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with random blow-ups

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Schrödinger operators on fractal lattices
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Abstract: Starting from a finitely ramified self-similar set \( X \) we can construct an unbounded set \( X_{<\infty} \) by blowing-up the initial set \( X \). We consider random blow-ups and prove elementary properties of the spectrum of the natural Laplace operator on \( X_{<\infty} \) (and on the associated lattice). We prove that the spectral type of the operator is almost surely deterministic with the blow-up and that the spectrum coincides with the support of the density of states almost surely (actually our result is more precise). We also prove that if the density of states is completely created by the so-called Neuman-Dirichlet eigenvalues, then almost surely the spectrum is pure point with compactly supported eigenfunctions.

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In this text we prove elementary results on spectral properties of Laplace operators on unbounded fractal lattices based on finitely ramified self-similar sets, and there continuous analogous. One of the main novelty in this text is to consider random blow-up, i.e., the unbounded lattice or fractal is constructed by blowing-up randomly the initial figure. The results of this text show that the natural Laplace operator on these random lattices share the same basic properties as ergodic families of random Schrödinger operators, as defined for example in [10], [2]. In particular, we consider the relations between the spectrum of the operator and two important measures: the density of states and the density of Neuman-Dirichlet eigenvalues (also called molecular states in [11]) which are eigenvalues associated with eigenfunctions with both Neuman and Dirichlet boundary condition. In [14] we investigated the relations between these measures and the dynamics of a certain renormalization map, which is a rational map of a compact Kähler manifold.

Starting from a finitely ramified self-similar set \( X \) (for example a p.c.f. self-similar set as in [4]) we can construct an increasing sequence \( X_{<n>} \) by blowing-up \( X \). The way the set \( X \) is blowed-up is represented by a sequence \( \omega = (\omega_1, \ldots, \omega_n, \ldots) \) in \( \{1, \ldots, N\}^\mathbb{N} \), where \( N \) is the number of cells in \( X \). The unbounded set \( X_{<\infty>} \) is defined as the union \( X_{<\infty>} = \cup_n X_{<n>} \). If \( H = H_{<0>} \) is the "natural" Laplace operator on \( X \) then we can define by scaling a sequence of operators \( H_{<n>} \) on \( X_{<n>} \) and \( H_{<\infty>} \) on \( X_{<\infty>} \). The density of states (resp. of Neuman-Dirichlet eigenvalues) is defined as the limit of the renormalized counting measures of the eigenvalues of \( H_{<n>} \) (resp. of the Neuman-Dirichlet eigenvalues of \( H_{<n>} \)). In section 2 we prove three elementary results. The first two are the counterpart of well-known properties of ergodic families of random Schrödinger operators. The third one is more specific to our situation since it involves the Neuman-Dirichlet spectrum which is empty in the case of Schrödinger operators on \( \mathbb{Z}^d \).

- In proposition (1) we prove that almost surely on \( \omega \), the support of the density of states is equal to the spectrum of the operator on \( X_{<\infty>} \) (actually, we can precise for which \( \omega \) this equality is always true).

- In proposition (2) we prove that the spectral type of the operator is almost surely deterministic, i.e., that there exists deterministic subsets \( \Sigma, \Sigma_{ac}, \Sigma_{sc}, \Sigma_{pp} \) such that almost surely in \( \omega, \Sigma, \Sigma_{ac}, \Sigma_{sc}, \Sigma_{pp} \) are respectively the spectrum, the absolutely continuous spectrum, the singular continuous spectrum and the pure point spectrum of the operator on \( X_{<\infty>} \).

- In proposition (3) we prove that if the density of states is completely created by the Neuman-Dirichlet eigenvalues (i.e., if the density of states is equal to the density of Neuman-Dirichlet eigenvalues) then the spectrum of the operator on \( X_{<\infty>} \) is pure point with compactly supported eigenfunctions, almost surely in \( \omega \). This result is important since in [14], theorem 4.1 and proposition 4.4, we proved that this happen exactly when the asymptotic degree of the renormalization map is smaller than \( N \), the number of cells in \( X \).

In section 3 we introduce several measures that generalize the density of states and the density of Neuman-Dirichlet eigenvalues to the different parts of the spectrum. The result of proposition 2 suggests that the right object to investigate is the almost
sure type of the spectrum of the operator on $X_{<\infty>}$. A step is done in this direction in [14], where the density of states and the density of N-D eigenvalues are computed in terms of a certain explicit renormalization map. Previously, for some particular examples the spectral properties has been investigated, cf [8], [16], [13]. In particular in [16], Teplyaev investigated the spectrum of the discrete Laplace operator on the lattice associated with the Sierpinski gasket for different blow-ups.

In the first section we briefly recall the notations, but we send to the main text [14] for precise definitions, examples and a more complete bibliography.

1 Notations

1.1 Self-similar sets and self-similar lattices.

We first briefly recall the notations and definitions of [14]. Suppose that $X$ is a finitely ramified self-similar set as defined in [14] and, to simplify notations, that $X$ has a geometrical embedding in $\mathbb{R}^d$. This means that $X$ is a proper compact connected subset of $\mathbb{R}^d$ and that there exists $N$ strictly contractive similarities $(\Psi_1, \ldots, \Psi_N)$, with distinct fixed points, such that

$$X = \bigcup_{i=1}^N \Psi_i(X),$$

and that there exists a subset $F$ of the set of fixed points $(x_1, \ldots, x_N)$ of $(\Psi_1, \ldots, \Psi_N)$ such that

$$\Psi_i(X) \cap \Psi_j(X) = \Psi_i(F) \cap \Psi_j(F), \quad \forall i \neq j.$$

Set $\Omega = \{1, \ldots, N\}^\mathbb{N}$ and fix an element $\omega$ of $\Omega$. We define the blow-up of $X$ as the sequence of increasing sets $X_{<n>}$ defined by $X_{<0>} = X$ and

$$X_{<n>} = \Psi_{\omega_1}^{-1} \circ \cdots \circ \Psi_{\omega_n}^{-1}(X).$$

We set $X_{<\infty>} = \bigcup_{n=0}^\infty X_{<n>}$. The boundary of $X_{<0>}$ is $X$ is defined as $\partial X = F$ and we set

$$\partial X_{<n>} = \Psi_{\omega_1}^{-1} \circ \cdots \circ \Psi_{\omega_n}^{-1}(F),$$

and $\partial X_{<\infty>} = \cap_n \cup_{m \geq n} \partial X_{<m>}$. We also set $X_{<n>}' = X_{<n>} \setminus \partial X_{<n>}$ (and similarly for $X_{<\infty>}$). Remark that

$$X_{<n+p>;'} = \bigcup_{j_1, \ldots, j_p} X_{<n+p>;j_1,\ldots,j_p},$$

where

$$X_{<n+p>;j_1,\ldots,j_p} = \Psi_{\omega_1}^{-1} \circ \cdots \circ \Psi_{\omega_{n+p}}^{-1}(\Psi_{j_1} \circ \cdots \circ \Psi_{j_p}(X)).$$

