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On the existence of global weak solutions to the Navier–Stokes equations for viscous compressible and heat conducting fluids

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Abstract

The purpose of this work is to investigate the problem of global existence of weak solutions to the Navier–Stokes equations for viscous compressible and heat conducting fluids. A certain class of density and temperature dependent viscosity and conductivity coefficients is considered. This result extends the work of P.–L. Lions [Compacité des solutions des équations de Navier–Stokes compressibles isentropiques. C.R. Acad. Sci. Paris, Sér I. 317, (1993), p. 115–120] restricted to barotropic flows, and provides weak solutions ”à la Leray” to a full compressible model – including the internal energy evolution equation with thermal conduction effects –. A partial answer is therefore given to this currently widely open problem, described for instance in [Mathematical topics in fluid dynamics, Vol.2, Compressible models. Oxford Science Publication, Oxford, (1998)]. The proof is based on the generalization to the temperature dependent case, of a new mathematical entropy equality derived by the authors in [Some diffusive capillary models of Korteweg type. C. R. Acad. Sciences, Paris, Section Mécanique. Vol. 332, no 11 (2004), p 881–886]. The actual construction of approximate solutions, based on additional regularizing effects such as capillarity, will be provided in [12].

Keywords: compressible Navier–Stokes equations, thermal conduction, global weak solutions.

AMS subject classification: 35Q30.

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1 Introduction

For its many applications in industry, the Navier–Stokes equations modelling viscous compressible and heat conducting fluids has been studied for the last fifty years both from a theoretical and a numerical point of view. Robustness and accuracy of numerical schemes, in particular in the presence of strong shocks or complex boundary conditions, as well as questions of existence, uniqueness and stability of solutions is still a very competitive research area. The present study addresses the theoretical problem of existence of so called ”weak solutions” for the full system, including the evolution equation of internal energy. This problem was described as widely open for instance in P.-L. Lions’ book [43].

The case of incompressible flows was considered by Jean Leray in the 30’s [41]. He proved the global in time existence of weak solutions to the Navier–Stokes equations in dimension $d = 2$ or $3$, and global regularity and uniqueness in dimension $d = 2$. A somehow simplified compressible case was considered by Pierre–Louis Lions in the 90’s [42] [43], decoupling the internal energy equation from the mass and momentum conservation equations: assuming barotropic pressure laws (temperature independent equations of state), he proved the global existence of weak solutions in the spirit of Jean Leray’s work. Roughly speaking, the construction of weak solutions is based upon estimates induced by the physical energy of the system supplemented by more subtle integrability bounds that allow to control density oscillations. Unfortunately, uniqueness and strong stability is often out of reach only with a weak solution approach. Refined functional analysis has been used for the last decades, ranging from Sobolev, Besov, Lorentz and Triebel spaces, . . . to describe the regularity and long time behavior of solutions to the incompressible model [40] [27] [15] [16] [17] [14] [26] . . . as well as the compressible one [55] [30] [53] [37] . . . The question of global regularity and uniqueness of solutions without restrictions on the size of the data is still open, both in dimension $d = 3$ in the incompressible and in the compressible case, and in dimension $d = 2$ in the compressible case because of the lack of estimates in regions where the density vanishes. Even though global existence of weak solutions provides little information about well posedness of the system, such an analysis has many practical interests. In addition to their physical relevance, since regularity of initial data is closely related to physically meaningful quantities, weak solutions stability properties are likely to provide evidence of stability of numerical schemes, which most of
the time do not preserve strong regularity estimates at the continuous limit.

The aim of this paper is to extend the work of P.-L. Lions \[43\] to the case when pressure laws both depend on density and temperature. We intend to prove the global existence of weak solutions for the full Navier–Stokes solutions modelling viscous compressible and heat conducting fluids in dimension \(d = 2\) or \(3\). The space domain is assumed to be either the whole space \(\Omega = \mathbb{R}^d\), or a box \(\Omega = T^d\) with periodic boundary conditions. The case of more general domains will be discussed at the end of this work and is far from being completely handled. An extension of this result involving capillarity effects is also given.

A compressible and heat conducting fluid governed by the Navier–Stokes equations satisfies the following system in \(\mathbb{R}_+ \times \Omega\)

\[
\begin{align*}
\partial_t \rho + \text{div} (\rho u) &= 0, \\
\partial_t (\rho u) + \text{div} (\rho u \otimes u) &= \text{div} \sigma + \rho f, \\
\partial_t (\rho E) + \text{div} (\rho u H) &= \text{div} ((\sigma + \rho I) \cdot u) + \text{div} (\kappa \nabla \theta) + \rho f_{\text{ext}} \cdot u,
\end{align*}
\]

where \(u \in \mathbb{R}^d\) denotes the velocity field of the fluid, \(\rho\) the density, \(\kappa\) the thermal conductivity coefficient, \(\sigma\) the stress tensor, \(p\) the pressure field, \(e\) the specific internal energy and \(h\) the specific enthalpy. The specific total energy is denoted by \(E\) and the associated specific enthalpy by \(H\). Finally, bulk forces are represented by a given \(d\)-component vector field \(f\). It may be decomposed into two components \(f = f_{\text{int}} + f_{\text{ext}}\), where \(f_{\text{int}}\) and \(f_{\text{ext}}\) respectively denote the internal forces (drag force, . . . ) and the external forces (gravitation, . . . ). As we shall see later on, the work of internal forces \(\rho f_{\text{int}} \cdot u\) appears in the internal energy equation and has to be non negative in order to be consistent with the second principle of Thermodynamics. On the other hand, no sign condition is required for the work of external forces \(\rho f_{\text{ext}} \cdot u\) which appears at the right hand side of the total energy equation (3).

Equations (1) (2) and (3) respectively express the conservation of mass, momentum and total energy. In order to close the system, two additional ingredients are necessary. First, the fluid is assumed to be Newtonian, so that there exists two viscosity coefficients \(\mu\) and \(\lambda\) such that

\[
\sigma = 2\mu D(u) + (\lambda \text{div} u - p) I,
\]
where $D(u)$ denotes the strain rate tensor, defined as the symmetric part of the velocity gradient $\nabla u$. As a second condition, a thermodynamic closure law provides the pressure $p$ and the internal energy $e$ as functions of the density $\rho$ and the temperature $\theta$

$$p = P(\rho, \theta) \quad \text{and} \quad e = E(\rho, \theta). \quad (5)$$

In order to be consistent with the second principle of Thermodynamics, which implies the existence of the entropy as a closed differential form in the energy balance, the following compatibility condition, called "Maxwell equation", between $P$ and $E$ has to be satisfied

$$P(\rho, \theta) = \rho^2 \frac{\partial E}{\partial \rho} \bigg|_\theta + \theta \frac{\partial P}{\partial \theta} \bigg|_\rho. \quad (6)$$

The entropy $s = S(\rho, e)$ is defined up to an additive constant by

$$\frac{\partial S}{\partial e} \bigg|_\rho = \frac{1}{\theta} \quad \text{and} \quad \frac{\partial S}{\partial \rho} \bigg|_\theta = -\frac{p}{\rho^2 \theta}. \quad (7)$$

Another important assumption on the entropy function is made

the entropy $S$ is a concave function of $(\rho^{-1}, e)$. \quad (8)\)

Property (8) ensures in particular the non negativity of the so called $C_v$ coefficient given by

$$C_v = \frac{\partial E}{\partial \theta} \bigg|_\rho = -\frac{1}{\theta^2} \frac{\partial^2 S}{\partial e^2} \bigg|_\rho. \quad (6)$$

System (1)(2)(3) is supplemented with initial conditions

$$\rho|_{t=0} = \rho_0, \quad \rho u|_{t=0} = m_0, \quad \rho E|_{t=0} = G_0 + \frac{|m_0|^2}{2\rho_0}. \quad (9)$$

The functions $\rho_0$, $m_0$, and $G_0$ are assumed to satisfy

$$\rho_0 \geq 0 \quad \text{a.e. on } \Omega, \quad \text{and} \quad \frac{|m_0|^2}{\rho_0} = 0 \quad \text{a.e. on } \{x \in \Omega \mid \rho_0(x) = 0\}, \quad (10)$$

and $G_0$ has to be taken in such a way that

$$G_0(x) \in \overline{\rho_0(x)|E(\rho_0(x), R_+)} \quad \text{for a.e. } x \in \Omega, \quad (11)$$
which allows to define the initial temperature $\theta_0$ on $\{x \in \Omega / \rho_0(x) \neq 0\}$, which is assumed to be non-negative

$$\theta_0(x) = \mathcal{E}(\rho_0(x), \cdot)^{-1}\left(\left\{G_0(x)/\rho_0(x)\right\}\right) \geq 0 \text{ a.e. on } \{x \in \Omega / \rho_0(x) \neq 0\}. \quad (12)$$

In most practical applications, the two viscosity coefficients $\mu$ and $\lambda$, as well as the thermal conductivity coefficient $\kappa$ are given functions of the density and temperature (the so-called Sutherland law is a popular example [13]). The question of global well-posedness of the above system is one of the main challenges since the end of the last century. Only partial results are currently available [30], [31] and the full problem is still widely open. See also [24] with viscosities depending on temperature.

2 Brief survey on weak solutions of (1)(2)(3)

Let us now review briefly the currently available results regarding the existence of weak solutions for the compressible Navier–Stokes equations in $\mathbf{T}^d$ or $\mathbf{R}^d$.

2.1 Barotropic flows

One of the main result of the nineties is due to P.-L. Lions [43], who proved the global existence of weak solutions "à la Leray" for the compressible Navier–Stokes system in the case of barotropic equations of state. It means that the pressure field $p$ is a function of the density $\rho$ only, which behaves like $\rho \mapsto \rho^\gamma$ at infinity, where $\gamma$ is a large enough coefficient. The viscosity coefficients $\mu$ and $\lambda$ are assumed to be constants satisfying $\mu > 0$ and $\lambda + 2\mu > 0$. Weak solutions in $\Omega = \mathbf{T}^d$ (periodic boundary conditions) are defined as solutions in the sense of distributions such that the physical energy is bounded. In the case of a perfect $\gamma$–type pressure law ($p(\rho) = a\rho^\gamma$ for some positive constant $a$), the energy inequality writes for all $t \geq 0$,

$$\int_\Omega \rho\left(\frac{|u|^2}{2} + e\right)(t, x) \, dx + \int_0^t \int_\Omega (\mu|\nabla u|^2 + (\lambda + \mu)|\text{div } u|^2) (s, x) \, dxds \leq \int_0^t \int_\Omega (\rho f \cdot u)(s, x) \, dxds + \int_\Omega \left(\frac{|m_0|^2}{2\rho_0} + \rho_0e_0\right)(x) \, dx \quad (13)$$
where
\[ e = \frac{a\rho^{\gamma-1}}{\gamma - 1} \]
denotes the specific internal energy. The initial conditions are assumed to satisfy
\[ \int_{\Omega} \left( \frac{|m_0|^2}{2\rho_0} + \rho_0 e_0 \right)(x) \, dx < +\infty. \]  
(14)

The existence of global weak solutions (solutions in the sense of distributions with bounded physical energy) was obtained by P.-L. Lions in [42], [43] for large enough exponents \( \gamma \): assuming that \( f \equiv 0 \) for simplicity, that the above integrability of initial data (14) holds and
\[ \gamma \geq \gamma_0, \quad \text{where} \quad \gamma_0 = \frac{3}{2} \text{ if } d = 2, \quad \gamma_0 = \frac{9}{5} \text{ if } d = 3, \]
he proved that there exists a global weak solution \((\rho, u)\) such that the density \( \rho \in L^\infty(\mathbb{R}_+; L^\gamma(\Omega)) \), the velocity \( u \in L^2(\mathbb{R}_+; (H^1(\Omega))^d) \). This solution also satisfies the following regularity: \( \rho \in C(\mathbb{R}_+; L^p(\Omega)) \) for all \( p < \gamma \), \( \rho|u|^2 \in L^\infty(\mathbb{R}_+; L^1(\Omega)) \), \( \rho u \in C(\mathbb{R}_+; L^{2\gamma/(\gamma - 1)}(\Omega) - \text{weak}) \) and \( \rho \in L^q((0, T) \times \Omega) \) for all \( T > 0 \), where \( q = \gamma - 1 + 2\gamma/d. \) Notice that this result is also proved in the whole space \( \mathbb{R}^d \) as well as in the case of a suitably smooth bounded domain with homogeneous Dirichlet boundary conditions on the velocity. This result has been recently extended to the somehow optimal case \( \gamma > d/2 \) in [23] using oscillation defect measures on density sequences associated with suitable approximate solutions.

