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Abstract

We consider elliptic and parabolic systems with multivalued $x$–dependent graphs. The existence of solutions for elliptic equation was established in [5] and [6]. We extend this result to the elliptic and parabolic systems, in particular to the systems describing a flow of non–Newtonian incompressible fluids. Contrary to these two papers we follow the spirit of the compactness method of J.L. Lions for variational-type operators, however, expanded on the framework of measure-valued solutions. The main concept consists in applying the relation between $x$–dependent maximal monotone graphs and 1-Lipschitz Carathéodory functions to introduce the generalized Young measures. The method was announced in the short note [12].

1 Introduction

Let $\Omega$ be an open and bounded subset of $\mathbb{R}^d$ with $\partial\Omega \in C^{0,1}$. Given $f : \Omega \to \mathbb{R}$ and $A(x) \subset \mathbb{R}^d \times \mathbb{R}^d$ a maximal monotone $x$–dependent graph, consider the elliptic problem in a form

$$- \text{div} \sigma(x) = f(x), \quad (\nabla u(x), \sigma(x)) \in A(x) \text{ in } \Omega, \quad u|_{\partial\Omega} = 0,$$

(1.1)

where $u : \Omega \to \mathbb{R}$, $\sigma : \Omega \to \mathbb{R}^d$. In the paper by Chiadô Piat et al. [5] a set of assumptions on $A(x)$ was stated, and the first proof of existence of weak solutions was achieved. The crucial point was defining the proper measurability of $A(x)$ with respect to $x$. Note that since $A(x)$ is multi-valued, there are many possible choices, see e.g. [2].

Recently an entirely new approach was proposed by Francfort et al. [6]. They reformulate the assumptions into a completely equivalent form, but omitting the use of multi-valued techniques. Our aim is to propose a different method of proving the existence of solutions, with the same assumptions as in [5], [6], only replacing monotonicity by strict monotonicity. It benefits with the additional information about the strong convergence of approximate solutions. We extend the result to the case of parabolic systems

$$u_t(t, x) - \text{div} \sigma(t, x) = f(t, x);$$

$$(\nabla u(t, x), \sigma(t, x)) \in A(t, x) \text{ in } I \times \Omega;$$

$$u(0, x) = u_0(x), \quad u|_{\partial\Omega} = 0,$$

(1.2)

where $I = [0, T)$, $T < \infty$ and $u_0$ is a given initial data.

Besides the abstract problems (1.1) and (1.2) we consider also the problems arising from fluid mechanics, in particular the non-Newtonian fluids with a Cauchy stress tensor in a form of maximal monotone operator. These kind of problems cover and extend the ones investigated in [16] - the steady flow of Bingham fluids and in [10] - the blood flow with a particular attention to the behavior of platelets and formation of the clot. The latter model was introduced by [1]. Both papers mentioned above do not consider $x$– or $(t, x)$–dependent tensors. In this case our method admits fully general maximal monotone Cauchy stress tensor, satisfying only the proper growth and coercivity conditions, which is excluded by the methods of [16] and [10]. Note that besides the weak solutions the paper [10] provides a new kind of measure–valued solutions without the monotonicity assumption.
The steady flow in both cases can be generally described by the system of equations

\[
\begin{align*}
\text{div } \sigma(x) &= \text{div} \left[ u(x) \otimes u(x) \right] + \nabla \pi(x) - f(x), \\
\text{div } u(x) &= 0, \\
(\nabla^s u(x), \sigma(x)) &\in \mathcal{A}(x) \quad \text{in } \Omega, \\
u_{|\partial \Omega} &= 0.
\end{align*}
\]  

(1.3)

Here \( u = (u_1, \ldots, u_d) \) is a velocity field, \( \pi \) denotes a pressure and \( f = (f_1, \ldots, f_d) \) are given body forces. By \( u \otimes u \) we denote the second order tensor (dyadic product) with the components \((u \otimes u)_{ij} = u_i u_j\), and \( \nabla^s \) denotes the symmetric gradient, i.e. \( \nabla^s = \frac{1}{2}(\nabla + \nabla^T) \).

Similarly, \( u \) and \( \pi \) capture unsteady flows if

\[
\begin{align*}
u_t(t, x) + \text{div} \left[ u(t, x) \otimes u(t, x) \right] + \nabla \pi(t, x) &= \text{div} \sigma(t, x) + f(t, x), \\
\text{div } u(t, x) &= 0, \\
(\nabla^s u(t, x), \sigma(t, x)) &\in \mathcal{A}(t, x) \quad \text{in } I \times \Omega, \\
u(0, x) &= u_0(x), \quad u_{|\partial \Omega} = 0,
\end{align*}
\]  

(1.4)

where \( u_0 = (u_{01}, \ldots, u_{0d}) \) is a given initial velocity and \( I = [0, T) \), with \( T < \infty \).

Our goal is to formulate an abstract theorem (Theorem 5.1), which establishes compactness criteria enough for providing the existence of weak solutions to all the problems (1.1)-(1.4). The formulation of this lemma was inspired by the compactness method of J.L. Lions for variational-type operators, see [14, Ch. 2.2, Thm. 2.8] and [4, Lem. 5]. Nevertheless, our proof strongly differs from the method of Lions. The crucial step is using Young measures tools in a non-standard setting.

We start with formulating the assumptions on the \( x \)-dependent maximal monotone graph \( \mathcal{A}(x) \). Here and subsequently \( \zeta, \eta \in \mathbb{R}^N \) and \( x \) will denote a point of \( \Omega \). Assume that \( \mathcal{A}(x) \subset \mathbb{R}^{N \times N} \) is an \( x \)-dependent graph with the following properties:

A1: for a.e. \( x \in \Omega \) and every \( e \in \mathbb{R}^N \) the set \( \{ d \in \mathbb{R}^N : (e, d) \in \mathcal{A}(x) \} \) is closed;

A2: \( \mathcal{A}(x) \) is maximal strictly monotone for a.e. \( x \in \Omega \); i.e. for every \((e_1, d_1), (e_2, d_2) \in \mathcal{A}(x)\) we have

\[
(d_1 - d_2) \cdot (e_1 - e_2) > 0 \quad \text{for } e_1 \neq e_2 \quad \text{(strict monotonicity)};
\]

and if \( (e, d) \in \mathbb{R}^N \times \mathbb{R}^N \) is such that

\[
(\bar{d} - d) \cdot (\bar{e} - e) \geq 0 \quad \text{for every } (\bar{e}, \bar{d}) \in \mathcal{A}(x)
\]

then we have \( (e, d) \in \mathcal{A}(x) \) (maximality);

A3: there exist \( p \in (1, \infty) \), nonnegative function \( m \in L^1(\Omega) \) and \( \alpha > 0 \) such that for a.e. \( x \in \Omega \) and every \((e, d) \in \mathcal{A}(x)\)

\[
d \cdot e \geq -m(x) + \alpha(|e|^p + |d|^{p'}) \tag{1.5}
\]

with \( \frac{1}{p} + \frac{1}{p'} = 1 \).
A4: for any closed set $C \subset \mathbb{R}^N$, 
\[
\{(x,e) \in \Omega \times \mathbb{R}^N : \text{there exists } d \in C \text{ s.t. } (e,d) \in A(x)\}
\]
is measurable with respect to the $\sigma$-algebra $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^N)$, where $\mathcal{L}(\Omega)$ denotes the $\sigma$-algebra of Lebesgue measurable subsets of $\Omega$ and $\mathcal{B}(\mathbb{R}^N)$ that of all Borel subsets of $\mathbb{R}^N$.

**Remark.** The assumptions A1-A4 correspond to [5] and [6] with the difference that the original assumption A2 involved the monotonicity condition, i.e.

\[(d_1 - d_2) \cdot (e_1 - e_2) \geq 0.
\]

We require stronger assumption on monotonicity, which benefits with the strong convergence of approximate solutions.

**Remark.** Without loss of generality we can replace $\Omega$ with $I \times \Omega$ and $x$ with $(t,x)$ in assumptions A1–A4. This does not make any difference, since $I \times \Omega$ is an open and bounded set in $\mathbb{R}^{d+1}$. This is indeed what we will mean, each time when we recall to assumptions A1–A4 in case of evolutionary problems.

The paper is organized as follows. In section 2 we introduce all the main results of our paper. In section 3 we recall some ideas of the paper [6], which connect $x$-dependent maximal monotone graph and Carathéodory functions. This relation will play a crucial role for generalization of the framework of measure-valued solutions to multi-valued operators. Next, we establish some properties of the approximations. The section 4 is dedicated to the investigation of the properties of weak limits of measure-valued functions and application to the construction of generalized Young measures to the $x$–dependent (multi–valued) maximal monotone operators. The central role in the structure of the paper plays Theorem 5.1 (the main results of section 5), which provides an abstract compactness criteria for sequences of the approximations. In the last section, basing on the result from section 5, we prove main Theorems 2.1–2.4. For elliptic equations (i.e. Theorem 2.1) we give two different proofs. First, we apply the div–curl lemma of the theory of compensated compactness. Next, we show that instead of using div–curl structure, we can benefit from the energy equality. This path allows to show that even solutions coming directly from Galerkin approximation scheme are strongly convergent. The proofs for parabolic system, stationary and evolutionary non–Newtonian flow base on energy equality. Note that in the latter cases div–curl structure is absent due to divergence free projection in non–Newtonian fluids or evolutionary structure for parabolic equations. The application of Theorem 5.1 follows the similar lines as the application of compactness criteria formulated in [11] for the stationary flow and in [18] for the non stationary flow, see also [10]. The proof of Theorem 2.2 is shifted to the end of section 6 and treated in a very short way, since it is essentially included in the proof of Theorem 2.4.