We call the $X_{<n+p>;j_1,\ldots,j_p}$ the $<n>$-cells of $X_{<n+p>}$ and we remark that

$$X_{<n>} = X_{<n+p>;\omega_{n+p},\ldots,\omega_{n+1}}.$$

Remark also that $X_{<n+p>;j_1,\ldots,j_p}$ is naturally isomorphic to $X_{<n>}$. This structure of increasing sets has a discrete counterpart. Set $F_{<0>} = F$ and

$$F_{<n>} = \Psi_{\omega_1}^{-1} \circ \cdots \circ \Psi_{\omega_n}^{-1}(\bigcup_{j_1, \ldots, j_n} \Psi_{j_1} \circ \cdots \circ \Psi_{j_n}(F)).$$
(Hence, $F_{<n>}$ is the union of the boundaries of the 0-cells $X_{<n>,j_1,...,j_n}$ of $X_{<n>}$.) The sequence $F_{<n>}$ is clearly increasing and we set $F_{<\infty>} = \cup_{n} F_{<n>}$. Similarly, we have

$$F_{<n+p>} = \cup_{j_1,...,j_p} F_{<n+p>,j_1,...,j_p},$$

where $F_{<n+p>,j_1,...,j_p}$ are the $<n>$-cells of $F_{<n+p>}$ defined by

$$F_{<n+p>,j_1,...,j_p} = \Psi_{\omega_1}^{-1} \circ \cdots \circ \Psi_{\omega_{n+p}}^{-1} \circ \Psi_{j_1} \circ \cdots \circ \Psi_{j_p}(\cup_{i_1,...,i_n} \Psi_{i_1} \circ \cdots \circ \Psi_{i_n}(F)).$$

We set $\partial F_{<n>} = \partial X_{<n>}$ and $\overset{\circ}{F}_{<n>} = F_{<n>} \setminus \partial F_{<n>}$ (and idem for $\partial F_{<\infty>}$).

### 1.2 Self-similar Laplacians.

We fix for the rest of the text two $N$-tuples $(\alpha_1, \ldots, \alpha_N) \in ]0,1[^N$ and $(\beta_1, \ldots, \beta_N) \in ]0,1[^N$ such that $\beta_1 + \cdots + \beta_N = 1$. The $N$-tuple $(\alpha_1, \ldots, \alpha_N)$, resp. $(\beta_1, \ldots, \beta_N)$ will represent the scaling in energy, resp. in measure in our structure. We set $\gamma_i = (\alpha_i \beta_i)^{-1}$ and we make the following assumption

(H) We suppose that $(\beta_1, \ldots, \beta_N)$ is proportional to $(\alpha_1^{-1}, \ldots, \alpha_N^{-1})$ so that $\gamma_i$ does not depend on $i$. We denote by $\gamma$ the common value of the $\gamma_i$.

### Construction in the discrete case.

To construct a discrete Laplace operator on the sequence of lattices $F_{<n>}$ we suppose given $A$, a non-negative symmetric endomorphism of $\mathbb{R}^F$ of the form

$$Af(x) = - \sum_{y \in F, y \neq x} a_{x,y}(f(y) - f(x)), \quad \forall f \in \mathbb{R}^F, \forall x \in F,$$

where $a_{x,y}$, $x \neq y$, are non negative reals such that $a_{x,y} = a_{y,x}$. We suppose moreover that $A$ is irreducible, i.e. that the graph on $F$ defined by strictly positive $a_{x,y}$ is connected. We suppose also given a strictly positive measure $b$ on $F$.

We denote by $A_{<n>,i_1,...,i_n}$ (resp. $b_{<n>,i_1,...,i_n}$) the copy of the operator $A$ (resp. of the measure $b$) on the cell $F_{<n>,i_1,...,i_n}$ (cf [14] for precise definition). Then we define the symmetric operator $A_{<n>}$ and the measure $b_{<n>}$ on $F_{<n>}$ by

$$A_{<n>} = \sum_{i_1,...,i_n=1}^{N} \alpha_{\omega_1} \cdots \alpha_{\omega_n} \alpha_{i_1}^{-1} \cdots \alpha_{i_n}^{-1} A_{<n>,i_1,...,i_n},$$

$$b_{<n>} = \sum_{i_1,...,i_n=1}^{N} \beta_{\omega_1}^{-1} \cdots \beta_{\omega_n}^{-1} \beta_{i_1} \cdots \beta_{i_n} b_{<n>,i_1,...,i_n}.$$

**Remark 1**: We see from the definition that the value of $A_{<n>}$ and $b_{<n>}$ depend on the $N$-tuples $(\alpha_1, \ldots, \alpha_N)$ and $(\beta_1, \ldots, \beta_N)$ only up to a constant.

Remark that $A_{<n>}$ and $b_{<n>}$ form an inductive sequence since if $\text{supp}(f) \subset F_{<p>}$ for $p \leq n$ then

$$A_{<n>} f = A_{<p>} f$$

and

$$\int f b_{<n>} = \int f b_{<p>}.$$

Therefore $A_{<n>}$ and $b_{<n>}$ can be extended respectively to a linear operator $A_{<\infty>}$ on $\mathbb{R}^{F_{<\infty>}}$ and to a measure $b_{<\infty>}$ on $F_{<\infty>}$. 

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Remark that since $X$ is connected, $A_{<n>}$ is irreducible, i.e. $A_{<n>} f = 0$ implies that $f$ is a constant function. Denote by $<\cdot, \cdot>$ the usual scalar product on $\mathbb{R}^{F_{<n>}}$. Let $H_{<n>}^+$ be the operator on $L^2(F_{<n>}, b_{<n>})$ defined by:

$$< A_{<n>} f, g >= - \int H_{<n>}^+ f g dB_{<n>} \quad \forall f, g \in \mathbb{R}^{F_{<n>}}.$$  \hspace{1cm} (4)

The operator $H_{<n>}^+$ is non-positive, self-adjoint on $L^2(F_{<n>}, b_{<n>})$. The operator with Dirichlet boundary condition, denoted $H_{<n>}^-$, is the self-adjoint operator on $\mathbb{R}^{F_{<n>}}$ defined as the restriction of $H_{<n>}^+$ to $\mathbb{R}^{F_{<n>}} \simeq \{ f \in \mathbb{R}^{F_{<n>}}, \ f|_{\partial F_{<n>}} = 0 \}$. To be coherent with the notations of the continuous case we sometimes write $D_{<n>}^+ = \mathbb{R}^{F_{<n>}}$ and $D_{<n>}^- = \mathbb{R}^{F_{<n>}}$ for the domains of $H_{<n>}^\pm$.  

If $K > 0$ is such that $< Af, f > \leq K \int f^2 db$ for all $f$ in $\mathbb{R}^F$ then it is easy to see from (2) and (3) and assumption (H) that the same inequality is true for $A_{<n>}$ and $b_{<n>}$ and for $A_{<\infty>}$ et $b_{<\infty>}$. Thus the sequence $H_{<n>}^\pm$ is uniformly bounded for the operator norm on $L^2(b_{<n>})$ and can be extended into a non-positive, self-adjoint operator $H_{<\infty>}^\pm$ on $D_{<\infty>}^\pm = L^2(b_{<\infty>})$. We define $H_{<\infty>}^-$ as the restriction of $H_{<\infty>}^+$ to $D_{<\infty>}^\pm = \{ f \in D_{<\infty>}^+, \ f|_{\partial F_{<\infty>}} = 0 \}$. Clearly, we have

$$< A_{<\infty>} f, g >= - \int H_{<\infty>}^\pm f g dB_{<\infty>}, \quad \forall f, g \in D_{<\infty>}^\pm.$$  

Remark that if $\partial F_{<\infty>} = \emptyset$ then the operators $H_{<\infty>}^+$ and $H_{<\infty>}^-$ are equal and in this case we simply write $H_{<n>}$ for $H_{<\infty>}^+ = H_{<\infty>}^-$. 