Another recent improvement was achieved in [6] [7] [11]. It still involves the case of barotropic equations of state, but also handles degenerate viscosity coefficients, possibly depending on the density \( \rho \). Although degenerate coefficients bring additional difficulties, since the velocity itself no longer belongs to \( L^2_{\text{loc}}(\mathbb{R}_+; H^1(\Omega)) \), this degeneracy provides a particular mathematical structure that yields global in time integrability properties on density gradients. This new structure was first discovered in [11] in the framework of capillary fluids, when the viscosity coefficient \( \mu \) is proportional to the density \( \rho \) and \( \lambda = 0 \). Later on, the authors realized that the derivation of this Sobolev bounds on the density still works without capillarity [6]. It was applied to prove global existence of weak solutions to the 2-dimensional viscous shallow water model [6], in which viscosity is proportional to the height function (the equivalent of the density in the compressible flow analogy). As a matter of
fact, such a viscosity law arises in the physically relevant derivation process of the model in the long wavelength limit [28]. Notice that the rigorous physical derivation yields a non zero drag force, corresponding to bottom friction in the underlying 3d free surface model [45]. This zero order term turned out to be necessary in the mathematical stability result for technical reasons in [6]. This assumption has recently been removed in [47] by assuming more integrability on the initial velocity than the only physically relevant bound, still using the structure discovered in [11].

Finally, the most recent work about the barotropic case involves general density dependent viscosity coefficients for which the new Sobolev bound on the density still holds. It writes as a constraint involving the shear and the bulk viscosity coefficients, and was published in [7].

This mathematical structure, which in fact reduces to a new mathematical entropy equality, is the key ingredient in this work to prove the global existence of weak solutions to the full compressible and heat conducting Navier–Stokes system. Note that such a structure has also been recently used in [47]. A stability result on barotropic compressible Navier–Stokes equations is proved for general viscosities satisfying the constraint published in [7], assuming gamma type pressure law $p(\rho) = \rho^\gamma$ with $\gamma \geq 1$, without adding friction forces as in the previous work of the authors on the shallow water model in [6]. For this, they consider initial data such that $\sqrt{\rho_0} u_0 \in L^{2+\delta}$ for some small enough $\delta$ and prove that this integrability is transported in time. Such a property is obtained by using the test function $|u|^\delta u$ combined with the estimate on $\nabla \varphi(\rho)$ deduced from the new mathematical entropy equality [11] for some suitable function $\varphi$ depending on $\mu$.

Here, for the reader’s convenience, the derivation of the new mathematical entropy equality is adapted to the temperature dependent case and detailed in Subsection 4.2. Note that global existence of weak solutions in 1D with degenerate viscous term for compressible Navier-Stokes equations has been extensively studied, see for instance [44], [36] and [58] for an overview. We refer the reader to the paper [34] where an interesting result is shown about the failure of continuous dependence on initial data for the Navier-Stokes equations of compressible flows with constant viscosities. Our result shows that viscosities depending on $\rho$ may give provide information in order to get stability results.

The common advantage of the theory of weak solutions is that it does not require any restrictions on the size of the data. But it guarantees global
existence only in the class of low regularity solutions, where other fundamental questions such as uniqueness or stability (well–posedness in the sense of Hadamard) remain largely open. In singular perturbations studies, such as low Mach number analysis, attention has to be paid to the possibly non empty set where the density \(\rho\) vanishes and on the low regularity of solutions, see for instance [10] and references cited therein. However, a weak solutions approach does not require to take care of the possible dependence of the time existence with respect to the coefficients, as it is for smooth solutions. Proving such a lower bound on the time existence is not an easy task, see for instance [1] [50].

2.2 Temperature dependent pressure laws

To the best of our knowledge, almost all existence results for the full compressible Navier–Stokes equations (including the temperature equation) involve simplified or restrictive assumptions. For instance, symmetries (plane, cylindric or spherical) allow to reduce to the one dimensional setting in space variables. However, it somehow hides the mathematical structure of the system, even though very interesting work has been done [31] for the last fifty years . . . Another approach for the multidimensional case \(d \geq 2\) consists in considering existence, uniqueness and stability results "in the small", i.e. for small enough time depending on the size of the initial data, or globally in time for initial data close enough to an equilibrium (small enough initial velocity and initial density close enough to a constant) [38], [57], [18], [30], [46], [55]. Let us remark that the work by D. Hoff [31] [32] provides results in very low regularity spaces for possibly discontinuous initial data, still in the framework of "small data".

The question of existence of so called “variational” solutions in dimension \(d \geq 2\) has been recently addressed in [25]. Let us emphasize that this work is the very first attempt towards the existence of weak solutions for the full compressible Navier–Stokes equations.

Such a remarkable existence result is obtained for specific pressure laws with constant \(C_v\) coefficient, given by the general equation \(p(\rho, \theta) = p_c(\rho) + \theta p_\theta(\rho)\) with specific behavior at infinity for \(p_c\), and restrictions on \(p_\theta\), which has to grow suitably slower than the "zero temperature" pressure \(p_c\) for large densities. Unfortunately, the perfect gas equation of state is not covered by this result, even in some restricted regime of densities. As a matter of fact, the method used in [25] basically relies on a perturbation of P.–L. Lions’ method
in the barotropic case. Namely the dominant role of the first, barotropic, pressure term $p_c$ is one of the key argument to obtain such an existence result. This restrictive assumption prevents from considering equations of state commonly used in realistic applications.

Moreover, in [25], the temperature equation is satisfied only as an inequality (which justifies the notion of variational solutions): more precisely, it satisfies

$$C_v \left( \partial_t (\rho \theta) + \text{div}(\rho \theta u) \right) - \text{div}(\kappa(\rho, \theta) \nabla \theta) \geq 2\mu D(u) : D(u) + \lambda |\text{div} u|^2 - \theta \left. \frac{\partial p}{\partial \theta} \right|_\rho \text{div} u,$$

which is not fully satisfactory from a physical viewpoint, even though it preserves the second principle of thermodynamics. As a matter of fact, the compactness on the temperature does not seem to be sufficient to pass to the limit in the energy equality. Such a possibly non zero "ghost entropy production" is by no means justified by physical considerations, since no discontinuities (shocks) are expected in the viscous and heat conducting case. Anyway, we have to note a recent interesting extension performed by in [24], namely temperature dependent viscosities. This case is the only result which is known with such dependency of viscosities.

3 Setup and main result

The present work addresses the question of global existence of weak solutions for the full Navier–Stokes compressible and heat conducting system, where the total energy conservation equation is satisfied in the sense of distributions. Specific assumptions are made on the density and temperature dependence of the thermal conductivity $\kappa$ and the viscosity coefficients $\lambda$ and $\mu$ and an internal force or order 0 with respect to the physical variables may be considered. In order to simplify the presentation, only equations of state close to ideal polytropic gas are considered in the main statement in the case of the dimension $d = 3$ in $T^3$ (bounded box $[0, 2\pi]^3$ with periodic boundary conditions) or the whole space $\mathbb{R}^3$. Moreover, complete proofs are given only in this 3d case, even though some of the steps use the most general framework. As a matter of fact, some suitable real gas equations of state and two dimensional flows may be considered; these questions will be discussed in Sections 9 and 10. Finally, further results taking account of capillarity
effects [39] will be developed in Section 11, as well as open questions such as the case of domains $\Omega$ with non zero curvature boundary.

### 3.1 Assumptions

This subsection deals with assumptions regarding physical coefficients, such as viscosity, thermal conductivity and equation of state.

First of all, the viscosity coefficients $\lambda$ and $\mu$ are assumed to be respectively $C^0(\mathbb{R}_+)$ and $C^0(\mathbb{R}_+) \cap C^1(\mathbb{R}_+)$ functions of the density only, such that $\mu(0) = 0$, and the following constraints are satisfied: there exists positive constants $c_0$, $c_1$, $A$ and $m > 1$, $(d - 1)/d < n < 1$ such that

\[
\text{for all } s > 0, \quad \lambda(s) = 2(s\mu'(s) - \mu(s)), \tag{15}
\]

\[
\text{for all } s < A, \quad \mu(s) \geq c_0 s^n \quad \text{and} \quad d\lambda(s) + 2\mu(s) \geq c_0 s^n, \tag{16}
\]

\[
\text{for all } s \geq A, \quad c_1 s^m \leq \mu(s) \leq \frac{s^m}{c_1} \quad \text{and} \quad c_1 s^m \leq d\lambda(s) + 2\mu(s) \leq \frac{s^m}{c_1}. \tag{17}
\]

Let us also recall that assumption (15) was already introduced in [7] in the framework of barotropic flows. The extension to both density and temperature dependence of the viscosity coefficients, which may allow to capture the case of Sutherland’s law [54], unfortunately seems out of reach in the present work.

Next, the thermal conductivity coefficient $\kappa$ is assumed to satisfy

\[
\kappa(\rho, \theta) = \kappa_0(\rho, \theta)(\rho + 1)(\theta^a + 1), \tag{18}
\]

where $a \geq 2$, and $\kappa_0$ is a $C^0(\mathbb{R}_+^2)$ function satisfying for all positive $\rho$ and $\theta$

\[
c_3 \leq \kappa_0(\rho, \theta) \leq \frac{1}{c_3}, \tag{19}
\]

for some positive constants $c_3$. Notice that reference density and temperature are taken equal to one in the above assumptions; non dimensional forms of the equations may indeed be considered, so that no generality is lost.

Finally, we assume that the equations of state (5) are of ideal polytropic gas type:

\[
p = \rho r \theta + p_c(\rho), \quad e = C_v \theta + e_c(\rho), \tag{20}
\]
where \( r \) and \( C_v \) are two constant positive coefficients. For convenience, we denote \( \gamma = 1 + r/C_v \). Moreover, the additional pressure and internal energy \( p_c \) and \( e_c \) are associated with the "zero Kelvin isothermal". We require that \( e_c \) is a \( C^2 \) non negative function on \( \mathbb{R}_+^* \) and that the following constraint is satisfied in order to respect assumption (6)

\[
p_c(\rho) = \rho^2 \frac{d e_c}{d\rho}(\rho).
\]  

(21)

We also require that there exists \( \rho_* > 0, \tau_* > 0, k > 1, \ell > 1, C_* > 0, C'_*, C''_* > 0 \) such that for all \( \rho \in (0, \rho_*) \),

\[
\frac{\rho^{-\ell-1}}{C_*} \leq p'_c(\rho) \leq C_*\rho^{-\ell-1}, \quad \frac{\rho^{-\ell-1}}{C'_*} \leq e_c(\rho) \leq C'_*\rho^{-\ell-1},
\]

where \( \ell \geq \frac{2n(3m-2)}{m-1} - 1 \),

(22)

and for all \( \rho > \rho_* \),

\[
-\frac{1}{\tau_*} \mu'(\rho) \leq p'_c(\rho) \leq C''_*\rho^{k-1}, \quad 0 \leq e_c(\rho) \leq C''_*\rho^{k-1},
\]

where \( k \leq \left( m - \frac{1}{2} \right) \frac{5(\ell + 1) - 6n}{\ell + 1 - n} \).