## 2 Main results

Assume $A(x) \subset \mathbb{R}^{d \times m} \times \mathbb{R}^{d \times m}$ satisfies the conditions A1–A4. By $\mathcal{D}(\Omega)$ we understand the space of all $C^\infty$-functions with compact support in $\Omega$. Moreover, $\mathcal{D}(-\infty,T;X)$ is the space of all $C^\infty$-functions with compact support from $(-\infty,T)$ to some space $X$. The following theorem was proved in [5] and [6]:

3
Remark. Note that the symmetric gradient $\nabla^s$ for all $\sigma \in L^p(\Omega, \mathbb{R}^{d \times m})$ such that $(\nabla u(x), \sigma(x)) \in \mathcal{A}(x)$ a.e. in $\Omega$

and

$$-\text{div} \sigma = f \quad \text{in } D'(\Omega).$$

Theorem 2.2. Assume that $A(t, x)$ satisfies conditions A1–A4 and let $u_0 \in L^2(\Omega, \mathbb{R}^m)$. Given $f \in L^p(I, W^{-1,p}(\Omega, \mathbb{R}^m))$ there exists a weak solution to system (1.2), i.e. functions $u \in L^\infty(I; L^2(\Omega, \mathbb{R}^m)) \cap L^p(I, W^{1,p}(\Omega, \mathbb{R}^m))$ and $\sigma \in L^p(I; L^p(\Omega, \mathbb{R}^{d \times m}))$ such that

$$(\nabla u(t,x), \sigma(t,x)) \in \mathcal{A}(t,x) \quad \text{a.e. in } I \times \Omega$$

and

$$\int_{I \times \Omega} \sigma(t,x) \cdot \nabla u(t,x) \, dxdt = \int_{I \times \Omega} u(t) \cdot \varphi_t \, dxdt + \int_{I \times \Omega} f(t) \cdot \varphi \, dxdt + \int \Omega u_0 \varphi(0) \, dx$$

for all $\varphi \in D(-\infty, T; D(\Omega)).$

Remark. Note that the symmetric gradient $\nabla^s$ belongs to the space

$$\mathbb{R}^{d \times d}_{\text{sym}} \cong \mathbb{R}^{\frac{d(d+1)}{2}},$$

where by $\mathbb{R}^{d \times d}_{\text{sym}}$ we mean the space of $d \times d$ symmetric matrices.

Before we state the results concerning non-Newtonian fluids, we describe the notation. Let $V = \{ u : u \in D(\Omega, \mathbb{R}^d), \text{ div } u = 0 \}$. By $W^{1,p}_{0, \text{div}}(\Omega, \mathbb{R}^d)$ we mean the closure of $V$ with respect to the norm $\|u\|_{W^{1,p}} = (\int_{\Omega} |\nabla u|^p \, dx)^{\frac{1}{p}}$ and by $L^2_{\text{div}}(\Omega, \mathbb{R}^d)$ we denote the closure of $V$ with respect to the standard $L^2$–norm.

Theorem 2.3. Let $p \geq \frac{3d}{d+2}$. Assume $A(x)$ satisfies assumptions A1–A4. Given $f \in (W^{1,p}_{0, \text{div}}(\Omega, \mathbb{R}^d))^*$, there exists a weak solution to the system (1.3), i.e. functions $u \in W^{1,p}_{0, \text{div}}(\Omega, \mathbb{R}^d)$ and $\sigma \in L^p(\Omega, \mathbb{R}^{d \times d})$ such that

$$(\nabla^s u(x), \sigma(x)) \in \mathcal{A}(x) \quad \text{a.e. in } \Omega$$

and for all $\varphi \in V$

$$\int \Omega \sigma \cdot \nabla^s \varphi \, dx = \int \Omega ((u \otimes u) \cdot \nabla \varphi + f \cdot \varphi) \, dx.$$
Remark. The assumptions for the exponent $p$ in the above theorem are not optimal. However, they are in accord with the classical results, cf. [14] and [13], but the combination of the Lipschitz truncations methods [7] together with the ideas developed in the current paper, allow to extend the range of $p$ up to $p \geq \frac{2d}{d+2}$, see [9].

**Theorem 2.4.** Let $p \geq \frac{3d+2}{d+2}$. Assume $A(t,x)$ satisfies assumptions A1–A4. Given $f \in L^p(I; (W^{1,p}_0(\Omega, \mathbb{R}^d))^*)$ and $u_0 \in L^2_{div}(\Omega; \mathbb{R}^d)$ there exists a weak solution to (1.4), i.e. functions $u \in L^\infty(I; L^2(\Omega, \mathbb{R}^d)) \cap L^p(I; W^{1,p}_0(\Omega, \mathbb{R}^d))$ and $\sigma \in L^p(I; L^p(\Omega, \mathbb{R}^{d\times d}))$ such that

$$(\nabla u(t,x), \sigma(t,x)) \in A(t,x) \text{ a.e. in } (0,T) \times \Omega$$

and for all $\varphi \in D(-\infty,T; V)$

$$\int_{(0,T)\times\Omega} \sigma \cdot \nabla \varphi dx dt = \int_{(0,T)\times\Omega} (u \cdot \varphi_t + (u \otimes u) \cdot \nabla \varphi + f \cdot \varphi) dx dt - \int_\Omega u_0 \cdot \varphi(0) dx.$$  

**Remark 2.5.** Note that $p \geq \frac{3d+2}{d+2}$ provided by Theorem 2.4 corresponds to the classical results by Ladyzhenskaya [13] and Lions [14] for single valued stress tensor.

### 3 Preliminary results

Consider a graph $A(x) \subset \mathbb{R}^N \times \mathbb{R}^N$ satisfying A1–A4. According to [6], for each such graph satisfying additionally the condition $(0,0) \in A(x)$, there exists a function $\phi: \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ such that

$$A(x) = \{(e,d) \in \mathbb{R}^N \times \mathbb{R}^N: d - e = \phi(x,d+e)\}$$

and the following conditions hold

1. $\phi$ is a Carathéodory function;
2. $\phi(x,\cdot)$ is a contraction for a.e. $x \in \Omega$;
3. if we define functions $d,e: \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ as

$$\begin{cases}
    d(x,\lambda) := \frac{1}{2}(\lambda + \phi(x,\lambda)) \\
    e(x,\lambda) := \frac{1}{2}(\lambda - \phi(x,\lambda))
\end{cases}$$

then for a.e. $x \in \Omega$ and all $\lambda \in \mathbb{R}^N$

$$d(x,\lambda) \cdot e(x,\lambda) \geq -m(x) + \alpha(|e(x,\lambda)|^p + |d(x,\lambda)|^{p'})$$

4. $\phi(x,0) = 0$ for a.e. $x \in \Omega$.

Moreover, for each graph $A(x) \subset \mathbb{R}^N \times \mathbb{R}^N$ satisfying A1–A4 there exists a function $a: \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ such that for all $\xi \in \mathbb{R}^N$ we have $(\xi, a(x,\xi)) \in A(x)$ and $a$ satisfies the following conditions
a1. a is measurable with respect to the \(\sigma\)-field generated by \(\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^N)\) (see e.g. [6, Remark 2.2] and [2, Thm. 8.1.4]);

a2. \(\text{Dom } a = \mathbb{R}^N\);

a3. a is strictly monotone, i.e. for every \(\xi_1, \xi_2 \in \mathbb{R}^N, \xi_1 \neq \xi_2\)
\[
(a(x,\xi_1) - a(x,\xi_2)) \cdot (\xi_1 - \xi_2) > 0;
\]

\[ (3.9) \]

a4. for all \(\xi \in \mathbb{R}^N\) the function a satisfies the following growth and coercivity conditions
\[
|a(x,\xi)| \leq c_1|\xi|^{p-1} + k(x)
\]
\[
a(x,\xi) \cdot \xi \geq c_2|\xi|^p - m(x),
\]

for a.e. \(x \in \Omega\), where \(c_1, c_2 > 0, k \in L^{p'}(\Omega)\) and \(m \in L^1(\Omega)\) are non-negative functions. This condition follows from (3.8) and the fact that \((\xi, a(x,\xi)) \in A(x)\).

For a given graph \(A(x)\) the relation between \(a\) and \(\phi\) is established by the following lemma:

**Lemma 3.1.** Let functions \(d\) and \(e\) be defined by \(\phi\) according to the formula (3.7). If there exists \(\xi\) such that \(\lambda = a(x,\xi) + \xi\), then
\[
d(x,\lambda) = a(x,\xi)
\]

and
\[
e(x,\lambda) = \xi
\]

for a.e. \(x \in \Omega\).