Note finally that the hypothesis (H) corresponds to a property of local invariance by translation of the operator $H_{<\infty>}^\pm$, as explained in [14] (this assumption is related to the lattice case condition introduced in [5], [6]).

**In the continuous case**

We know that there exists a unique positive measure on $X$, which is self-similar with respect to the weights $(\beta_1, \ldots, \beta_N)$, i.e. which satisfies

$$\int_X f dm = \sum_{i=1}^N \beta_i \int_X f \circ \Psi_idm.$$  

We suppose given on $X$ a local, regular, conservative Dirichlet form $(a, D)$ on $L^2(X, m)$, self-similar with respect to the weights $(\alpha_1, \ldots, \alpha_N)$, as defined in [12]. Essentially, this means that $a$ satisfies

$$a(f, f) = \sum_{i=1}^N (\alpha_i)^{-1} a(f \circ \Psi_i, f \circ \Psi_i).$$

On $X_{<n>}$ we define the measures $m_{<n>}$ and the Dirichlet form $(a_{<n>}, D_{<n>})$ by scaling by

$$\int_{X_{<n>}} f dm_{<n>} = \beta_\omega^{-1} \ldots \beta_{\omega_n}^{-1} \int_X f \circ \Psi_{\omega_1}^{-1} \circ \ldots \circ \Psi_{\omega_n}^{-1} dm, \quad \forall f \in C^0(X_{<n>}),$$

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and

\[ \mathcal{D}_{<n>} = \{ f \in L^2(m_{<n>}), \text{ s.t. } f \circ \Psi_{\omega_1}^{-1} \circ \cdots \circ \Psi_{\omega_n}^{-1} \in \mathcal{D} \}, \]

\[ a_{<n>}(f) = \alpha_{\omega_1} \cdots \alpha_{\omega_n} a(f \circ \Psi_{\omega_1}^{-1} \circ \cdots \circ \Psi_{\omega_n}^{-1}). \]

If \( f \) in \( \mathcal{D}_{<n+p>} \) is such that \( \text{supp}(f) \subset X_{<n>} \) then we see that \( a_{<n+p>}(f, f) = a_{<n>}(f, f) \) and \( \int f dm_{<n>} = \int f dm_{<n+p>} \). Hence, we see that \( m_{<n>} \) can be extended to a measure \( m_{<\infty>} \) on \( X_{<\infty>} \), and we set

\[ \mathcal{D}_{<\infty>} = \{ f \in L^2(X_{<\infty>}, m_{<\infty>}), \sup_{n} a_{<n>}(f_{|X_{<n>}}, f_{|X_{<n>}}) < \infty \}; \]

and \( a_{<\infty>}(f,f) = \lim_{n \to \infty} a_{<n>}(f_{|X_{<n>}}, f_{|X_{<n>}}) \) on \( \mathcal{D}_{<\infty>} \). We set \( \mathcal{D}_{<\infty>}^- = \{ f \in \mathcal{D}_{<\infty>}, f_{|\partial X_{<n>}} = 0 \} \) and \( \mathcal{D}_{<\infty>}^+ = \mathcal{D}_{<\infty>} \) (and ident for \( \mathcal{D}_{<\infty>}^- \)). We define \( H_{<\infty>}^\pm \) as the infinitesimal generators of \( (a_{<\infty>}, \mathcal{D}_{<\infty>}^\pm) \) and \( (a_{<\infty>}, \mathcal{D}_{<\infty>}^-) \).

We refer to [14] for examples.

1.3 The density of states and the density of Neuman-Dirichlet eigenvalues.

We denote both in the continuous case and in the lattice case by \( \nu_{<n>}^\pm \) the counting measure of the eigenvalues of the operators \( H_{<n>}^\pm \) (in the lattice \( \nu_{<n>}^\pm \) is a finite sum of Dirac masses, in the continuous case it is a countable sums of Dirac masses accumulating at infinity). As usual, the density of states, that we denote \( \mu \), is defined as the limit (when it exists and is the same for Neuman and Dirichlet boundary condition)

\[ \mu = \lim_{n \to \infty} \frac{1}{N^n} \nu_{<n>}^\pm. \]

The existence of this measure is proved in [3], [5]. (Remark that despite the terminology, \( \mu \) is a measure which does not necessarily have a density).

We say that a function \( f \) is a Neuman-Dirichlet (N-D for short) eigenfunction of \( H_{<n>} \) with eigenvalue \( \lambda \) if it is both an eigenfunction of \( H_{<n>}^- \) and \( H_{<n>}^+ \), i.e. in the lattice case this means that \( f \) is \( \mathcal{D}_{<n>}^- \), i.e. \( f_{|\partial X_{<n>}} = 0 \), and that

\[ \langle A_{<n>} f, g \rangle = -\lambda \int fg \, db_{<n>}, \quad \forall g \in \mathcal{D}_{<n>}^+, \quad \mathbb{R}^{F_{<n>}}, \]

and in the continuous case that \( f \) is in \( \mathcal{D}_{<n>}^- \), and that

\[ a_{<n>}(f, g) = -\lambda \int fg dm_{<n>}, \quad \forall g \in \mathcal{D}_{<n>}^+. \]

We denote by \( \nu_{<n>}^{ND} \) the counting measure of the N-D eigenvalues of \( H_{<n>} \) (counted with multiplicity) and by \( E_{<n>}^{ND} \) the subspace of \( \mathcal{D}_{<n>}^+ \) generated by the N-D eigenfunctions.

Remark that any function \( f \) of \( E_{<n>}^{ND} \), when extended by 0 to \( F_{<n+p>} \) (resp. \( X_{<n+p>} \)) is a N-D eigenfunction of \( H_{<n+p>} \). When extended by 0 to \( F_{<\infty>} \) (resp. \( X_{<\infty>} \)) it is an eigenfunction of \( H_{<\infty>}^\pm \) and \( H_{<\infty>}^- \), with compact support. We denote by \( \mathcal{H}_{ND} \) the closure in \( \mathcal{D}_{<\infty>}^+ \) of the space \( \cup_{n} E_{<n>}^{ND} \).

It is easy to see that

\[ \nu_{<n+1>}^{ND} \geq N \nu_{<n>}^{ND}. \]
Indeed, if \( f \) is a N-D eigenfunction of \( H_{<n>} \) then we can construct \( N \) copies of \( f \) on the \( N < n > \)-cells of \( F_{<n+1>} \). Precisely, for all \( i = 1, \ldots, N \) we consider the function \( f_i \) on \( \mathbb{R}^{F_{<n+1>}} \) which is the copy of \( f \) on \( F_{<n+1>,i} \) and equal to 0 on \( F_{<n+1>} \setminus F_{<n+1>,i} \). These functions form a orthogonal family of N-D eigenfunctions of \( H_{<n+1>} \) with same eigenvalues (by the hypothesis (H)). Thus, the limit

\[
\frac{1}{N^n \nu^{N_D}_{<n>}}
\]

exists and is called the density of N-D eigenvalues and we denote it by \( \mu^{N_D} \).