(23)

Let us comment the assumptions on the zero Kelvin isothermal curve: first, the cold component of the pressure and internal energy may vanish away from zero under assumptions (22) and (23), so that the usual perfect polytropic gas equation of state is recovered. Moreover, the physical relevance of the compressible Navier–Stokes equations is very questionable in regions where both density and temperature are close to zero: at zero Kelvin, the medium is not only unlikely to be in a liquid or gas state (elasticity and plasticity has to be considered for such solid materials [29]), but also the rarefied regime of vanishing densities violates the assumptions on the mean free path of particles suitable for fluid models. As a consequence, assumptions (22) and (23) may be viewed as mathematical assumptions designed to preserve stability properties.

Finally, the extension to more general equations of state will be discussed in Section 9.
3.2 Definition of weak solutions

Before stating the main existence theorem, the precise definition of weak solutions has to be introduced, together with the required corresponding regularity. Basically, the equations of mass, momentum, and energy conservation are considered in the sense of distributions with initial data of arbitrary size of finite physical energy and entropy, with suitably smooth initial density.

We shall say that \((\rho, u, \theta)\) is a weak solution on \((0, T)\) of Equations (1)(2)(3) and initial conditions (9) if the following three conditions are fulfilled:

— The following regularity properties hold

\[
\begin{align*}
\rho \text{e} \quad \text{and} \quad \rho |u|^2 &\in L^\infty(0, T; L^1(\Omega)), \\
\nabla \mu(\rho) &\in L^\infty(0, T; L^2(\Omega)^d), \\
(\rho^{\alpha/2} + \rho^{\beta/2})\nabla u &\in L^2(0, T; L^2(\Omega)^{d\times d}), \\
(1 + \sqrt{\rho})\nabla \theta^{\alpha/2} \quad \text{and} \quad (1 + \sqrt{\rho})\frac{\nabla \theta}{\theta} &\in L^2(0, T; L^2(\Omega)^d),
\end{align*}
\]

for \(\alpha \geq 2\). Finally, one has, for some large enough positive number \(\sigma\),

\[
\rho \quad \text{and} \quad \rho E \in C([0, T]; H^{-\sigma}(\Omega)), \quad \rho u \in C([0, T]; H^{-\sigma}(\Omega)^d).
\]

— The initial conditions (9) hold in \(D'(\Omega)\), and (10)–(12) are satisfied.

— Equations (1)(2)(3) hold in \(D'((0, T) \times \Omega)\).

Weak solutions are called ”global” as soon as existence holds for all positive time \(T\).

3.3 Main theorem

We establish the following existence result

**Theorem 3.1** Let us assume that the viscosity, thermal conduction and equation of state satisfy (15), (16), (17), (18), (19) and (20). The initial data \((\rho_0, m_0, G_0)\) are taken in such a way that (10), (11) and (12) are satisfied and such that

\[
\mathcal{H}(0) = \int_\Omega \left( G_0 + \frac{|m_0|^2}{2\rho_0} \right) \, dx < +\infty,
\]

12
that the initial density $\rho_0$ satisfies for some positive constant $\rho_\infty$

\[
\rho_0 - \rho_\infty \in L^1(\Omega), \quad \rho_0 \log \frac{\rho_\infty}{\rho_0} \in L^1(\Omega) \quad \rho_0 e_c(\rho_0) \in L^1(\Omega),
\]

and

\[
\nabla \mu(\rho_0) \overline{\rho_0} \in L^2(\Omega)^d,
\]

and that the initial entropy density $s_0 = C_v \log(\theta_0/\rho_0^\infty)$ satisfies

\[
\rho_0 s_0 \in L^1(\Omega).
\]

Then, there exists a global in time weak solution to (1)(2)(3) and (9).

In order to prove the global existence of weak solutions, the first step is to obtain suitable a priori bounds on $(\rho, u, \theta)$, and next to consider sequences $(\rho_n, u_n, \theta_n)$ of uniformly bounded weak solutions constructed from an adapted approximation process, see [43] of [11]. Such sequences may be built by using the regularization of capillary effects, since the mathematical entropy discovered in [7] can be adapted to the capillary case. The uniform bounds on the sequences $(\rho_n, u_n, \theta_n)_{n \in \mathbb{N}}$ write as follows: for all $T > 0$, there exists $C_T$ such that for all $n \in \mathbb{N}$

\[
\sup_{t \in (0,T)} \int_\Omega \rho_n \left(e_c(\rho_n) + C_v \theta_n + \frac{|u_n|^2}{2}\right)(t, \cdot) \leq \int_\Omega \left(G_0^n + \frac{|m^n_0|^2}{2\rho_0^n}\right) \leq C_T, \tag{31}
\]

\[
\int_0^T \int_\Omega \frac{1}{\theta_n} \left(2\mu S(u_n):S(u_n) + \left(\lambda(\rho_n) + \frac{2\mu(\rho_n)}{d}\right)|\text{div} u_n|^2\right)(s, \cdot) ds
\]

\[
+ \int_0^T \int_\Omega \frac{\kappa(\rho_n, \theta_n)}{\theta_n^2} |\nabla \theta_n|^2(s, \cdot) ds \leq C_{1,T} \int_\Omega \left(\rho_0^n(s_0^n)^2 + \rho_0^n \log \frac{\rho_\infty}{\rho_0^n}\right) \leq C_T. \tag{32}
\]

\[
\sup_{t \in (0,T)} \int_\Omega \left(\frac{1}{2} \rho_n |u_n|^2 + \rho_n e_c(\rho_n) + \frac{1}{2} \rho_n |u_n + 2\nabla \varphi(\rho_n)|^2\right)(t, \cdot)
\]

\[
+ \int_0^T \int_\Omega \left(2\mu(\rho_n) |\nabla u_n|^2 + \lambda(\rho_n)|\text{div} u_n|^2 + \frac{\rho_{nt} \theta_n}{\mu'(\rho_n)} |\nabla \varphi(\rho_n)|^2
\]

\[
+ c_0 |\nabla \zeta(\rho_n)|^{-\epsilon} \right)(s, \cdot) ds \leq C_{2,T} \int_\Omega \left(\frac{|m^n_0|^2}{2\rho_0^n} + \rho_0^n e_c(\rho_0^n)
\]

\[
+ \frac{|m^n_0 + 2\nabla \mu(\rho_0^n)|^2}{2\rho_0^n}\right) + C_{2,T} \leq C_T, \tag{33}
\]

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ζ being taken such that ζ(ρ) = ρ for ρ ≤ ρ∗/2 and ζ(ρ) = 0 for ρ ≥ ρ∗.

The final step is to obtain compactness on (ρ, u, e) in suitably strong topologies and prove that the limit (ρ, u, e) satisfies Equations (1)(2)(3) and initial conditions (9) in the distribution sense.

4 Energy estimates

This section is devoted to a priori estimates on solutions of system (1)(2) (3). Estimates based on physical energy and entropy are supplemented with extra bounds derived from a new structure which was first introduced for barotropic flows in [7]. Let us emphasize that the constraint involving the viscosity coefficients (15) is a key assumption for obtaining such estimates.

In order to avoid additional technical difficulties, only equations of state of ideal polytropic type (20) (21) (22) (23) are assumed in Sections 4 and 7. Extension to real gas equations of state, two dimensional domains, bounded domains and capillarity effects will be considered in Sections 9, 10 and 11.

4.1 Physical energy

First, the physical energy equality

\[ \frac{d}{dt} \int_\Omega \rho \left( e + \frac{|u|^2}{2} \right) (t, x) \, dx = \int_\Omega \rho f_{ext} \cdot u \]

is obtained in a classical way by multiplying the momentum equation by \( u \) and by using the energy equation. This gives, integrating in time from 0 to \( t > 0 \)

\[ \int_\Omega \rho \left( e + \frac{|u|^2}{2} \right) (t, x) \, dx \leq \int_\Omega \left( G_0 + \frac{|m_0|^2}{2p_0} \right) (x) \, dx \]

\[ + \int_0^t \int_\Omega (\rho f_{ext} \cdot u)(s, x) \, dx \, ds. \]

As can be checked easily, the only energy estimates (35) are not sufficient to build up a reasonable theory of solutions in the sense of distributions since viscous stresses dissipate in internal energy. This is a major difference with the barotropic case, in which the viscous dissipation naturally provides a \( H^1 \) bound in space on the velocity \( u \).
4.2 On a density related velocity and associated energy

The last remark of the preceding subsection motivates additional investigations on the mathematical structure of compressible viscous and heat conducting flows. Dimensional analysis may be used as a starting point of the discussion: as a matter of fact, apart from the velocity \( u \) attached to fluid particles, another velocity scale may be built depending on the density heterogeneities. As a matter of fact, the usual definition of the Reynolds number scales the velocity order of magnitude as \( \mu/\rho L \), where \( \mu \) and \( \rho \) denote typical dynamic viscosity and density scales, and \( L \) the typical size of the domain. However, the length scale \( L_\rho = \rho/\|\nabla \rho\| \) characterizing the density variations may also be used, leading to the velocity \( \mu \|\nabla \rho\| \), which turns out to play an essential role from a mathematical and physical point of view. For instance, the multiphase turbulent flow community often uses such a drift velocity \( \nu_t \) \( \nabla \rho \) to correct the velocity in drag forces [22], \( \nu_t \) denoting the turbulent kinematic viscosity. Indeed, as sometimes observed in some other research fields, the engineering community introduced this useful concept three decades ago [35] with practical motivations.

Let us recall that the following computations have been derived in [7] in the case of barotropic equations of state. In the present situation covering general temperature dependent pressure laws, the following identities hold

**Lemma 4.1**

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \rho |u|^2 + \int_\Omega 2\mu(\rho)D(u) : D(u) + \int_\Omega \lambda(\rho)|\nabla u|^2 \\
+ \chi \frac{d}{dt} \int_\Omega |\nabla \mu(\rho)|^2 = \int_\Omega p(\rho, \theta) \text{div} u + \int_\Omega \rho f \cdot u. \tag{36}
\]

and

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \rho|u + 2
\nabla \varphi(\rho)|^2 + \int_\Omega 2\mu(\rho)A(u) : A(u) \\
+ \chi \frac{d}{dt} \int_\Omega |\nabla \mu(\rho)|^2 + 2\chi \int_\Omega \mu'(\rho)|\Delta \mu(\rho)|^2 \\
= \int_\Omega p(\rho, \theta) \text{div} u - 2 \int_\Omega \nabla p(\rho, \theta) \cdot \nabla \varphi(\rho) + \int_\Omega \rho f \cdot \nabla \varphi(\rho), \tag{37}
\]

where \( A(u) = (\nabla u - \text{tr} \nabla u)/2 \) denotes the skew symmetric part of \( \nabla u \).
Notice that the above lemma holds whenever surface tension effects are present ($\chi > 0$) or not ($\chi = 0$).

