$ in $\mathcal{T}$ and denote by $\text{ht}(v)$ the distance of $v$ to the root. The leaves of $\mathcal{T}$ are those vertices that do not belong to the interior of any path leading from one vertex to another, and the skeleton of the tree is the set of non-leaf vertices. The branchpoints (hubs) are the vertices $b$ so that there exist $v \neq b, w \neq b$ such that $[[\text{root}, v]] \cap [[\text{root}, w]] = [[\text{root}, b]]$. Call $\mathcal{H}(\mathcal{T})$ the set of hubs of $\mathcal{T}$. With each realization of $\mathcal{T}$ is associated a probability measure $\mu$, called the mass measure, that is supported by $\mathcal{T}$ and that attributes zero mass to the skeleton. This measure allows to evaluate the magnitude of hubs as follows. For every $b \in \mathcal{H}(\mathcal{T})$, consider the fringe subtree $\mathcal{T}_b$ rooted at $b$, i.e. the subset $\{v \in \mathcal{T} : b \in [[\text{root}, v]]\}$. Then one can define the local time, or width of the hub $b$ as the limit

$$L(b) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mu\{v \in \mathcal{T}_b : d(v, b) < \varepsilon\}$$

which exists a.s. and is positive (see Proposition 3 below).

Now given a realization of $\mathcal{T}$ and for every $b \in \mathcal{H}(\mathcal{T})$ take a standard exponential random variable $e_b$, so that the variables $e_b$ are independent as $b$ varies (notice that $\mathcal{H}(\mathcal{T})$ is countable). For all $t \geq 0$ define an equivalence relation $\sim_t$ on $\mathcal{T}$ by saying that $v \sim_t w$ if and only if the path $[[v, w]]$ does not contain any hub $b$ for which $e_b < tL(b)$. Alternatively, following more closely the spirit of Aldous-Pitman’s fragmentation, we can also say that we consider Poisson point process $(b(t), t \geq 0)$ on the set of hubs with intensity $dt \otimes \sum_{b \in \mathcal{H}(\mathcal{T})} L(b)\delta_b(dv)$, and for each $t$ we let $v \sim_t w$ if and only if no atom of the Poisson process that has appeared before time $t$ belongs to the path $[[v, w]]$. We let $\mathcal{T}_1, \mathcal{T}_2, \ldots$ be the distinct equivalence classes for $\sim_t$, ranked according to the decreasing order of their $\mu$-masses (provided these are well-defined quantities). It is easy to see that these sets are trees (in the same sense as $\mathcal{T}$), and that the families $(\mathcal{T}_i^t, i \geq 1)$ are nested as $t$ varies, that is, for every $t' > t$ and $i \geq 1$, such that $\mathcal{T}_i^t \subset \mathcal{T}_j^{t'}$. If we let $F^+(t) = (\mu(\mathcal{T}_1^t), \mu(\mathcal{T}_2^t), \ldots)$, $F^+$ is thus a fragmentation process in the sense that $F^+(t')$ is obtained by splitting at random the elements of $F^+(t)$. We mention that the fragmentation $F^+$ is also considered and studied in the work in preparation [1], with independent methods.

Let

$$D_\alpha = \frac{\alpha(\alpha - 1) \Gamma\left(1 - \frac{1}{\alpha}\right)}{\Gamma(2 - \alpha)} = \frac{\alpha^2 \Gamma\left(2 - \frac{1}{\alpha}\right)}{\Gamma(2 - \alpha)}.$$  

We now state our main result, see the following section for definitions and properties of stable subordinators.
**Theorem 1.** The process $F^+$ is a self-similar fragmentation with index $1/\alpha \in (1/2, 1)$ and erosion coefficient $c = 0$. Its dislocation measure $\nu_\alpha$ is characterized by

$$\nu_\alpha(G) = D_\alpha E \left[ T_1 G(T_1^{-1} \Delta T_{[0,1]}) \right]$$

for any positive measurable function $G$, where $(T_x, 0 \leq x \leq 1)$ is a stable subordinator with index $1/\alpha$, characterized by the Laplace transform

$$E[\exp(-\lambda T_1)] = \exp(-\lambda^{1/\alpha}) \quad \lambda \geq 0,$$

and $\Delta T_{[0,1]}$ is the sequence of the jumps of $T$, ranked by decreasing order of magnitude.

Comparing this result with [22, Theorem 1], we see that the dislocation measure is the same as that of $F^-$, and our question admits a positive answer.

Let us now present a second motivation for studying the fragmentation $F^+$. As the rest of the paper will show, our proofs involve a lot the theory of Lévy processes, and compared with the study of $F^-$, which made a consequent place to combinatoric tree structures, the study of $F^+$ will be mainly “analytic”. The fact that Lévy processes may be involved in fragmentation processes is not new. According to [7] and [21], adding a drift to a certain class of Lévy processes allows to construct interesting fragmentations related to the entrance boundary of the stochastic additive coalescent. Here, rather than adding a drift, which by analogy between [4] and [7] amounts to cut the skeleton of a continuum random tree with a homogeneous Poisson process, we will perform a “removing the jumps” operation analog to our inhomogeneous cutting on the hubs of the tree.

Precisely, let $(X_s, s \geq 0)$ be the canonical process in the Skorokhod space $D([0, \infty))$ and let $P$ be the law of the stable Lévy process with index $\alpha \in (1, 2)$, upward jumps only, characterized by the Laplace exponent

$$E[\exp(-\lambda X_1)] = \exp(\lambda^\alpha).$$

As we will see in the following section, we may define the law $N^{(1)}$ of the excursion with unit duration of this process above its infimum process. Under this law, $X_s = 0$ for $s > 1$, so we let $\Delta X_{[0,1]}$ be the sequence of the jumps $\Delta X_s = X_s - X_{s-}$ for $s \in (0, 1]$, ranked in decreasing order of magnitude. Consider the following marking process on the jumps: conditionally on $X$, let $(\epsilon_s, s : \Delta X_s > 0)$ be a family of independent random variables with standard exponential distribution, indexed by the countable set of jump-times of $X$. For every $t \geq 0$ let

$$Z^{(t)}_s = \sum_{0 \leq u \leq s} \Delta X_u 1_{\{\epsilon_u < t \Delta X_u\}}.$$

That is, each jump with magnitude $\Delta$ is marked with probability $1 - \exp(-t \Delta)$ independently of the other jumps and consistently as $t$ varies, and $Z^{(t)}$ is the process that sums the marked jumps. We will see that $Z^{(t)}$ is finite a.s., so we may define $X^{(t)} = X - Z^{(t)}$ under $N^{(1)}$. Let

$$\Delta X^{(t)} = \inf_{0 \leq u \leq s} X^{(t)}_u, \quad 0 \leq s \leq 1,$$

and let $F^2(t)$ be the sequence of lengths of the constancy intervals of the process $\Delta X^{(t)}$, ranked in decreasing order.


**Theorem 2.** The process \((F^y(t), t \geq 0)\) has the same law as \((F^+(t), t \geq 0)\).

We organize the paper as follows. In Sect. 2 we recall some facts about Lévy processes, excursions, and conditioned subordinators that will be crucial for our study. In Sect. 3 we give the rigorous description of Duquesne and Le Gall’s Lévy trees, and rephrase the definition of \(F^+\) given above in terms of a partition of the unit interval associated to a certain marked excursion of a stable Lévy process. Sections 4 and 5 are then respectively dedicated to the study of \(F^+\) and \(F^y\). Asymptotic results are finally given concerning the behavior at small and large times, and that of the small fragments of \(F^+\) in Sect. 6.

## 2 Some facts about Lévy processes

### 2.1 Stable processes, inverse subordinators

Let \((X_s, s \geq 0)\) be the canonical process in the Skorokhod space \(\mathbb{D}([0, \infty))\) of càdlàg paths on \([0, \infty)\). We fix \(\alpha \in (1, 2)\). Let \(P\) be the law on \(\mathbb{D}([0, \infty))\) that makes \(X\) the spectrally positive stable process with index \(\alpha\), that is, \(X\) has independent and stationary increments under \(P\), it has only positive jumps, and its marginal law at some \((\text{and then all}) \ s > 0\) has Laplace transform given by the Lévy-Khintchine formula:

\[
E[e^{-\lambda X_s}] = \exp(s\lambda^\alpha) = \exp\left(t \int_0^\infty \frac{C_\alpha \, dx}{x^{1+\alpha}} (e^{-\lambda x} - 1 + \lambda x)\right), \quad \lambda \geq 0,
\]

(2)

where \(C_\alpha = \alpha(\alpha - 1)/\Gamma(2 - \alpha)\). A fundamental property of \(X\) under \(P\) is the *scaling property*

\[
\left(\frac{1}{\lambda^{1/\alpha}} X_{\lambda s}, s \geq 0\right) \overset{d}{=} (X_s, s \geq 0) \quad \text{for all } \lambda > 0.
\]

We let \((p_s(x), s > 0, x \in \mathbb{R})\) be the density with respect to Lebesgue measure of the law \(P(X_s \in dx)\), which is known to exist and to be jointly continuous in \(s\) and \(x\).

Denote by \(\underline{X}\) the infimum process of \(X\) defined by

\[
\underline{X}_s = \inf_{0 \leq u \leq s} X_u, \quad s \geq 0.
\]

Let \(T\) be the right-continuous inverse of the increasing process \(-\underline{X}\) defined by

\[
T_x = \inf\{s \geq 0 : \underline{X}_s < -x\}.
\]

Then it is known that under \(P\), \(T\) is a stable subordinator with index \(1/\alpha\), that is, an increasing Lévy process with Laplace exponent

\[
E[e^{-\lambda T_x}] = \exp(-x\lambda^{1/\alpha}) = \exp\left(-x \int_0^\infty \frac{c_\alpha \, dy}{y^{1+1/\alpha}} (1 - e^{-\lambda y})\right) \quad \text{for } \lambda, x \geq 0,
\]

where \(c_\alpha = (\alpha \Gamma(1 - 1/\alpha))^{-1}\). We denote by \((q_x(s), x, s > 0)\) the family of densities with respect to Lebesgue measure of the law \(P(T_x \in ds)\), by [6, Corollary VII.1.3] they are given by

\[
q_x(s) = \frac{x}{s} p_s(-x).
\]

(3)
We also introduce the notations \( P^s \) for the law of of the processes \( X \) under \( P \), killed at time \( s \), and \( P(-x, \infty) := P^T_x \) for the law of the process killed when it first hits \(-x\).

Let us now discuss the conditioned forms of distributions of jumps of subordinators. An easy way to obtain regular versions for these conditional laws is developed in [24, 25]. First, we define the size-biased permutation of the sequence \( \Delta T_{[0,x]} \) of the ranked jumps of \( T \) in the interval \([0, x]\) as follows. Write \( \Delta T_{[0,x]} = (\Delta_1(x), \Delta_2(x), \ldots) \) with \( \Delta_1(x) \geq \Delta_2(x) \geq \ldots \), and recall that \( T_x = \sum_i \Delta_i(x) \). Then let \( 1^* \) be a r.v. such that

\[
P(1^* = i | \Delta T_{[0,x]}) = \frac{\Delta_i(x)}{T_x}
\]

for all \( i \geq 1 \), and set \( \Delta_i^*(x) = \Delta_i(1^*) \). Recursively, let \( k^* \) be such that

\[
P(k^* = i | \Delta T_{[0,x]}, (j^*, 1 \leq j \leq k - 1)) = \frac{\Delta_i(x)}{T_x - \Delta_1^*(x) - \cdots - \Delta_{k-1}^*(x)}
\]

for \( i \geq 1 \) distinct of the \( j^* \), \( 1 \leq j \leq k - 1 \), and let \( \Delta_k^*(x) = \Delta_{k^*}(1^*) \). Then

**Lemma 1.** (i) For \( k \geq 1 \),

\[
P(\Delta_k^*(x) \in dy | T_x, (\Delta_j^*(x), 1 \leq j \leq k - 1)) = \frac{c\alpha x a(y)}{s y^{-\alpha} a_x(s)} dy
\]

where \( s = T_x - \Delta_1^*(x) - \cdots - \Delta_{k-1}^*(x) \).

(ii) Consequently, given \( T_x = t \), \( \Delta_1^*(x) = y \), the sequence \( (\Delta_2^*(x), \Delta_3^*(x), \ldots) \) has the same law as \( (\Delta_1^*(x), \Delta_2^*(x), \ldots) \) given \( T_x = t - y \). Conversely, if we are given a random variable \( Y \) with same law as \( \Delta_1^*(x) \) given \( T_x = t \) and, given \( Y = y \), a sequence \( (Y_1, Y_2, \ldots) \) with same law as \( (\Delta_1^*(x), \Delta_2^*(x), \ldots) \) given \( T_x = t - y \), then \( (Y, Y_1, Y_2, \ldots) \) has same law as \( (\Delta_1^*(x), \Delta_2^*(x), \ldots) \) given \( T_x = t \).

This gives a regular conditional version for \( (\Delta_i^*(x), i \geq 1) \) given \( T_x \), and thus induces a conditional version for \( \Delta T_{[0,x]} \) given \( T_x \) by ranking.

### 2.2 Marked processes

We are now going to enlarge the original probability space to mark the jumps of the stable process. We let \( M_X \) be the law of a sequence \( \mathbf{e} = (e_s, s : \Delta X_s > 0) \) of independent standard exponential random variables, indexed by the (countable) set of times where the canonical process \( X \) jumps. We let \( \mathbb{P}(dX, d\mathbf{e}) = P(dX) \otimes M_X(d\mathbf{e}) \). This probability allows to mark the jumps of \( X \), precisely we say that a jump occurring at time \( s \) is marked at level \( t \geq 0 \) if \( e_s < t \Delta X_s \). Write

\[
Z_s^{(t)} = \sum_{0 \leq u \leq s} \Delta X_u 1_{\{e_u < t \Delta X_u\}}
\]

for the cumulative process of marked jumps at level \( t \). We also let \( X^{(t)} = X - Z^{(t)} \). We know that the process \( (\Delta X_s, s \geq 0) \) of the jumps of \( X \) is under \( P \) a Poisson point process with
intensity \( C_\alpha x^{1-\alpha}dx \) on \((0, \infty)\), it is then standard that the process \((\Delta Z_s^{(t)}, s \geq 0)\) is a Poisson point process with intensity \( C_\alpha x^{1-\alpha}(1 - e^{-t_x})dx \), meaning that under \( P \), \( Z^{(t)} \) is a subordinator with no drift and Lévy measure \( C_\alpha x^{-\alpha-1}(1 - e^{-t_x})dx \), more precisely its Laplace transforms are given by

\[
E[e^{-\lambda Z_s^{(t)}}] = \exp \left( -s \int_0^\infty C_\alpha (1 - e^{-t_x}) \frac{1 - e^{-\lambda x}}{x^{\alpha+1}} dx \right) = \exp(-s(\lambda + t)^\alpha + s\lambda^\alpha + st^\alpha).
\]

We denote by \((\rho_s^{(t)}(x), s, x \geq 0)\) the densities of the laws \( P(Z_s^{(t)} \in dx) \). It can be checked by [26, Proposition 28.3] from the expression of the Lévy measure of \( Z_s^{(t)} \) that these densities exist and are jointly continuous. Likewise, the process \( X^{(t)} \) is under \( P \) a Lévy process with Lévy measure \( C_\alpha e^{-t_x}x^{-\alpha-1}dx \), and the Laplace transform of \( X_s^{(t)} \) is given by

\[
E[e^{-\lambda X_s^{(t)}}] = \exp \left( s\lambda (\lambda + t)^{-\alpha-1} + s \int_0^\infty C_\alpha e^{-t_x} \frac{dx}{x^{\alpha+1}}(e^{-\lambda x} - 1 + \lambda x) \right) = \exp(s(\lambda + t)^\alpha - st^\alpha),
\]

which is obtained by dividing the Laplace exponent of \( X_s \) by that of \( Z_s^{(t)} \).