**Proof.** By (3.6) we have
\[
a(x,\xi) - \xi = \phi(x,\xi + a(x,\xi)) \text{ for a.e. } x \in \Omega \text{ and } \xi \in \mathbb{R}^N.
\]

Therefore
\[
d(x,\lambda) + e(x,\lambda) = a(x,\xi) + \xi,
\]
\[
d(x,\lambda) - e(x,\lambda) = a(x,\xi) - \xi.
\]

The above system of equations implies that
\[
\xi = e(x,\lambda),
\]
\[
a(x,\xi) = d(x,\lambda).
\]

By \(B(x_0,r)\) we mean the ball in \(\mathbb{R}^N\) (with Euclidean metric) of radius \(r\), centered in \(x_0\). Let then \(\varepsilon > 0\) and \(\varphi^\varepsilon(x) := \varepsilon^{-N}\varphi(x/\varepsilon)\) for a fixed nonnegative radially symmetric function \(\varphi\) of class \(C^\infty_0(B(0,1))\) with \(\int \varphi(\xi) d\xi = 1\). Define
\[
a^\varepsilon(x,\xi) = (a(x,\cdot) * \varphi^\varepsilon)(\xi),
\]

where \(a\) is such that: \((\xi, a(x,\xi)) \in A(x)\) for all \(\xi \in \mathbb{R}^N\) and it fulfils a1.– a4.
Remark 3.2. With regularization given by formula (3.13) we obtain a Carathéodory function. Measurability in \( x \) follows from interpolation by continuous functions \( a^\varepsilon \delta \) (the upper index \( \delta \) corresponds to the regularization in the \( x \) variable) and Fubini Theorem (note that \( a \) is a locally integrable function on \( \Omega \times \mathbb{R}^N \)). Note also that the function \( a^\varepsilon \) is independent of the choice of \( a \) and depends only on \( A(x) \).

Lemma 3.3. If \( a \) satisfies conditions (3.9) and (3.10), then \( a^\varepsilon \) satisfies the strict monotonicity condition

\[
(a^\varepsilon(x, \xi_1) - a^\varepsilon(x, \xi_2)) \cdot (\xi_1 - \xi_2) > 0 \quad \text{for all} \quad \xi_1 \neq \xi_2. \tag{3.14}
\]

Moreover, for \( \varepsilon < \varepsilon_0 \leq 1 \), \( \varepsilon_0 = \varepsilon_0(c_1, c_2, p) \), there exist positive constants \( \tilde{c}_1 \), \( \tilde{c}_2 \) and nonnegative functions \( \tilde{m} \in L^1(\Omega) \) and \( k \in L^p(\Omega) \) such that

\[
|a^\varepsilon(x, \xi)| \leq \tilde{c}_1|\xi|^{p-1} + \tilde{k}(x),
\]

\[
a^\varepsilon(x, \xi) \cdot \xi \geq \tilde{c}_2|\xi|^p - \tilde{m}(x), \tag{3.15}
\]

for a.e. \( x \in \Omega \) and for all \( \xi \in \mathbb{R}^N \).

Proof. We start with the monotonicity condition. We have

\[
(a^\varepsilon(x, \xi_1) - a^\varepsilon(x, \xi_2)) \cdot (\xi_1 - \xi_2)
\]

\[
= \left( \int_{\mathbb{R}^N} [a(x, \xi_1 - \zeta) - a(x, \xi_2 - \zeta)]\varphi^\varepsilon(\zeta)d\zeta \right) \cdot (\xi_1 - \xi_2)
\]

\[
= \int_{\mathbb{R}^N} [a(x, \xi_1 - \zeta) - a(x, \xi_2 - \zeta)] \cdot [(\xi_1 - \zeta) - (\xi_2 - \zeta)]\varphi^\varepsilon(\zeta)d\zeta.
\]

Since \( a \) is strictly monotone, the integral with respect to a probability measure is positive.

For the growth condition we have

\[
|a^\varepsilon(x, \xi)| = |\int_{\mathbb{R}^N} a(x, \zeta)\varphi^\varepsilon(\zeta - \xi)d\zeta|
\]

\[
\leq \sup_{\zeta \in B(\xi, \varepsilon)} |a(x, \zeta)| \leq \sup_{\zeta \in B(\xi, \varepsilon)} (c_1|\zeta|^p + k(x))
\]

\[
\leq c_1(\varepsilon + |\xi|)^{p-1} + k(x)
\]

\[
\leq \tilde{c}_1|\xi|^{p-1} + \tilde{k}(x).
\]

For the coercivity condition first estimate

\[
a^\varepsilon(x, \xi) \cdot \xi = \int_{\mathbb{R}^N} a(x, \xi - \zeta)\varphi^\varepsilon(\zeta) \cdot (\xi - \zeta)d\zeta + \int_{\mathbb{R}^N} a(x, \xi - \zeta)\varphi^\varepsilon(\zeta) \cdot \zeta d\zeta
\]

\[
\geq \int_{\mathbb{R}^N} \left( c_2|\zeta| - m(x) \right)\varphi^\varepsilon(\zeta)d\zeta + I^\varepsilon
\]

\[
\geq c_2|\xi| - \varepsilon|\xi|^p - m(x) - |I^\varepsilon|.
\]

We have

\[
|I^\varepsilon| \leq \int_{\mathbb{R}^N} (c_1|\xi - \zeta|^{p-1} + k(x))\varphi^\varepsilon(\zeta) \cdot \zeta d\zeta \leq \varepsilon(c''_1|\xi|^{p-1} + k''(x)). \tag{3.17}
\]
If $|\xi| \geq 1$, then
\[ |I^c| \leq \varepsilon \left( c'' |\xi|^p + k''(x) \right), \]
on otherwise
\[ |I^c| \leq k''(x). \]
We may thus write
\[ |I^c| \leq \varepsilon \left( c'' |\xi|^p + \max \left( k''(x), k'''(x) \right) \right) \]
for all $\xi$. (3.18)

To continue the estimate (3.16) we consider two cases. First we assume that $|\xi| \geq 2$. Then, since $\varepsilon \leq 1$,
\[ (c_2(|\xi| - |\varepsilon|)^p - m(x)) \geq (c_2(|\xi| - 1)^p - m(x)) \geq \frac{c_2}{2^p} |\xi|^p - m(x). \] (3.19)
In the case $|\xi| < 2$ we have to notice that $(|\xi| - 1)^q$ is bounded from below, thus we can adjust the constants $c_2$ and $c_3$ such that
\[ (c_2(|\xi| - 1)^p - m(x)) \geq c'_2 \cdot 2^p - m'(x) \geq c'_2 |\xi|^p - m'(x). \] (3.20)
Combining (3.17)–(3.20) and taking $\varepsilon < \varepsilon_0 \leq 1$ with $\varepsilon_0$ such that
\[ \min \left( \frac{c_2}{2^p}, c'_2 \right) - \varepsilon_0 c''_1 > 0, \]
yields the assertion.

\[ \square \]

4 Weak-\(*\) limits of measure–valued functions

The Young measures, in a classical understanding, are the limits of the sequences of single distributed measures. Since for our considerations we will handle the sequences of general probability measures, then the classical results need the extensions. The forthcoming theorem is a modification of fundamental theorem on Young measures, cf. [3]. We will use the standard notation i.e.: $\mathcal{M}(\mathbb{R}^N)$ will denote the space of bounded Radon measures, $\mathcal{P}(\mathbb{R}^N)$ the subset of probability measures, $L^\infty_w(\Omega; \mathcal{M}(\mathbb{R}^N))$ the space of essentially bounded, measurable in the sense of Pettis functions with values in the space of Radon measures. We will follow Young measures theory notation, indicating the dependence on an argument in the lower index for the (Radon) measure–valued functions.

**Theorem 4.1.** Let $\Omega$ be an open and bounded subset of $\mathbb{R}^d$. Assume that for every $x \in \Omega$ there exists a sequence of probability measures $\nu^j_x$ on $\mathbb{R}^N$ such that for every $j$ the mapping $\nu^j : \Omega \to \mathcal{M}(\mathbb{R}^N)$ is weak-*$\ast$ measurable. Assume $\nu : \Omega \to \mathcal{M}(\mathbb{R}^N)$ to be such that $\nu^j \rightharpoonup \ast \nu$ in $L^\infty_w(\Omega, \mathcal{M}(\mathbb{R}^N))$.

If the sequence $\nu^j$ satisfies the tightness condition:
\[ \lim_{M \to \infty} \sup_j \left| \{ x \in \Omega : \text{supp}(\nu^j_x) \setminus B(0, M) \neq \emptyset \} \right| \to 0, \] (4.21)

then:
1. \( \|\nu_x\|_{\mathcal{M}(\mathbb{R}^N)} = 1 \ a.e \ in \ \Omega \). Moreover, for every \( f \in L^\infty(\Omega; C_b(\mathbb{R}^N)) \)

\[
\int_{\mathbb{R}^N} f(x,\lambda) d\nu_x^j(\lambda) \rightarrow \int_{\mathbb{R}^N} f(x,\lambda) d\nu_x(\lambda) \quad \text{in} \quad L^\infty(\Omega); \tag{4.22}
\]

2. for every measurable subset \( A \subset \Omega \) and for every Carathéodory function \( f \) (measurable in the first, and continuous in the second variable) such that

\[
\lim_{R \rightarrow 0} \sup_{j \in \mathbb{N}} \int_A \int_{\{\lambda \in \mathbb{R}^N : |f(x,\lambda)| > R\}} |f(x,\lambda)| d\nu_x^j(\lambda) dx = 0 \tag{4.23}
\]

we have

\[
\int_{\mathbb{R}^N} f(x,\lambda) d\nu_x^j(\lambda) \rightharpoonup \int_{\mathbb{R}^N} f(x,\lambda) d\nu_x(\lambda) \quad \text{in} \quad L^1(A). \]

Proof. The forthcoming proof comes from [8], nevertheless we add it for completeness of the paper.