2 Statements and proofs of the results.

We state 3 elementary results on the spectrum of these operators and their relations with the density of states and the density of N-D eigenvalues. The first one is the counterpart of a classical result for random Schrödinger operators (but nevertheless never appeared in the literature). For convenience, we suppose here the existence of the density of states.

We denote by \( \Sigma^{\pm} \) the topological spectrum of the operators \( H^{\pm}_{<\infty>} \) (and we simply write \( \Sigma \) when \( \partial F_{<\infty>} = \emptyset \)). We recall that the essential spectrum is obtained from the spectrum by removing all isolated points corresponding to eigenvalues with finite multiplicity, we denote it by \( \Sigma_{ess}^{\pm} \).

**Proposition 1** For both the discrete and the continuous case we have the following:

i) If the boundary set \( \partial X_{<\infty>} = \partial F_{<\infty>} \) is empty then \( \text{supp} \mu = \Sigma = \Sigma_{ess}^{+} = \Sigma_{ess}^{-} \).

ii) Otherwise we just have \( \text{supp} \mu = \Sigma_{ess}^{+} = \Sigma_{ess}^{-} \). Moreover, the eigenvalues eventually lying in \( \Sigma^{\pm} \setminus \text{supp}(\mu) \) have multiplicity 1.

**Remark 2**: We are in case i) for almost all blow-up \( \omega_i \) for the product of the uniform measure on \( \{1, \ldots, N\} \).

**Remark 3**: In [13] we proved i) in the case of Nested fractals for equal weights \( \alpha_i = \alpha \) and \( \beta_i = \frac{1}{\alpha} \). For this class of self-similar sets, due to symmetry arguments, we proved that the equality \( \text{supp} \mu = \Sigma^{+} = \Sigma^{-} \) is true even in the case ii). In [15], we plan to show that in the case of the unit interval blowed-up to the half-line \( \mathbb{R}_+ \) (by the constant blow-up \( \omega_k = 1 \)) the spectrum of the operator can be pure point with isolated eigenvalues of multiplicity 1 lying in the complement of \( \text{supp} \mu \) and accumulating on \( \text{supp} \mu \). Therefore in this case the equality \( \Sigma_{ess}^{\pm} = \text{supp} \mu \) is satisfied by not \( \Sigma^{\pm} = \text{supp} \mu \).

**Proof**: The proof is similar to that of [13], except that we must be careful with the inhomogeneous weights \( \alpha_i, \beta_i \) and that we must prove the extra result ii). We consider first the discrete case. We know from section 1 that the norm of the operator \( H^{\pm}_{<\infty>} \) in \( L^2(F_{<\infty>}, b_{<\infty>}) \) is finite, say smaller than a real \( K > 0 \).

By classical arguments we know that \( \text{supp}(\mu) \subset \Sigma_{ess}^{\pm} \). We denote by \( P^{\pm}_{<\infty>}(d\lambda) \) and \( F^{\pm}_{<n>}(d\lambda) \) the spectral resolution of the operators \( H^{\pm}_{<\infty>} \) and \( H^{\pm}_{<n>} \) resp. on \( \mathcal{D}^{\pm}_{<\infty>} \) and \( \mathcal{D}^{\pm}_{<n>} \). We first prove i). We suppose that \( \partial F_{<\infty>} = \emptyset \) and we let \( \lambda \in \Sigma \) and \( \epsilon > 0 \). We choose \( f \) in \( P^{\pm}_{<\infty>}([\lambda - \epsilon/4, \lambda + \epsilon/4])(\mathcal{D}^{\pm}_{<\infty>}) \) such that \( \int f^2 db_{<\infty>} = 1 \). For all \( \eta > 0 \) we can find \( n_0 \) such that \( \int_{\partial F_{<\infty>}} |f|^2 db_{<\infty>} \leq \eta \). We define \( \tilde{f} \) by \( \tilde{f} = f \) on \( \partial F_{<n_0>} \) and \( \tilde{f} = 0 \) on \( F_{<\infty>} \setminus \partial F_{<n_0>} \). We have easily

\[
\int |H_{<\infty>}(\tilde{f}) - \lambda \tilde{f}|^2 db_{<\infty>} \leq K^2 \eta + \epsilon^2/16 + \lambda^2 \eta.
\]
and \( \int |\tilde{f}|^2 db_{<\infty} \geq 1 - \eta \).

Set \( \tilde{f} = \frac{f}{\int |f|^2 db_{<\infty}} \). For \( p_0 \) large enough \( \tilde{f} \) is in \( \mathcal{D}_{H_{<\infty}^{<n_0+p_0}}^+ \cap \mathcal{D}_{H_{<\infty}^{<n_0+p_0}}^- \) (precisely, it is sufficient that \( \partial F_{<n_0+p_0} \cap \partial F_{<n_0} = \emptyset \), which is possible since \( \partial F_{<\infty} = \emptyset \)). We can choose \( \eta \) such that

\[
\int |H_{<\infty} \tilde{f} - \lambda \tilde{f}|^2 db_{<\infty} \leq \varepsilon^2 / 4.
\]

Then we proceed as in lemma 2.1. of [13]. Precisely, equation (5) and the fact that \( \tilde{f} \) is in \( \mathcal{D}_{H_{<\infty}^{<n_0+p_0}}^+ \cap \mathcal{D}_{H_{<\infty}^{<n_0+p_0}}^- \), imply that

\[
\| P_{<n_0+p_0}^\pm ([\lambda - \epsilon, \lambda + \epsilon])(\tilde{f}) \| \geq \frac{1}{4}.
\]

At each level \( < n_0 + p_0 + k > \) we can make \( N^k \) copies of \( \tilde{f} \) on the \( < n_0 + p_0 > \)-cells of \( F_{<n_0+p_0+k>} \). This implies that

\[
\int_{[\lambda - \epsilon, \lambda + \epsilon]} \mu_{<n_0+p_0+k>} = \text{Tr}(P_{<n_0+p_0+k>}^\pm ([\lambda - \epsilon, \lambda + \epsilon])) \geq \frac{1}{4} N^k.
\]

This implies that \( \mu([\lambda - \epsilon, \lambda + \epsilon]) \geq \frac{1}{4(\epsilon^2 + \eta^2)} > 0 \). Hence, we proved that \( \lambda \in \text{supp}(\mu) \).

To prove ii) it is enough to prove that if \( \lambda \in \Sigma \) is such that for all \( \epsilon > 0 \) \( \dim P_{<\infty}^\pm ([\lambda - \epsilon, \lambda + \epsilon])(\mathcal{D}_{<\infty}^\pm) \geq 2 \) then \( \lambda \) is in supp(\( \mu \)). We do the proof for the Neumann boundary condition, the proof for the Dirichlet boundary condition being identical. If \( \partial F_{<\infty} \neq \emptyset \) then \( \partial F_{<\infty} \) contains a unique point that we denote \( z_0 \). Let \( \epsilon > 0 \) and suppose that \( \dim P_{<\infty}^\pm ([\lambda - \epsilon, \lambda + \epsilon])(\mathcal{D}_{<\infty}^+) \geq 2 \). We can find \( f \) with \( L^2 \) norm 1, in \( P_{<\infty}^\pm ([\lambda - \epsilon/4, \lambda + \epsilon/4])(\mathcal{D}_{<\infty}^+) \) such that \( f(z_0) = 0 \). Then, exactly as previously we can construct \( \hat{f} \) with norm 1, proportional to \( f \) on \( F_{<n_0}, \) null outside, and such that \( \int |H_{<\infty} \hat{f} - \lambda \hat{f}|^2 db_{<\infty} \leq \varepsilon^2 / 4 \). Moreover, we see that \( \hat{f} \) is in \( \mathcal{D}_{H_{<n_0+1}}^+ \cap \mathcal{D}_{H_{<n_0+1}}^- \) (indeed, \( \partial F_{<n_0+1} \cap \partial F_{<n_0} = \{ z_0 \} \) and \( \hat{f}(z_0) = 0 \)). At this point the proof goes exactly as before.