We now state an absolute continuity result that is analogous to Cameron-Martin’s formula for Brownian motion with drift.

**Proposition 2.** For every \( t, s \geq 0 \), we have the following absolute continuity relation: for every positive measurable functional \( F \),

\[
E[F(X^{(t)}_u, 0 \leq u \leq s)] = E[exp(-st^\alpha - tX_s)F(X_u, 0 \leq u \leq s)].
\]

**Proof.** By the expression for the Laplace exponent of \( X^{(t)} \), we get

\[
E[e^{-\lambda X_s^{(t)}}] = e^{-st^\alpha} E[e^{-(\lambda + t)X_s}],
\]

hence giving \( P(X^{(t)}_s \in dx) = e^{-st^\alpha - tx} P(X_s \in dx) \). The result easily follows by the Markov property. \( \square \)

As a first consequence, it immediately follows that \( X^{(t)} \) also have jointly continuous densities under \( P \), which are given by

\[
p_s^{(t)}(x) = \frac{P(X_s^{(t)} \in dx)}{dx} = \exp(-st^\alpha - tx)p_s(x).
\]

We let \( X^{(t)} \) be the infimum process of \( X^{(t)} \) and \( T^{(t)} \) the right-inverse process of \( -X^{(t)} \), defined as we did above define \( \underline{X} \) and \( T \).

It is easily obtained that for every \( t \geq 0 \), the process \((X, Z^{(t)})\) is again a Lévy process under the law \( P \). We will also denote by \( P^s, P^{(-\infty, \infty)} \) the laws derived from \( P^s \) and \( P^{(-\infty, \infty)} \) by marking the jumps with \( M_X; Z^{(t)} \) and \( X^{(t)} \) are then defined as before.
2.3 Bridges, excursions

For \( r \in \mathbb{R} \) and \( s > 0 \) we will denote by \( P_{0 \to r}^s \) the law of the stable bridge from 0 to \( r \) with length \( s \), so the family \( (P_{0 \to r}^s, r \in \mathbb{R}) \) forms a regular conditional version for \( P^s(x|X_s = r) \). By [18], a regular version (which is the one we will always consider) is obtained as the unique law on the Skorokhod space \( D([0, s]) \) that satisfies the following absolute continuity relation: for every \( a \in (0, s) \) and any continuous functional \( F \),

\[
P_{0 \to r}^s(F(X_u, 0 \leq u \leq s - a)) = E \left[ F(X_u, 0 \leq u \leq s - a) \frac{p_u(r - X_{s-a})}{p_s(r)} \right].
\]

(4)

We let \( P_{0 \to r}^s \) be the marked analog of \( P_{0 \to r}^s \) on an enriched probability space. Notice that Proposition 2 immediately implies that the laws bridges for the process \( X^{(0)} \) under \( \mathbb{P} \) are the same as those of \( X \). Stable bridges from 0 to 0 satisfy the following scaling property: under \( P_{0 \to 0}^u \), the process \( (u^{-1/\alpha}X_{us}, 0 \leq s \leq 1) \) has law \( P_{0 \to 0}^1 \).

Lemma 2. The following formula holds for any positive measurable \( f, g, H \):

\[
E_{0 \to 0}^1 \left[ H(X) \sum_{0 \leq s \leq 1} \Delta X_s f(s)g(\Delta X_s) \right] = \int_0^1 ds f(s) \int_0^\infty dx \frac{C_\alpha}{x^\alpha} \frac{p_1(-x)}{p_1(0)} g(x) E_{0 \to -x}^1 [H(X \oplus (s, x))],
\]

where \( X \oplus (s, x) \) is the process \( X \) to which has been added a jump at time \( s \) with magnitude \( x \). Otherwise said, a stable bridge from 0 to 0 together with a jump \( (s, \Delta X_s) \) picked according to the \( \sigma \)-finite measure \( m(ds, dx) = \sum_{u: \Delta X_u > 0} \Delta X_u \delta_{(u, \Delta X_u)}(ds, dx) \) is obtained by taking a stable bridge from 0 to \(-x\) and adding a jump with magnitude \( x \) at time \( s \), where \( s \) is uniform in \((0, 1)\) and \( x \) is independent with \( \sigma \)-finite "law" \( C_\alpha p_1(-x)p_1(0)^{-1}x^{-\alpha}dx \).

Proof. By the Lévy-Itô decomposition of Lévy processes, one can write, under \( P \), that \( X_s \) is the compensated sum

\[
X_s = \lim_{\varepsilon \to 0} \left( \sum_{0 \leq u \leq s} \Delta X_u 1_{\{\Delta X_u > \varepsilon\}} - (\alpha - 1)^{-1} C_\alpha \varepsilon^{1-\alpha} s \right), \quad s \geq 0,
\]

where \( (\Delta X_u, u \geq 0) \) is a Poisson point process with intensity \( C_\alpha x^{-\alpha-1}dx \), and where the convergence is almost sure. By the Palm formula for Poisson processes, we obtain that for positive measurable \( f, g, h, H \):

\[
E_{0 \to 0}^1 \left[ h(X_1)H(X) \sum_{0 \leq s \leq 1} \Delta X_s f(s)g(\Delta X_s) \right] = \int_0^1 ds f(s) \int_0^\infty dx \frac{C_\alpha}{x^\alpha} g(x) E_{0 \to -x}^1 [h(x + X_1)H(X \oplus (s, x))].
\]

The result is then obtained by disintegrating with respect to the law of \( X_1 \). \( \square \)

We now state a useful decomposition of the stable bridge from 0 to 0. Recall \( (\rho_{s}^{(t)}(x), x \geq 0) \) is the density of \( Z_s^{(t)} \) under \( \mathbb{P} \) and that \( X_1^{(t)} + Z_1^{(t)} = X_1 \), which is a sum of two independent variables. From this we conclude that \( (p_1(0)^{-1}p_1^{(t)}(x)p_1^{(t)}(x), x \geq 0) \) is a probability density on \( \mathbb{R}_+ \).
Lemma 3. Take a random variable \( Z \) with law \( P(Z \in dz) = p_{1}^{(t)}(-z)p_{1}^{(t)}(z)p_{1}(0)^{-1}dz \). Conditionally on \( Z = z \), take \( X' \) with law \( P_{0 \rightarrow z} \) and \( Z \) with law \( P_{1}(Z(t) \in \cdot | Z_{1}^{[t]} = z) \), independently. That is, \( Z \) is the bridge of \( Z^{(t)} \) with length 1 from 0 to \( z \). Then \( X' + Z \) has law \( P_{0 \rightarrow 0}^{1} \).

Remark. The definition for the bridges of \( Z^{(t)} \) under \( P_{1}^{1} \) has not been given before. One can either follow an analogous definition as (4), or use Lemma 1 about conditioned jumps of subordinators. Precisely, take \( (\Delta_{i}, i \geq 1) \) a sequence whose law is that of the jumps \( \Delta T_{[0,1]} \) of \( T \) under \( P \) before time 1, ranked in decreasing order, and conditioned by \( T_{1} = z \), in the sense of Lemma 1. Take also a sequence \( (U_{i}, i \geq 1) \) of independent uniformly distributed random variables on \([0, 1]\), independent of \( \Delta T_{[0,1]} \). Then one checks from the \( \text{Lévy-Itô} \) decomposition for \( \text{Lévy processes} \) that the law \( Q_{z} \) of the process \( Z_{s} = \sum \Delta_{i} 1_{\{s \geq U_{i}\}} \), with \( 0 \leq s \leq 1 \), defines as \( z \) varies a regular version of the conditional law \( P_{1}(Z(t) \in \cdot | Z_{1}^{[t]} = z) \).

Proof. Recall that under \( P_{1}^{1} \), \( X \) can be written as \( X^{(t)} + Z^{(t)} \) with \( X^{(t)} \) and \( Z^{(t)} \) independent. Consequently, for \( f \) and \( G \) positive continuous, we have

\[
E^{1}[f(X_{1})G(X)] = E^{1}[f(X_{1}^{(t)} + Z_{1}^{(t)})G(X^{(t)} + Z^{(t)})]
\]

so

\[
\int_{\mathbb{R}} dx p_{1}(x) f(x) E_{0 \rightarrow x}^{1}[G(X)] = \int_{\mathbb{R}} dx p_{1}(x) \int_{0}^{\infty} dz \frac{p_{1}^{(t)}(x-z)p_{1}^{(t)}(z)}{p_{1}(x)} f(z) \times E^{1}[G(X^{(t)} + Z^{(t)})|X_{1}^{(t)} = x-z, Z_{1}^{(t)} = z].
\]

Thus, for (Lebesgue) almost every \( x \), the bridge with law \( P_{0 \rightarrow x}^{1} \) is obtained by taking a bridge of \( X^{(t)} \) (or \( X \) by previous remarks) from 0 to \( -Z_{x} \) and an independent bridge of \( Z^{(t)} \) from 0 to \( Z_{x} \), where \( Z_{x} \) is a r.v. with law \( dz p_{1}(x)^{-1} p_{1}^{(t)}(x-z)p_{1}^{(t)}(z) \) on \( \mathbb{R}_{+} \). We extend this result to every \( x \in \mathbb{R} \) by an easily checked continuity result for the laws of bridges which stems from (4) and the continuity of the densities. Taking \( x = 0 \) gives the result.

We now turn our attention to excursions. The fact that \( X \) has no negative jumps implies that \( -\overline{X} \) is a local time at 0 for the reflected process \( X - \overline{X} \). Let \( N \) be the Itô excursion measure of \( X - \overline{X} \) away from 0, so that the path of \( X - \overline{X} \) is obtained by concatenation of the atoms of a Poisson measure with intensity \( N(dX) \otimes dt \) on \( \mathbb{D}[0, \infty) \times \mathbb{R}_{+} \), where \( \mathbb{D}[0, \infty) \) denotes the Skorokhod space of paths that are killed at some time \( \zeta \). Under \( N \), almost every path \( X \) starts at 0, is positive on an interval \( (0, \zeta) \) and dies at the first time \( \zeta(X) \in (0, \infty) \) it hits 0 again. We let \( N \) be the enriched law with marked jumps. It follows from excursion theory that the \( \text{Lévy process} \) \((X, Z^{(t)})\) under \( P \) is obtained by taking a Poisson point measure \( \sum_{i \in I} \delta_{X^{i}, e^{i}, s^{i}} \) with intensity \( N(dX, de) \otimes ds \), writing \( Z^{(t),i} \) for the cumulative process of marked jumps for \( X^{i} \) and letting

\[
X_{s} = -s^{i} + X^{i} \left( s - \sum_{j: s^{i} < s^{j}} \zeta_{j}(X^{j}) \right)
\]

and

\[
Z_{s}^{(t)} = \sum_{j: s^{j} < s^{i}} Z_{s^{j}}^{(t),j} + Z^{(t),i} \left( s - \sum_{j: s^{i} < s^{j}} \zeta_{j}(X^{j}) \right),
\]

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whenever $\sum_{j:j<s} \zeta_j(X^j) \leq s \leq \sum_{j:j<s} \zeta_j(X^j)$.

If $X$ is stopped at some time $s$, for any $u \in [0, s]$ we define the rotated process

$$V_uX(r) = (X_{r+u} - X_u)1_{\{0 \leq u \leq s-u\}} + (X(r-s+u) + X_s - X_u)1_{\{s-u \leq r \leq s\}}.$$ 

Let $m_s = -X_s$ and suppose that this minimum is attained only once on $[0, s]$. We define the Vervaat transform of $X$ as $VX = V_{T(m_s)}X$, the rotation of $X$ at the time where it attains its infimum. Provided that $X_0 = 0$ and $X_s = X_{s-} = 0$ (say that $X$ is a bridge), $VX$ is then an excursion-like function, starting and ending at 0, and staying positive in the meanwhile.

We will denote by $N^{(v)}$ the law of $VX$ under $P_{0 \to v}$ and $N^{(v)}$ the corresponding “marked” version. Call it the law of the excursion of $X$ with duration $v$. The “Vervaat theorem” in [13] shows that $N^{(v)}$ is indeed a regular conditional version for the “law” $N(\cdot | \zeta = v)$: for any positive measurable functional $F$ and function $f$,

$$N(F(X_s, 0 \leq s \leq \zeta)f(\zeta)) = \int_{\mathbb{R}_+} f(\zeta) N(\zeta \in dv)N^{(v)}(F(X_s, 0 \leq s \leq v)).$$

As for bridges, we also have the scaling property at the level of conditioned excursions: under $N^{(v)}$, $(v^{-1/\alpha} X_{s v}, 0 \leq s \leq 1)$ has law $N^{(1)}$. Notice also (either by Vervaat’s theorem or directly, using Proposition 2) that the excursions of $X^{(t)}$ under $P$, conditioned to have a fixed duration $v$ are the same as that of $X$ under $N^{(v)}$.

## 3 The stable tree

### 3.1 Height Process, width process

We now introduce the rigorous definition and useful properties of the stable tree. This section is mainly inspired by [16, 15]. With the notations of section 2, and for $t \geq 0$, let $R^{(t)}$ be the time-reversed process of $X$ at time $t$:

$$R^{(t)}_s = X_t - X_{(t-s)-} \quad 0 \leq s \leq t.$$ 

It is standard that this process has the same law as $X$ killed at time $t$ under $P$. Let $\overline{R}^{(t)}$ be its supremum process, and $\widehat{L}^{(t)}$ be the local time process at level 0 of the reflected process $\overline{R}^{(t)} - R^{(t)}$. We let $H_t = \widehat{L}^{(t)}$. The normalization for $\widehat{L}^{(t)}$ is chosen so that

$$H_t = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t 1\{|\overline{R}^{(t)}_s - R^{(t)}_s| \leq \varepsilon\}ds,$$

in probability for every $t$. It is proved in [16] that $H$ admits a continuous modification, which is the one we are going to work with from now on. It has to be noticed that $H_t$ is not a Markov process, except in the case where $X$ is Brownian motion. As a matter of fact, it can be noticed that $H$ admits infinitely many local minima attaining the same value as soon as $X$ has jumps. To see this, consider a jump time $t$ of $X$, and let $t_1, t_2 > t$ so that $\inf_{t \leq u \leq t_1} X_u = X_{t_1}$ and $X_{t_1} < X_{t_2} < X_t$, $i \in \{1, 2\}$. Then it is easy to see that $H_t = H_{t_1} = H_{t_2}$ and that one may
in fact find an infinite number of distinct \( t_i \)'s satisfying the properties of \( t_1, t_2 \). On the other hand, it is not difficult to see that \( H_t \) is a local minimum of \( H \), see Proposition 3 below.

It is shown in [16] that the definition of \( H \) still makes sense under the \( \sigma \)-finite measure \( N \) rather than the probability law \( P \). The process \( H \) is then defined only on \([0, \zeta] \), and we call it the excursion of the height process. Using the scaling property, one can then define the height process under the laws \( N^{(v)} \). Call it the law of the excursion of the height process with duration \( v \).