**POINT 1.** For positive \( M \) we define a function \( g_M \) of the form

\[
g_M(\lambda) = \begin{cases} 
 1 & \text{for } |\lambda| \leq M, \\
 2 - \frac{|\lambda|}{M} & \text{for } M \leq |\lambda| \leq 2M, \\
 0 & \text{for } |\lambda| \geq 2M.
\end{cases}
\]

Notice that

\[
\int_{\Omega} \int_{\mathbb{R}^N} g_M(\lambda) d\nu_x^j(\lambda) dx \geq |\{x \in \Omega : \text{supp} \nu_x^j \subset B(0,M)\}| \tag{4.24}
\]

and obviously

\[
|\{x \in \Omega : \text{supp} \nu_x^j \subset B(0,M)\}| = |\Omega| - |\{x \in \Omega : \text{supp} \nu_x^j \setminus B(0,M) \neq \emptyset\}|.
\]

Let \( j \rightarrow \infty \) in (4.24)

\[
\lim_{j \rightarrow \infty} \int_{\Omega} \int_{\mathbb{R}^N} g_M(\lambda) d\nu_x^j(\lambda) dx \geq |\Omega| - \sup_{j \in \mathbb{N}} |\{x \in \Omega : \text{supp} \nu_x^j \setminus B(0,M) \neq \emptyset\}|.
\]

Using the weak-* convergence of \( \nu_x^j \) in \( L^\infty(\Omega, \mathcal{M}(\mathbb{R}^N)) \), letting \( M \rightarrow \infty \) in the above statement and using condition (4.21) we conclude that

\[
\lim_{M \rightarrow \infty} \int_{\Omega} \int_{\mathbb{R}^N} g_M(\lambda) d\nu_x(\lambda) dx \geq |\Omega|. \tag{4.25}
\]

The lower semicontinuity of the norm with respect to the weak-* convergence in the space \( L^\infty(\Omega, \mathcal{M}(\mathbb{R}^N)) \) provides that inequality (4.25) is indeed an equality. This ensures that \( \nu_x \) is a probability measure for a.a. \( x \in \Omega \).

To prove (4.22) take an arbitrary \( f \in L^\infty(\Omega; C_b(\mathbb{R}^N)) \). Then for all \( \varphi \in L^1(\Omega) \) it holds

\[
\left| \int_{\Omega} \varphi(x) \int_{\mathbb{R}^N} (f(x,\lambda) g_M(\lambda) - f(x,\lambda)) d\nu_x^j(\lambda) dx \right| \\
\leq \|f\|_{L^\infty(\Omega; C_b(\mathbb{R}^N))} \int_{\{x \in \Omega : \text{supp} (\nu_x^j) \setminus B(0,M) \neq \emptyset\}} |\varphi(x)| dx.
\]
Another usage of condition (4.21) yields
\[
\lim_{M \to \infty} \sup_{j \in \mathbb{N}} \left| \int_{\Omega} \varphi(x) \int_{\mathbb{R}^N} (g_M(\lambda)f(x, \lambda) - f(x, \lambda))d\nu_x^j(\lambda)dx \right| = 0 \quad (4.26)
\]
and obviously
\[
\lim_{M \to \infty} \left| \int_{\Omega} \varphi(x) \int_{\mathbb{R}^N} (g_M(\lambda)f(x, \lambda) - f(x, \lambda))d\nu_x(\lambda)dx \right| = 0. \quad (4.27)
\]

Then
\[
\lim_{j \to \infty} \left| \int_{\Omega} \varphi(x) \left( \int_{\mathbb{R}^N} f(x, \lambda)d\nu_x^j(\lambda) - \int_{\mathbb{R}^N} f(x, \lambda)d\nu_x(\lambda) \right) dx \right| \\
\leq \lim_{j \to \infty} \left| \int_{\Omega} \varphi(x) \int_{\mathbb{R}^N} (f(x, \lambda) - f(x, \lambda)g_M(\lambda))d\nu_x^j(\lambda)d\nu_x(\lambda) \right| \\
+ \lim_{j \to \infty} \left| \int_{\Omega} \varphi(x) \left( \int_{\mathbb{R}^N} f(x, \lambda)g_M(\lambda)d\nu_x^j(\lambda) - \int_{\mathbb{R}^N} f(x, \lambda)g_M(\lambda)d\nu_x(\lambda) \right) dx \right| \\
+ \lim_{j \to \infty} \left| \int_{\Omega} \varphi(x) \int_{\mathbb{R}^N} (f(x, \lambda)g_M(\lambda) - f(x, \lambda))d\nu_x(\lambda)dx \right| = I_1 + I_2 + I_3.
\]

Conditions (4.27) and (4.26) provide that $I_1$ and $I_3$ vanish as $M \to \infty$. Since $f g_M$ is in $L^1(\Omega; C_0(\mathbb{R}^N))$, which is a separable space and predual to $L^\infty(\Omega, \mathcal{M}(\mathbb{R}^N))$, the weak-$*$ convergence of the sequence $\{\nu_x^j\}$ allows us to conclude that $I_2 = 0$.

**Point 2.** Consider an arbitrary Carathéodory function $f(x, \lambda)$ such that condition (4.23) holds. Apply the truncation $T_n(f(x, \lambda))$ in the form
\[
T_n(f(x, \lambda)) = \begin{cases} 
  f(x, \lambda) & \text{for } |f(x, \lambda)| \leq n, \\
  \frac{f(x, \lambda)}{|f(x, \lambda)|}n & \text{for } |f(x, \lambda)| > n.
\end{cases}
\]

By $A$ we denote the measurable set described by the second part of Lemma 4.1. According to an obvious observation that $|T_n \circ f - f| \leq |f|$ and by (4.23) it follows that
\[
\lim_{n \to \infty} \sup_{j \in \mathbb{N}} \int_{A} \int_{\mathbb{R}^N} |(T_n \circ f)(x, \lambda) - f(x, \lambda)|d\nu_x(\lambda)dx = 0. \quad (4.28)
\]

Moreover, $|T_n \circ f - f| = |f| - |T_n \circ f|$. The monotone convergence theorem implies that for a.e. $x$ in $\Omega$
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |(T_n \circ f)(x, \lambda)|d\nu_x(\lambda) = \int_{\mathbb{R}^N} |f(x, \lambda)|d\nu_x(\lambda).
\]

Let $\varphi$ be an arbitrary test function from $L^\infty(A)$. We will use the standard notation $\varphi^+ = \max\{\varphi, 0\}$, $\varphi^- = \max\{-\varphi, 0\}$. Again, the monotone convergence theorem allows us to claim that
\[
\lim_{n \to \infty} \int_A \varphi^+(x) \int_{\mathbb{R}^N} |(T_n \circ f)(x, \lambda)|d\nu_x(\lambda)dx = \int_A \varphi^+(x) \int_{\mathbb{R}^N} |f(x, \lambda)|d\nu_x(\lambda)dx
\]
and
\[
\lim_{n \to \infty} \int_A \varphi^-(x) \int_{\mathbb{R}^N} |(T_n \circ f)(x, \lambda)|d\nu_x(\lambda)dx = \int_A \varphi^-(x) \int_{\mathbb{R}^N} |f(x, \lambda)|d\nu_x(\lambda)dx.
\]
Since 
\[
\sup_{n \in \mathbb{N}} \int_{A} \varphi(x) \int_{\mathbb{R}^N} |(T_n \circ f)(x, \lambda)| d\nu_x(\lambda) dx < \infty,
\]
it follows that
\[
\lim_{n \to \infty} \int_{A} \varphi(x) \int_{\mathbb{R}^N} |(T_n \circ f)(x, \lambda) - f(x, \lambda)| d\nu_x(\lambda) dx = 0. \tag{4.29}
\]

To finish the proof we estimate as follows
\[
\begin{align*}
&\lim_{j \to \infty} \left| \int_{A} \varphi(x) \left( \int_{\mathbb{R}^N} f(x, \lambda) d\nu_j^2(\lambda) - \int_{\mathbb{R}^N} f(x, \lambda) d\nu_x(\lambda) \right) dx \right| \\
&\leq \lim_{j \to \infty} \left| \int_{A} \varphi(x) \int_{\mathbb{R}^N} \left( f(x, \lambda) - (T_n \circ f)(x, \lambda) \right) d\nu_j^2(\lambda) dx \right| \\
&+ \lim_{j \to \infty} \left| \int_{A} \varphi(x) \left( \int_{\mathbb{R}^N} (T_n \circ f)(x, \lambda) d\nu_j^2(\lambda) - \int_{\mathbb{R}^N} (T_n \circ f)(x, \lambda) d\nu_x(\lambda) \right) dx \right| \\
&+ \lim_{j \to \infty} \left| \int_{A} \varphi(x) \int_{\mathbb{R}^N} \left( (T_n \circ f)(x, \lambda) - f(x, \lambda) \right) d\nu_x(\lambda) dx \right| = J_1 + J_2 + J_3.
\end{align*}
\]
The terms $J_1, J_3$ vanish as $n \to \infty$ by (4.28) and (4.29) respectively. Since $(T_n \circ f)(x, \lambda)$ is a Carathéodory function and $T_n \circ f \in L^\infty(\Omega \times \mathbb{R}^N)$, we have $T_n \circ f \in L^\infty(\Omega; C_b(\mathbb{R}^N))$. Hence condition (4.22) with the function $T_n \circ f$ instead of $f$ allows us to conclude that also $J_2$ vanishes as $n \to \infty$.

\[\square\]

Remark 4.2. Note that $\int_{A} \int_{\mathbb{R}^N} |f(x, \lambda)|^p d\nu_j^2(\lambda) dx \leq C$ (where $C$ does not depend on $j$ and $p > 1$) implies the integrability condition (4.23), indeed
\[
\int_{\{x \in \mathbb{R}^N : |f(x, \lambda)| > R\}} |f(x, \lambda)| d\nu_j^2(\lambda) \leq \frac{1}{R^{p-1}} \int_{\mathbb{R}^N} |f(x, \lambda)|^p d\nu_j^2(\lambda).
\]

For the proof of the following lemma see e.g. [17, Corollary 3.2].