The proof in the continuous case is more technical since we cannot just approximate the function \( f \) by its restriction to \( X_{<n>} \) (which is in general not in the domain of \( H_{<\infty} \)), but we need to approximate it smoothly as it is done in [13]. We safely leave the details to the reader since the proof given for the discrete case for ii) and the arguments developed in [13] give easily the result. \( \square \)

The following results concern the Lebesgue decomposition of the spectrum. Let us first recall some elementary notions. We denote by \( P_{<\infty}^\pm(d \lambda) \) the spectral resolution of the self-adjoint operator \( H_{<\infty}^\pm \) on the Hilbert space \( \mathcal{D}_{<\infty}^\pm \). If \( f \) is in \( \mathcal{D}_{<\infty}^\pm \), remind that the spectral measure of \( f \) is defined as the measure \( \sigma^\pm(f)(d \lambda) \) on \( \mathbb{R} \) given by

\[
\sigma^\pm(f)(A) = \| P_{<\infty}^\pm(A)(f) \|^2,
\]

for any borelian \( A \subset \mathbb{R} \), where \( \| \| \) denotes the \( L^2 \) scalar product associated with \( b_{<\infty} \) on \( F_{<\infty} \) in the discrete setting, and \( m_{<\infty} \) on \( X_{<\infty} \) in the continuous setting. We denote by \( \sigma_{ac}^\pm(f)(d \lambda), \sigma_{sc}^\pm(f)(d \lambda), \sigma_{pp}^\pm(f)(d \lambda) \) respectively the absolutely continuous, the singular continuous, the purely pointual part of the Lebesgue decomposition of the
measure $\sigma^\pm(f)(d\lambda)$. Remind that the Hilbert space $D_{<\infty}^\pm$ can be decomposed into three orthogonal Hilbert subspaces (cf for example [2])

$$D_{<\infty}^\pm = H_{ac}^\pm \oplus H_{sc}^\pm \oplus H_{pp}^\pm,$$

such that $f$ is in $H_{ac}^\pm$, $H_{sc}^\pm$, $H_{pp}^\pm$ iff its spectral measure is respectively absolutely continuous, singular continuous or purely punctual. The Lebesgue decomposition of the spectrum is the closed sets $\Sigma_{ac}^\pm$, $\Sigma_{sc}^\pm$, $\Sigma_{pp}^\pm$ equal to the topological spectrum of the restriction of $H_{<\infty}^\pm$ to the subspaces $H_{ac}^\pm$, $H_{sc}^\pm$, $H_{pp}^\pm$. It is clear that the subspaces $H_{ND}$ generated by the Neuman-Dirichlet eigenfunctions is included in both $H_{ac}^\pm$ and $H_{pp}^\pm$. It is then natural to define $\tilde{H}_{pp}^\pm$ as the orthogonal supplement of $H_{ND}$ in $H_{pp}^\pm$ and to define $\tilde{\Sigma}_{pp}^\pm$ and $\Sigma_{ND}$ as the topological spectrum of $H_{<\infty}^\pm$, restricted respectively to $\tilde{H}_{pp}^\pm$ and $H_{ND}$. It is clear by definition that $\Sigma_{ND} = \text{supp}\mu^\text{ND}$ (and this is true for any blow-up $\omega$).

As pointed out, the infinite lattices $F_{<\infty}$ (or the unbounded set $X_{<\infty}$) are not isomorphic for different blow-up $\omega$. Hence, the spectral properties of $H_{<\infty}^\pm$ depends a priori on $\omega$: to show this dependence we sometimes write $P_{<\infty}^\pm(\omega, d\lambda)$, $\Sigma_{sc}^\pm(\omega)$, $\cdots$.

We endow $\Omega = \{1, \ldots, N\}^\mathbb{N}$ with the product of the uniform measure on $\{1, \ldots, N\}$. In the next two propositions we give almost sure results on the blow-up. The following result is the analogous of a result initially due to Pastur, [9], for random Schrödinger operators.

**Proposition 2** There exist deterministic sets $\Sigma$, $\Sigma_{ac}$, $\Sigma_{sc}$, $\Sigma_{pp}$, $\tilde{\Sigma}_{pp}$ and $\Sigma_{ND}$ such that for almost all $\omega$ in $\Omega$ (for the product of the uniform measure on $\{1, \ldots, N\}$) we have

$$\Sigma^\pm(\omega) = \Sigma, \quad \Sigma_{sc}^\pm(\omega) = \Sigma_{sc}, \quad \tilde{\Sigma}_{pp}^\pm(\omega) = \tilde{\Sigma}_{pp}.$$

**Remark 4**: As we pointed out, $\Sigma_{ND}$ is constant in $\omega$ and equal to $\text{supp}\mu^\text{ND}$. From proposition 1 we see that the almost sure spectrum $\Sigma$ is equal to $\text{supp}\mu$.

**Remark 5**: The structure of the spectrum can really depend on $\omega$. In a forthcoming paper, [15], we plan to prove that for a self-similar Sturm-Liouville operator on $[0, 1]$, the spectrum is continuous for a typical blow-up $\omega$, but can be pure point for a particular $\omega$.

**Proof.** Remark first that for almost all blow-up $\omega$, $\partial F_{<\infty}^\pm = \emptyset$. It is thus enough to consider only $\omega$ such that $\partial F_{<\infty}^\pm = \emptyset$. Let $P_{<\infty}^\pm(d\lambda, \omega)$ denotes the spectral resolution of $H_{<\infty}^\pm(\omega)$. Denote by $\tilde{H}$ the orthogonal supplement of $H_{ND}$ in $D_{<\infty}^\pm$. For a function $f$ in $D_{<\infty}^\pm$, we denote by $\tilde{\sigma}(f)(d\lambda, \omega)$ the spectral measure of the projection of $f$ on $\tilde{H}$. We denote by $\tilde{\sigma}_{ac}(f)(d\lambda, \omega)$, $\tilde{\sigma}_{sc}(f)(d\lambda, \omega)$, $\tilde{\sigma}_{pp}(f)(d\lambda, \omega)$, resp. the absolutely continuous, the singular continuous and the purely punctual part of the Lebesgue decomposition of the measure $\tilde{\sigma}(f)$. Remark that $\sigma_{ac}(f) = \sigma_{ac}(f) + \sigma_{sc}(f) = \sigma_{ac}(f) + \sigma_{pp}(f)$ is the sum of $\tilde{\sigma}_{pp}(f)$ plus the spectral measure of the projection of $f$ on $H_{ND}$.