The key tool for defining the local time of hubs is the local time process of the height process. We will denote by \( (L^t_s,t,s \geq 0) \). It can be obtained a.s. for every fixed \( s,t \) by

\[
L^t_s = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t 1_{\{t < H_u \leq t + \varepsilon\}} du.
\]

That is, \( L^t_s \) is the density of the occupation measure of \( H \) at level \( t \) and time \( s \). For \( t = 0 \), one gets \( (L^0_s, s \geq 0) = (X_s, s \geq 0) \), which is a reminiscent of the fact that the excursions of the height process are in one-to-one correspondence with excursions of \( X \) with the same lengths.

It is again possible to define the local time process under the excursion measures \( N \) and \( N^{(v)} \). Duquesne and Le Gall [16] have shown that under \( P \), the process \( (L^t_s, t \geq 0) \) has the law of the continuous-stable branching process starting at \( x > 0 \), with stable \( (\alpha) \) branching mechanism. One can get interpretations for the process \( (L^t_s, t \geq 0) \) under the measure \( N \) or of \( (L^t_s, t \geq 0) \) under \( N^{(v)} \) in terms of conditioned continuous-state branching processes, see [22].

### 3.2 The tree structure

Let us motivate the term of “height process” for \( H \) by embedding a tree inside \( H \), following [20, 2]. Consider the height process \( H \) under the law \( N^{[1]} \). We can define a pseudo metric \( D \) on \([0, 1] \) by letting \( D(s, s') = H_s + H_{s'} - 2 \inf_{u \in [s, s']} H_u \) (with the convention that \([s, s'] = [s', s] \) if \( s' < s \)). Let \( s \equiv s' \) if and only if \( D(s, s') = 0 \).

**Definition 1.** The stable tree \((\mathcal{T}, d)\) is the quotient of the pseudo-metric space \(([0, 1], D)\) by \( \equiv \). The root of \( \mathcal{T} \) is the equivalence class of \( 0 \). The mass measure \( \mu \) is the Borel measure induced on \( \mathcal{T} \) by Lebesgue’s measure on \([0, 1] \) (so its support is \( \mathcal{T} \)).

In the sequel, we will often identify \( \mathcal{T} \) with \([0, 1] \), even if the correspondence is not one-to-one. Some comments on this definition. First, the way the tree is embedded in the function \( H \) can seem quite intricate. It is not difficult, however, to see what its “marginals” look like. For any finite set of vertices \( s_1, s_2, \ldots, s_k \in [0, 1] \), one recovers the structure of the subtree spanned by the root and \( s_1, s_2, \ldots, s_k \), according to the following simple rules:

- The height of \( s \) is \( \text{ht}(s) = H_s \).
- The common ancestor of \( s_1, \ldots, s_k \) is \( b = b(s_1, \ldots, s_k) \in [\min_{1 \leq i \leq k} s_i, \max_{1 \leq i \leq k} s_i] \) such that \( H_b = \inf \{ H_s : s \in [\min_{1 \leq i \leq k} s_i, \max_{1 \leq i \leq k} s_i] \} \).

Notice that all such \( b \) are equivalent with respect to \( \equiv \). The fact that \((\mathcal{T}, d)\) is indeed a tree (a complete metric space such that the only path leading from a vertex to another is the geodesic)
is intuitive and proven in [17]. It follows from the construction of “marginals” of \( \mathcal{T} \) in [16] that
given \( \mu, \mu \)-a.e. vertex is a leaf of \( \mathcal{T} \).

We now relate properties on the stable tree to path properties of the underlying Lévy process we
started with to construct the height process. We understand here that \( X \) and \( H \) are defined
under \( N^{(1)} \). Recall that \( \mathcal{T}_b \) stands for the fringe subtree rooted at \( b \).

**Proposition 3.** (i) Each hub \( b \in \mathcal{H}(\mathcal{T}) \) with height \( h \), is encoded by exactly one time \( \tau(b) \) in
\([0,1]\) such that \( L(b) = \Delta X_{\tau(b)} > 0 \), and \( L(b) \) is given by (1) a.s.

(ii) If \( \sigma(b) = \inf\{s \geq \tau(b) : X_s = X_{\tau(b)-}\} \), then \( \mathcal{T}_b = [\tau(b), \sigma(b)] \equiv \).

(iii) More precisely, let \( \mathcal{T}_b^{(1)}, \mathcal{T}_b^{(2)} \ldots \) be the connected components of \( \mathcal{T}_b \setminus \{b\} \), arranged
in decreasing order of mass. Let \( ([\tau(i), \sigma(i)], i \geq 1) \) be the constancy intervals of the infimum
process of \( (X_s - X_{\tau(b)}, \tau(b) \leq s \leq \sigma(b)) \), and ranked in decreasing order of length. Then
\( \mathcal{T}_b^{(i)} = (\tau(i), \sigma(i)) \equiv \).

**Proof.** (i) Working first under \( P \), fix \( \ell > 0 \) and let \( \tau_\ell = \inf\{s \geq 0 : \Delta X_s > \ell\} \). Then \( \tau_\ell \) is a
stopping time for the natural filtration associated to \( X \), as well as \( \sigma_\ell = \inf\{s > \tau_\ell : X_s = X_{\tau_\ell}\} \).

By the Markov property, the process \( X_{[\tau_\ell, \sigma_\ell]} = (X_{\tau_\ell+s} - X_{\tau_\ell}, 0 \leq s \leq \sigma_\ell - \tau_\ell) \) is independent
of \( (X_{\tau_\ell+s} - X_{\tau_\ell}, s \geq 0) \), which has the same law as \( X \), and of \( (X_s, 0 \leq s \leq \tau_\ell) \) conditionally
on its final jump \( \Delta X_{\tau_\ell} \). Now if we remove this jump, that is, if we let \( \tilde{X}_s, 0 \leq s \leq \tau_\ell \) be the
modification of \( (X_s, 0 \leq s \leq \tau_\ell) \) that is left-continuous at \( \tau_\ell \), then \( \tilde{X} \) has the law of a stable
Lévy process killed at some independent exponential time, and conditioned to have jumps with
magnitude less than \( \ell \). Also, conditionally on \( \Delta X_{\tau_\ell} = x \), \( X_{[\tau_\ell, \sigma_\ell]} \) has the law \( P^{(-x, \infty)} \) of the
stable process killed when it first hits \(-x\). Hence, by the additivity of the local time and the
definition of \( H \), one has that for every \( s \in [\tau_\ell, \sigma_\ell] \), \( H_s = H_{\tau_\ell} + \tilde{H}_{s-\tau_\ell} \), where \( \tilde{H} \) is an independent
copy of \( H \), killed when its local time at 0 attains \( x \). Consequently, one has \( H_s \geq H_{\tau_\ell} \) for every
\( s \in [\tau_\ell, \sigma_\ell] \) and \( H_{\sigma_\ell} = H_{\tau_\ell} \), moreover, one has that for every \( \varepsilon > 0 \),
\[
\inf_{(\tau_\ell-\varepsilon) \vee 0 \leq s \leq \tau_\ell} H_s \vee \inf_{\sigma_\ell \leq s \leq \sigma_\ell+\varepsilon} H_s < H_{\tau_\ell},
\]
as a consequence of the following fact. By the left-continuity of \( X \) at \( \tau_\ell \), for any \( \varepsilon > 0 \) we may
find \( s \in [\tau_\ell - \varepsilon, \tau_\ell] \) such that \( \inf_{u \in [s, \tau_\ell]} X_u = X_s \). This implies \( H_s = H_{\tau_\ell} - \tilde{L}^{(\tau_\ell)}_{s-\tau_\ell} \) and this last
term is a.s. strictly less than \( H_{\tau_\ell} \) because 0 is a.s. not a holding point for \( \tilde{L}^{(\tau_\ell)}_{s-\tau_\ell} \), \( 0 \leq s \leq \tau_\ell \).

This last fact is obtained by a time-reversal argument, using the fact that the points of increase
of the local time \( \tilde{L}^{(\tau_\ell)} \) correspond to that of the supremum process of \( R^{(\tau_\ell)} \). Moreover, the fact
that \( X \) has only positive jumps under \( P \) implies that for some suitable \( \varepsilon' > 0 \), one can find some
\( s' \in [\sigma_\ell, \sigma_\ell + \varepsilon'] \) and some \( s'' \in [\tau_\ell - \varepsilon, \tau_\ell] \) such that \( H_u \geq H_{s'} = H_{s''} \) for every \( u \in [s', s''] \), and
such that again \( \inf_{u \in [s', s'']} X_u = X_{s''} \). Thus the claimed inequality. In terms of the structure of
the stable tree, (6) implies that a node \( b \) of the tree is present at height \( h \), which is encoded
by all the \( s \in [\tau_\ell, \sigma_\ell] \) such that \( H_s = H_{\tau_\ell} \), i.e. such that \( X_s = \inf_{u \in [\tau_\ell, s]} X_u \) (there is always
an infinite number of them). By definition, the mass measure of the vertices in \( \mathcal{T}_b \) at distance
less than \( \varepsilon \) of \( b \) is exactly the Lebesgue measure of \( \{s \in [\tau_\ell, \sigma_\ell] : \tilde{H}_s-\tau_\ell < \varepsilon\} \). Thus by (5) we
can conclude that \( \tilde{L}(b) \) defined at (1) exists and equals \( \tilde{L}_{\tau_\ell-\tau_\ell}^{(\tau_\ell)} = x \) where \( \tilde{L} \) is the local time
associated to \( \tilde{H} \). The same argument allows to handle the second, third, ... jumps that are
> \( \ell \). Letting \( \ell \downarrow 0 \) implies that to any jump of \( X \) with magnitude \( x \) corresponds a hub of the
stable tree with local time $x$. By excursion theory and scaling, the same property holds under $N$ and $N^{(1)}$.

Conversely, suppose that $b$ is a node in the stable tree. This means that there exist times $s_1 < s_2 < s_3$ such that $H_{s_1} = H_{s_2} = H_{s_3}$ and $H_s \geq H_{s_1}$ for every $s \in [s_1, s_3]$. Let

$$\tau(b) = \inf\{s \leq s_2 : H_s = H_{s_2} \text{ and } H_u \geq H_{s_2} \forall u \in [s, s_2]\}$$

and

$$\sigma(b) = \sup\{s \geq s_2 : H_s = H_{s_2} \text{ and } H_u \geq H_{s_2} \forall u \in [s_2, s]\}$$

(which are not stopping times). If $\Delta X_{\tau(b)} > 0$, we are in the preceding case. Suppose that $\Delta X_{\tau(b)} = 0$, then by the same arguments as above, $X_s \geq X_{\tau(b)}$ for $s \in [\tau(b), \sigma(b)]$, else we could find some $s' \in [\tau(b), \sigma(b)]$ such that $H_{s'} < H_{\tau(b)}$. Also, the points $s \in [\tau(b), \sigma(b)]$ such that $H_s = H_{\tau(b)}$ must then satisfy $X_s = X_{\tau(b)}$ (else there would be a strict increase of the local time of the reversed process). This implies that $X_{\tau(b)}$ is a local infimum of $X$, attained at $s$. By standard considerations, such local infima cannot be attained more than three times on the interval $[\tau(b), \sigma(b)]$, a.s. But if it was attained exactly three times, then the node would have degree 3, which is impossible according to the analysis of $F^-$ in [22], which implies that all hubs of the stable tree have infinite degree.

Assertion (ii) follows easily from this, and (iii) comes from the fact that the points $u \in [\tau(b), \sigma(b)]$ with $H_u = H_{\tau(b)}$ are exactly those points where $\inf_{r \in [\tau(b), u]} X_r = X_u$, and the definition of the mass measure on $\mathcal{T}$.

\[\square\]

### 3.3 A second way to define $F^+$

We will now give some elementary properties of $F^+$ and rephrase its definition directly from the excursion of the underlying stable excursion $X$ rather than the tree itself. First recall that given $\mathcal{T}$, we can define a marking procedure on $\mathcal{H}(\mathcal{T})$ by taking a Poisson process $(b(t), t \geq 0)$ with intensity $dt \otimes \sum_{b \in \mathcal{H}(b)} L(b) \delta_b (dv)$, and by saying that $b$ is marked at level $t$ if $b \in \{b(s), 0 \leq s \leq t\}$. Let us state a useful lemma.

**Lemma 4.** Let $s \in [0, 1]$, and write $v(s)$ for the vertex of $\mathcal{T}$ encoded by $s$. Then almost-surely,

$$\sum_{b \in \mathcal{H}(\mathcal{T}) \cap [[\text{root}, v]]} L(b) < \infty.$$ 

In particular, almost surely, for every hub $b \in \mathcal{H}(\mathcal{T})$ and $t \geq 0$, there is a finite number of marked hubs at level $t$ on the path $[[\text{root}, v]]$.

**Proof.** Let $s$ be the leftmost time in $[0, 1]$ that encodes $v$. It follows from Proposition 3 (ii) (and the fact that a.s. under $P$, every excursion of $R^{(s)}$ below $\overline{R}^{(s)}$ ends by a jump) that the hubs $b$ in the path $[[\text{root}, v]]$ are all encoded by the times $s' < s$ such that $\overline{R}^{(s)}$ jumps at time $s - s'$. This jump corresponds to a jump of the reversed process $R^{(s)}$, whose magnitude $\Delta R^{(s)}_{s-s'} \geq \Delta \overline{R}^{(s)}_{s-s'}$ equals $L(b)$ by Proposition 3 (i). Therefore, we have to show that the sum of these jumps is
finite a.s. By excursion theory and time-reversal, it suffices to show that under $P$, letting $\overline{X}$ be the supremum process of $X$,

$$\sum_{0 \leq s' \leq s: \Delta X_{s'} > 0} \Delta X_{s'} < \infty, \quad s \geq 0.$$ 

Now by excursion theory and basic fluctuation theory (see e.g. the proof of [16, Lemma 1.1.2]), after appropriate time-change, the jumps $\Delta X_{s'}$ above a previous supremum form a Poisson point process with intensity $C_\alpha x^{-\alpha} dx$, so the sum defines a time-changed subordinator, which is a.s. finite at all times. The statement on hubs follows since for any hub $b$ encoded by a jump-time $\tau(b)$, there is a rational number $r' \in [\tau(b), \sigma(b)]$ which encodes some vertex $v$ in the fringe subtree rooted at $b$. \hfill \Box

By definition, two vertices $v, w \in \mathcal{T}$ satisfy $v \sim_t w$ if and only if $\{b(s): 0 \leq s \leq t\} \cap [[v, w]] = \emptyset$. Let $\mathcal{H}_t = \{b(s): 0 \leq s \leq t\}$. For $b \in \mathcal{H}_t$, let $\mathcal{T}_b^1, \mathcal{T}_b^2, \ldots$ be the connected components of $\mathcal{T}_b \setminus \{b\}$ ranked in decreasing order of total mass. We know that these trees are encoded by intervals of the form $(\tau_i(b), \sigma_i(b))$ whose union is $[\tau(b), \sigma(b)] \setminus \{u: u \equiv b\}$. Define

$$C(t, b, i) = \mathcal{T}_b^i \setminus \bigcup_{b' \in \mathcal{H}_t \cap \mathcal{T}_b^i} \mathcal{T}_{b'}.$$ 

Plainly, $C(t, b, i)$ is an equivalence class for $\sim_t$ for every $b \in \mathcal{H}_t$ and $i \geq 1$. By (iii) in Proposition 3, with obvious notations,

$$C(t, b, i) \equiv (\tau_i(b), \sigma_i(b)) \setminus \bigcup_{b' \in \mathcal{T}_b^i \cap \mathcal{H}_t} [\tau(b'), \sigma(b')].$$

We also let $C(t, \emptyset)$ be the set of vertices whose path to the root does not cross any marked hub at level $t$, which is equivalent to $[0, 1] \setminus \bigcup_{b \in \mathcal{H}_t} [\tau(b), \sigma(b)]$. Then $C(t, \emptyset)$ is also an equivalence class for $\sim_t$. Intuitively, the classes $C(t, \emptyset)$ and $C(t, b, i)$ for $b$ a hub are the equivalence classes for $\sim_t$ that have a positive weight. We will see later that the rest is a set of leaves of mass zero.