Lemma 4.3. Suppose that a sequence $z^j$ of measurable functions generates the Young measure $\nu : \Omega \to \mathcal{M}(\mathbb{R}^N)$. Then
\[
z^j \to z \text{ in measure iff } \nu_x = \delta_{\{z(x)\}} \text{ a.e.}
\]

Theorem 4.4. Assume that $\nu$ is a weak-* limit of a sequence of weak-* measurable mappings $\nu^j : \Omega \to \mathcal{M}(\mathbb{R}^N)$, such that $\nu^j(x) = \nu^j_x$ is a probability measure on $\mathbb{R}^N$ for every $x \in \Omega$ and condition (4.21) holds. Let $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a nonnegative Carathéodory function. Then
\[
\liminf_{j \to \infty} \int_{\Omega} \int_{\mathbb{R}^N} f(x, \lambda) d\nu^j_x(\lambda) dx \geq \int_{\Omega} \int_{\mathbb{R}^N} f(x, \lambda) d\nu_x(\lambda) dx.
\]

Proof. Take the truncation $T_n(f(x, \lambda))$ in the form
\[
T_n(f(x, \lambda)) = \begin{cases} f(x, \lambda) & \text{for } f(x, \lambda) \leq n, \\
n & \text{for } f(x, \lambda) > n. \end{cases}
\]
Then
\[
\liminf_{j \to \infty} \int_{\Omega} \int_{\mathbb{R}^N} f(x, \lambda) d\nu_x^j(\lambda) dx \geq \liminf_{j \to \infty} \int_{\Omega} \int_{\mathbb{R}^N} (T_n \circ f)(x, \lambda) d\nu_x^j(\lambda) dx.
\]

By (4.22) we have
\[
\liminf_{j \to \infty} \int_{\Omega} \int_{\mathbb{R}^N} (T_n \circ f)(x, \lambda) d\nu_x^j(\lambda) dx = \lim_{j \to \infty} \int_{\Omega} \int_{\mathbb{R}^N} (T_n \circ f)(x, \lambda) d\nu_x^j(\lambda) dx = \int_{\Omega} \int_{\mathbb{R}^N} (T_n \circ f)(x, \lambda) d\nu_x(\lambda) dx.
\]

The sequence of functions \(x \mapsto \int_{\mathbb{R}^N} (T_n \circ f)(x, \lambda) d\nu_x\) is pointwise increasing and for a.e. \(x\) it converges to \(\int_{\mathbb{R}^N} f(x, \lambda) d\nu_x\). Thus by the monotone convergence theorem we have
\[
\liminf_{j \to \infty} \int_{\Omega} \int_{\mathbb{R}^N} f(x, \lambda) d\nu_x^j(\lambda) dx \geq \int_{\Omega} \int_{\mathbb{R}^N} f(x, \lambda) d\nu_x(\lambda) dx.
\]

We formulate now a theorem, which will play a crucial role in the proof of the main technical Theorem 5.1 in the forthcoming section. The results of weak$$^*$$ convergence of measure-valued functions are now applied to the very particular case, namely to the sequences given by the approximations of \(x\)-dependent maximal monotone graph.

**Theorem 4.5.** Let \(z^\varepsilon : \Omega \to \mathbb{R}^N\) be a bounded sequence in \(L^p(\Omega, \mathbb{R}^N)\). Assume \(a^\varepsilon\) to be the sequence of Carathéodory regularizations given by (3.13) of the maximal strictly monotone graph \(A(x)\), which fulfills conditions A1-A4. Then there exists a measure-valued function \(\nu \in L^\infty(\Omega; \mathcal{M}(\mathbb{R}^N))\), with \(\nu_x \in \mathcal{P}(\mathbb{R}^N)\) a.e. in \(\Omega\), such that
\[
a^\varepsilon(\cdot, z^\varepsilon) \rightharpoonup \sigma \quad \text{in} \quad L^p(\Omega, \mathbb{R}^d) \quad \text{with} \quad \sigma(x) = \int_{\mathbb{R}^N} d(x, \lambda) d\nu_x(\lambda), \quad (4.30)
\]
\[
z^\varepsilon \rightharpoonup z \quad \text{in} \quad L^p(\Omega, \mathbb{R}^N) \quad \text{with} \quad z(x) = \int_{\mathbb{R}^N} e(x, \lambda) d\nu_x(\lambda), \quad (4.31)
\]
and
\[
\liminf_{\varepsilon \to 0} \int_{\Omega} a^\varepsilon(x, z^\varepsilon(x)) \cdot z^\varepsilon(x) \psi(x) dx \geq \int_{\Omega} \int_{\mathbb{R}^N} d(x, \lambda) \cdot e(x, \lambda) d\nu_x(\lambda) \psi(x) dx, \quad (4.32)
\]
for every nonnegative \(\psi \in L^\infty(\Omega)\) and the functions \(d(x, \cdot), e(x, \cdot)\) related to \(A(x)\) by (3.7).

**Proof of Theorem 4.5.** Observe that
\[
a^\varepsilon(x, z^\varepsilon(x)) = \int_{\mathbb{R}^N} a(x, \xi) \varphi^\varepsilon(z^\varepsilon(x) - \xi) d\xi = \int_{\mathbb{R}^N} a(x, \xi) d\mu_x^\varepsilon(\xi). \quad (4.33)
\]

By \(\mu_x^\varepsilon\) we denote a probability measure that is absolutely continuous with respect to the Lebesgue measure with density \(\varphi^\varepsilon(z^\varepsilon(x) - (\cdot))\). We also have
\[
\int_{\mathbb{R}^N} \xi d\mu_x^\varepsilon(\xi) = \int_{\mathbb{R}^N} \xi \varphi^\varepsilon(z^\varepsilon(x) - \xi) d\xi = \int_{\mathbb{R}^N} (z^\varepsilon(x) - \xi) \cdot \varphi^\varepsilon(\xi) d\xi.
\]
By radial symmetry of $\varphi$ 
\[ \int_{\mathbb{R}^N} \xi \cdot \varphi^{\varepsilon}(\xi) \, d\xi = 0 \]
and thus 
\[ z^{\varepsilon}(x) = \int_{\mathbb{R}^N} \xi \, d\mu_x^{\varepsilon}(\xi). \quad (4.34) \]
To obtain a characterization of $a^{\varepsilon}(x, z^{\varepsilon}(x)) \cdot z^{\varepsilon}(x)$ we start with 
\[ a^{\varepsilon}(x, z^{\varepsilon}(x)) \cdot z^{\varepsilon}(x) \rightleftharpoons \left( \int_{\mathbb{R}^N} a(x, \xi) \varphi^{\varepsilon}(z^{\varepsilon}(x) - \xi) \, d\xi \right) \cdot z^{\varepsilon}(x) \]
\[ = \int_{\mathbb{R}^N} a(x, \xi) \cdot z^{\varepsilon}(x) \varphi^{\varepsilon}(z^{\varepsilon}(x) - \xi) \, d\xi. \]
On the other hand 
\[ \int_{\mathbb{R}^N} a(x, \xi) \cdot \xi \, d\mu_x^{\varepsilon}(\xi) = \int_{\mathbb{R}^N} a(x, \xi) \cdot \xi \varphi^{\varepsilon}(z^{\varepsilon}(x) - \xi) \, d\xi. \]
Hence 
\[ \left| a^{\varepsilon}(x, z^{\varepsilon}(x)) \cdot z^{\varepsilon}(x) - \int_{\mathbb{R}^N} a(x, \xi) \cdot \xi \, d\mu_x^{\varepsilon}(\xi) \right| \]
\[ \leq \int_{\mathbb{R}^N} |a(x, \xi) \cdot (z^{\varepsilon}(x) - \xi) \varphi^{\varepsilon}(z^{\varepsilon}(x) - \xi)| \, d\xi \]
\[ \leq C\varepsilon \sup_{\xi \in B(z^{\varepsilon}(x), r)} |a(x, \xi)| \leq C\varepsilon (|z^{\varepsilon}(x)| + \varepsilon)^{p-1} + k(x), \]
where the inequality follows from (3.10). Thus we obtain 
\[ a^{\varepsilon}(x, z^{\varepsilon}(x)) \cdot z^{\varepsilon}(x) = \int_{\mathbb{R}^N} a(x, \xi) \cdot \xi \, d\mu_x^{\varepsilon}(\xi) + c_\varepsilon(x), \quad (4.35) \]
with $\|c_\varepsilon\|_{L^p} \leq C\varepsilon$.