Consider $x$ in $F_{<\infty}^\pm$, we first prove that the map $\omega \to \tilde{\sigma}(\delta_x)(d\lambda, \omega)$, where $\delta_x$ is the Dirac function at $x$, is measurable in $\omega$ (the $\sigma$-field on the set of non-negative measures on $\mathbb{R}$ is the smallest $\sigma$-field such that $\mu \to \mu(A)$ is measurable for any Borelian $A$). Indeed, denote by $E_{<\infty}^\pm$ the orthogonal supplement of the $L^2$ scalar product for $b_{<n_r}$ of $E_{<n_r}^\pm$ in $D_{<\infty}^\pm$ and for $x$ in $F_{<\infty}$ and $m \geq n$, by $\tilde{\sigma}_{<m_r}(\delta_x)(d\lambda, \omega)$ the spectral measure of the projection of $\delta_x$ on $E_{<m_r}^\pm$ for the operator $H_{<m_r}^\pm$. It is clear that $\tilde{\sigma}(\delta_x)(d\lambda, \omega)$ is the limit of $\tilde{\sigma}_{<m_r}(\delta_x)(d\lambda, \omega)$ when $m$ goes to infinity. But $\tilde{\sigma}_{<m_r}(\delta_x)(d\lambda, \omega)$ depends
on $\omega$ only by the finite sequence $(\omega_1, \ldots, \omega_m)$ (which gives the position of $x \in F_{<n>}$ as a point in $\hat{F}_{<m>}$), and hence, is measurable in $\omega$. This implies the measurability of $\omega \rightarrow \hat{\sigma}(\delta_x)(d\lambda, \omega)$. Proceeding exactly like in lemma V.15 of [2], we know that the components $\hat{\sigma}_{nc}(\delta_x)(d\lambda, \omega)$, $\hat{\sigma}_{sc}(\delta_x)(d\lambda, \omega)$, $\hat{\sigma}_{pp}(\delta_x)(d\lambda, \omega)$ of the Lebesgue decomposition of $\hat{\sigma}(\delta_x)(d\lambda, \omega)$ are measurable.

Consider now the family of "translations" in $\Omega$ defined by

$$\tau_{i_1, \ldots, i_k}(\omega) = (\omega_1 + i_1[N], \ldots, \omega_k + i_k[N], \omega_{k+1}, \ldots),$$

where $\omega_j + i_j[N]$ is the value of $\omega_j + i_j$ modulo $N$. It is clear that $\{\tau_{i_1, \ldots, i_k}\}_{i_1, \ldots, i_k \in \{1, \ldots, N\}^k}$ is a measure preserving ergodic family of transformations of $\Omega$. Writing

$$\{\omega \text{ s.t. } \lambda \in \Sigma.(\omega)\} = \bigcap_{\lambda', \lambda'' \in \mathbb{Q}} \{\omega \text{ s.t. } \exists n, \exists x \in F_{<n>}, \sigma.(\delta_x)(|\lambda', \lambda''|, \omega) > 0\},$$

(and idem for $\Sigma_{pp}(\omega)$) we know that the set $\{\omega \text{ s.t. } \lambda \in \Sigma.(\omega)\}$ is measurable and $\tau$ invariant, and thus is of measure 0 or 1. We define the deterministic set $\Sigma = \{\lambda \in \mathbb{Q} \text{ s.t. } \mathbb{P}(\lambda \in \Sigma.(\omega)) = 1\}$, where $\mathbb{P}$ denotes the expectation with respect to $\omega$. It is clear, since $\Sigma.(\omega)$ is closed, that we have $\Sigma.(\omega) = \Sigma$ for almost all $\omega$. □

**Proposition 3** If the density of states is completely created by the N-D eigenvalues, i.e. if $\mu^{ND} = \mu$ then for almost all $\omega \in \Omega$ the set of N-D is complete i.e. $\mathcal{H}_{ND} = \mathcal{D}^{+}_{<\infty>}(\omega) = \mathcal{D}^{-}_{<\infty>}(\omega)$. In this case, the spectrum of $H_{<\infty>}$ is then pure point with compactly supported eigenfunctions.

**Remark 6**: When this last equality is satisfied then necessarily $\partial X_{<\infty>} = \partial F_{<\infty>} = \emptyset$. In particular, this means that if $\partial F_{<\infty>} \neq \emptyset$ then there is a component of the spectrum in the complement of $\mathcal{H}_{ND}$ (for the Sierpinski gasket, it is known from [16] that for Neumann boundary condition, this component is also pure point, but the answer is not known for the Dirichlet boundary condition).

**Remark 7**: It is known that the equality $\mu^{ND} = \mu$ is satisfied for Nested fractals (cf [13, 14]). In [13] we precised an almost sure class of blow-ups for which the set of N-D eigenfunctions is complete (called asymmetrical blow-ups). In particular for the Sierpinski gasket this is true as soon as $\partial F_{<\infty>} = \emptyset$ (but this was known since [16]).

Proof: Denote by $\mathbb{P}$ and $\mathbb{E}$ the probability and the expectation with respect to the blow-up for the product of the uniform measure on $\{1, \ldots, N\}$. We first prove the result in the discrete case. We must prove that

$$\mathbb{P}(\cup F_{<n>}) = \mathcal{D}^{+}_{<\infty>},$$

$$\Leftrightarrow \mathbb{P}(\forall f \in \mathcal{D}_{<\infty>} \text{ with compact support}, \lim_{n \to \infty} \|P_{\hat{E}^{+}_{<n>}} f\|^2_{b_{<n>}} = 0) = 1,$$

where $P_{\hat{E}^{+}_{<n>}}$ is the orthogonal projection on the space $\hat{E}^{+}_{<n>}$ generated by the "Neumann only" eigenfunctions, i.e. the orthogonal supplement in $\mathcal{D}^{+}_{<\infty>}$ of $E_{<n>}^{ND}$. But this is again equivalent to

$$\forall k_0, \forall f \in \mathcal{D}^{-}_{<k_0>}, \mathbb{P}(\lim_{n \to \infty} \|P_{\hat{E}^{+}_{<n>}} f\|^2_{b_{<n>}} = 0) = 1\Leftrightarrow \forall k_0, \forall f \in \mathcal{D}^{-}_{<k_0>}, \forall \varepsilon > 0, \lim_{n \to \infty} \mathbb{P}(\|P_{\hat{E}^{+}_{<n>}} f\|^2_{b_{<n>}} \geq \varepsilon) = 0.$$
Let us now make a remark: for different blow-ups $\omega$, the sets $F_{<n>}^\omega$ are isomorphic but the measures $b_{<n>}^\omega$ are not equal but differ from a constant multiple. On $F_{<n>}^\omega$, we introduce the measure $\tilde{b}_{<n>}^\omega$, independant of $\omega$, by

$$\int_{F_{<n>}^\omega} f d\tilde{b}_{<n>}^\omega = \sum_{i_1, \ldots, i_n} \beta_{i_1} \cdots \beta_{i_n} \int_F f|_{F_{<n>}^\omega, i_1, \ldots, i_n} \, db.$$ 

Remark that $b_{<n>}^\omega = \beta_{\omega_1}^{-1} \cdots \beta_{\omega_n}^{-1} b_{<n>}$. Choose now a basis $g_1, \ldots, g_{\dim E_{<n>}^+}$ of $E_{<n>}^+$, orthonormal for the scalar product associated with $\tilde{b}_{<n>}^\omega$.