Let us now translate the relation $\sim_t$ in terms of the stable excursion $X$ under $\mathbb{N}^{(1)}$. Let $s, s' \in [0, 1]$ encode respectively the vertices $v \neq w \in \mathcal{T}$. Again by Proposition 3 (ii), the branchpoint $b(v, w)$ of $v$ and $w$ is encoded by the largest $u$ such that the processes $(\overline{R}_{s-u+r}^{(s)}, 0 \leq r \leq u)$ and $(\overline{R}_{s'-u+r}^{(s')}, 0 \leq r \leq u)$ coincide. Let $u(s, s')$ be the jump-time of $X$ that encodes this branchpoint. Then $v \sim_t w$ if and only if the (left-continuous) processes $(\overline{R}_{u}, u(v, w) \leq r \leq s)$ and $(\overline{R}_{u}, u(v, w) \leq r \leq s')$ never jump at times when marked jumps at level $t$ for $X$ occur.

In particular, we may rewrite the equivalence classes $C(t, b, i)$ and $C(t, \emptyset)$ as follows. Let $z_1^t \geq z_2^t \geq \ldots \geq 0$ be the marked jumps of $X$ at level $t$ under $\mathbb{N}^{(1)}$, ranked in decreasing order, and let $\tau_1^t, \tau_2^t, \ldots$ the corresponding jump times (i.e. such that $\Delta Z_{\tau_i^t} = z_i^t$). For every $i$, let

$$\sigma_i^t = \inf\{s > \tau_i^t: X_s = X_{\tau_i^t -} = X_{\tau_i^t} - z_i^t\}$$

be the first return time to level $X_{\tau_i^t -}$ after time $\tau_i^t$. Define the intervals $I_i^t = [\tau_i^t, \sigma_i^t]$ (so $I_i^t / \equiv$ is the fringe subtree of the marked hub that has width $z_i^t$). Moreover, for each $i$, the jump
with magnitude \( z_t \) gives rise to a family of excursions of \( X \) above its minimum. Precisely, let \( (X_{i,1}^t, X_{i,2}^t, \ldots) \) the sequence of excursions above its infimum of the process

\[
X_i^t(s) = X_{\tau_i^t + s} - X_{\tau_i^t}, \quad \tau_i^t \leq s \leq \sigma_i^t,
\]

where the \( (X_{i,j}^t, j \geq 1) \) are arranged by decreasing order of duration. Let also \( I_{i,j}^t = [\tau_{i,j}, \sigma_{i,j}] \) be the interval in which \( X_{i,j}^t \) appears in \( X \), so that \( \bigcup_j I_{i,j}^t = I_{i,j}^t \). Consider the set

\[
C_{i,j}^t = I_{i,j}^t \setminus \bigcup_{k: I_k^t \subseteq I_{i,j}^t} I_k^t.
\]

By Lemma 4, there exists some set of indices \( k' \) such that \( I_{k'}^t \subseteq I_{i,j}^t \) and so that the \( I_{k'}^t \)'s are maximal with this property. Let also \( I_t^0 = [\tau_t^0, \sigma_t^0] \) be the interval in which \( X_t^0 \) appears in \( X \), so that \( S_j^t I_t^0 = I_t^0 \). Consider the set

\[
C_t^0 = I_t^0 \setminus \bigcup_{k: I_k^0 \subseteq I_t^0} I_k^0.
\]

By Lemma 4, there exists some set of indices \( k' \) such that \( I_{k'}^0 \subseteq I_t^0 \) and so that the \( I_{k'}^0 \)'s are maximal with this property. Writing \( C_t^0 = [0, 1] \setminus \bigcup_{i=1}^\infty I_i^t \), we finally get (identifying Borel subsets of \([0, 1]\) with Borel subsets of \( T \)).

**Lemma 5.** The sets \( C_t^0 \) and \( C_{i,j}^t \), for \( i, j \geq 1 \), are a relabeling of the sets \( C(t, \emptyset) \) and \( C(t, b, i) \).

Notice also that another consequence of Lemma 4 is that \( F^+ \) is continuous in probability at time 0. Indeed, as \( t \downarrow 0 \), the component \( C(t, \emptyset) \) of the fragmented tree containing the root increases to \( C(0+, \emptyset) \). Suppose \( \mu(C(0+, \emptyset)) < 1 \) with positive probability. Given \( T \) take \( L_1, L_2, \ldots \) independent with law \( \mu \). By the law of large numbers, with positive probability a positive proportion of the \( L_i \)'s are separated from the root at time \( 0+ \). However, as a consequence of Lemma 4, a.s. for every \( n \geq 1 \) and \( t \) small enough, there is no marked hub on the paths \([\text{root}, L_i]\), \( 1 \leq i \leq n \), hence a contradiction.

## 4 Study of \( F^+ \)

The goal of this section is to study the fragmentation \( F^+ \) through the representation given in the last section. The first step is to study the behavior of the excursion on the equivalence classes \( C_{i,j}^t \) and \( C_t^0 \) defined previously.

### 4.1 Self-similarity

This section is devoted to the proof that \( F^+ \) is a self-similar fragmentation with index \( 1/\alpha \) and no erosion.

Let us first introduce some notation. Let \( (f(x), 0 \leq x \leq \zeta) \) be a càdlàg function on \([0, \zeta]\) with \( \zeta \in [0, \infty) \). By convention we let \( f(\zeta) = f(x) \) for \( x > \zeta \). We define the unplugging operation UNPLUG as follows. Let \( ([a_n, b_n], n \geq 1) \) be a sequence of disjoint closed intervals with
non-empty interior, such that $0 < a_n < b_n < \zeta$ for every $n$. Define the increasing continuous function

$$x^{-1}(s) = s - \sum_{n \geq 1} (s \wedge b_n - a_n)^+, \quad s \geq 0,$$

where $a^+ = a \vee 0$ and where the sum converges uniformly on $[0, \zeta]$ since $\sum_n (b_n - a_n) < \zeta$. Define $x$ as the right-continuous inverse of $x^{-1}$, then $f \circ x$ is càdlàg (notice that $(f \circ x)(s-1) = f(x(s-1))$, call it UNPLUG($f, [a_n, b_n], n \geq 1$) (the action of UNPLUG is to remove the bits of the path of $f$ that are included in $[a_n, b_n]$). We say that the intervals $[a_n, b_n]$ are separated if $x(a_n) < x(a_m)$ for every $n \neq m$ such that $a_n < a_m$. This is equivalent to the fact that for every $n \neq m$ with $a_n < a_m$, the set $[a_n, a_m] \setminus \bigcup_n [a_n, b_n]$ has positive Lebesgue measure. Last, if we are given intervals $[a_n, b_n]$ that are not overlapping (i.e. such that $a_n < a_m < b_n < b_m$ does not happen for $n \neq m$, though we might have $[a_n, b_n] \subset [a_m, b_m]$), but such that there is a subsequence $[a_{\phi(n)}, b_{\phi(n)}], n \geq 1$ of maximal intervals that covers $\bigcup_n [a_n, b_n]$, we similarly define the unplugging operation by simply ignoring the non-maximal intervals.

**Lemma 6.** Let $([a_n, b_n], n \geq 1)$ be a sequence of separated intervals. Then as $N \to \infty$, UNPLUG($f, [a_n, b_n], 1 \leq n \leq N$) converges to UNPLUG($f, [a_n, b_n], n \geq 1$) in the Skorokhod space.

**Proof.** Define $x_N^{-1}$ as above by truncating the sum at $N$. The separation of intervals ensures that every jump of $x$ corresponds to a jump of $x_N$ for some large $N$. Since $f \circ x$ is càdlàg with duration $\zeta' = \zeta - \sum_n (b_n - a_n)$, for every $N$ we may find a sequence of times $0 = s_0 < s_1 < s_2 < \ldots < s_{k(N)} = \zeta'$ such that the oscillation

$$\omega(f \circ x, [s_i, s_{i+1}]) = \sup_{s, s' \in [s_i, s_{i+1}]} |f \circ x(s) - f \circ x(s')| < 2^{-N}.$$ 

Let also $s_i^N$ be the corresponding times for $f \circ x_N$, that is, $s_i^N = x_N^{-1}(x(s_i))$. We build a time change $\lambda_N$ (a strictly increasing continuous function) by setting $\lambda_N(s_i) = s_i^N$ for $1 \leq i \leq k(N)$, and interpolating linearly between these times. Then $|\lambda_N(s_i) - s_i| < \sum_{n>N} (b_n - a_n) \to 0$, and it follows that $\lambda_N$ converges pointwise and uniformly to the identity function of $[0, \zeta']$. On the other hand, we have that $f \circ x(s_i) = f \circ x_N \circ \lambda_N(s_i)$, and for $s \in (s_i, s_{i+1})$,

$$|f \circ x_N \circ \lambda_N(s) - f \circ x(s)| \leq \omega(f \circ x, [s_i, s_{i+1}]) + \sup_{n>N} (\omega(f, [a_n, b_n]) + f(a_n) - f(a_n-)).$$

We can conclude that $f \circ x_N \circ \lambda_N$ converges uniformly to $f \circ x$ since the oscillation $\omega(f, [a_n, b_n])$ converges to 0 uniformly in $n \geq N$ as $N \to \infty$, as does the jump $f(a_n) - f(a_n-)$. \hfill \square

Under the law $\mathbf{P}^{(-z, \infty)}$ under which $X$ is killed when it first attains $-z$, for every $t > 0$ we let $z^t_1 \geq z^t_2 \geq \ldots \geq 0$ be the marked jumps of $X$ at level $t$, ranked in decreasing order of magnitude, and $\tau^t_1$ be the time of occurrence of the jump with magnitude $z^t_1$, while $\sigma^t_1$ is the first time after $\tau^t_1$ when $X$ hits level $X_{\tau^t_1}$ (notice that $\tau^t_1, \sigma^t_1$ are not stopping times). Similarly as before, we let $I^t_i = [\tau^t_i, \sigma^t_i]$.

**Lemma 7.** For every $z, t \geq 0$, the process UNPLUG($X, (I^t_i : i \geq 1)$) has same law as $X^{(t)}$ under $\mathbf{P}$, killed when it first hits $-z$.

Part of this lemma is that it makes sense to apply the unplugging operation with the intervals $I^t_i$, that is, that these intervals admit a separated covering maximal subfamily.
Proof. The fact that the intervals $I^t_i$ admit a covering maximal subfamily is obtained by reusing the proof of Lemma 4 and an argument in the preceding section. Next, write $X = X^{(i)} + Z^{(i)}$. For $a > 0$, let $\tau^t_{i,a}$ be the time of the first jump of $Z^{(i)}$ that is $> a$, and let $\sigma^t_{i,a} = \inf \{ u \geq \tau^t_{i,a} : X_u = X_{\tau^t_{i,a} + a} \}$. Recursively, let $\tau^t_{i+1,a} = \inf \{ u \geq \tau^t_{i+1,a} : \Delta Z^{(i)}_u > a \}$ and $\sigma^t_{i+1,a} = \inf \{ u \geq \tau^t_{i+1,a} : X_u = X_{\tau^t_{i+1,a} + a} \}$. Let $Z^{(t,a)}_s = \sum_{u \leq s} \Delta Z^{(i)}_u 1_{(\Delta Z^{(i)}_u \leq a)}$. The $\tau^t_{i,a}$'s are stopping times for the filtration generated by $(X^{(i)}, Z^{(t)})$, as well as the $\sigma^t_{i,a}$'s. By a repeated use of the Markov property at these times we get

$$\text{UNPLUG}(X; (I^t_i : z^t_i > a)) \overset{d}{=} X^{(i)} + Z^{(t,a)},$$

where this last process is killed at the time $T^{(t,a)}_2$ when it first hits $-z$. In particular, $T^t_i - \sum_i (\sigma^t_{i,a} - \tau^t_{i,a})$ has the same law as $T^{(t,a)}_2$, which converges in law to $T^{(t)}_2$ as $a \downarrow 0$ because $Z^{(t,a)}$ converges to 0 uniformly on compacts, and $X^{(i)}$ enters $(-\infty, -z)$ immediately after $T^{(t)}_2$ by the Markov property and the fact that 0 is a regular point for Lévy processes with infinite total variation. Therefore, writing $|I^t_i|$ for the Lebesgue measure of $I^t_i$, $T^t_i - \sum_k |I^t_k|$ (where the sum is over the $I^t_k$ that are maximal) has same law as $T^{(t)}_2$, and in particular it is nonzero a.s. Now to check that the intervals $I^t_i$ are separated (we are only interested by those which are maximal), consider two left-ends of such intervals such as $\tau^t_{i,a} < \tau^t_{j,a}$ (where $a$ is small enough). The regularity of 0 for the Lévy process $X$ implies that $\inf_{s \in [\sigma^t_{i,a}, \tau^t_{j,a}]} X_s < X_{\sigma^t_{i,a}}$, so by the same arguments as above and the Markov property at $\sigma^t_{i,a}$, there exists a (random) $\varepsilon^t_{i,j} > 0$ such that given $\varepsilon^t_{i,j},$

$$\tau^t_{j,a} - \sigma^t_{i,a} - \sum_{I^t_k \in [\sigma^t_{i,a}, \tau^t_{j,a}]} |I^t_k| 1_{(I^t_k \text{ maximal})}$$

is stochastically larger than $T^{(t,a)}_{z^t_i}$. This ensures the a.s. separation of the $I^t_i$'s, so the a.s. convergence of $\text{UNPLUG}(X, (I^t_i : z^t_i > a))$ to $\text{UNPLUG}(X, (I^t_i, i \geq 1))$ as $a \downarrow 0$ comes from Lemma 6. Identifying the limiting law follows from the above discussion. \qed

Let now $X^t_i(s) = X_{\tau^t_i + s} - X_{\tau^t_i}$ for $0 \leq s \leq \sigma^t_i - \tau^t_i$ and $i \geq 0$, where by convention $\tau^t_0 = 0$, and $\sigma^t_0 = T_i$. We write $-\tau^t_i + I^t_k = [0, \sigma^t_k - \tau^t_k]$. The next lemma does most of the job to extract the different tree components of the logged stable tree at time $t$.

**Lemma 8.** (i) Under the law $P^{(-1,\infty)}$, as $a \downarrow 0$, the processes $\text{UNPLUG}(X^t_i, (-\tau^t_i + I^t_k, k : I^t_k \subseteq I^t_i \text{ and } z^t_i > a), i \geq 1$ converge in $D$ to the processes $Y^t_i = \text{UNPLUG}(X^t_i, (-\tau^t_k + I^t_k, k : I^t_k \subseteq I^t_i), i \geq 1$.