Denote $g_x(\xi) = a(x, \xi) + \xi$. We define a measure $\nu_x^{\varepsilon} \in \mathcal{M}(\text{im } g_x)$ by the formula 
\[ \nu_x^{\varepsilon}(S) = \mu_x^{\varepsilon}(g_x^{-1}(S)) \quad \text{for } S \subset \text{im } g_x. \quad (4.36) \]
For every $\varepsilon$ the measure $\nu_x^{\varepsilon}$ is a probability measure on $\text{im } g_x$. We will consider the measures $\nu_x^{\varepsilon}$ as the measures defined on the whole $\mathbb{R}^N$. By Lemma 3.1, using (4.33) – (4.35) we obtain 
\[ a^{\varepsilon}(x, z^{\varepsilon}(x)) = \int_{\text{im } g_x} d(x, \lambda) \, d\nu_x^{\varepsilon}(\lambda) = \int_{\mathbb{R}^N} d(x, \lambda) \, d\nu_x^{\varepsilon}(\lambda), \]
\[ z^{\varepsilon}(x) = \int_{\text{im } g_x} e(x, \lambda) \, d\nu_x^{\varepsilon}(\lambda) = \int_{\mathbb{R}^N} e(x, \lambda) \, d\nu_x^{\varepsilon}(\lambda) \quad (4.37) \]
and 
\[ a^{\varepsilon}(x, z^{\varepsilon}(x)) \cdot z^{\varepsilon}(x) = \int_{\mathbb{R}^N} d(x, \lambda) \cdot e(x, \lambda) \, d\nu_x^{\varepsilon}(\lambda) + c_\varepsilon(x). \quad (4.38) \]
To complete the proof we will use Theorem 4.1. First, however, we assemble the properties of $\nu^{\varepsilon}$ introduced by (4.36) to ensure that such defined family of measures satisfies the assumptions of Theorem 4.1. These properties are collected in the following three claims.
Claim 1. For every \( \varepsilon > 0 \) the mapping \( x \mapsto \nu_x^\varepsilon \) belongs to \( L_\infty^w(\Omega; \mathcal{M}(\mathbb{R}^N)) \).

Proof. We have to show that for every \( f \in L^1(\Omega; C_0(\mathbb{R}^N)) \) the mapping

\[
x \mapsto \int_{\mathbb{R}^N} f(x, \lambda) d\nu_x^\varepsilon(\lambda)
\]

is measurable. Define \( F(x, \xi) = \varphi^\varepsilon(z^\varepsilon(x) - \xi) \cdot f(x, g_x(\xi)) \). The function \( F \) is measurable with respect to \( L(\Omega) \otimes \mathcal{B}(\mathbb{R}^N) \) and integrable. It follows from the Fubini Theorem that the mapping

\[
x \mapsto \int_{\mathbb{R}^N} F(x, \xi) d\xi
\]

is measurable. Note that

\[
\int_{\mathbb{R}^N} F(x, \xi) d\xi = \int_{\mathbb{R}^N} f(x, \lambda) d\nu_x^\varepsilon(\lambda),
\]

hence the claim follows. \( \square \)

Claim 2. The sequence of mappings \( \nu^\varepsilon \) satisfies the tightness condition (4.21).

Proof. Define a function

\[
\gamma^\varepsilon(x) = \max_{\lambda \in \text{supp } \nu_x^\varepsilon} |\lambda|.
\]

From the definition of \( \nu_x^\varepsilon \)

\[
\max_{\lambda \in \text{supp } \nu_x^\varepsilon} |\lambda| = \max_{\xi \in \text{supp } \mu_x^\varepsilon} |g_x(\xi)|.
\]

Since \( \text{supp } \mu_x^\varepsilon \subset B(z^\varepsilon(x), \varepsilon) \), we have \( \|\gamma^\varepsilon\|_{L^p} \leq C \) and hence the claim follows. \( \square \)

Claim 3. The family of measures \( \nu^\varepsilon \) with the functions \( d(x, \lambda) \) and \( e(x, \lambda) \) defined by (3.7) satisfy condition (4.23) on the set \( A = \Omega \).

Proof. By Remark 4.2 it suffices to show that

\[
\int_A \int_{\mathbb{R}^N} |d(x, \lambda)|^{p'} d\nu_x^\varepsilon(\lambda) dx \leq C \quad (4.39)
\]

and

\[
\int_A \int_{\mathbb{R}^N} |e(x, \lambda)|^{p'} d\nu_x^\varepsilon(\lambda) dx \leq C. \quad (4.40)
\]

The monotonicity condition (3.9), together with Lemma 3.1 and the fact that \((0, 0) \in A(x)\), implies that \( d(x, \lambda) \cdot e(x, \lambda) \geq 0 \). Moreover, by definitions of measures \( \mu_x^\varepsilon, \nu_x^\varepsilon \) and the uniform boundedness of \( z^\varepsilon \) in \( L^p(\Omega, \mathbb{R}^N) \) we have

\[
\int_\Omega \int_{\mathbb{R}^N} d(x, \lambda) \cdot e(x, \lambda) d\nu_x^\varepsilon(\lambda) dx \leq C
\]

for some constant \( C \) independent of \( \varepsilon \). Therefore inequalities (4.39) and (4.40) follow from condition (3.8). \( \square \)
The Banach–Alaoglu Theorem yields the existence of a weak-$^*$ measurable mapping $\nu \in L^\infty_\text{w}(\Omega, \mathcal{M}(\mathbb{R}^N))$ such that $\nu^\varepsilon \rightharpoonup \nu$ and $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^N)} \leq 1$. Theorem 4.1 together with Claim 1 and Claim 2 imply that $\nu_x$ is a probability measure on $\mathbb{R}^N$ for a.e. $x \in \Omega$. Applying Claim 3, the second part of Theorem 4.1 and using (4.37) we obtain

$$\sigma(x) = \int_{\mathbb{R}^N} d(x, \lambda) \, d\nu_x(\lambda),$$
$$z(x) = \int_{\mathbb{R}^N} e(x, \lambda) \, d\nu_x(\lambda).$$

By (4.38) and Theorem 4.4 we obtain

$$\liminf_{\varepsilon \to 0} \int_{\Omega} a^\varepsilon(x, z^\varepsilon(x)) \cdot z^\varepsilon(x) \psi(x) \, dx \geq \int_{\Omega} \int_{\mathbb{R}^N} d(x, \lambda) \cdot e(x, \lambda) \, d\nu_x(\lambda) \psi(x) \, dx$$
for every nonnegative $\psi \in L^\infty(\Omega)$. That completes the proof of Theorem 4.5.

5 The compactness criteria

The present section is delivered to established an abstract theorem which will be the main technical block in the proof of Theorems 2.1-2.4, given in the next section. The forthcoming theorem presents the common structure of all the considered problems.

**Theorem 5.1.** Assume $z^\varepsilon : \Omega \to \mathbb{R}^N$ to be a sequence of measurable functions. Let $a^\varepsilon(\cdot, \cdot)$ be a sequence of Carathéodory regularizations given by formula (3.13) of the maximal strictly monotone graph $A(x)$, which fulfills conditions A1-A4. Assume moreover that the following conditions are satisfied:

$$a^\varepsilon(\cdot, z^\varepsilon) \rightharpoonup \sigma \quad \text{in} \quad L^{p'}(\Omega, \mathbb{R}^N), \quad z^\varepsilon \rightharpoonup z \quad \text{in} \quad L^p(\Omega, \mathbb{R}^N),$$

and one of these two conditions is satisfied:

$$\int_{\Omega} \sigma(x) \cdot z(x) \psi(x) \, dx = \lim_{\varepsilon \to 0} \int_{\Omega} a^\varepsilon(x, z^\varepsilon(x)) \cdot z^\varepsilon(x) \psi(x) \, dx$$

(5.41)

for every $\psi \in C^\infty_0(\Omega)$ or

$$\int_{\Omega} \sigma(x) \cdot z(x) \, dx \geq \liminf_{\varepsilon \to 0} \int_{\Omega} a^\varepsilon(x, z^\varepsilon(x)) \cdot z^\varepsilon(x) \, dx.$$  

(5.42)

Then, at least for a subsequence, we have

$$z^\varepsilon \to z \quad \text{a.e. in} \quad \Omega$$

(5.43)

and

$$(z(x), \sigma(x)) \in A(x) \quad \text{for a.e.} \quad x \in \Omega.$$  

(5.44)
Proof. Fix $x \in \Omega$ and define a function

$$h(x, \lambda) := [d(x, \lambda) - a(x, z(x))] \cdot [e(x, \lambda) - z(x)].$$

Since $A(x)$ is strictly monotone, $h(x, \cdot) \geq 0$, moreover

$$\text{supp } h(x, \cdot) = \{ \lambda \in \mathbb{R}^N : e(x, \lambda) \neq z(x) \}.$$

The properties of $d$ and $e$ provide that $h$ is a Carathéodory function and satisfies the proper growth conditions. Note that by (4.31)

$$a(x, z(x)) \int_{\mathbb{R}^N} e(x, \lambda) - z(x) \, d\nu_x(\lambda) = 0$$

and thus (4.30) implies that

$$\int_{\mathbb{R}^N} h(x, \lambda) d\nu_x(\lambda) = \int_{\mathbb{R}^N} d(x, \lambda) \cdot e(x, \lambda) \, d\nu_x(\lambda) - \sigma(x) \cdot z(x).$$

Each of the conditions (5.41) and (5.42) implies that

$$\int_{\Omega} \int_{\mathbb{R}^N} h(x, \lambda) d\nu_x(\lambda) \, dx \leq 0,$$

thus, since $h(x, \cdot)$ is nonnegative,

$$\int_{\mathbb{R}^N} h(x, \lambda) d\nu_x(\lambda) = 0 \text{ for a.e. } x \in \Omega.$$

Therefore $\text{supp } h_x \cap \text{supp } \nu_x = \emptyset$ for a.e. $x \in \Omega$. This implies that

$$\text{supp } \nu_x \subset \{ \lambda \in \mathbb{R}^N : e(x, \lambda) = z(x) \} \quad (5.45)$$

and

$$\langle z(x), e(x, \lambda) \rangle \in A(x) \quad \nu_x \text{-a.e. } \lambda. \quad (5.46)$$

Now we define a measure $\mu_x \in \mathcal{M}(\mathbb{R}^N)$ as follows:

$$\mu_x(A) = \nu_x(B) \quad \text{where } B \text{ is such that } A = e(x, B).$$

By (4.31) and Lemma 3.1 we get

$$z(x) = \int_{\mathbb{R}^N} \xi \, d\mu_x(\xi).$$

On the other hand

$$\text{supp } \mu_x \subset \{ \xi \in \mathbb{R}^N : \xi = z(x) \} = \{ z(x) \},$$

thus $\mu_x = \delta_{\{z(x)\}}$. It follows from Lemma 4.3 that $z^\varepsilon \to z$ in measure on $\Omega$.