**Proof.**

**Energy equality** (36). Let us multiply the momentum equation by $u$ and the mass conservation equation by $|u|^2/2$. We test the mass conservation equation by $-\chi \mu'(\rho) \Delta \mu(\rho)$ and we integrate by parts the first term with the time derivative. We get Equality (36) by summing the previous three equalities.

**Additional equality** (37). From the mass conservation equation, we deduce that

$$\frac{\partial_t \varphi(\rho)}{\rho} + u \cdot \nabla \varphi(\rho) + \varphi'(\rho) \rho \text{div} u = 0.$$ (38)

This gives, differentiating this equation with respect to the space variable $x_i$ ($1 \leq i \leq d$)

$$\frac{\partial_t \partial_i \varphi(\rho)}{\rho} + (u \cdot \nabla) \partial_i \varphi(\rho) + (\partial_i u \cdot \nabla) \varphi(\rho) + \partial_i (\varphi'(\rho) \rho \text{div} u) = 0.$$ (39)

Let us multiply this equation by $\rho \partial_i \varphi(\rho)$ and sum over $i$. By using the mass equation, this gives

$$\frac{1}{2} \frac{d}{dt} \int_\Omega \rho |\nabla \varphi(\rho)|^2 + \int_\Omega \rho \nabla \varphi(\rho) \otimes \nabla \varphi(\rho) : \nabla u$$

$$+ \int_\Omega \nabla (\varphi'(\rho) \rho \text{div} u) \cdot \nabla \mu(\rho) = 0.$$ (40)

By multiplying the momentum equation by $\nabla \mu(\rho)/\rho$, we get

$$\int_\Omega \left( \partial_t u + u \cdot \nabla u \right) \cdot \nabla \mu(\rho) + 2 \int \mu(\rho) D(u) \cdot \left( \frac{\nabla \nabla \mu(\rho)}{\rho} \right)$$

$$- \frac{\nabla \mu(\rho) \otimes \nabla \rho}{\rho^2} \right) + \chi \int \mu'(\rho) |\Delta \mu(\rho)|^2 + \int \nabla p(\rho, \theta) \cdot \frac{\nabla \rho}{\rho} \mu'(\rho)$$

$$+ 2 \int \nabla \left( (\mu(\rho) - \mu'(\rho) \rho \text{div} u) \cdot \frac{\nabla \mu(\rho)}{\rho} \right) = \int_\Omega \rho f \cdot \nabla \varphi(\rho).$$ (41)

Integrating by parts, Equation (40) may be rewritten under the form

$$\frac{1}{2} \frac{d}{dt} \int_\Omega \rho |\nabla \varphi(\rho)|^2 + \int_\Omega \frac{\mu(\rho) \nabla \mu(\rho) \cdot \nabla u \cdot \nabla \rho}{\rho^2} - \int_\Omega \frac{\mu(\rho) \nabla u : \nabla \nabla \mu(\rho)}{\rho}$$
\[- \int_\Omega \frac{\mu(\rho) \nabla \text{div} \mathbf{u} \cdot \nabla \mu(\rho)}{\rho} + \int_\Omega \nabla (\varphi'(\rho) \rho \text{div} \mathbf{u}) \cdot \nabla \mu(\rho) = 0. \tag{42}\]

Summing Equation (41) to Equation (42) multiplied by 2, we get
\[
\int_\Omega (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) \cdot \nabla \mu(\rho) - 2 \int_\Omega \frac{\mu(\rho) \nabla \text{div} \mathbf{u} \cdot \nabla \mu(\rho)}{\rho} + \chi \int_\Omega \mu'(\rho) |\Delta \mu(\rho)|^2
+ \frac{1}{2} \frac{d}{dt} \int_\Omega 2 \rho |\nabla \varphi(\rho)|^2 + 2 \int_\Omega \nabla (\mu(\rho) - \mu'(\rho) \rho \text{div} \mathbf{u}) \cdot \nabla \mu(\rho) \rho
+ 2 \int_\Omega \nabla (\mu'(\rho) \text{div} \mathbf{u}) \cdot \nabla \mu(\rho) + \int_\Omega \nabla \rho(p, \theta) \cdot \frac{\nabla \rho}{\rho} \mu'(\rho) = \int_\Omega \rho \mathbf{f} \cdot \nabla \varphi(\rho). \tag{43}\]

By splitting the terms involving \(\text{div} \mathbf{u}\) and by summing them, we get
\[
\int_\Omega (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) \cdot \nabla \mu(\rho) + \chi \int_\Omega \mu'(\rho) |\Delta \mu(\rho)|^2
+ \frac{1}{2} \frac{d}{dt} \int_\Omega 2 \rho |\nabla \varphi(\rho)|^2 + \int_\Omega \nabla \rho(p, \theta) \cdot \frac{\nabla \rho}{\rho} \mu'(\rho) = \int_\Omega \rho \mathbf{f} \cdot \nabla \varphi(\rho). \tag{44}\]

Let us now look at the first term of (44). We get
\[
\int_\Omega (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) \cdot \nabla \mu(\rho)
= \frac{d}{dt} \int_\Omega \mathbf{u} \cdot \nabla \mu(\rho) - \int_\Omega \mathbf{u} \cdot \nabla \partial_t \mu(\rho) + \int_\Omega \mathbf{u} \cdot \nabla \mu(\rho).
\]

By using now the mass equation and integrating by parts the last two terms, this gives
\[
\int_\Omega (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) \cdot \nabla \mu(\rho) = \frac{d}{dt} \int_\Omega \mathbf{u} \cdot \nabla \mu(\rho)
- \int_\Omega \mu'(\rho) \text{div}(\rho \mathbf{u}) \text{div} \mathbf{u} - \int_\Omega \mu(\rho) \text{div} \mathbf{u} - \int_\Omega \mu(\rho) \sum_{i,j} \partial_i u_j \partial_j u_i.
\]

Integrating by parts the third term, we get
\[
\int_\Omega (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) \cdot \nabla \mu(\rho) = \frac{d}{dt} \int_\Omega \mathbf{u} \cdot \nabla \mu(\rho)
- \int_\Omega (\rho \mu'(\rho) - \mu(\rho)) |\text{div} \mathbf{u}|^2 - \int_\Omega \mu(\rho) \sum_{i,j} \partial_i u_j \partial_j u_j.
\]
Adding the above identity with (44), we get the following equality
\[
\frac{d}{dt} \int_{\Omega} u \cdot \nabla \mu(\rho) - \int_{\Omega} (\rho \mu'(\rho) - \mu(\rho)) |\text{div} u|^2 - \int_{\Omega} \mu(\rho) \sum_{ij} \partial_i u_j \partial_j u_i \\
+ \chi \int_{\Omega} \mu'(\rho) |\Delta \mu(\rho)|^2 + \frac{1}{2} \frac{d}{dt} \int_{\Omega} 2\rho |\nabla \varphi(\rho)|^2 + \int_{\Omega} \nabla p(\rho, \theta) \cdot \frac{\nabla \rho}{\rho} \mu'(\rho)
\]
\[
= \int_{\Omega} \rho f \cdot \nabla \varphi(\rho).
\]
By summing this last equation multiplied by 2 to the energy estimate (36), this gives (37).

### 4.3 Temperature estimates

Let us give there some temperature estimates that will be used to construct weak solutions.

Two different reformulations of the internal energy equations will lead to useful bounds on the temperature. The first one writes as

\[
C_v \left( \partial_t (\rho \theta) + \text{div} (\rho \theta u) + \Gamma \rho \theta \text{div} u \right) \\
= \rho f_{\text{int}} \cdot u + 2 \mu D(u) : D(u) + \lambda |\text{div} u|^2 + \text{div} (\kappa \nabla \theta),
\]

(45)

where the deviatoric part \( S(u) \) of the stain rate tensor \( D(u) \) is defined its the zero trace component: \( S(u) = D(u) - (\text{div} u) I/d \). Moreover, Maxwell’s equation (6) has been used, the Grüneisen coefficient \( \Gamma \) being defined by \( \Gamma = \rho^{-1} \partial p/\partial e |_{\rho} \) (see Section 9 for details about thermodynamical coefficients).

As a direct consequence of equation (46), we obtain the a.e. non negativity of the temperature \( \theta \) on \( \mathbb{R}_+ \times \Omega \), recalling that (17) holds. Indeed, using the assumption that \( \theta_0 \geq 0 \) a.e. on \( \Omega \) and the fact that the first two terms of the right hand side of (45) are non negative, the minimum principle formally applies to the temperature (recalling that the \( C_v \) coefficient is positive).

The second form of the internal energy equations is the most physically relevant, since it involves the specific entropy \( s \). Indeed, we deduce from the definition of \( s \) that it satisfies formally

\[
\theta (\partial (\rho s) + \text{div} (\rho us)) = \rho f_{\text{int}} \cdot u + 2 \mu D(u) : D(u) + \lambda |\text{div} u|^2 + \text{div} (\kappa \nabla \theta),
\]

(47)
hence dividing equation (47) by $\theta$ and integrating over $\Omega$, we end up with
\[
\int_\Omega \frac{1}{\theta} (\rho f_{int} \cdot u + 2\mu D(u) : D(u) + \lambda |\text{div} u|^2) + \int_\Omega \frac{\kappa}{\theta^2} |\nabla \theta|^2 = \frac{d}{dt} \int_\Omega \rho s. \tag{48}
\]
Therefore, the following proposition is proved, denoting $\omega_+ = \max(\omega, 0)$ and $\omega_- = \max(-\omega, 0)$ the positive and negative parts of $\omega \in \mathbb{R}$.

**Proposition 4.2** Assume that $\rho_0(s_0) \in L^1(\Omega)$. Then, for all $t > 0$, one has
\[
\int_0^t \int_\Omega \frac{1}{\theta} (\rho f_{int} \cdot u + 2\mu D(u) : D(u) + \lambda |\text{div} u|^2) + \int_0^t \int_\Omega \frac{\kappa}{\theta^2} |\nabla \theta|^2 \leq \int_\Omega (\rho s(t, \cdot) + \rho_0(s_0)). \tag{49}
\]

In the case of equations of state (20), the physical entropy writes in terms of $\rho$ and $\theta$ as $s = C_v \log(\theta/\rho^\gamma)$ (identical to the perfect polytropic gas case), where $\Gamma = \gamma - 1$. Introducing $\rho_\infty > 0$, we may write
\[
\rho s = C_v \rho \log \frac{\theta}{\rho_\infty} + \Gamma C_v \rho \log \frac{\rho_\infty}{\rho},
\]
so that
\[
\rho s \leq C_v \rho \theta + \Gamma C_v \rho \log \frac{\rho_\infty}{\rho},
\]
and the first term of right hand side of (49) is estimated by
\[
\int_\Omega \rho s_+(t, \cdot) \leq \int_\Omega C_v \rho \theta + \int_\Omega \Gamma C_v \rho \log \frac{\rho_\infty}{\rho}.
\]
It remains to control the last above term, which is done using the mass conservation equation in a renormalized way with $\beta_\infty(\rho) = \rho \log(\rho_\infty/\rho)$
\[
\partial_t \beta_\infty(\rho) + \text{div} (\beta_\infty(\rho) u) - \rho \text{div} u = 0,
\]
so that the right hand side of (49) is estimated by
\[
\int_\Omega \rho_0(s_0) + \int_\Omega \rho C_v \theta(t, \cdot) + \Gamma C_v \left( \int_\Omega \beta_\infty(\rho_0) + \int_0^t \int_\Omega \rho |\text{div} u| \right). \tag{50}
\]
The last term above can be estimated by the left hand side of (49) as follows
\[
\int_0^t \int_\Omega \rho |\text{div } u| \leq \int_0^t \int_\Omega \frac{\rho^{1/2}}{(d\lambda + 2\mu)^{1/2}} \frac{(d\lambda + 2\mu)^{1/2}}{\theta^{1/2}} |\text{div } u| \sqrt{\rho \theta},
\] (51)
and using the inequality \(ab \leq \varepsilon a^2 + C_\varepsilon b^2\) together with the bound of \(\rho \theta\) in \(L^\infty(0,T;L^1(\Omega))\) and assumptions (16) and (17) that ensures that \(s \mapsto s/(d\lambda(s) + 2\mu(s))\) belongs to \(L^\infty(\mathbb{R}_+)\) (since \(n \leq 1\) and \(m \geq 1\)).