We have

$$\mathbb{P}(\|P_{E_{<n>}^+} f\|_{b_{<n>}^\omega}^2 \geq \epsilon) \leq \frac{1}{\epsilon} \mathbb{E}(\|P_{E_{<n>}^+} f\|_{\tilde{b}_{<n>}^\omega}^2) = \mathbb{E}(\beta_{\omega_1}^{-1} \cdots \beta_{\omega_n}^{-1} \|P_{E_{<n>}^+} f\|_{\tilde{b}_{<n>}^\omega}^2) = \sum_{i=1}^{\dim E_{<n>}^+} \mathbb{E}(\beta_{\omega_1}^{-1} \cdots \beta_{\omega_n}^{-1} | f, g_i >_{b_{<n>}^\omega}^2).$$

Now, we average on the blow-up: to do this we consider that the cell $F_{<k_0}>^\omega$ is the subcell $F_{<n>, w_{<n>}}^{\omega}, \ldots, w_{k_0+1}$ of $F_{<n>}^\omega$ and we average on the position of $F_{<k_0>^\omega}$. We denote by $f_{w_{<n>}, \ldots, w_{k_0+1}}$ the function with value $f$ on the cell $F_{<n>}, \ldots, w_{k_0+1}$ and 0 outside. The last expression is equal to

$$\frac{1}{N^n} \sum_{\omega_1, \ldots, \omega_k} \beta_{\omega_1}^{-1} \cdots \beta_{\omega_{k_0}^{-1} \sum_{\omega_{k+1}, \ldots, \omega_n} b_{\omega_{k_0+1}}^{-1} \cdots b_{\omega_n}^{-1} < f_{\omega_{k_0+1}, \ldots, \omega_n}, g_i >_{b_{<n>}}.$$ 

But the $f_{\omega_{k_0+1}, \ldots, \omega_n}$ are orthogonal and

$$\|f_{\omega_{k_0+1}, \ldots, \omega_n}\|_{b_{<n>}}^2 = \beta_{\omega_{k_0+1}}^{-1} \cdots \beta_{\omega_n}^{-1} \|f\|_{\tilde{b}_{<k_0>}}^2.$$ 

This implies that

$$\sum_{\omega_{k_0+1}, \ldots, \omega_n} \beta_{\omega_{k_0+1}}^{-1} \cdots \beta_{\omega_n}^{-1} | f_{\omega_{k_0+1}, \ldots, \omega_n}, g_i >_{b_{<n>}}^2 \| f \|_{\tilde{b}_{<k_0>}}^2 \leq \| f \|_{b_{<k_0>}}^2,$$

for all $i$, and thus that

$$\mathbb{E}(\|P_{E_{<n>}^+} f\|_{b_{<n>}}^2) \leq \frac{\dim E_{<n>}^+}{N^{n-k_0}} \frac{1}{N^{k_0}} \sum_{\omega_1, \ldots, \omega_k} \beta_{\omega_1}^{-1} \cdots \beta_{\omega_{k_0}}^{-1} \| f \|_{\tilde{b}_{<k_0>}}^2 = \frac{\dim E_{<n>}^+}{N^{n-k_0}} \mathbb{E}(\| f \|_{\tilde{b}_{<k_0>}}^2),$$

which goes to 0 when $n$ goes to infinity, by hypothesis. Thus, we proved that almost surely on the blow-up we have $\cup E_{<n>}^\omega = D_{<\infty>}^\omega$. When this situation is satisfied then necessarily $D_{<\infty>}^\omega = D_{<\infty>}$ and hence $\partial F_{<\infty>}$ is pure point since the Neumann-Dirichlet eigenfunctions form a dense set of compactly supported eigenfunctions.

To prove the result in the continuous case we must consider the space $E_{<n>}^+(\lambda)$, $\lambda \leq 0$, generated by the "Neuman only" eigenfunctions with eigenvalues larger than $\lambda$. 

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i.e. the orthogonal supplement in $P_{<n>}^+([\lambda,0])(D_{<n>}^+)$ of $E_{<n>}^{ND} \cap P_{<n>}^+([\lambda,0])(D_{<n>}^+)$. The space $E_{<n>}^+(\lambda)$ is finite dimensional and we prove exactly in the same way that almost surely on the blow-up for any function $f$ with compact support $\|P_{E_{<n>}^+(\lambda)}f\|^2$ converges to 0. □

3 About a ”Lebesgue decomposition” of the density of states

3.1 The discrete case

To avoid confusion, we precise that we do not consider here the Lebesgue decomposition of the measure $\mu$, but a decomposition of the measure $\mu$ into three (or four) parts corresponding to the Lebesgue decomposition of the spectral measures. As pointed out in remark 4, we proved that the measures $\mu$ and $\mu^{ND}$ are related to the almost sure spectrum and to the Neuman-Dirichlet spectrum by $\text{supp}(\mu) = \Sigma$, $\Sigma^{ND} = \text{supp}\mu^{ND}$.

In [14], we were able to compute these two measures in terms of a certain renormalization map that we explicitly defined. The aim of this section is to introduce some measures generalizing the measures $\mu$ and $\mu^{ND}$ to the different parts of the Lebesgue decomposition of the spectrum. We are not able to say much about these measures, but we think they are central notions. In particular, we think that the central question is wether it is possible to compute these measures in terms of the renormalization map introduced in [14].

Let us start by a lemma.

Lemma 0.1 The measures $\mu$ and $\mu^{ND}$ satisfy:

$$\mu(d\lambda) = \mathbb{E} \left( \sum_{x \in F_{<0>}^+} \frac{b_{<0>}(x)}{(b_{<\infty>}^+(x))^2}\sigma(\delta_x)(d\lambda, \omega) \right),$$

$$\mu^{ND}(d\lambda) = \mathbb{E} \left( \sum_{x \in F_{<0>}} \frac{b_{<0>}(x)}{(b_{<\infty>}^+(x))^2}\sigma^{ND}(\delta_x)(d\lambda, \omega) \right),$$

where $\mathbb{E}$ denotes the expectation with respect to the blow-up $\omega \in \{1, \ldots, N\}^\mathbb{N}$, and $\sigma(\delta_x)$ and $\sigma^{ND}(\delta_x)$ denotes respectively the spectral measure of the Dirac function $\delta_x$ at $x$ and the spectral measure of the projection of $\delta_x$ to the subspace $\mathcal{H}_{ND}$ generated by the Neuman-Dirichlet eigenfunctions.

N.B.: Remind that for almost all $\omega$, $\partial F_{<\infty>^+} = \emptyset$ so that the boundary condition does not matter when we take expectation and we can simply write $\sigma(\delta_x)(d\lambda, \omega)$ for $\sigma^\pm(\delta_x)(d\lambda, \omega)$.