Since maximal monotone operators are convex-valued, hence, from (5.46),

$$\langle z(x), \sigma(x) \rangle \in A(x) \text{ for a.e. } x \in \Omega.$$

$$\square$$
6 Proofs of main theorems

Proof of Theorem 2.1: There exists a weak solution \( u^\varepsilon \) to the problem

\[
-\text{div} \ a^\varepsilon (x, \nabla u^\varepsilon) = f, \\
\left. u^\varepsilon \right|_{\partial \Omega} = 0.
\]

The energy estimates and conditions on \( a \) yield also a uniform bound on \( W_0^{1,p} \) norm of solutions

\[
\|u^\varepsilon\|_{W_0^{1,p}} \leq C.
\]

Moreover,

\[
\int_{\Omega} |a^\varepsilon(x, \nabla u^\varepsilon)|^{p'} \, dx \leq c_1 \int_{\Omega} (k(x) + c_1 |\nabla u^\varepsilon|^{p-1})^{p'} \, dx \\
\leq c \int_{\Omega} (k^{p'}(x) + |\nabla u^\varepsilon|^p) \, dx = c' + \int_{\Omega} |\nabla u^\varepsilon|^p \, dx. \tag{6.47}
\]

Therefore \( \nabla u^\varepsilon(\cdot) \) and \( a^\varepsilon(\cdot, \nabla u^\varepsilon(\cdot)) \) are bounded sequences in \( L^p(\Omega, \mathbb{R}^{d \times m}) \) and \( L^{p'}(\Omega, \mathbb{R}^{d \times m}) \) respectively, and up to subsequences we have

\[
\nabla u^\varepsilon \to \nabla u \quad \text{in} \quad L^p(\Omega, \mathbb{R}^{d \times m}),
\]

\[
a^\varepsilon(\cdot, \nabla u^\varepsilon) \to \sigma \quad \text{in} \quad L^{p'}(\Omega, \mathbb{R}^{d \times m}),
\]

where \( \sigma \) is a measurable function in \( L^{p'}(\Omega, \mathbb{R}^{d \times m}) \). Div–curl lemma of the theory of compensated compactness yields

\[
\lim_{\varepsilon \to 0} \int_{\Omega} a^\varepsilon(x, \nabla u^\varepsilon(x)) \cdot \nabla u^\varepsilon(x) \psi(x) \, dx = \int_{\Omega} \sigma(x) \cdot \nabla u(x) \psi(x) \, dx
\]

for every \( \psi \in C_0^\infty(\Omega) \). Hence, the application of Theorem 5.1 for \( z^\varepsilon = \nabla u^\varepsilon \) completes the proof.

Another interesting question is whether it is possible to prove the existence of weak solutions directly from Galerkin approximation. Below we positively answer that question.

Alternative proof of Theorem 2.1. Set \( \varepsilon = 1/n \) for \( n \in \mathbb{N} \). As previously, we regularize the function \( a \) by convolution with \( \varphi^{1/n} \). By the Galerkin approximation we obtain a sequence of functions satisfying

\[
\int_{\Omega} a^{1/n}(x, \nabla u^n(x)) \nabla \omega_k(x) \, dx = \int_{\Omega} f(x) \omega_k(x) \, dx \quad \text{for} \quad k = 1, \ldots, n.
\]

Hence it holds

\[
\int_{\Omega} a^{1/n}(x, \nabla u^n(x)) \nabla u^n(x) \, dx = \int_{\Omega} f(x) u^n(x) \, dx,
\]

which allows to conclude the uniform bound on \( W_0^{1,p} \) norm. As previously, up to subsequences we have

\[
\nabla u^n \to \nabla u \quad \text{in} \quad L^p(\Omega, \mathbb{R}^{d \times m}),
\]

\[
a^{1/n}(\cdot, \nabla u^n) \to \sigma \quad \text{in} \quad L^{p'}(\Omega, \mathbb{R}^{d \times m}),
\]
hence
\[ \lim_{n \to \infty} \int_{\Omega} a_1^{1/n}(x, \nabla u^n) \cdot \nabla u^n(x) \, dx = \lim_{n \to \infty} \int_{\Omega} f(x) u^n(x) \, dx = \int_{\Omega} f(x) u(x) \, dx. \]

Fix \( n \). For \( k \geq n \) the function \( u^n \) belongs to the space \( V^k = \text{span} \{\omega_1, \ldots, \omega_k\} \), so
\[ \int_{\Omega} a_k(x, \nabla u^k(x)) \cdot \nabla u^n(x) \, dx = \int_{\Omega} f(x) u^n(x) \, dx. \]

Passing to the limit with \( k \) we obtain
\[ \int_{\Omega} \sigma(x) \cdot \nabla u^n(x) \, dx = \int_{\Omega} f(x) u^n(x) \, dx. \]

Then passing to the limit with \( n \) we get
\[ \int_{\Omega} \sigma(x) \cdot \nabla u(x) \, dx = \int_{\Omega} f(x) u(x) \, dx = \lim_{n \to \infty} \int_{\Omega} a_1^{1/n}(x, \nabla u^n(x)) \cdot \nabla u^n(x) \, dx. \]

As previously we take \( z^n = \nabla u^n \). Again, by Theorem 5.1 we finish the proof. \( \Box \)

**Proof of Theorem 2.3:** To prove the existence of weak solutions we may start from Galerkin approximation as in the alternative proof of Theorem 2.1. Nevertheless, for the simplicity of notation we will start with weak solutions to the \( \alpha^{-}\)regularized problem, i.e. a function \( u^\varepsilon \in W^{1,p}_0(\Omega, \mathbb{R}^d) \) such that
\[
\int_{\Omega} a^\varepsilon(x, \nabla^s u^\varepsilon) \cdot \nabla^s \psi \, dx = \int_{\Omega} (u^\varepsilon \otimes u^\varepsilon) \cdot \nabla \psi \, dx + \int_{\Omega} f \cdot \psi \, dx \quad (6.48)
\]
for all \( \psi \in \mathcal{V} \). For an existence result in such a case we refer to [7, Thm. 1.1].

The growth conditions and Korn’s inequality (cf. [15, p. 196]) imply that
\[
\|u^\varepsilon\|_{W^{1,p}_0(\Omega, \mathbb{R}^d)}^p \leq c(\|f\|_{W^{-1,p'}(\Omega)}^p + |\Omega|). \quad (6.49)
\]

If \( p \geq \frac{3d}{d+2} \), we may use \( u^\varepsilon \) as a test function in (6.48). Since \( u^\varepsilon \in W^{1,p}_0(\Omega, \mathbb{R}^d) \),
\[
\int_{\Omega} (u^\varepsilon \otimes u^\varepsilon) \nabla u^\varepsilon \, dx = 0. \quad (6.50)
\]

Hence
\[
\int_{\Omega} a^\varepsilon(x, \nabla^s u^\varepsilon) \cdot \nabla^s u^\varepsilon \, dx = \int_{\Omega} f \cdot u^\varepsilon \, dx. \quad (6.51)
\]

Letting \( \varepsilon \to 0 \), from (6.50) we have (at least for a subsequence)
\[
u^\varepsilon \rightharpoonup u \quad \text{in} \quad W^{1,p}_0(\Omega, \mathbb{R}^d). \quad (6.52)
\]

Moreover, notice that \( W^{1,p}(\Omega, \mathbb{R}^d) \hookrightarrow L^2(\Omega, \mathbb{R}^d) \) if \( p > \frac{2d}{d+2} \). This provides that
\[
u^\varepsilon \rightharpoonup u \quad \text{in} \quad L^2(\Omega, \mathbb{R}^d). \quad (6.53)
\]

The strong convergence yields
\[
\int_{\Omega} \nabla u^\varepsilon \cdot \nabla \psi \to \int_{\Omega} \nabla u \cdot \nabla \psi \quad \text{for all} \quad \psi \in \mathcal{D}(\Omega). \quad (6.54)
\]
We conclude existence of a subsequence and some \( \sigma \in L^p(\Omega, \mathbb{R}^{d \times d}) \) such that
\[
a^\varepsilon(x, \nabla^s u^\varepsilon) \rightharpoonup \sigma \quad \text{in} \quad L^p(\Omega, \mathbb{R}^{d \times d}). \tag{6.55}
\]
So we can state the limit identity for all \( \psi \in \mathcal{V} \)
\[
\int_\Omega \sigma \cdot \nabla^s \psi \, dx = \int_\Omega (u \otimes u \cdot \nabla \psi - f \cdot \psi) \, dx. \tag{6.56}
\]
Since \( \mathcal{V} \) is dense in \( W^{1,p}_{0, \text{div}}(\Omega, \mathbb{R}^d) \) and for \( p \geq \frac{3d}{d+2} \) all the integrals are well defined with \( \psi \in W^{1,p}_{0, \text{div}}(\Omega, \mathbb{R}^d) \), notice that for these \( p \) the limit identity (6.56) also holds for all \( \psi \in W^{1,p}_{0, \text{div}}(\Omega, \mathbb{R}^d) \).