(ii) The process $Y^t_i$ has the same law as $z^t_i + X^{(i)}$ under $P$, killed when it first hits 0, and these processes are independent conditionally on $(z^t_i, i \geq 1)$.

(iii) The sum of the durations of $Y^t_i, i \geq 1$ equals $T_i$ a.s.

**Proof.** (i) Fix $a > 0$, we modify slightly the notations of the preceding proof by letting $\tau^{t,a}_{1} < \cdots < \tau^{t,a}_{k(a)}$ be the times when $Z^{(i)}$ accomplishes a jumps that is $> a$, and letting $\sigma^{t,a}_{i,a} = \inf \{ u \geq \tau^{t,a}_{i,a} : X_u = X_{\tau^{t,a}_{i,a} + a} \}$. Let also $\tau^{t,a}_{0,a} = 0, \sigma^{t,a}_{0,a} = T_i$. Write $I^{t,a}_i = [\tau^{t,a}_{i,a}, \sigma^{t,a}_{i,a}]$, and let $X^{t,a}_i(s) = X_{\tau^{t,a}_{i,a} + s} - X_{\tau^{t,a}_i} \text{ for } 0 \leq s \leq \sigma^{t,a}_{i,a} - \tau^{t,a}_{i,a}$. By the Markov property at times $\tau^{t,a}_{i,a}, \sigma^{t,a}_{i,a}$, we obtain that for every $i$, $X^{t,a}_i$ is independent of $\text{UNPLUG}(X, I^{t,a}_i)$ given the jump
\[ \Delta X_{i,t^a}. \] By a repeated use of the Markov property, we obtain the independence of the processes

\[ \text{UNPLUG}(X_{i,t^a}^i, (R_{i,t}^j : I_{i,t}^j \subseteq I_{i,t}^k)) \] given \((\Delta X_{i,t^a}, 1 \leq i \leq k(a))\), and moreover, the law of \(X_{i,t^a}^i\) given \(\Delta X_{i,t^a}\) is that of \(X\) under \(P\), killed when it first hits \(-\Delta X_{i,t^a}\). Letting \(a \downarrow 0\) and applying Lemma 7 finally gives the convergence to the processes \(\text{UNPLUG}(X_{i,t}^i, (I_{i,t}^k : I_{i,t}^k \subseteq I_{i,t}))\), as well as the conditional independence and the distribution of the processes, giving also (ii).

(iii) Let us introduce some extra notation. Say that the marked jump with magnitude \(z_i^j\) is of the \(j\)-th kind if and only if the future infimum process \((\inf_{s \leq u \leq t} X_u, 0 \leq s \leq t)\) accomplishes exactly \(j\) jumps at times that correspond to marked jumps of \(X\). Write \(|I_t^i|\) for the duration of \(X_t^i\) and let \(A_j\) be the set of indices \(i\) such that \(\tau_i^j\) is a jump time of the \(j\)-th kind. By a variation of Lemma 4 already used above, every marked jump is of the \(j\)-th kind for some \(j\) a.s. By Lemma 7 the duration of \(Y_{t}^i\) is \(T_1 - \sum_{j \in A_j} |I_t^i|\), similarly, one has that if \(i \in A_j\), the duration of \(Y_t^i\) equals \(|I_t^i| - \sum_{k \in A_{j+1}} |I_k^i| 1_{(I_k^i \subseteq I_t^i)}\). Therefore, proving that the sum of durations of \(Y_t^i\) equals \(T_1\) amounts to showing that \(\sum_{j \in A_j} |I_t^i| \to 0\) in probability as \(j \to \infty\). But the sum of the marked jumps is finite a.s., since conditionally on a marked jump \(z_i^j\), the duration of the corresponding \(X_t^i\) has same law as \(T_{z_i^j}\), and since we have independence as \(i\) varies. Hence this sum is (conditionally on \((z_i^j, i \geq 1)\)) equal in law to \(\sum_{j \in A_j} z_i^j\) under \(P\), and it converges to 0. \(\Box\)

**Lemma 9.** The process \((F^+(t), t \geq 0)\) is a Markovian self-similar fragmentation with index 1/\(\alpha\). Its erosion coefficient is 0

**Proof.** For every \(v > 0\), define the processes \(X_t^i\) under \(N^{(v)}\) as in the preceding section, replacing the duration 1 by \(v\). By virtue of the Lemma 8 and by excursion theory, we obtain that for almost every \(v > 0\), and for all \(t\) in a dense countable subset of \(\mathbb{R}_+\), under \(N^{(v)}\), the processes \(\text{UNPLUG}(X_t^i, (k : I_k^i \subseteq I_t^i \text{ and } z_k^i > v))\) converge as \(a \downarrow 0\) to processes \(Y_t^i\) that are independent conditionally on the \(z_k^i\)'s and on their durations, and whose durations sum to \(v\) (by convention we let \(X_0^i = X\)). By scaling, this statement remains valid for \(v = 1\). We then extend it to all \(t \geq 0\) by a continuity argument. The case \(t = 0\) is obvious, so take \(t_0 > 0\) and \(t \uparrow t_0\) in the dense subset of \(\mathbb{R}_+\). Almost surely, \(t_0\) is not a time at which a new hub is marked, so \(X_t^{t_0} = X_t^i\) for \(t\) close enough of \(t_0\), and by Lemma 6 and the fact that \(\{I_t^i, i \geq 0\} \subset \{I_t^{t_0}, i \geq 0\}\) for \(t \leq t_0\),

\[ Y_i^{t_0} = \text{UNPLUG}(X_i^{t_0}, I_{i,t}^k \subseteq I_{i,t_0}^k) = \lim_{t \uparrow t_0} \text{UNPLUG}(X_t^i, I_k^i \subseteq I_t^i). \]

Now write \(Y_{i,j}^{t_0}\) for the excursions of \(Y_t^i\) above its infimum, ranked in decreasing order of durations. Then by the same arguments as in the proof of Lemma 7, the joint law of the durations of \(Y_t^i, Y_{i,j}^t, i \geq 1, j \geq 1\) equals the law of \((|C_{i,j}^1|, |C_{i,j}^2|, i \geq 1, j \geq 1)\) with notations above. Hence, by Lemma 5 and the fact that excursions of \(X^{(v)}\) with prescribed duration are stable excursions, it holds that conditionally on \(F^+(t) = (x_1, x_2, \ldots)\), the excursions \(Y_{i,j}^{t_0}\) are independent stable excursions with respective durations \(x_1, x_2, \ldots\).

Now let \(\sim_{t_0}^{i,j}\) be the equivalence relation defined for the excursion \(Y_{i,j}^{t_0}\) in a similar way as \(\sim_i\) for the normalized excursion of \(X\). Write also \(j_i(u) = u - \sum_{k : t_k \subseteq t_i, t_k < u} |I_k^i|\) for \(u \in [0, 1]\). Then it is clear that if \(x, y \in \mathcal{C}_{i,j}^t\), one has also \(x \sim_{t_0}^{i,j} y\) if and only if \(j_i(x) \sim_{t_0}^{i,j} j_i(y)\). By the scaling property, a stable excursion \(\varepsilon^x\) with duration \(x\) where every jump with magnitude \(\ell\) is marked with probability \(1 - \exp(-t^\ell\ell)\) is obtained by taking a normalized excursion \((\varepsilon_s^x, 0 \leq s \leq 1)\), marking every jump with magnitude \(\ell\) independently with probability \(1 - \exp(t^\ell x^{1/\alpha})\), and
then letting \( \varepsilon^x_s = x^{1/\alpha} \varepsilon^x_{s/x} \) for \( 0 \leq s \leq x \); the marked jumps of \( \varepsilon^x \) occurring at the times \( sx \) whenever \( s \) is a marked jump time for \( \varepsilon^x \). This means that given \( F^+(t) = (x_1, \ldots) \), the process \( (F^+(t+t'), t' \geq 0) \) has the same law as \( (x_1 F^+, x_2 F^+, \ldots) \) where the \( F^+ \)'s are independent copies of \( F^+ \). This entails both the Markov property and the self-similar property, the self-similarity index being \( 1/\alpha \). Moreover, Lemma 8 (iii) shows that the sum of durations of \( Y_{i,j}^t \) is 1 a.s. under \( N^{[1]} \), so \( \sum_i F^+_i(t) = 1 \) a.s. and the erosion coefficient must be 0 according to [10].

To conclude, we notice that the previous result of continuity in probability of \( F^+ \) at time 0 extends to any time \( t \geq 0 \) by the self-similar fragmentation property.

\[ \square \]

### 4.2 Splitting rates and dislocation measure

To complete the study of the characteristics of \( F^+ \), we must identify the dislocation measure. This is done by computing the splitting rate of the stable tree, that is, the rate at which the tree with mass 1 instantaneously splits into a sequence of subtrees with given masses \( s_1 \geq s_2, \ldots \) with \( \sum s_i = 1 \), by analogy with the splitting rate of the Brownian CRT in [3]. Recall that our marking process on the hubs of the tree amounts to taking a Poisson process with intensity \( m(dv) = \sum L(b) \delta_b(dv) \) on \( T \), where the sum is over hubs \( b \) in \( T \). For \( v \in T \), let \( T_1(v), T_2(v), \ldots \) be the tree components of the forest obtained when removing \( v \), arranged by decreasing order of masses, and let

\[ r(ds) = \mathbb{E}m\{v \in T : (\mu(T_1(v)), \mu(T_2(v)), \ldots) \in ds\} \]

be the rate at which a \( m \)-picked vertex splits \( T \) into trees with masses in a volume element \( ds \). It is quite intuitive that the splitting rate equals the dislocation measure of \( F^+ \), and Theorem 1 reduces to the following lemmas:

**Lemma 10.** The splitting rate \( r(ds) \) equals the dislocation measure of \( F^+ \).

**Proof.** For \( t \geq 0 \) we let \( T(t) \) be the forest obtained by our logging procedure of the stable tree at time \( t \). Let \( n \geq 2 \), and consider \( n \) leaves \( L_1, \ldots, L_n \in T \) that are independent and distributed according to the mass measure \( \mu \), conditionally on \( \mu \) (we are implicitly working on an enlarged probability space). Write \( \Pi_n(t) \) for the partition of \( [n] = \{1, \ldots, n\} \) obtained by letting \( i \) and \( j \) be in the same block of \( \Pi_n(t) \) if and only if \( L_i \) and \( L_j \) belong to the same tree component of \( T(t) \). For \( K > 2 \) let \( \Lambda^n_K(t) \) be the event that at time \( t \), the leaves \( L_1, \ldots, L_n \) are all contained in tree components of \( T(t) \) with masses \( > 1/K \). Write \( \mathcal{P}_n^\pi \) for the set of partitions \( \pi \) of \( [n] = \{1, \ldots, n\} \) with at least two non-void blocks \( A_1, \ldots, A_k \) (for some arbitrary ordering convention). Given \( F^+(t) = s = (s_1, s_2, \ldots) \), the probability that \( \Pi_n(t) \) equals some partition \( \pi \in \mathcal{P}_n^\pi \) and that \( \Lambda^n_K(t) \) happens is

\[ G_K(s) = P(\Pi_n(t) = \pi, \Lambda^n_K(t) | F^+(t) = s) = \sum_{i_1, \ldots, i_k} \prod_{j=1}^k s_{i_j}^{A_j}, \]

the sum being over pairwise distinct \( i_j \)'s such that \( s_{i_j} > 1/K \). This last function is continuous and null on a neighborhood of \( (1, 0, \ldots) \), so Proposition 1 (which use is enabled by Lemma 9)
We claim that knowing this quantity for every $n, \pi, K$ characterizes $\nu$. One can obtain this by first letting $K \to \infty$ by monotone convergence, and then using an argument based on exchangeable partitions as in [19, p. 378] (a Stone-Weierstrass argument can also work).

On the other hand, for any $b$ in the set $B(\mathcal{T})$ of branchpoints of $\mathcal{T}$, let $\pi^b_n$ be the partition of $[n]$ obtained by letting $i$ and $j$ be in the same block if and only if $b$ is not on the path from $L_i$ to $L_j$. Let also $\mathcal{T}_i(b)$ be the tree component of the forest obtained by removing $b$ from $\mathcal{T}$ that contains $L_i$. For $K \in (2, \infty]$ and $\pi \in \mathcal{P}_n$, let $\Psi^b_K(\pi)$ be the set of branchpoints $b \in \mathcal{T}$ such that $\pi^b_n = \pi$ and such that $\mu(\mathcal{T}_i(b)) > 1/K$ for $1 \leq i \leq n$, and let $\Psi^b_K = \bigcup_{\pi \in \mathcal{P}_n} \Psi^b_K(\pi)$. Recall that we may construct the fragmentation $F^+$ by cutting the stable tree at the points of a Poisson point process $(b(s), s \geq 0)$ with intensity $ds \otimes m(db)$. Now for $\Pi_n(t) = \pi$ to happen, it is plainly necessary that at least one $b(s)$ falls in $\Psi^b_n$ for some $s \in [0, \bar{t}]$, if in addition $\Lambda_n^b(t)$ happens then no $b(s), 0 \leq s \leq t$ must fall in $\Psi^b_n \setminus \Psi^b_K$. Therefore,

$$
P(\Pi_n(t) = \pi, \Lambda_n^b(t)) = P(\exists s \in [0, \bar{t}]: b(s) \in \Psi^b_n, \text{ and } b(s) \in \Psi^b_K(\pi), \Lambda_n^b(t)) + R(t),$$

where the residual $R(t)$ is bounded by the probability that $b(s)$ falls in $\Psi^b_n$ for at least two $s \in [0, \bar{t}]$. Hence $R(t) = o(t)$ by standard properties of Poisson processes provided we can show that $E[m(\Psi^b_n)] < \infty$. This could be shown using the forthcoming lemma, but we may also just notice that if $E[m(\Psi^b_n)]$ was infinite, then there would be arbitrarily many $b(s), 0 \leq s \leq t$ falling in $\Psi^b_n \setminus \Psi^b_K$ for some appropriately large $K$, and the above probability would be 0, which is impossible from the beginning of this proof and since $F^+$ is a self-similar fragmentation with nonzero dislocation measure (because it has erosion coefficient 0 and it is not constant). On the other hand, conditionally on the event on the right-hand side of (8), the $b(s), 0 \leq s \leq t$ that do not fall in $\Psi^b_n$ (call them $b'(s)$) form an independent Poisson point process with intensity $m(\cdot \cap B(\mathcal{T}) \setminus \Psi^b_n)$. Therefore, the size of the tree component of the forest obtained when removing the points $b'(s), 0 \leq s \leq t$ that contains $L_1$ converges a.s. to 1 as $t \downarrow 0$ (so it also contains the other $L_i$'s for small $t$ a.s.), as it is stochastically bigger that the component of $\mathcal{T}(t)$ containing $L_1$, and since $F^+(t) \to (1, 0, \ldots)$ in probability as $t \downarrow 0$. It follows that one can remove $\Lambda_n^b(t)$ from the right-hand side of (8), and basic properties of Poisson measures finally give

$$t^{-1}P(\Pi_n(t) = \pi, \Lambda_n^b(t)) \to E[m(\Psi^b_K(\pi))],$$

which is equal to $\int_S r(ds) \sum_{k=1}^{s_K} \prod_{j=1}^k s_{ij}^{\#A_j}$ since $L_i$ belongs to $B \subset \mathcal{T}$ with probability $\mu(B)$ that is equal to the Lebesgue measure of the subset of $[0, 1]$ encoding $B$. Identifying with (7) gives the claim.