Proof: We first present the proof for $\mu$. Remind that $\sigma^{+}_{<\infty>^+}(\delta_x)(d\lambda, \omega)$ is the spectral measure of the Dirac function $\delta_x$ for the operator $H_{<n>}^+$ on $D_{<n>}^+$ (it depends on $(\omega_1, \ldots, \omega_n)$ via the position of $F_{<0>}$ in $F_{<n>}$). It is clear that for all $\omega$ such that $\partial F_{<\infty>^+} = \emptyset$ and all $x \in F_{<0>}^+$ we have

$$\frac{b_{<0>}(x)}{(b_{<\infty>}^+(x))^2}\sigma(\delta_x)(d\lambda, \omega) = \lim_{n \to \infty} \frac{b_{<0>}(x)}{(b_{<\infty>}^+(x))^2}\sigma^{+}_{<n>^+}(\delta_x)(d\lambda, \omega).$$

Choose now a basis of eigenfunctions of $H_{<n>}^+$, $g_1^+, \ldots, g_{|F_{<n>}^+|}$ for the eigenvalues $\lambda_1^+ = 0 > \cdots \geq \lambda_1^{|F_{<n>}^+|}$. Assume moreover that these eigenfunctions are orthonormal.
for the $l^2$ scalar product associated with the measure $\tilde{b}_{n>}$ on $F_{n>}$, introduced in the proof of proposition 3 (and remind that $b_{n>} = \beta_{w_1} \cdots \beta_{w_n}^{-1} \tilde{b}_{n>}$. Each point in $F_{n>}$ can be labelled in a natural way (and in a non unique way) by a point $(\omega_1, \ldots, \omega_n, z)$ in $\{1, \ldots, N\} \times F$. In the following, by abuse of notations, we simply write $(\omega_1, \ldots, \omega_n, z)$ for the corresponding point in $F_{n>}$. We have

$$
\mathbb{E}\left( \sum_{x \in F_{n>}} \frac{b_{n>}(x)}{(b_{n>}(x))^2} \sigma_{n>}^-(\delta_x)(d\lambda, \omega) \right) = \mathbb{E}\left( \sum_{x \in F_{n>}} \frac{b_{n>}(x)}{(b_{n>}(x))^2} \left| \sum_{i=1}^{\lfloor F_{n>}\rfloor} \frac{1}{\| g_i \|_{b_{n>}}^2} \delta_{b_{n>}}^+(i) \right|^2 (d\lambda) \right) = \mathbb{E}\left( \sum_{x \in F_{n>}} \sum_{i=1}^{\lfloor F_{n>}\rfloor} b_{n>}(x) \beta_{w_1} \cdots \beta_{w_n} g_i^+(x) \right)^2 \delta_{b_{n>}}^+(d\lambda) = \frac{1}{N^n} \left( \sum_{i=1}^{\lfloor F_{n>}\rfloor} \delta_{b_{n>}}^+(d\lambda) \right) = \frac{1}{N^n} \nu_{n>}(d\lambda).
$$

This proves the formula concerning $\mu$. The proof for the formula for $\mu^{ND}$ is similar: on just has to replace $\sigma_{n>}^-(\delta)$ by $\sigma_{n>}^-(\delta_x)$, the spectral measure of the projection of $\delta_x$ on $E_{n>}^{ND}$. □

**Definition 0.1** We introduce the measures $\mu^{ac}$, $\mu^{sc}$, $\mu^{pp}$, $\tilde{\mu}^{pp}$, by

$$
\mu^+(d\lambda) = \mathbb{E}\left( \sum_{x \in F_{n>}} \frac{b_{n>}(x)}{(b_{n>}(x))^2} \sigma_{n>}^-(\delta_x)(d\lambda, \omega) \right),
$$

and idem for $\tilde{\mu}^{pp}$. Remark that we have

$$
\mu = \mu^{ac} + \mu^{sc} + \mu^{pp},
\mu^{pp} = \tilde{\mu}^{pp} + \mu^{ND}.
$$

Then we have the following easy proposition:

**Proposition 4** We have

$$
\Sigma = \text{supp}(\mu),
\Sigma_{pp} = \text{supp}(\tilde{\mu}^{pp}),
$$

where $\Sigma_{ac}, \Sigma_{sc}, \Sigma_{pp}, \Sigma_{pp}$ are the almost sure components of the spectrum introduced in proposition 2.

Proof: By the definition itself it is clear that $\Sigma_c \subset \text{supp}(\mu)$. Reciprocally, if $\Sigma \cap \chi' \chi'' \neq \emptyset$ this means that there exist $n$ and $x$ in $F_{n>}$ such that $\sigma_{n>}^-(\delta_x)(\chi', \chi'', \omega) > 0$ for a set
of blow-up $\omega$ of positive measure. But the point $x$ in $F_{<n>}$ is in $F_{<0>}$ for all $\omega$ starting from a certain sequence $(\omega_1, \ldots, \omega_n)$. This means that for a set of positive measure of blow-up $\omega$, there exists $x$ in $F_{<0>}$ such that $\sigma_\omega(\delta_x)([\lambda', \lambda''](\omega)) > 0$. This immediately implies that $\mu^i([\lambda', \lambda'']) > 0$ and thus that $\text{supp}\mu^i = \Sigma$. $\square$

In [14] we were able to compute the measures $\mu$ and $\mu^{ND}$ and to characterize the equality $\mu = \mu^{ND}$. The natural question (but certainly difficult) is whether it is possible to compute the different measures $\mu^{ac}$, $\mu^{sc}$, $\mu^{pp}$, $\tilde{\mu}^{pp}$ in terms of the renormalization map we introduced in [14]. This would give an information stronger than the Lebesgue decomposition of the spectrum. In particular it would be interesting to understand the measure $\tilde{\mu}^{pp}$, or the set $\Sigma^{pp}$, which corresponds to the pure point part not induced by the N-D spectrum. There are very few examples where the spectral properties of the operator $H_{<\infty>}$ are understood. There is the case where $\mu^{ND} = \mu$ (including the Sierpinski gasket and the nested fractals, cf [16], [13]), corresponding more or less to the case where the asymptotic degree $d_\infty$ is smaller than $N$ (cf [14]). In this case we know that the only non-empty component in the spectrum is the Neumann-Dirichlet component. We can prove also that in the case of a self-similar Sturm-Liouville operator, the spectrum is continuous, i.e. $\Sigma^{pp} = \emptyset$. We do not know any example where $\Sigma^{pp}$ is non-empty.

Of course, what make things easy for the measure $\mu$ and $\mu^{ND}$ is that they have an expression as a limit of counting measures related to the operators on finite level $F_{<n>}$. There is no corresponding expression for the other parts of the density of states $\mu^{ac}$, $\mu^{sc}$, $\tilde{\mu}^{pp}$.

### 3.2 The continuous case

Of course, there are similar formulas for the continuous setting. We give without proof the correct definitions for the measures $\mu^{ac}$, $\mu^{sc}$, $\mu^{pp}$, $\tilde{\mu}^{pp}$. Denote by $R_{X_{<\infty>}} : L^2(X_{<\infty>}) \to L^2(X_{<\infty>})$ the operator of restriction to $X_{<0>}$ defined by $R_{X_{<\infty>}}(f)(x) = 1_{\{x \in X_{<0>}\}}f(x)$. We also denote respectively by $P^{ac}_{<\infty>}(d\lambda, \omega)$, $P^{sc}_{<\infty>}(d\lambda, \omega)$, $P^{pp}_{<\infty>}(d\lambda, \omega)$, $P^{ND}_{<\infty>}(d\lambda, \omega)$, $P^{ac}_{<\infty>}(d\lambda, \omega)$, the composition of the spectral resolution $P_{<\infty>}(d\lambda, \omega)$ with the projection on respectively $H_{ac}$, $H_{sc}$, $H_{pp}$, $H_{ND}$, $H_{pp}$. Then we define

$$\mu^i = \mathbb{E}(\text{Trace}(R_{X_{<\infty>}} \circ P_{<\infty>}(d\lambda, \omega) \circ R_{X_{<\infty>}PURE})$$

and idem for $\tilde{\mu}^{pp}$. It is not difficult to prove that these measures have the same properties as in the discrete case (i.e. the analogous of lemma 0.1 and proposition 4 are satisfied).

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**References**