Let \( \varepsilon \to 0 \) in (6.51). Then
\[
\lim_{\varepsilon \to 0} \int_\Omega a^\varepsilon(x, \nabla^s u^\varepsilon) \cdot \nabla^s u^\varepsilon \, dx = \int_\Omega f \cdot u \, dx.
\]
Now, using (6.56) tested with \( \psi = u \), we claim that
\[
\lim_{\varepsilon \to 0} \int_\Omega a^\varepsilon(x, \nabla^s u^\varepsilon) \cdot \nabla^s u^\varepsilon \, dx = \int_\Omega \sigma \cdot \nabla^s u \, dx. \tag{6.57}
\]
Applying Theorem 5.1 with \( z^\varepsilon = \nabla^s u^\varepsilon \), \( N = \frac{d(d+1)}{2} \) we complete the proof. \(\square\)

**Proof of Theorem 2.4:** To prove the existence of weak solutions we might again start with Galerkin approximation (as in the alternative proof of Theorem 2.1). For the simplicity of notation we will however start with a weak solution to \( a^\varepsilon \)–regularized problem, i.e. a function \( u^\varepsilon \in L^p(I; W^{1,p}_{0, \text{div}}(\Omega, \mathbb{R}^d)) \cap L^\infty(I; L^2(\Omega, \mathbb{R}^d)) \) such that

\[
\int_{I \times \Omega} u^\varepsilon_t \cdot \psi \, dx \, dt = \int_{I \times \Omega} (u^\varepsilon \otimes u^\varepsilon \cdot \nabla \psi - a^\varepsilon(\tau, x, \nabla^s u^\varepsilon) \cdot \nabla^s \psi - f \cdot \psi) \, dx \, dt. \tag{6.58}
\]
for all \( \psi \in \mathcal{D}(-\infty, T; \mathcal{D}(\Omega)) \). Note that the existence of solution to the regularized problem in the special case for a \( p \)–laplacian operator (thus without dependence on \( t \) and \( x \)) was proved by Lions [14, Thm. 5.1], with the help of Minty-Browder method. However, the generalization given by expression (6.58) does not influence the proof, since \( a^\varepsilon \) is a pseudomonotone operator.

The coercivity conditions, Korn’s and Young’s inequality imply that
\[
\frac{1}{2} ||u^\varepsilon(t)||_{L^2}^2 + \int_{(0,t) \times \Omega} a^\varepsilon(\tau, x, \nabla^s u^\varepsilon) \cdot \nabla^s u^\varepsilon \, dx \, d\tau \leq \frac{1}{2} ||u^\varepsilon_0||_{L^2}^2 + \int_{(0,t) \times \Omega} f \cdot u^\varepsilon \, dx \, d\tau, \tag{6.59}
\]
which yields the uniform estimate
\[
||u^\varepsilon||_{L^\infty(I; L^2(\Omega, \mathbb{R}^d))}^2 + ||u^\varepsilon||_{L^p(I; W^{1,p}_{0, \text{div}}(\Omega, \mathbb{R}^d))}^p \leq c. \tag{6.60}
\]
Notice that for \( p \geq \frac{3d+2}{d} \) the right hand side of (6.58) is a linear bounded functional on \( L^p(I; W^{1,p}_{0, \text{div}}(\Omega, \mathbb{R}^d)) \), see [15, Lemma 2.44] for detailed estimates. Thus \( u^\varepsilon_1 \) is an element of \( L^p(I; (W^{1,p}_{0, \text{div}}(\Omega, \mathbb{R}^d))^*) \), which provides that (6.58)
holds for all $\psi \in L^p(I; W^{1,p}_{0,\text{div}}(\Omega, \mathbb{R}^d))$. Hence it is allowed to test (6.58) with a $u^\varepsilon$. Since $u^\varepsilon \in L^p(I; W^{1,p}_{0,\text{div}}(\Omega, \mathbb{R}^d))$ and $u^\varepsilon_t \in L^p(I; (W^{1,p}_{0,\text{div}}(\Omega, \mathbb{R}^d))^*)$ (and are even uniformly bounded), for all $0 \leq s \leq t \leq T$ it holds that, cf. [19, Prop. 1.5.8],

$$\int_s^t \langle u^\varepsilon_t(\tau), u^\varepsilon(\tau) \rangle d\tau = \frac{1}{2} \| u^\varepsilon(t) \|_{L^2}^2 - \frac{1}{2} \| u^\varepsilon(s) \|_{L^2}^2.$$  

Thus we obtain

$$\frac{1}{2} \| u^\varepsilon(t) \|_{L^2}^2 + \int_{(0,t) \times \Omega} a^\varepsilon(\tau, x, \nabla^s u^\varepsilon) \cdot \nabla^s u^\varepsilon \, dx \, d\tau = \frac{1}{2} \| u^\varepsilon_0 \|_{L^2}^2 + \int_{(0,t) \times \Omega} f \cdot u^\varepsilon \, dx \, d\tau.$$  

Estimate (6.60) implies that, at least for a subsequence,

$$u^\varepsilon \rightharpoonup u \quad \text{in} \quad L^\infty(I; L^2(\Omega, \mathbb{R}^d)), \quad u^\varepsilon \rightarrow u \quad \text{in} \quad L^p(I; W^{1,p}_{0,\text{div}}(\Omega, \mathbb{R}^d)).$$

Since $W^{1,p}_{0,\text{div}}(\Omega, \mathbb{R}^d) \hookrightarrow L^2_{\text{div}}(\Omega, \mathbb{R}^d) \hookrightarrow (W^{1,p}_{0,\text{div}}(\Omega, \mathbb{R}^d))^*$, the Aubin-Lions lemma [15, p. 36] yields

$$u^\varepsilon \rightarrow u \quad \text{in} \quad L^p(I; L^2(\Omega, \mathbb{R}^d)), \quad \text{(6.62)}$$

which provides the limit passage

$$\int_{(0,t) \times \Omega} u^\varepsilon \otimes u^\varepsilon \cdot \nabla \psi \, dx \, d\tau \rightarrow \int_{(0,t) \times \Omega} u \otimes u \cdot \nabla \psi \, dx \, d\tau.$$  

for all $\psi \in \mathcal{D}(-\infty, T; \mathcal{V})$. Therefore we may write

$$\int_{(0,t) \times \Omega} u^\varepsilon \cdot \nabla \psi \, dx \, d\tau = \int_{(0,t) \times \Omega} (u \otimes u \cdot \nabla \psi - \sigma \cdot \nabla^s \psi - f \cdot \psi) \, dx \, d\tau.$$  

(6.63)

Notice that the strong convergence (6.62) implies that

$$u^\varepsilon(t) \rightarrow u(t) \quad \text{in} \quad L^2(\Omega, \mathbb{R}^d) \quad \text{for a.a.} \ t \in I$$

and one can easily show that

$$u^\varepsilon(t) \rightarrow u(t) \quad \text{in} \quad L^2(\Omega, \mathbb{R}^d) \quad \text{for all} \ t \in I,$$  

hence

$$\lim_{\varepsilon \rightarrow 0} \| u^\varepsilon(t) \|_{L^2} \geq \| u(t) \|_{L^2} \quad \text{for all} \ t \in I.$$  

(6.65)

In the similar way as before we can show that (6.61) holds for $u$. We test (6.63) with $u$, obtaining

$$\int_s^t \langle u_t(\tau), u(\tau) \rangle d\tau = \frac{1}{2} \| u(t) \|_{L^2}^2 - \frac{1}{2} \| u(s) \|_{L^2}^2,$$

and in particular

$$\frac{1}{2} \| u(t) \|_{L^2}^2 + \int_{(0,t) \times \Omega} \sigma \cdot \nabla^s u \, dx \, d\tau = \frac{1}{2} \| u_0 \|_{L^2}^2 + \int_{(0,t) \times \Omega} f \cdot u \, dx \, d\tau.$$  

(6.66)
Let $\varepsilon \to 0$ in (6.61). Using lower semicontinuity of the norm (6.65) we conclude that
\[
\limsup_{\varepsilon \to 0} \int_{(0,t) \times \Omega} a^\varepsilon(\tau, x, \nabla^s u^\varepsilon) : \nabla^s u^\varepsilon \, dx d\tau \leq \int_{(0,t) \times \Omega} f \cdot u \, dx d\tau - \frac{1}{2} \|u(t)\|_2^2 + \frac{1}{2} \|u_0\|_2^2.
\]
Applying energy equality (6.66) leads to
\[
\limsup_{\varepsilon \to 0} \int_{(0,t) \times \Omega} a^\varepsilon(\tau, x, \nabla^s u^\varepsilon) : \nabla^s u^\varepsilon \, dx d\tau \leq \int_{(0,t) \times \Omega} \sigma \cdot \nabla^s u \, dx d\tau. \quad (6.67)
\]
Applying Theorem 5.1 with $z^\varepsilon = \nabla^s u^\varepsilon$ and $N = \frac{d(d+1)}{2}$, after replacing $\Omega$ by $(0, t) \times \Omega$ and $a^\varepsilon(x, \cdot)$ by $a^\varepsilon(\tau, x, \cdot)$ in the statement of the theorem we finish the proof.

**Proof of Theorem 2.2:** The proof of Theorem 2.2 follows the same path as the proof of Theorem 2.4. Note that since the nonlinear term $u \otimes u$ is absent, the usual estimates are enough to claim that the time derivative $u^\varepsilon_t$ is in $L^p(I; W^{-1,p'}(\Omega, \mathbb{R}^m))$ for all $p > 1$.

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