**Lemma 11.** One has $r(ds) = \nu_\alpha(ds)$ with the notations of Theorem 1.

**Proof.** We must see what is the effect of splitting $\mathcal{T}$ at a hub $b$ picked according to $m(db)$. Recall that $m$ picks a hub proportionally to its local time, and that hubs are in one-to-one correspondence with jumps of the stable excursion with duration 1. More precisely, if $b$ is the hub that has been picked and with the notations $\tau(b), \sigma(b)$ above, the masses of the tree components obtained when removing $b$ are equal to the lengths of the constancy intervals of the infimum process of $(X_{\tau(b)}+s - X_{\tau(b)}, 0 \leq s \leq \sigma(b) - \tau(b))$, and the extra term $1 - (\sigma(b) - \tau(b))$. 20
Now by Vervaat’s theorem, we may suppose that the excursion is the Vervaat transform of a stable bridge and that the marked jump in the excursion corresponds to a jump \((s, \Delta X_s)\) of the bridge picked according to the \(\sigma\)-finite measure \(\sum_{u: \Delta X_u > 0} \Delta X_u \delta_{u, \Delta X_u}(ds, dx)\). By Lemma 2, the bridge has the same law as \(X \oplus (s, x)\), where \((s, x)\) is independent of \(X\), with a certain \(\sigma\)-finite “law” and \(X\) has law \(P^1_{0 \to -x}\). Now a bridge is invariant under cyclic shift, so we may suppose without change that the extra jump with size \(x\) occurs at time 0. This shows that the sizes of the components of the split CRT have the same law as the sequence constituted of \(1 - T_x\) and the lengths of the constancy intervals of the infimum process of \((X_u, 0 \leq u \leq T_x)\), under the law \(P^1_{0 \to -x}\).

It is now easy that conditionally on \(x, T_x = t\) these constancy intervals have the same law as \(\Delta T_{[0,x]}\) given \(T_x = t\) under \(P\) (one checks that \((X_u, 0 \leq u \leq T_x)\) is the first-passage bridge with law \(P^1_{0 \to -x}\) below). The law of \(1 - T_x\) given \(x\) is simply obtained by using the definition of bridges and the Markov property: for \(a < 1\) and positive measurable \(f\),

\[
E^1_{0 \to -x}[f(1 - T_x) 1_{\{T_x < a\}}] = E^1[f((1 - T_x) 1_{\{T_x < s\}} p_1(-x)^{-1} p_{1-a}(-x - X_a))]
\]

\[
= \int_0^s ds q_x(s) f(1 - s) p_1(-x)^{-1} \int dy p_{1-s}(y) p_{1-a}(-y)
\]

\[
\xrightarrow{a \to 1} \int_0^1 ds q_x(s) f(1 - s) p_{1-s}(0) p_1(-x)^{-1}.
\]

In the last integral, change variables \(1 - s \to s\), use \(p_1(-x) = x^{-1} q_x(1)\), check by scaling that \(p_4(0) = s^{-1/\alpha} p_1(0)\), and conclude by identifying with Lemma 1 that \(1 - T_x\) under \(P^1_{0 \to -x}\) has same law as a size-biased pick from \(\Delta T_{[0,x]}\) given \(T_x = 1\) under \(P\) (notice that in particular we must have \(p_1(0) = c_\alpha\)). By Lemma 1 (ii), it follows that given the local time \(x\) of the marked hub \(b\), the law of the sizes of the stable tree split at this hub is the same as that of \(\Delta T_{[0,x]}\) given \(T_x = 1\) under \(P\).

Putting pieces together and recalling the distribution of the marked jump \(x\) from Lemma 2 we obtain the formula

\[
r(ds) = \int_0^\infty dx \frac{C_\alpha p_1(-x)}{x^\alpha p_1(0)} P(\Delta T_{[0,x]} \in ds | T_x = 1).
\]

By using the scaling property for \(T\) and its density \((q_x(1) = x^{-\alpha} q_1(x^{-\alpha}))\), formula (3) and a change of variables, we obtain

\[
r(ds) = \int_0^\infty dx \frac{C_\alpha q_1(x^{-\alpha})}{c_\alpha x^{2\alpha+1}} P(T_1^{-1} \Delta T_{[0,1]} \in ds | T_1 = x^{-\alpha})
\]

\[
= \alpha^{-1} c_\alpha^{-1} C_\alpha \int_0^\infty du u q_1(u) P(T_1^{-1} \Delta T_{[0,1]} \in ds | T_1 = u),
\]

which gives the desired formula, after checking that \(\alpha^{-1} c_\alpha^{-1} C_\alpha = D_\alpha\). \(\square\)

## 5 Study of \(F^1\)

Recall the construction of \(F^1\) from Sect. 1. As noticed above, this fragmentation process somehow generalizes the one considered in \([7, 21]\) (we could actually build it in an analogous way
for a large class of Lévy processes with no negative jumps, though the resulting fragmentations would not be self-similar due to the absence of scaling. Notice that none of the fragmentation of [21] are self-similar, but for the Brownian case. The reason for this was a lack of a Girsanov-type theorem saying that a Lévy process plus drift has a law that is absolutely continuous with the initial process, but for the Brownian case. Here, this is fixed by Lemma 2, but where the operation is removing jumps rather than adding a drift.

5.1 The self-similar fragmentation property

For any \( t' > t \geq 0 \) let \( \mu_t(x, ds) \) be a kernel from \( \mathbb{R}_+^2 \) to \( S \) defined as follows: \( \mu_t(x, ds) \) is the law of the ranked lengths of the constancy intervals of the process \( X^{(t)} \) under \( N^{(x)} \). Moreover, define \( F_i^{t,1} \) exactly as \( F_i \), but where \( X \) is under the law \( P^{(-1, \infty)} \). In particular, \( F_i^{t,1}(t) \) is not \( S \)-valued (the sum of its components is random).

**Proposition 4.** (i) The processes \( F_i^{t,1} \) and \( F_i^t \) enjoy the fragmentation property, with fragmentation kernel \( \mu_t(x, ds) \). That is, conditionally on \( F_i^{t,1}(t) = (x_1, x_2, \ldots) \) (resp. \( F_i^t(t) \)), \( F_i^{t,1}(t + t') \) (resp. \( F_i^t(t + t') \)) has the same law as the decreasing rearrangement of independent sequences \( s_i \) with respective laws \( \mu_{t'}(x_i, ds) \).

(ii) The process \( F_i^t \) is a self-similar fragmentation with index \( 1/\alpha \), and no erosion.

The fact that \( F_i^t \) is a fragmentation process directly comes from the fact that the processes \( X^{(t')} - X^{(t)} = Z^{(t')} - Z^{(t)} \) are non-increasing. We now prove the fragmentation property. The key lies in a Skorokhod-like relation that is analogous to that in [7] and generalized in [21].

**Lemma 12.** For every \( t, t' \geq 0 \) with \( t < t' \) and \( s \geq 0 \), one has

\[
X^{(t)}_s = \inf_{0 \leq u \leq s} (X^{(t')}_u + (Z^{(t')}_u - Z^{(t)}_u)).
\]

The proof can be done following exactly the same lines as in [7, Lemma 2]. As a consequence, we obtain that the sigma-field \( \mathcal{G}_t = \sigma\{X^{(t)}, (Z^{(s)}, 0 \leq s \leq t)\} \) induces a filtration, with respect to which \( F_i^{t,1} \) is adapted.

The end of the proof of the fragmentation property in Proposition 4 also goes as in [7]. For any variable \( K \) that is \( \mathcal{G}_t \)-measurable, the excursions of \( X^{(t)} \) above its infimum and before time \( T^{(t)}_K \) are independent excursions conditionally on \( \mathcal{G}_t \), respectively conditioned to have durations \( \ell_{1,x}^{(t)}, \ell_{2,x}, \ldots \), where the last family is the decreasing sequence of constancy intervals of \( X^{(t)} \) before time \( T^{(t)}_K \). Take \( K = X^{(t)}_{T^{(t)}_1} \), which is measurable with respect to \( \mathcal{G}_t \) by virtue of the Skorokhod property. Then \( T^{(t)}_K = T_1 \), which gives readily that conditionally on \( \mathcal{G}_t \), the excursions of \( X^{(t)} \) above \( X^{(t)}_{T^{(t)}_1} \) are independent with durations \( (F_i^{t,1}(t), i \geq 0) \).

To conclude, it remains to notice that the lack of memory of the exponential law implies that the jumps that are unmarked at time \( t \) but that are marked at time \( t + t' \) can be obtained also by marking with probability \( 1 - e^{-t' \ell} \) any unmarked jump at time \( t \) that has magnitude \( \ell \). Thus, conditionally on \( F_i^{t,1}(t) \), we obtain a sequence with the same law as \( F_i^{t,1}(t + t') \) by taking independent sequences \( (s_i, i \geq 1) \) with laws \( \mu_{t'}(F_i^{t,1}(t), ds) \) and rearranging, as claimed. This remains true for \( F_i^t \) by excursion theory and scaling.
To show the self-similarity for $F^\alpha$, it then suffices to check, using the scaling property of the excursions of stable processes, that $\mu_\alpha(x, ds)$ is the image of $\mu_{t^{\alpha}/\alpha}(1, ds)$ by $s \mapsto xs$. The fact that $F^\alpha$ has no erosion again comes from the fact that $\sum_i F^\alpha_i(t) = 1$ a.s.

5.2 The semigroup

According to the preceding section, and since plainly there is no loss of mass in the fragmentation $F^\alpha$ (so the erosion coefficient is 0), proving Theorem 2 requires only to check that the dislocation measure of $F^\alpha$ equals that of $F^+$. It is intuitively straightforward that this is the case, by looking at the procedure we use for deleting jumps, and indeed we could easily follow the same lines as above and compute a “splitting rate” for the bridge, when the “first” marked jump is deleted. However, a nice feature of this fragmentation is that we can compute explicitly its semigroup (hence that of $F^+$), as will follow. The semigroup then gives enough information to re-obtain the dislocation measure, and this will prove Theorem 2. Recall from Sect. 2 that $\rho^{(t)}_1$ is the density of $Z_1^{(t)}$ under $\mathbb{P}$.

Proposition 5. The semigroup of $F^1$ is given by

$$
P(F^1(t) \in ds) = \int_0^\infty dz \frac{p^{(t)}_1(-z) \rho^{(t)}_1(z)}{p_1(0)} P(\Delta T_{[0,z]} \in ds| T_z = 1).
$$

We will need a couple of intermediate lemmas. Since $Z^{(t)}$ is non-decreasing, under the law $\mathbb{N}^{(1)}$, the process $X^{(t)}$ starts at 0 and hits $-Z^{(t)}_1$ at time 1 for the first time. Since we are interested in the constancy intervals of $X^{(t)}$, and thanks to Vervaat’s theorem, we would like to relate these constancy intervals to the bridge of $X$. We now work under the law of the bridge with unit duration $\mathbb{P}_{1_{[0,\infty)}}$, so we may suppose that the excursion of $X$ with duration 1 is equal to the Vervaat transform $VX$. Let $m = -X_1$ be the absolute value of the minimum of $X$, and $\tau_2 = T_{m-}$ be the (a.s. unique) time when $X$ attains this minimum, so $VX = V^{\tau_2}X$. Decompose $X$ as $X^{(t)} + Z^{(t)}$ where $Z^{(t)}$ is the cumulative process of marked jumps. Then $VX = V^{\tau_2}X^{(t)} + V^{\tau_2}Z^{(t)}$, and by independence of the marking procedure of jumps we can consider that $V^{\tau_2}Z^{(t)}$ is the cumulative process of marked jumps for the excursion $VX$. The problem is now to describe the law of lengths of the constancy intervals of the process $\overline{V^{\tau_2}X^{(t)}}$. Let $m^{(t)} = -\overline{X^{(t)}}$ be the absolute value of the minimum of $X^{(t)}$ and $\tau_3 = T^{(t)}_{m^{(t)}-}$ be the (a.s. unique) time when $X^{(t)}$ attains this minimum. Let also $\tau_1 = T^{(t)}_{m^{(t)}-Z^{(t)}_1}$ be the first time when $X^{(t)}$ attains the value $Z_1^{(t)} - m^{(t)}$. The following lemma is somehow “deterministic”. For $a < b$, write $X_{[a,b]}$ for the process $(X_{a+s} - X_s, 0 \leq s \leq b - a)$.

Lemma 13. One has $\tau_1 \leq \tau_2 \leq \tau_3$ a.s., and the sequence of lengths of the constancy intervals of $\overline{V^{\tau_2}X^{(t)}}$, ranked in decreasing order, are equal to that of the process $\overline{X^{(t)}_{[\tau_1,\tau_3]}}$, to which has been added (at the appropriate rank) the extra term $1 - \tau_3 + \tau_1$.

Proof. Since $Z^{(t)}$ is an increasing process, one has $X^{(t)}_{\tau_2} = X_{\tau_2} - Z^{(t)}_{\tau_2} \leq X_s - Z^{(t)}_s$ for any $s \leq \tau_2$. Hence, $X^{(t)}_{\tau_2} = \overline{X^{(t)}_{\tau_2}}$ which implies $\tau_2 \leq \tau_3$. On the other hand, one has $-m(t) = X_{\tau_3} - Z^{(t)}_{\tau_3} \geq -m - Z^{(t)}_1$ and thus $m(t) - Z^{(t)}_1 \leq m$, implying $\tau_1 \leq \tau_2$.
For convenience, if \((f(x), 0 \leq x \leq \zeta)\) and \((f'(x), 0 \leq x \leq \zeta')\) are two càdlàg functions, we let \(f \bowtie f'\) be the concatenation of the paths of \(f\) and \(f'\), defined by
\[
f \bowtie f'(s) = \begin{cases} 
f(s) & \text{if } 0 \leq s < \zeta \\
f'(s - \zeta) + f(\zeta) & \text{if } \zeta \leq s \leq \zeta + \zeta'.
\end{cases}
\]
We let \(Y^1 = X^{(t)}_{[0,\tau_2]}\), \(Y^2 = X^{(t)}_{[\tau_2,\tau_3]}\) and \(Y^3 = X^{(t)}_{[\tau_3,1]}\), so \(X^{(t)} = Y^1 \bowtie Y^2 \bowtie Y^3\), and \(V^{(t)} X^{(t)} = Y^2 \bowtie Y^3 \bowtie Y^1\).

Observing that \(Y_3\) is non-negative, we obtain that \(V^{(t)} X^{(t)} = Y^2 \bowtie Y^3 = 0_{[0,1-\tau_3]}\) where \(0_{[0,a]}\) is the null process on \([0,a]\). Since the final value of \(Y_3\) is \(m(t) - Z^{(t)}_1\), we obtain that
\[
V^{(t)} X^{(t)} = Y^2 \bowtie 0_{[0,1-\tau_3]} \bowtie 0_{[0,\tau_1]} \bowtie X^{(t)}_{[\tau_1,\tau_2]} \bowtie Y^2 \bowtie 0_{[0,1-\tau_3+\tau_1]} \bowtie X^{(t)}_{[\tau_1,\tau_3]}.
\]
It follows that the constancy intervals of \(V^{(t)} X^{(t)}\) are the same as those of \(X^{(t)}\), except for the first and last constancy intervals of \(X^{(t)}\) which are merged to form the constancy interval with length \(1 - \tau_3 + \tau_1\).