Then, if \(\rho_0(s_0)\) and \(\rho_0 \log(\rho_\infty/\rho_0)\) belong to \(L^1(\Omega)\), then the components of the following three quantities \((d\lambda + 2\mu)^{1/2} |\text{div } u| \sqrt{\rho \theta}\) are a priori bounded in \(L^2((0,T) \times \Omega)\) as soon as \(\rho s_+\) is bounded in \(L^\infty(0,T;L^1(\Omega))\). Remark that the hypothesis on \(\rho_0 \log(\rho_\infty/\rho_0)\) is easily deduced from (29). Note that the last two bounds involving the temperature gradient provide the following useful estimates
\[
(\sqrt{\rho} + 1) \nabla \theta^\beta \in L^2((0,T) \times \Omega)^d \text{ for all } \beta \text{ such that } 0 \leq \beta \leq a/2 \quad (52)
\]
since \(a \geq 2\).

Another estimate derived from (45) will be useful when dealing with compactness properties of sequences of approximate solutions.

**Lemma 4.3** Let \(\theta\) be the temperature associated with the full compressible Navier–Stokes equations (1), (2), (3) and (9). Then, for any nondecreasing concave function \(f\) from \(\mathbb{R}_+\) to \(\mathbb{R}\), one has
\[
\int_\Omega \frac{f'(\theta)}{C_v} (\rho f_{\text{int}} \cdot u + 2\mu D(u): D(u) + \lambda |\text{div } u|^2) - \int_\Omega \kappa(\rho, \theta) \frac{f''(\theta)}{C_v} |\nabla \theta|^2 \leq \frac{d}{dt} \int_\Omega \rho \left( f(\theta) - \log \frac{\rho}{\rho_\infty} \right) + \int_\Omega \rho |(\Gamma f'(\theta) - 1) \text{div } u|. \quad (53)
\]

**Proof.** Estimates (53) on the temperature field are obtained by multiplying equation (45) by \(f'(\theta)/C_v\) and using the mass conservation equation for dealing with the ”\(\rho \text{div } u\)’’ term in the right hand side of the obtained estimate.

Let us observe that the classical physical entropy estimate (48) is recovered in the perfect gas equation of state up to a multiplicative constant by taking \(f(\theta) = C_v \Gamma^{-1} \log \theta\) (in that case, \(\Gamma = \gamma - 1\) and the additional term in the right hand side of (53) vanishes). Still in the perfect polytropic gas case, the entropy is given by \(s = C_v \log(\theta/\rho^\gamma - 1)\), which simplifies much the analysis of entropy regularity.
4.4 Consequences

The additional estimates are deduced from Lemma 4.1 and Lemma 4.3, namely estimates (36), (37) taking $\chi = 0$ and (53):

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |u|^2 + \int_{\Omega} 2\mu(\rho) D(u) : D(u) + \int_{\Omega} \lambda(\rho) |\text{div} u|^2 \bigg|_{u} = \int_{\Omega} p(\rho, \theta) \text{div} u + \int_{\Omega} \rho f \cdot u.$$  \hspace{0.5cm} (54)

and

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |u + 2\nabla \varphi(\rho)|^2 + \int_{\Omega} 2\mu(\rho) A(u) : A(u) = -\int_{\Omega} \nabla p(\rho, \theta) \cdot (u + 2\nabla \varphi(\rho)) + \int_{\Omega} \rho f \cdot \nabla \varphi(\rho),$$  \hspace{0.5cm} (55)

where $A(u) = (\nabla u - i\nabla u)/2$ denotes the skew symmetric part of $\nabla u$, and $\varphi$ is defined up to a constant by $\varphi'(\tau) = \mu'(\tau)/\tau$ ($\tau > 0$). Adding (54) and (55), we have to control the following terms

$$\int_{\Omega} p \text{div} u, \quad \int_{\Omega} \nabla p \cdot \nabla \varphi(\rho), \quad \int_{\Omega} \rho f \cdot u, \quad \int_{\Omega} \rho f \cdot \nabla \varphi(\rho).$$  \hspace{0.5cm} (56)

4.4.1 Control of $\int_{\Omega} \nabla p \cdot \nabla \varphi$

In order to bound such an integral, the thermodynamical properties of the fluid have to be used. Defining the "hot" pressure and energy components as $p_h(\rho, \theta) = p(\rho, \theta) - p_c(\rho)$, and $e_h(\rho, \theta) = e(\rho, \theta) - e_c(\rho)$, i.e. the pressure and energy associated with non zero temperature effects, we introduce the Gr"uneisen coefficient $\Gamma$, the volume expansion coefficient at constant pressure $\alpha_p$, the compressibility at constant temperature $Z$, the specific heat at constant pressure $C_p$ and the auxiliary thermodynamical coefficient $r$

$$\Gamma = \frac{1}{\rho} \frac{\partial p}{\partial e} \bigg|_{\rho}, \quad \alpha_p = \frac{\partial \rho}{\partial \theta} \bigg|_{p_h}, \quad Z = \frac{p_h}{\rho} \frac{\partial \rho}{\partial p_h} \bigg|_{\theta}, \quad C_p = \frac{\partial (e_h + p_h)}{\partial \theta} \bigg|_{p_h}, \quad r = \frac{\Gamma C_v}{\alpha_p}.$$

(57)

Notice that in the case of an equation of state satisfying (20) (21)(22)(23) with coefficients $\gamma > 1$ and $C_v$, the preceding five coefficients are constant and given by $\Gamma = \gamma - 1$, $\alpha_p = 1$, $Z = 1$, $C_p = \gamma C_v$ and $r$ denotes the classical
perfect gas constant \( r = R/M \), where \( R \sim 8.31 \) and \( M \) denotes the molar mass of the gas). Then, tedious but straightforward computations lead to the differential identity valid for any equation of state

\[
\nabla p = \nabla p_c + r (\theta \nabla \rho + \alpha_p \rho \nabla \theta).
\]

Let us also observe that \( p = p_c + Z \rho r \theta \), which may be used later on.

Hence, expanding as in (58) the pressure gradient in the first term of (56), we recover on the one hand the following integral at the left hand side of (55)

\[
\int r \theta \mu' \left( \frac{\nabla \rho}{\rho} \right)^2,
\]

which may be used later on for the study of compactness properties. The cold component of the pressure also yields the following integral for some positive constant \( c_0 \)

\[
\int p_c' \mu' \left( \frac{\nabla \rho}{\rho} \right)^2 \geq c_0 \int r \theta \left( \frac{\nabla \zeta}{\theta} \right)^2 - \frac{1}{\tau_s} \int \left( \frac{\nabla \mu}{\rho} \right)^2,
\]

the lower bound being deduced from assumption (23) and \( \zeta \) being taken such that \( \zeta(\rho) = \rho \) for \( \rho \leq \rho_\ast/2 \) and \( \zeta(\rho) = 0 \) for \( \rho \geq \rho_\ast \). On the other hand, one has to control

\[
\int r \alpha_p \mu' \nabla \theta \cdot \nabla \rho,
\]

which may be done by using Cauchy–Schwartz type inequalities:

\[
|r \alpha_p \mu' \nabla \theta \cdot \nabla \rho| \leq \kappa(\rho, \theta) \frac{\nabla \theta^2}{\theta^2} + C \frac{\rho \theta^2 \alpha_p^2}{\kappa(\rho, \theta)} \frac{\nabla \mu^2}{\rho}
\]

so that, integrating (62) over \( \Omega \), we deal with the second term of the right hand side by a Gronwall–type argument, since (in view of Assumptions (18) (19) on \( \kappa \))

\[
\rho \theta^2 \alpha_p^2 \leq C \kappa(\rho, \theta),
\]

whereas the first is already controlled since, due to (52),

\[
\int_0^T \int \kappa \frac{\nabla \theta^2}{\theta^2} \leq C_T.
\]
4.4.2 Control of $\int_\Omega p \text{div} u$

It remains to bound the second term of (56), namely

$$\int_\Omega p(\rho, \theta) \text{div} u = \int_\Omega \rho^2 \frac{d\varepsilon}{d\rho}(\rho) \text{div} u + \int_\Omega Z \rho \theta \text{div} u,$$

$$= -\frac{d}{dt} \int_\Omega \rho \varepsilon(\rho) + \int_\Omega Z \rho \theta \text{div} u.$$

In the case of a perfect gas equation of state (constant $r$ and $Z = 1$), we split the integrated expression into bounded and unbounded densities (recalling that $n \leq 1 \leq m$)

$$\left| \int_\Omega \rho r \theta \text{div} u \right| \leq C r \|(d\lambda + 2\mu)^{1/2} \text{div} u\|_{L^2(\Omega)}$$

$$\times \left( \|\rho^{2/5} \theta\|_{L^2(\Omega)} \|\rho^{(6-5n)/10} 1_{\rho < A}\|_{L^\infty(\Omega)} + A^{-m/2} \|\rho 1_{\rho \geq A}\|_{L^3(\Omega)} \|\theta\|_{L^6(\Omega)} \right), \quad (63)$$

$$\leq C r \|(d\lambda + 2\mu)^{1/2} \text{div} u\|_{L^2(\Omega)}$$

$$\times \left( \|\rho \theta\|_{L^1(\Omega)}^{2/5} \|\theta\|_{L^6(\Omega)}^{3/5} A^{(6-5n)/10} \right.$$

$$\left. + A^{(4-5m)/2} \|\rho 1_{\rho > A}\|_{L^{6m-3}(\Omega)} \|\theta\|_{L^6(\Omega)} \right), \quad (64)$$

Now using Inequality (85) proved in Section 6, we deduce that (where $\eta = 1$ when $\Omega = T^d$, $\eta = 0$ otherwise)

$$\left| \int_\Omega \rho r \theta \text{div} u \right| \leq C \|(d\lambda + 2\mu)^{1/2} \text{div} u\|_{L^2(\Omega)} \left( \|\rho \theta\|_{L^1(\Omega)} + \|\theta\|_{L^6(\Omega)} \right)$$

$$\times \left( 1 + \|\theta\|_{L^6(\Omega)} \left( \frac{\|\nabla \mu(\rho)\|}{\sqrt{\rho}} \|\theta\|_{L^2(\Omega)} + \eta \|\rho\|_{L^1(\Omega)} \right) \right),$$

$$\leq \varepsilon \|(d\lambda + 2\mu)^{1/2} \text{div} u\|_{L^2(\Omega)}^2 + \frac{C}{\varepsilon} \|\rho \theta\|_{L^1(\Omega)}^2$$

$$+ \frac{C}{\varepsilon} \left( 1 + \|\theta\|_{L^6(\Omega)}^2 \right) \left( \frac{\|\nabla \mu(\rho)\|}{\sqrt{\rho}} \|\theta\|_{L^2(\Omega)}^2 + \eta \|\rho\|_{L^1(\Omega)}^2 \right),$$

for all positive $\varepsilon$. Using (52) together with Sobolev embeddings (see Section 4.3), we deduce that $t \mapsto \|\theta(t, \cdot)\|_{L^6(\Omega)}^2$ is a priori bounded in $L^1_{\text{loc}}(\mathbb{R}^+)$. Taking
ε small enough in order to absorb $\varepsilon \| (d\lambda + 2\mu)^{1/2} \text{div} u \|^2_{L^2(\Omega)}$ by the left hand side coming from viscous dissipation in (54), observing that $\rho \theta$ is already known to belong to $L^\infty_{\text{loc}}(\mathbb{R}_+; L^1(\Omega))$, the third term will be estimated by a Gronwall type lemma.

4.4.3 A priori estimates

Let us summarize the a priori bounds previously obtained: given any $T > 0$, the "generic" constant $C$ depends only on $T$ and on the bounds (28)–(30) on the initial data when exterior bulk forces vanish $f_{\text{ext}} \equiv 0$. First, the right hand side of the total energy (34) estimate only involve the initial data, and therefore do not need additional work. As a matter of fact, one has

$$\int_\Omega \rho \left( e_v(\rho) + C_v \theta + \frac{|u|^2}{2} \right) (t, \cdot) \leq \int_\Omega \left( G_0 + \frac{|m_0|^2}{2\rho_0} \right).$$  \hspace{1cm} (65)$$

On the other hand, the entropy estimate (49) combined with (50) and (51) yields

$$\int_0^t \int_\Omega \frac{1}{\theta} \left( \rho f_{\text{int}} \cdot u + 2\mu S(u) : S(u) + \left( \lambda + \frac{2\mu}{d} \right) |\text{div} u|^2 \right) (s, \cdot) ds$$

$$\quad + \int_0^t \int_\Omega \frac{\kappa |\nabla \theta|^2}{\theta^2} (s, \cdot) ds \leq \frac{1}{2} \int_0^t \int_\Omega \frac{1}{\theta} \left( \lambda + \frac{2\mu}{d} \right) |\text{div} u|^2 (s, \cdot) ds$$

$$\quad + C \int_0^t \int_\Omega C_v \rho \theta (s, \cdot) ds + \int_\Omega \rho C_v \theta (t, \cdot)$$

$$\quad + \int_\Omega \rho_0 (s_0)_- + \Gamma C_v \int_\Omega \beta_\infty (\rho_0).$$  \hspace{1cm} (66)$$

As a consequence, summing (65) and 1/2 times (66), we deduce that for all $T > 0$, there exists $C_T > 0$ such that for all $t \in (0, T)$, one has

$$\int_\Omega \rho \left( e_v(\rho) + C_v \theta + \frac{|u|^2}{2} \right) (t, \cdot) + \int_0^t \int_\Omega \frac{\kappa |\nabla \theta|^2}{\theta^2} (s, \cdot) ds$$

$$\quad + \int_0^t \int_\Omega \frac{1}{\theta} \left( \rho f_{\text{int}} \cdot u + 2\mu S(u) : S(u) + \left( \lambda + \frac{2\mu}{d} \right) |\text{div} u|^2 \right) (s, \cdot) ds$$

$$\leq C_T \int_\Omega \left( G_0 + \frac{|m_0|^2}{2\rho_0} + \rho_0 (s_0)_- + \Gamma C_v \beta_\infty (\rho_0) \right).$$  \hspace{1cm} (67)$$

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The other main estimate is obtained by adding (54) and (55), and using estimates (60) (62) derived in Subsections 4.4.1 and 4.4.2

\[
\int_\Omega \left( \frac{1}{2} \rho |u|^2 + \rho e_c(\rho) + \frac{1}{2} \rho |u + 2 \nabla \varphi(\rho)|^2 \right) (t, \cdot) + \int_0^t \int_\Omega \left( 2 \mu(\rho) |\nabla u|^2 + \lambda(\rho) |\nabla u|^2 + \rho u \cdot f_{int} + \frac{\rho \theta}{\mu'(\rho)} |\nabla \varphi(\rho)|^2 + c_0 |\nabla \zeta(\rho)^{-\frac{4+\beta-n}{2}}|^2 \right) (s, \cdot) ds
\]

\[
\leq \int_0^t \left( \frac{|m_0|^2}{2 \rho_0} + \rho_0 e_c(\rho_0) + \frac{|m_0 + 2 \nabla \mu(\rho_0)|^2}{2 \rho_0} \right)
\]

\[
+ C \int_0^t \int_\Omega \left( 1 + \frac{1}{T_*} + \|\theta(s, \cdot)\|_{L^2(\Omega)}^2 \right) \rho |\nabla \varphi(\rho)|^2 (s, \cdot) ds
\]

\[
+ C \int_0^t \int_\Omega \left( 1 + \|\theta(s, \cdot)\|_{L^2(\Omega)} \right) \|\rho(s, \cdot)\|_{L^1(\Omega)}^2 + C \int_0^t \|\rho \theta(s, \cdot)\|_{L^1(\Omega)} ds
\]

\[
+ \frac{1}{2} \int_0^t \int_\Omega \left( \lambda + \frac{2 \mu}{d} \right) |\nabla u|^2 (s, \cdot) ds,
\]

(68)

where again \( \eta = 0 \) in the whole space case and \( \eta = 1 \) otherwise. Gronwall’s lemma over \( (0, T) \) therefore yields the a priori estimates in the case when \( f_{int} \equiv 0 \)

\[
\| \sqrt{\rho} u \|_{L^\infty(0,T;L^2(\Omega))} \leq C_T, \quad \| \rho^{-\frac{3}{2}} |\nabla \mu(\rho)| \|_{L^\infty(0,T;L^2(\Omega))} \leq C_T, \quad (69)
\]

\[
\| (\rho^2 + \rho^2) \nabla u \|_{L^2((0,T) \times \Omega)} \leq C_T, \quad \| \rho^2 + \rho^2 |\nabla u|^2 \|_{L^2((0,T) \times \Omega)} \leq C_T, \quad (70)
\]

\[
\left\| \left( r \theta \right)^{\frac{1}{2}} (\rho^2 + \rho^2) \nabla u \right\|_{L^2((0,T) \times \Omega)} \leq C_T, \quad \| \rho e_c(\rho) \|_{L^\infty(0,T;L^1(\Omega))} \leq C_T, \quad (71)
\]

\[
\left\| \sqrt{\frac{\rho \theta}{\mu'(\rho)}} \nabla \varphi(\rho) \right\|_{L^2((0,T) \times \Omega)} \leq C_T, \quad \| \rho \theta \|_{L^\infty(0,T;L^1(\Omega))} \leq C_T, \quad (72)
\]

\[
\| \nabla \zeta(\rho)^{-\frac{4+\beta-n}{2}} \|_{L^2((0,T) \times \Omega)} \leq C_T, \quad \| (1 + \rho) \frac{3}{2} \nabla \theta^\beta \|_{L^2((0,T) \times \Omega)} \leq C_T \quad (73)
\]

for all \( \beta \in [0, a/2] \).

Without additional assumptions either on the zero temperature internal energy curve or on the internal forces \( f_{int} \), such bounds do not seem to be sufficient as such to pass to the limit in non linear terms in the sense of
distributions, when it comes to compactness of sequences of approximate solutions. In particular the flux of kinetic energy requires that $\rho^{1/3}u$ belongs at least to $L^3$ locally in space and time. This problem may be overcome by two different strategies.

The first one is to make assumptions on the zero Kelvin isothermal of the equation of state in the neighborhood of small densities. The physical justification is that the fluid assumption is no longer relevant in a regime associated with zero Kelvin temperature (since matter generally change to solid state) and small densities (rarefied situation). This is the default assumption considered in the proof.

Another way to solve the problem is to take account of more Physics with an additional internal force in the momentum equation. For instance, a drag force of the form

$$f_{\text{int}} = -|u|^{b-2}u,$$

may be considered in the momentum equation, where $3 < b < 4$ has to be suitably chosen (see Section 8).

\section{Some integrability Lemmas}

As mentioned in \cite{43}, the lack of a priori bounds on solutions to the full compressible Navier–Stokes equations is the main difficulty to prove the existence of global in time weak solutions. Indeed the only natural a priori estimates are not sufficient, since the energy equation does not make sense so far even in the distribution theory framework.

This difficulty has been circumvented in \cite{25} \cite{24} by restricting the generality of the constitutive law for large densities and for large (and small) temperatures, and defining variational solutions for which the energy equation becomes an inequality in the sense of distributions. However, this result requires significant restrictions on the equation of state, in particular the ideal gas case is not covered.

This section is devoted to the local integrability analysis of the various energy fluxes such as $\rho u|u|^2$, $\rho u e$, $u \nu \nabla \theta$, assuming several integrability properties on the density $\rho$, $u$ and $\theta$, that will be proved in the next section. One of the key points is the additional integrability obtained on $\rho$. 26
5.1 Kinetic energy flux

Let us begin with some bounds on the velocity with density dependent weights.

**Lemma 5.1** Let $\Omega$ be either the whole space $\mathbb{R}^3$ or the three dimensional periodic box $T^3$ and let $T > 0$. Let $u$ be a vector field over $(0, T) \times \Omega$ such that $u \in L^q(0, T; L^2_{\text{loc}}(\Omega))$, $\sqrt{\rho} u \in L^2(0, T; L^2_{\text{loc}}(\Omega))$, and $\rho \in L^\infty(0, T; L^p_{\text{loc}}(\Omega))$ such that

$$q_1 \in (1, 2), \quad \frac{1}{p} + \frac{2q_1}{q_2(q_1 - 1)} < 1. \quad (75)$$

Then, there exists $\delta > 3$ such that $\rho^{1/3} u \in L^\delta((0, T) \times B)$ for all bounded subset $B$ in $\Omega$.

**Proof:** the estimate simply relies on the observation that

$$\rho^{1/3}|u| = \rho^{1/3 - \alpha} \rho^\alpha |u|^{2\alpha} |u|^{1 - 2\alpha} \quad \text{where} \quad \alpha \in [0, 1/3] \quad \text{has to be chosen.}$$

Then, given a bounded subset $B$ in $\Omega$, one has the estimate

$$\|\rho^{1/3} u\|_{L^s(0, T; L^r(B))} \leq \|\rho\|_{L^\infty(0, T; L^p(B))} \|\sqrt{\rho} u\|_{L^2(0, T; L^2(B))} \|u\|_{L^{q_1}(0, T; L^{q_2}(B))}^{1 - 2\alpha},$$

where $s$ and $r$ are given by

$$\frac{1}{s} = 1 - 2\alpha \quad \text{and} \quad \frac{1}{r} = \frac{1}{p} \left(\frac{1}{3} - \alpha\right) + \alpha + \frac{1 - 2\alpha}{q_2},$$

hence $s > 3$ if

$$\alpha > \frac{3 - q_1}{6},$$

and $r > 3$ if

$$3\alpha(pq_2 - 2p - q_2) < pq_2 - 3p - q_2.$$ Such an $\alpha$ exists if condition (75) is satisfied, which completes the proof of Lemma 5.1.