The rest of the section is devoted to the study of these constancy intervals. Recall from Lemma 3 that under \(P^{1}_{0 \to -v}\), the process \(X^{(t)}\) has law \(P^{1}_{0 \to -z}\), where \(Z\) is an independent random variable with law \(P(Z \in dz) = p_1(0)^{-1}p^{(t)}_1(-z)\rho^{(t)}_1(z)dz\). It thus suffices to analyze the constancy intervals of \(X^{(t)}_{[\tau_1,\tau_3]}\) under the law \(P^{v}_{0 \to -z}\) for fixed \(z > 0\), where we now call \(m = -X^{(t)}_{\tau_1}\), \(\tau_3\) the time when \(X\) first hits level \(-m\) and \(\tau_3\) the first time when \(X\) attains level \(-m\).

For \(z > 0\), let \((P^{v}_{0 \to -z}, v > 0)\) be a regular version of the conditional law \(P^{-z,\infty}[T_z = v]\). Call this the law of the first-passage bridge from 0 to \(-z\) with length \(v\). A consequence of the Markov property is

**Lemma 14.** Let \(a, b > 0\). For (Lebesgue) almost every \(v > 0\), under the law \(P^{v}_{0 \to -(a+b)}\), the law of \(T_a\) is given by
\[
P^{v}_{0 \to -(a+b)}(T_a \in ds) = ds \frac{q_a(s)q_b(v-s)}{q_{a+b}(v)}.
\]
Moreover, conditionally on \(T_a\), the paths \((X_s, 0 \leq s \leq T_a)\) and \((X_s + T_a - a, 0 \leq s \leq T_{a+b} - T_a)\) are independent with respective laws \(P^{T_a}_{0 \to a}\) and \(P^{v-T_a}_{0 \to b}\).

We also state a generalization of Williams’ decomposition of the excursion of Brownian motion at the maximum, given in Chaumont [12]. We need to make a step out of the world of probability and consider \(\sigma\)-finite measures instead of probability laws. Recall that \(m_v = -X_v\) is the absolute value of the minimum before time \(s\), and with our notations \(T_{m(v)}-\) is the first time (and a.s. last before \(v\)) when \(X\) attains this value. Write
\[
X_s = X_v, \quad 0 \leq s \leq T_{m(v)}-,
\]
\[
\Delta X_s = m_v + X_{s+T_{m(v)}-}, \quad 0 \leq s \leq v - T_{m(v)}-.
\]
for the pre- and post- minimum processes of \(X\) before time \(v\). Then by [12],

**Lemma 15.** One has the identity for \(\sigma\)-finite measures
\[
\int_0^\infty dv P^v(\Delta X_v \in d\omega, X_s \in d\omega') = \int_0^\infty dx P^{(v,x,\infty)}(d\omega) \otimes \int_0^\infty du N^{>u}(d\omega'),
\]
where \(N^{>u}(d\omega') := N^u(d\omega', \zeta > u)\). This in turn determines entirely the laws \(P^v\) for \(v > 0\).
Loosely speaking, if \( v \) is “random” with “law” the Lebesgue measure on \((0, \infty)\), the pre- and post-minimum processes are independent with respective “laws” \( \int_0^\infty dx P^{(x, \infty)}(d\omega) \) and \( \int_0^\infty du N^{>u}(dX') \), where \( N^{>u} \) is the finite measure characterized by \( N^{>u}(F(X)) = N(F(X_u, 0 \leq s \leq u), \zeta(X) > u) \). As a consequence of this identity, we have that under \( P^v \) for some fixed \( v > 0 \), conditionally on \( m_v \) and \( T_{m_v-} = \tau \), the processes \( X \) and \( \Delta X \) are independent with respective laws \( P^\tau_{0_{1-m_v}(\omega)}(d\omega) \) and \( (N^{>\tau-\tau}(1))^{-1} N^{>\tau-\tau}(d\omega') \).

**Lemma 16.** Let \( z > 0 \). Under the probability \( P^{\tau}_{0_{1-m_v}} \), conditionally on \( \tau_3 - \tau_1 = t \), the ranked sequence of lengths of the constancy intervals of the infimum process of \( (X_{s+\tau_1}, 0 \leq s \leq \tau_3 - \tau_1) \) have the same law as \( \Delta T_{[0,z]} \) given \( T_z = t \) under \( P \).

**Proof.** We first condition by the value of \((m, \tau_3)\). Then by Lemma 15 the path \( \Delta X \) has the law \( P^\tau_{0_{1-m_v}} \) of the first-passage bridge from 0 to \(-z\) with lifetime \( \tau_3 \). Applying Lemma 14 and the Markov property we obtain that conditionally on \( \tau_1 \) the path \((X_{s+\tau_1} + m - z, 0 \leq s \leq \tau_3 - \tau_1)\) is a first passage bridge ending at \(-z\) at time \( \tau_3 - \tau_1 \). Since it depends only on \( \tau_3 - \tau_1 \), we have obtained the conditional distribution given \( \tau_3 - \tau_1 \). Hence, the sequence defined in the lemma’s statement has the same conditional law as the ranked lengths of the constancy intervals of the infimum process of such a first-passage bridge, that is, it has the same law as \( \Delta T_{[0,z]} \) given \( T_z = \tau_3 - \tau_1 \), with \( T' \) as in the statement.

The last lemma gives an explicit form for the law of the remaining length \( 1 - \tau_3 + \tau_1 \) under \( P^{\tau}_{0_{1-m_v}} \).

**Lemma 17.** One has

\[
P^{\tau}_{0_{1-m_v}}(1 - \tau_3 + \tau_1 \in ds) = ds \frac{c_\alpha z_q_z(1 - s)}{s^{1/\alpha} q_z(1)},
\]

which is the law of a size-biased pick of the sequence \( \Delta T_{[0,z]} \) given \( T_z = 1 \) under \( P \).

**Proof.** By Lemma 15, if \( s \) is “distributed” according to Lebesgue measure on \( \mathbb{R}_+ \), then under \( P^s \), the processes \( X \) and \( \Delta X \) are independent with respective “laws” \( \int_0^\infty dx P^{(x, \infty)}(d\omega) \) and \( \int_0^\infty du N^{>u}(d\omega') \). Our first task is to disintegrate these laws to obtain a relation under \( P^{\tau}_{0_{1-m_v}} \). Let \( H \) and \( H' \) be two continuous bounded functionals and \( f \) be continuous with a compact support on \((0, \infty)\). Then, letting \( T'_x = \inf\{s \geq 0 : \omega(s) < \cdot\} \),

\[
\int_0^\infty ds f(s) E[sH(X_s)(X_{s+z}) \mathbb{1}_{|X_{s+z}| < \varepsilon}]
= \int_0^\infty dx \int_0^\infty du \int P^{(x, \infty)}(d\omega) N^{>u}(d\omega') f(T'_x + u) H(\omega) H'(\omega') \frac{1_{|\Omega(x+u(1-s))| < \varepsilon}}{P(|X_{T'_x+u+z}| < \varepsilon)}
= \int_0^\infty du \int N^{>u}(d\omega') H'(\omega') \int_0^\infty dx \frac{1_{|\Omega(x+u(1-s))| < \varepsilon}}{2\varepsilon} \int P^{(x, \infty)}(d\omega) \frac{f(T'_x + u) H(\omega)}{(2\varepsilon)^{-1} P(|X_{T'_x+u+z}| < \varepsilon)}.
\]

The measure \((2\varepsilon)^{-1} 1_{|\Omega(x+u(1-s))| < \varepsilon}dx\) converges weakly as \( \varepsilon \to 0 \) to the Dirac mass at \( z + \omega_u' \). Recall that the family of probability measures \( P^{(x, \infty)} \) is continuous as \( x \) varies. Since \( f \) has compact support, we can restrain \( T'_x + u \) to stay in a compact set. Then, the denominator in
the last integral, which converges to $p_{T_z^\omega + u}(-z)$, remains bounded and converges uniformly in $x$ and $u$. Then the boundedness of $H$ implies that the two last integrals converge to
\[
\int P^{(-z-\omega',\infty)}(d\omega) \frac{f(T_{z+\omega'(u)}^\omega \omega' \omega' + u)}{p_{T_z^\omega + u}(-z)}.
\]

Now, the measure $N^\omega$ is a finite measure, so the fact that $u$ actually stays in a compact set and the fact that the two last integrals above remain bounded allow to apply the dominated convergence theorem to obtain
\[
\int \limits_0^\infty ds f(s) P_0^{s} \left( (H(X)H'(X')) \right) = \int \limits_0^\infty du \int \limits_{N^\omega \leq u}(d\omega') H'(\omega') \int P^{(-z-\omega',\infty)}(d\omega) \frac{f(T_{z+\omega'(u)}^\omega \omega' \omega' + u)}{p_{T_z^\omega + u}(-z)}.
\]

Now we disinegrate this relation by taking $f(s) = (2\varepsilon)^{-1} \mathbb{1}_{[1-\varepsilon,1+\varepsilon]}(s)$, so a similar argument as above gives that the left hand side converges to $P_0^{1} \left( (H(X)H'(X')) \right)$ as $\varepsilon \downarrow 0$, whereas the right hand side is
\[
\int \limits_0^\infty du \int \limits_{N^\omega \leq u}(d\omega') H'(\omega') \int P^{(-z-\omega'(u),\infty)}(d\omega) \frac{1_{[1-\varepsilon,1+\varepsilon]}(T_{z+\omega'(u)}^\omega \omega' \omega' + u)}{2\varepsilon p_{T_z^\omega + u}(-z)}.
\]

The third integral may be rewritten as
\[
P_0^{1} \left( (H(X)H'(X')) \right) \frac{p_1(-z)^{-1}}{2\varepsilon} \int \limits_0^1 du N^\omega \left[ H'(\omega') q_{z+\omega'(u)}(1-u) E_{01-\omega'(u)}^{1-u} \left[ H(\omega) \right] \right] - \frac{H(\omega)}{p_{T_z^\omega + u}(-z)} \left[ T_{z+\omega'(u)}^\omega \omega' \omega' + u - 1 \right] < \varepsilon
\]

with a slightly improper writing (the $\omega$’s should not appear in the expectation, but we keep them to keep the distinction with the expectation with respect to $\omega'$). Similar arguments as above imply that the limit we are looking for is
\[
P_0^{1} \left( (H(X)H'(X')) \right) = p_1(-z)^{-1} \int \limits_0^1 du N^\omega \left[ H'(\omega') q_{z+\omega'(u)}(1-u) E_{01-\omega'(u)}^{1-u} \left[ f(u + T_{\omega'(u)}^\omega) \right] \right].
\]

This in turn completely determines the law of the bridge by a monotone class argument. A careful application of the above identity thus gives
\[
E_{01-\omega'(u)}^{1-u} \left[ f(1 - (\tau_3 - \tau_1)) \right] = p_1(-z)^{-1} \int \limits_0^1 du N^\omega \left[ q_{z+\omega'(u)}(1-u) E_{01-\omega'(u)}^{1-u} \left[ f(u + T_{\omega'(u)}^\omega) \right] \right].
\]

Applying Lemma 14 to the rightmost expectation term, this is equal to
\[
p_1(-z)^{-1} \int \limits_0^1 du N^\omega \left[ q_{z+\omega'(u)}(1-u) \int \limits_0^{1-u} dv \frac{q_{\omega'(u)}(v)q_{z}(1-u-v)}{q_{\omega'(u)}+z(1-u)} f(u+v) \right] = p_1(-z)^{-1} \int \limits_0^1 du \int \limits_U df(s) q_{z}(1-s) N^\omega \left[ q_{\omega'(u)}(s-u) \right] = z q_{z}(1)^{-1} \int \limits_0^1 df(s) q_{z}(1-s) \int \limits_0^s du N^\omega \left[ q_{\omega'(u)}(s-u) \right].
\]
It remains to compute the second integral. Using scaling identities for $N^{>u}$ and $q_x(s)$ we have

\[
\int_0^s du N^{>u} [q_{\omega'(u)}(s-u)] = \int_0^1 dr N^{>sr} [q_{\omega'(sr)}(s(1-r))] \\
= s^{-1/\alpha} \int_0^1 dr s^{1/\alpha} N^{>sr} [q_{s^{-1/\alpha}\omega'(sr)}(1-r)] \\
= s^{-1/\alpha} \int_0^1 dr N^{>r} [q_{\omega'(r)}(1-r)],
\]

where the last equality is an easy consequence of the scaling property. Finally, the integral in the right hand side does not depend on $s$, we call it $c$ and obtain

\[
E_{0\to-z}^1[f(1-(\tau_3-\tau_1))]=\int_0^1 ds f(s) \frac{czq_z(1-s)}{s^{1/\alpha}q_z(1)}.
\]

So we necessarily have $c = c_\alpha$, and the claim follows. \qed

**Proof of Proposition 5.** The proof is now easily obtained by combining the last lemmas. Under $P_{0\to-z}$ conditionally on $Z_1^{(t)} = z$, the law of the lengths of constancy intervals of $\frac{Y_{t\to-z}X^{(t)}}{Z_1}$ is obtained by adjoining the term $1-(\tau_3-\tau_1)$ to a sequence which, conditionally on $1-(\tau_3-\tau_1) = t$, has same law as $\Delta T_{0,z}$ given $T_z = 1 - t$ under $P$ (Lemma 16). By Lemma 17, $1 - (\tau_3 - \tau_1)$ has itself the law of a size-biased pick from $\Delta T_{0,z}$ given $T_z = 1$ under $P$, so Lemma 1 shows the whole sequence has the law of $\Delta T_{0,z}$ given $T_z = 1$. Last, by Lemma 3, $Z_1^{(t)}$ has density $p_1^{(t)}(-z)p_1^{(t)}(z)p_1(0)^{-1}dz$, entailing the claim. \qed

### 5.3 Proof of Theorem 2

To recover the dislocation measure of $F^t$, we use the following variation of Proposition 1 and [23, Corollary 1]. For details on size-biased versions of measures on $S$, see e.g. [14], which deals with probability measures, but the results we mention are easily extended to $\sigma$-finite measures.

**Proposition 6.** Let $(F(t), t \geq 0)$ be a ranked self-similar fragmentation with characteristics $(\beta, 0, \nu)$, $\beta \geq 0$. For every $t$, let $F_*(t)$ be a random size-biased permutation of the sequence $F(t)$ (defined on a possibly enlarged probability space). Let $G$ be a continuous bounded function on the set of non-negative sequences with sum $\leq 1$, depending only on the first $I$ terms of the sequence, with support included in a set of the form $\{s_i \in [\eta, 1-\eta], 1 \leq i \leq I\}$. Then

\[
\frac{1}{t} E[G(F_*(t))] \to \nu_*(G),
\]

where $\nu_*$ is the size-biased version of $\nu$ characterized by

\[
\nu_*(G) = \int_S \nu(ds) \sum_{j_1, \ldots, j_I} G(s_{j_1}, \ldots, s_{j_I}) \frac{s_{j_1}}{1-s_{j_1}} \cdots \frac{s_{j_I}}{1-s_{j_1} - \cdots - s_{j_I}},
\]

where the sum is on all possible distinct $j_1, \ldots, j_I$. Moreover, $\nu$ can be recovered from $\nu_*$.  