An application of the above lemma provides a constraint between the coefficients $m$, $n$ and $\ell$. Indeed, one may write $\nabla u = \rho^{-n/2} \rho^n/2 \nabla u$, so that the following estimate holds for all bounded subset $B$ of $\Omega$

$$\|\nabla u\|_{L^q(0, T; L^p(B))} \leq C_B \left(1 + \|
abla \zeta(\rho)^{-\frac{n}{2}}\|_{L^2((0, T); L^2(B))}\right) \|\rho^{\frac{n}{2}} \nabla u\|_{L^2((0, T) \times \Omega)},$$

$$\leq C_B \left(1 + \|
abla \zeta(\rho)^{-\frac{n+1}{2}}\|_{L^2((0, T) \times \Omega)}\right) \times \|\rho^{\frac{n}{2}} \nabla u\|_{L^2((0, T) \times \Omega)}, \quad (76)$$
where
\[ j = \frac{\ell + 1 - n}{n}, \quad q_1 = \frac{2j}{j+1} = 2 \left( 1 - \frac{n}{\ell+1} \right) \quad \text{and} \quad \frac{1}{q_3} = \frac{1}{6j} + \frac{1}{2}, \]
so that Lemma 5.1 can be applied with \( q_2 = (3q_3)/(3 - q_3) = 3q_1 \). Then, taking \( p = 6m - 3 \), condition (75) is satisfied if
\[ \ell > \frac{2n(3m-2)}{m-1} - 1. \]
(77)
Notice that since \( m > 1 \) and \( n > 2/3 \), one always has \( \ell > 3 \).

As a byproduct of the preceding analysis, we also derive useful bounds for subsequent estimates on energy fluxes. More precisely, one has for all bounded subset \( B \) in \( \Omega \)
\[ \| \rho^{-\ell} v \|_{L^s(0,T;L^r(B))} \leq C \| \rho^{-\ell} \|_{L^{\infty}(0,T;L^1(B))} \| \rho^{-\ell} \|_{L^{j_1}(0,T;L^{r_1}(B))} \| u \|_{L^{q_1}(0,T;L^{q_2}(B))}, \]
(78)
where
\[ j_1 = \frac{\ell + 1 - n}{2\ell}, \quad \frac{1}{s} = \frac{1}{6j_1} + \frac{1}{q_1} = \frac{5\ell + 3}{6(\ell + 1 - n)}, \]
and \( \frac{1}{r} = \frac{2}{3} + \frac{1}{18j_1} + \frac{1}{3q_1} = \frac{17\ell + 15 - 12n}{18(\ell + 1 - n)} \).

Notice that \( s > 1 \) and \( r > 1 \) if and only if \( \ell > 3(2n - 1) \), which is already satisfied if (77) holds.

In order to bound the energy fluxes such as \( \rho u e_c(\rho) \) and \( p_c(\rho) u \), it remains to control \( \rho^k u \) in \( L^k(0,T;L^j_{\text{loc}}(\Omega)) \) for some \( \delta > 1 \). Since \( u \) is bounded in \( L^{q_1}(0,T;L^{q_2}_{\text{loc}}(\Omega)) \) and \( \rho^k \) bounded in \( L^{\infty}(0,T;L^{6m-3/k}_{\text{loc}}(\Omega)) \), it suffices to require that \((6m-3)/k > q_2/(q_2 - 1)\), in other words
\[ k < \left( m - \frac{1}{2} \right) \frac{5(\ell + 1) - 6n}{\ell + 1 - n}, \]
which writes exactly as condition (23).

### 5.2 Convection flux of internal energy

In order to bound the first part of the internal energy flux \( \rho u \theta \) in \( L^\delta_{\text{loc}}(R^+ \times \Omega) \) for some \( \delta > 1 \), Lemma 5.1 may be used by observing that:
\[ \rho u \theta = \rho^{1/3} u \rho^{2/3} \theta, \]
so that it suffices to prove that $\rho^{3/2}$ is bounded in $L^1((0, T) \times B)$ for all $T > 0$ and all bounded subset $B$ of $\Omega$. This bound is clear since $\theta^{3/2}$ is bounded in $L^{2a/3}(0, T; L^{2a}(B))$ with $a \geq 2$ and $\rho$ is bounded in $L^\infty((0, T) \times \Omega) + L^\infty((0, T; L^{6m-3}(B)))$ ($2a/3 \geq 4/3$ and $1/(6m-3) + 1/(2a) < 1$ since $m > 1$).

The other part of the internal energy flux is of the form $\rho \partial_c(\rho) u$ or $p_c(\rho) u$, and is therefore bounded in $L^2_{\text{loc}}(\mathbb{R}^+ \times \Omega)$ for some $\delta > 1$ in view of inequality (78).

### 5.3 Heat flux

In order to give a local integrability result on the heat flux $\kappa \nabla \theta$, preliminary bounds deduced from Lemma 4.3 are needed.

As a matter of fact, we consider the nonnegative concave nondecreasing function $f$ in Lemma 4.3 defined by $f'(\theta) = 1/\theta c$, with some suitable coefficient $c > 0$. One gets

$$
\int_\Omega \frac{\theta^c}{C_v} \left( \rho f_{\text{int}} \cdot u + 2\mu D(u) : D(u) + \lambda |\text{div} u|^2 \right) + \int_\Omega \frac{c}{C_v} \kappa(\rho, \theta) \theta^{-c-1} |\nabla \theta|^2
\leq \frac{d}{dt} \int_\Omega \rho \left( \theta^{1-c} - \log \rho \right) + \int_\Omega \rho |(\Gamma)^{1-c} - 1| \text{div} u| \tag{79}
$$

Now using the fact that the conductivity $\kappa$ is of the form $(\rho, \theta) \mapsto \kappa_o(\rho, \theta)(1 + \rho)(\theta^a + 1)$ for some $\kappa_o$ bounded from below, we conclude that the term generated by the thermal conductivity in the left hand side of (79) is bounded from above by

$$
\int_\Omega \kappa_o(\rho, \theta)(\rho + 1) |\nabla \theta^{a-c+1/2}|^2.
$$

**Control of $\int \rho f'(\theta) \theta \text{div} u$.** The derivation of estimates on this term easily follows the same lines as $\int \rho \text{div} u$ using (52) with $\beta = 1 - c$.

**Lemma 5.2** Let $\Omega$ be either the whole space $\mathbb{R}^3$ or a three dimensional periodic and let $T > 0$. Let $\theta$ satisfying the hypothesis of Lemma 6.3 and $\kappa$ be given by (18) and $(\sqrt{\theta} + 1) \nabla \theta^{a-c+1/2} \in L^2(0, T; L^2(\Omega))$, $\theta^{a-c+1/2} \in L^2(0, T; L^3(\Omega))$ and $\rho \in L^\infty(0, T; L^{6m-3}(\Omega))$. Then, we get, for some $p > 1$,

$$
\kappa(\rho, \theta) \nabla \theta \in L^p(0, T; L^p(\Omega)).
$$
Proof: Let us study the term \( \rho \theta^a \nabla \theta \). The other term \( \theta^a \nabla \theta \) is treated in a similar way. The other terms \( (\rho + 1) \nabla \theta \) is straightforward. The term \( \rho \theta^a \nabla \theta \) may be written

\[
\rho \theta^a \nabla \theta = \rho^{1/2} \theta^{(a+c+1)/2} \rho^{1/2} \theta^{(a-c-1)/2} \nabla \theta,
\]

so that it remains to control \( \rho^{1/2} \theta^{(a+c+1)/2} \) in \( L^p(0, T; L^p(\Omega)) \) for some \( p > 2 \), i.e. \( \rho \theta^{a+c+1} \) in \( L^p(0, T; L^p(\Omega)) \) for some \( p > 1 \). Using the \( L^\infty(0, T; L^1(\Omega)) \) bound on \( \rho \theta \) and writing \( \rho \theta^{a+c+1} \) as \( (\rho \theta)^\beta \rho^{1-\beta} \theta^{a+c+1-\beta} \), one gets, using the previous lemma, a \( L^p(0, T; L^q(\Omega)) \) bound on \( \rho \theta^{a+c+1} \), where

\[
\frac{1}{p} = \frac{a + c + 1 - \beta}{a - c + 1}, \quad \frac{1}{q} = \beta + \frac{2(a + c + 1 - \beta)}{3(a - c + 1)} + \frac{1 - \beta}{6m - 3}.
\]

As a result, \( p > 1 \) if and only if

\[
2c < \beta \tag{80}
\]

and \( q > 1 \) if and only if

\[
\beta((a-c)(6m-4)+2m-2) < 2(a+c+1)(1-2m)+(6m-4)(a-c+1). \tag{81}
\]

Concerning the term \( (\rho + 1) \nabla \theta \), the regularity is straightforward. Indeed \( (\sqrt{\rho} + 1) \nabla \theta \in L^2(0, T; L^2(\Omega)) \) and \( \sqrt{\rho} \in L^\infty(0, T; L^{12m-6}(\Omega)) \).

6 Auxiliary a priori estimates

This section is devoted to auxiliary a priori estimates that are used in Section 4, using mathematical tools such as Sobolev embeddings and interpolation tools. Additional useful bounds on derivatives of the density \( \rho \), velocity \( u \) and temperature \( \theta \) and precise dependence on the initial data is given. This will allow us to use the previous section to construct global weak solutions of the full compressible Navier–Stokes equations.

6.1 Bounds on the density

In this section, \( L^p \) bounds on the density are derived from estimates on the gradient of suitable functions of \( \rho \). In both cases (whole space and periodic domain), we shall use Hypothesis (17) on \( s\mu'(s) \). The above property can be
used to prove that for any $C^1$ function $\beta : \mathbb{R}^*_+ \to \mathbb{R}_+$ such that for all $s > 0$, \( \sqrt{s} \beta'(s) \leq c_0 \mu'(s) \) for some positive constant $c_0$, the a priori estimate holds

$$\| \nabla \beta(\rho) \|_{L^2(\Omega)} \leq c_0 \left\| \frac{\nabla \mu(\rho)}{\sqrt{\rho}} \right\|_{L^2(\Omega)}.$$  \hfill (82)

In particular, (82) may be applied to control the density close to vacuum by choosing a smooth $\beta$ function vanishing for large enough $s$ and such that $\beta(s) = s^{-1/2}$ when $s \in (0, B)$.

In order to deal with large or vanishing densities $\rho$, the whole space and the periodic box will be treated separately.

The periodic case: the mean value $\overline{f}$ of $f \in D'(\Omega)$ will be denoted by

$$\overline{f} = \frac{1}{|T^d|} \int_{T^d} f(x) \, dx.$$  

The main a priori estimates in the periodic case read as follows

**Lemma 6.1** Let $\Omega$ be the three dimensional periodic box $T^3$ and let $T > 0$. One has for all $\rho \in L^1(\Omega)$ such that $\rho^{-1/2} \nabla \mu(\rho) \in L^2(\Omega)$

$$\| \rho^{m-1/2} \chi_{\rho > A} \|_{L^6(\Omega)} \leq C \left( \left\| \frac{\nabla \mu(\rho)}{\sqrt{\rho}} \right\|_{L^2(\Omega)} + \| \rho \|_{L^1(\Omega)}^{m-1/2} \right),$$  \hfill (83)

$$\| \rho^{m-1/2} \chi_{\rho < A} \|_{L^6(\Omega)} \leq C \left( \left\| \frac{\nabla \mu(\rho)}{\sqrt{\rho}} \right\|_{L^2(\Omega)} + \| \rho \|_{L^1(\Omega)}^{m-1/2} \right).$$  \hfill (84)

**Proof:** let us consider (82) with $\beta \in C^1(0, +\infty)$ such that $\beta = \xi^{m-1/2}$, where $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth function such that $\xi(s) = s$ for $s \geq A$ and $\xi(s) = 0$ for $s \leq A/2$. Then, Sobolev embeddings yield

$$\| \beta(\rho) - \overline{\beta(\rho)} \|_{L^6(\Omega)} \leq C \left\| \frac{\nabla \mu(\rho)}{\sqrt{\rho}} \right\|_{L^2(\Omega)},$$

so that it remains to estimate $|\overline{\beta(\rho)}|$. When $m - 1/2 \leq 1$, Jensen’s inequality provides

$$|\overline{\beta(\rho)}| \leq C \| \rho \|_{L^1(\Omega)}^{m-1/2},$$
whereas if \( m > 3/2 \), basic Hölder type interpolation provides for some \( \alpha \in (0, 1) \)
\[
|\beta(\rho)| \leq \|\beta(\rho)\|_{L^6(\Omega)}^{1-\alpha} \|\xi(\rho)\|_{L^1(\Omega)}^{(1-\alpha)(m-1/2)},
\]
\[
\leq C_e \|\rho\|_{L^1(\Omega)}^{m-1/2} + \|\beta(\rho)\|_{L^6(\Omega)},
\]
which allows to conclude taking small enough \( \varepsilon > 0 \).