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**Proof of Theorem 2.** Let \( G \) be a function of the form \( G(x) = f_1(x_1) \ldots f_l(x_k) \) for \( x = (x_1, x_2, \ldots) \) and \( \sum_i x_i \leq 1 \), with \( f_1, \ldots, f_l \) continuous bounded functions on \([0, 1]\) that are null on a set of the form \([0, 1] \setminus \{1\} \). Let \( \Delta^* T_{[0, z]} \) be the sequence of the jumps of \( T \) on the interval \([0, z]\), listed in size-biased order (which involves some enlargement of the probability space). Using Lemma 1, it is easy that \( z \mapsto E[G(\Delta^* T_{[0, z]})) T_z = 1] \) is a continuously differentiable function with derivative bounded by some \( M > 0 \). Let also \( F^+_z(t) \) be the sequence \( F^+(t) \) listed in size-biased order. Now by Proposition 5,

\[
E \left[ \frac{G(F^+_z(t))}{t} \right] = \frac{1}{t} E \left[ e^{-u \alpha \log Z^+_1(t)} p_1(Z^+_1(t)) p_1(0)^{-1} E \left[ G(\Delta^* T_{[0, Z^+_1(t)]}) T_{Z^+_1(t)} = 1 \right] \right].
\]

Consider a function \( f(t, z) \) that is continuous in \( t \) and \( x \) and null at \((t, 0)\) for every \( t \geq 0 \). Then the compensation formula applied the subordinator \( Z(t) \) between times 0 and 1 gives

\[
\frac{1}{t} E[f(t, Z^+_1(t))] = \frac{1}{t} \int_0^1 dx \int C_{ \alpha}(1 - e^{-t s}) s^{-\alpha - 1} ds E[f(t, Z^+_x(t)) - f(t, Z^+_x(t))]
\]

\[
\to_{t \to 0} C_{ \alpha} \int_0^1 dx \int s^{-\alpha} ds f(0, s) = C_{ \alpha} \int s^{-\alpha} ds f(0, s),
\]

as soon as we may justify the convergence above. Take

\[
f(t, z) = \exp(-u \alpha \log t \alpha) p_1(-z) p_1(0)^{-1} E[G(\Delta^* T_{[0, z]})) T_z = 1],
\]

then we have to check that \( s^{-\alpha} E[f(t, Z^+_x(t)) - f(t, Z^+_z(t))] \) is bounded independently on \( x \in [0, 1] \). By the hypotheses on \( G \), it is again true that \( z \mapsto f(t, z) \) is a continuously differentiable function with uniformly bounded derivative, when \( t \) stays in a neighborhood of 0. Hence the expectation above is bounded by \( (M' s \wedge M'') s^{-\alpha} \) for some \( M', M'' > 0 \), which allows to apply the dominated convergence theorem. By Proposition 6, we obtain that

\[
t^{-1} E[G(F^+_z(t))] \to_{t \to 0} \int_s \nu_+(ds) G(s_{j_1}, \ldots, s_{j_l}) \sum_{j_1, \ldots, j_l} s_{j_1} \frac{s_{j_2}}{1 - s_{j_1}} \cdots \frac{s_{j_l}}{1 - s_{j_{l-1}}} = C_{ \alpha} \int_0^\infty ds \frac{s^{-\alpha} p_1(s)}{p_1(0)} E[G(\Delta^* T_{0, s})] T_s = 1],
\]

allowing to conclude with the same computations as in the proof of Lemma 11. \( \square \)

## 6 Asymptotics and small fragments

In this section we discuss asymptotic results for \( F^+ \). We first state a result on the asymptotic behavior for small times.

### 6.1 Small-time asymptotics

**Proposition 7.** Let \( (Z(x, x \geq 0) \) be a stable subordinator with Laplace exponent \( \alpha \lambda^{\alpha - 1} \). Denote by \( \Delta_1, \Delta_2, \ldots \) the ranked jumps of \( (T_x, 0 \leq x \leq Z_1) \), where \( T \) is as before the stable \( 1/\alpha \) subordinator, which is taken independent of \( Z \). Then

\[
t^{\alpha/(1-\alpha)} (1 - F^+_1(t), F^+_2(t), F^+_3(t), \ldots) \overset{d}{\to} (T_{Z_1}, \Delta_1, \Delta_2, \ldots).
\]

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Notice that the limiting sequence differs from the jumps of \((T_{Z^t}, 0 \leq x \leq 1)\), which by Bochner’s subordination is a stable \(1 - 1/\alpha\) subordinator. However, the first component is equal in law to \(\sigma_1\) where \(\sigma\) is a stable \(1 - 1/\alpha\) subordinator. We first need the

**Lemma 18.** Let \(Z^{(t)}_1\) have the law \(p^{(t)}_1(s)ds\) above, then

\[
t^{1/(1-\alpha)}T^{(t)}_1 \xrightarrow{d} Z_1.
\]

**Proof.** Recall that \(Z^{(t)}_1\) is a subordinator with characteristic exponent given by

\[
\mathbb{E}[e^{-\lambda Z^{(t)}_1}] = \exp \left( - \int_0^\infty \frac{C_\alpha(1 - e^{-tx})}{x^{\alpha+1}}(1 - e^{-\lambda x}) \right).
\]

Therefore, evaluating the Laplace exponent at the point \(t^{1/(1-\alpha)}\lambda\), changing variables and using dominated convergence entails

\[
\mathbb{E}[\exp(-\lambda t^{1/(1-\alpha)} Z^{(t)}_1)] \xrightarrow{t \to 0^+} \exp \left( - \int_0^\infty \frac{C_\alpha dy}{y^{\alpha}}(1 - e^{-\lambda y}) \right).
\]

Thus the convergence to some limiting \(Z_1\). Using now the explicit value for \(C_\alpha\), we see that the Laplace exponent of \(Z_1\) has to be \(\alpha\lambda^{\alpha-1}\), as claimed. \(\square\)

The proof of Proposition 7 follows the same lines as for Proposition 6 in [22], so we will only sketch it. One first begins with proving that if \(Z\) is as in Lemma 3 a random variable distributed according to the law that has density \(\rho^{(t)}_1(z)p^{(t)}_1(-z)dz/p_1(0)\), then \(t^{1/(1-\alpha)}Z\) converges in law to \(Z_1\). This is a consequence of the preceding lemma, since as \(t \to 0\), \(Y^{(t)}\) converges to \(Y\), so one can write

\[
\mathbb{E}[g(t^{1/(1-\alpha)}Z)] = \mathbb{E}[g(t^{1/(1-\alpha)}Z^{(t)}_1)p^{(t)}_1(-Z^{(t)}_1)/p_1(0)],
\]

where \(Z^{(t)}_1\) is distributed as above. By Skorokhod’s embedding theorem, we may suppose that \(t^{1/(1-\alpha)}Z^{(t)}_1\) converges a.s. to its limit in law \(Z_1\), So it remains to show that a.s. \(p^{(t)}_1(-Z^{(t)}_1) \to p_1(0)\) as \(t \to 0\) to apply dominated convergence, and this is done by recalling that \(p^{(t)}_1(z) = e^{-t\alpha+bz}p_1(z)\). Then one reasons by induction just as in [22, Proposition 6], using the explicit form of the semigroup of \(F^+\).

### 6.2 Large-time asymptotics

By a direct application of Theorem 3 in [11], one gets the large \(t\) asymptotic behavior for \(F^+\). Recall that the Gamma law with parameters \((a, b)\) is the law with density proportional to \(x^{a-1}e^{-bx}\) on \(\mathbb{R}_+\). The moments of this law are given, for \(r > -a\), by

\[
\frac{b^a}{\Gamma(a)} \int_0^\infty x^{r+a-1}e^{-bx}dx = \frac{\Gamma(a+r)}{\Gamma(a)b^r}.
\]

**Proposition 8.** Define

\[
\rho_t(dy) = \sum_{i=1}^{\infty} F_i(t)\delta_{\alpha F_i(t)}(dy),
\]

then \(\rho_t\) is a probability measure that converges in law as \(t \to \infty\) to the deterministic Gamma law with parameters \((1-1/\alpha, (\Gamma(1/\alpha)/\alpha\Gamma(1+1/\alpha))^a)\).
Proof. We know by [11, Theorem 3] that $\rho_t$ converges to some probability $\rho_\infty$ that is characterized by its moments,
\[ \int_0^\infty y^{k/\alpha} \rho_\infty(dy) = \frac{\alpha(k-1)!}{\Phi'(0+)\Phi\left(\frac{1}{\alpha}\right)\ldots\Phi\left(\frac{k-1}{\alpha}\right)} \]
for every $k \geq 1$, where $\Phi$ is the Laplace exponent of a subordinator related to a tagged fragment of the process $F^+$. This exponent depends only on the dislocation measure (and not the index), so it is the same as for $F_-$ in [22]. By taking the explicit value of $\Phi$ (Section 3.2 therein), we easily get
\[ \int_0^\infty y^{k/\alpha} \rho_\infty(dy) = \left(\frac{\alpha\Gamma\left(1+\frac{1}{\alpha}\right)}{\Gamma\left(\frac{1}{\alpha}\right)}\right)^k \frac{\Gamma\left(1+\frac{k-1}{\alpha}\right)}{\Gamma\left(1-\frac{1}{\alpha}\right)}. \]
Replacing $k$ by $\alpha k$, one can recognize the moments of the law Gamma with the claimed parameters.

### 6.3 Small fragments

Another quantity of interest is the mass of small fragments, that is studied in [9]:
\[ M(\varepsilon, t) = \sum_{i \geq 1} F_i(t)1_{\{F_i(t) < \varepsilon\}}. \]

**Proposition 9.** For every $t$, with probability 1,
\[ \frac{M(\varepsilon, t)}{\varepsilon^{1-1/\alpha}} \xrightarrow{\varepsilon \to 0} \frac{\Gamma\left(1+\frac{1}{\alpha}\right)\Gamma\left(2-\alpha\right)}{\alpha} \int_0^t ds \sum_{i=1}^\infty (F_i(s))^{2/\alpha}. \]

This is also the limit of $(\alpha - 1)\varepsilon^{1/\alpha} N(\varepsilon, t)$ as $\varepsilon \to 0$, where
\[ N(\varepsilon, t) = \#\{i : F_i(t) > \varepsilon\}. \]

**Proof.** Recall from Theorem 1 and section 3 in [9] that as soon as one can prove that the function
\[ f(\varepsilon) = \int_S \sum_{i=1}^\infty s_i 1_{\{s_i < \varepsilon\}} \nu_\alpha(ds) \]
is regularly varying at 0+ with exponent $1 - \beta$ for some $\beta \in (0, 1)$, and that
\[ g(\varepsilon) = \int_S \left(\sum_{i=1}^\infty s_i 1_{\{s_i < \varepsilon\}}\right)^2 \nu_\alpha(ds) \xrightarrow{\varepsilon \to 0} O(f(\varepsilon)^2), \]
then one obtain that for every $t \geq 0$, $M(\varepsilon, t)/f(\varepsilon)$ converges a.s. to $\int_0^t \sum_{i \geq 1} (F_i(s))^{(1/\alpha)+\beta} ds.$

We thus compute the quantities $f(\varepsilon)$ and $g(\varepsilon)$. First, denoting by $\Delta_i$ the $i$-th largest jump of $T$
before time 1, one obtains by conditioning on $T_1$, using Lemma 1 and changing variables, that

$$f(\varepsilon) = E \left[ \sum_{i=1}^{\infty} \Delta_i 1_{\{\Delta_i, T_1 < \varepsilon\}} \right]$$

$$= \int_0^\infty q_1(u) du E \left[ \sum_{i=1}^{\infty} \Delta_i 1_{\{\Delta_i < \varepsilon u\}} | T_1 = u \right]$$

$$= \int_0^\infty q_1(u) du \int_0^{\varepsilon u} \frac{c_\alpha q_1(u - y)}{uy^{1/\alpha}q_1(u)} dy$$

$$= c_\alpha \varepsilon^{1-1/\alpha} \int_0^\infty \frac{dx}{x^{1/\alpha}} \int_x^\infty \frac{du}{u} q_1(u - \varepsilon x)$$

We can pass to the limit by dominated convergence in the second integral. Indeed, it is equal to

$$\int_0^\infty \frac{dx}{x^{1/\alpha}} E \left[ \frac{1_{\{T_1 > (1 - \varepsilon) x\}}}{T_1 + \varepsilon x} \right]$$

so when $\varepsilon < 1/2$ say we can bound the integrand by

$$\frac{1}{x^{1/\alpha}} E \left[ \frac{1_{\{T_1 > x/2\}}}{T_1} \right]$$

which is easily checked to be integrable. So this integral goes to

$$\int_0^\infty \frac{dx}{x^{1/\alpha}} \int_x^\infty \frac{du}{u} q_1(u) \left( \frac{1}{u} \right) = E \left[ \frac{1}{(1 - \frac{1}{\alpha}) T_1^{1/\alpha}} \right] = \frac{1}{(1 - \frac{1}{\alpha}) \Gamma(1 + \frac{1}{\alpha})}.$$ 

This gives that $f(\varepsilon)$ is regularly varying at $0+$ with exponent $1 - 1/\alpha$, and more precisely

$$\frac{f(\varepsilon)}{\varepsilon^{1-1/\alpha}} \xrightarrow{\varepsilon \to 0} \frac{c_\alpha}{(1 - \frac{1}{\alpha}) \Gamma(1 + \frac{1}{\alpha})} = \frac{\alpha}{\Gamma(1 + \frac{1}{\alpha}) \Gamma(2 - \alpha)}.$$ 

On the other hand, we may compute $g(\varepsilon)$ as follows. It is similarly equal to

$$E \left[ \frac{1}{T_1} (\Delta_i 1_{\Delta_i < \varepsilon T_1})^2 \right] = \int_0^\infty u^{-1} q_1(u) du E \left[ \frac{1_{\{\Delta_i < \varepsilon u, \Delta_i' < \varepsilon u\}}}{T_1 = u} \right]$$

$$= \int_0^\infty u^{-1} q_1(u) \int_0^{\varepsilon u} \frac{c_\alpha q_1(u - x)}{ux^{1/\alpha}q_1(u)} \left( \frac{x}{u} + \frac{u - x}{u} \right) \int_0^{\varepsilon u} \frac{c_\alpha q_1(u - y)}{(u - x)y^{1/\alpha}q_1(u - x)} dy.$$

Here, we have denoted by $\Delta_i^*$ and $\Delta_i'$ the jumps of $T$ that straddle $U_i T_1$ and $U_i' T_1$ for two independent variables $U_i, U_i'$ independent of $T$ and uniform on $(0, 1)$. As such, $\Delta_i^*$ is the size-biased pick of section 2, but notice that $\Delta_i'$ may equal $\Delta_i^*$. Precisely, given $T_1 = u$ and $\Delta_i^* = x$, $\Delta_i'$ equals $x$ with probability $x/u$, and it equals the second size-biased pick $\Delta_2^*$ of the jumps of $T$ with probability $(u - x)/u$. Combined with Lemma 1, this explains the last identity. We may now finish the computation in a similar way as above, to obtain that $g(\varepsilon) = O(\varepsilon^{2-2/\alpha}) = O(f(\varepsilon)^2)$ as $\varepsilon \to 0$ (after changing variables in the last displayed expression, the term $x/u$ introduces a term of order $\varepsilon$, and the second integral term introduces a term of order $\varepsilon^{1-1/\alpha}$). This entails the claim.
The second part of the statement follows directly from this discussion and Theorem 1 in [9]. □

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References