In order to prove (84), a similar approach may be applied with some smooth function \( \xi: \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \xi(s) = s \) for \( s \in [0, A] \) and \( \xi(s) = 0 \) for \( s \geq 2A \). Taking \( \beta = \xi^{n-1/2} \), one deduces from Jensen’s inequality that
\[
|\beta(\rho)| \leq C \|\rho\|_{L^1(\Omega)}^{1/2},
\]
so that using again Sobolev embeddings (82) allows to prove (84).

The whole space case: in the whole space \( \Omega = \mathbb{R}^3 \), one has

**Lemma 6.2** Let \( \Omega \) be the whole space \( \mathbb{R}^3 \) and let \( T > 0 \). For all \( \rho \) satisfying \( \rho^{-1/2} \nabla \mu(\rho) \in L^2(\Omega) \), one has
\[
\|\rho^{n-1/2} 1_{\rho > A}\|_{L^6(\Omega)} \leq C \left\| \frac{\nabla \mu(\rho)}{\sqrt{\rho}} \right\|_{L^2(\Omega)} \quad \text{(85)}
\]
\[
\|\rho^{n-1/2} 1_{\rho < A}\|_{L^6(\Omega)} \leq C \left\| \frac{\nabla \mu(\rho)}{\sqrt{\rho}} \right\|_{L^2(\Omega)} \quad \text{(86)}
\]

**Proof:** inequality (85) is a consequence of the 3d Sobolev embeddings in (82) by taking suitable function \( \beta = \xi^{m-1/2} \) in (82) (where \( \xi: \mathbb{R}_+ \to \mathbb{R}_+ \) satisfies \( \xi(s) = s \) for \( s \geq A \) and \( \xi(s) = 0 \) for \( s \leq A/2 \)). The proof of (86) follows the same lines as in the periodic case: introducing \( \xi: \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \xi(s) = s \) for \( s \in [0, A] \) and \( \xi(s) = 0 \) for \( s \geq 2A \), Sobolev embeddings allows to conclude.

Let us observe that the preceding bounds imply that \( \rho^{-1/2} \mu(\rho) \) is bounded in \( L^6(B) \) for all bounded subset \( B \) of \( \Omega \), as soon as the right hand sides of Lemmas 6.1 and 6.2 are bounded.
6.2 Bounds on the temperature

Let us consider the perfect polytropic gas case, that means $p = \rho r \theta$ and $e = C_v \theta$. For the sake of simplicity, we focus on the dimension $d = 3$ in the periodic and whole space setting. The cases of $d = 2$ induce the same bounds up to the fact that bounds only hold locally in space when $\Omega = \mathbb{R}^2$.

Lemma 6.3 Let $\Omega$ be either the whole space $\mathbb{R}^3$ or a three-dimensional periodic box and let $T > 0$. Let $\theta$ be a vector field over $(0, T) \times \Omega$ such that $(\sqrt{\rho} + 1)|\nabla \theta|^{\alpha/2}$ and $(\sqrt{\rho} + 1)|\log \theta$ belong to $L^2((0, T); L^2(\Omega))$ with $\alpha \geq 2$, $\rho e \in L^\infty(0, T; L^1(\Omega))$, and let $\rho$ satisfying hypothesis of Lemma 6.2. Then $\theta^{(\alpha-c+1)/2}$ belongs to $L^2(0, T; L^6(\Omega))$ for all $0 < c \leq 1$.

Proof: In the periodic case, the first estimates on $\theta$ follows the same lines of the estimate on $u$ since for a perfect gas $\rho e = C_v \rho \theta$ belongs to $L^\infty(0, T; L^1(\Omega))$. Using the perfect gas assumption, one gets

$$|\theta| \leq C |\bar{e}| \leq \frac{C}{\rho} \left( \|\rho e\|_{L^2(\Omega)} \|\nabla \theta\|_{L^2(\Omega)} \right)$$

which belongs to $L^2(0, T)$. Since the norm $f \mapsto \|\nabla f\|_{L^2(\Omega)} + |\bar{f}|$ is equivalent to the $H^1$ norm $f \mapsto \|f\|_{H^1(\Omega)}$ and

$$\int_\Omega \rho_0 \bar{f} \leq \int_\Omega \rho f - \int_\Omega \rho (f - \bar{f}),$$

we get that the $H^1$ norm $f \mapsto \|f\|_{H^1(\Omega)}$ is equivalent to the following norm

$$f \mapsto \|\nabla f\|_{L^2(\Omega)} + \left| \int_\Omega \rho f \right|,$$

using that $\rho \in L^\infty(0, T; L^{q/(q-1)}(\Omega))$. This gives

$$\|\theta^{\alpha/2}\|_{H^1(\Omega)} \leq c \left( \|\nabla \theta^{\alpha/2}\|_{L^2(\Omega)} + \left| \int_\Omega \rho \theta^{\alpha/2} \right| \right) \leq c \left( \|\nabla \theta^{\alpha/2}\|_{L^2(\Omega)} + \|\rho \theta\|_{L^1(\Omega)}^{\alpha} \|\rho\|_{L^{q/(q-1)}(\Omega)}^{1-\alpha} \|\theta^{\alpha/2}\|_{L^{p'}(\Omega)} \right),$$

where

$$\alpha + \frac{1-\alpha}{q'} + \frac{1}{p'} = 1,$$

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and $\alpha$ is chosen small enough in order to ensure the property $(a/2 - \alpha)2p/a < 2d/(d - 2)$ (indeed if $\alpha = 0$, $p' = q'(q' - 1)$ and $q'(q' - 1) < 2d/(d - 2)$ if $q' > 2d/(d + 2)$. This is the case since $\rho \in L^\infty(0, T; L^{q/2}(\Omega) \cap L^1(\Omega))$. Then we deduce
\[\|\theta^{a/2}\|_{H^1(\Omega)} \leq c \left( \|\nabla\theta^{a/2}\|_{L^2(\Omega)} + \|\theta^{a/2}\|_{H^1(\Omega)} \right),\]
with $\theta = 2(a/2 - \alpha)/a < 1$. Hence we get the conclusion.

The same estimate may be obtained in the same manner on $\theta^{(a-c+1)/2}$ for all $0 < c < 1$ using the bound obtained on $\nabla\theta^{(a-c+1)/2}$.

7 Compactness of sequences of approximate solutions

Given the preceding $a$ priori bounds, we now intend to study the compactness of sequences of approximate solutions $(\rho_n, u_n, e_n)$ and pass to the limit in nonlinear terms. The actual construction of suitably smooth approximate solutions is not detailed in this work, since classical approximation procedures can be applied to this problem. As an example of approximate solutions, a numerical finite difference scheme can be designed in a compatible way with the mathematical structure discovered in the present paper. This is the purpose of a forthcoming work [8].

Notice that the next subsections make use of estimates derived in Section 4, and that only the three dimensional cases will be detailed. Adaptations to the two dimensional framework will be discussed at the end of the paper, with the particular case of the whole 2d space and the lack of compactness at infinity of $H^1(R^2)$ functions (local bounds in space have to be used in the compactness analysis).

7.1 Compactness of the density $(\rho_n)_{n \in \mathbb{N}}$.

From the uniform estimates derived in Section 4, we deduce that the sequence $(\rho_n)_{n \in \mathbb{N}}$ is uniformly bounded in $L^\infty(0, T; L^{6m-3}(B))$ for all bounded subset $B$ of $\Omega$. Up to the extraction of a subsequence, one may assume that $(\rho_n)_{n \in \mathbb{N}}$ converges weakly to some $\rho$ in $L^2_{loc}(R_+ \times \Omega)$ ($6m - 3 \geq 3$ since $m \geq 1$).

In order to prove the strong compactness on the density sequence, we shall
use the transport equation satisfied by \((\tilde{\mu}(\rho_n))_{n\in\mathbb{N}}\), where \(\tilde{\mu}(s) = s^n + sm\):

\[
\partial_t(\tilde{\mu}(\rho_n)) + \text{div} (\tilde{\mu}(\rho_n)u_n) + \frac{1}{2}\tilde{\lambda}(\rho_n)\text{div} u_n = 0,
\]

where \(\tilde{\lambda}(s) = 2((m - 1)\rho^n + (n - 1)\rho^m)\). As a consequence, for all compactly supported \(\phi \in C^\infty(\Omega)\), \((\partial_t(\phi \tilde{\mu}(\rho_n)))_{n\in\mathbb{N}}\) is bounded in \(L^2(0,T;H^{-\sigma_0}(\Omega))\) for some positive \(\sigma_0\). As a matter of fact, the sequence \((\sqrt{\rho_n}u_n)_{n\in\mathbb{N}}\) is bounded in \(L^\infty(0,T;L^2(\Omega))\) and \((\rho_n^{-1/2}\tilde{\mu}(\rho_n))_{n\in\mathbb{N}}\) is bounded in \(L^\infty(0,T;L^6(\Omega))\), so that \((\tilde{\mu}(\rho_n)u_n)_{n\in\mathbb{N}}\) is bounded in \(L^\infty(0,T;L^{3/2}(B))\) for all bounded subset \(B\) in \(\Omega\). On the other hand, \((\sqrt{\lambda}(\rho_n))\text{div} u_n)_{n\in\mathbb{N}}\) is bounded in \(L^2((0,T)\times\Omega)\), and the sequence \((\sqrt{\lambda}(\rho_n))_{n\in\mathbb{N}}\) is bounded in \(L^\infty(0,T;L^{6(2m-1)/m}(B))\) and therefore in \(L^\infty(0,T;L^6(B))\) for all bounded subset \(B\) in \(\Omega\), so that \((\lambda(\rho_n)\text{div} u_n)_{n\in\mathbb{N}}\) is bounded in \(L^2(0,T;L^{3/2}(B))\).

The space compactness on the sequence \((\phi \tilde{\mu}(\rho_n))_{n\in\mathbb{N}}\) may be used as follows:

\[
\nabla(\phi \tilde{\mu}(\rho_n)) = \tilde{\mu}(\rho_n)\nabla \phi + \phi \frac{\nabla \tilde{\mu}(\rho_n)}{\sqrt{\rho_n}}\sqrt{\rho_n},
\]

so that \(\nabla(\phi \tilde{\mu}(\rho_n))\) is bounded in \(L^\infty(0,T;L^{3/2}(\Omega))\) (recalling that \(12m - 6 \geq 6\)).

It follows that denoting \(\tilde{\mu}\) a weak limit in \(L^2_{loc}((0,T)\times\Omega)\) of \((\tilde{\mu}(\rho_n))_{n\in\mathbb{N}}\), we deduce from the above space and time compactness that up to the extraction of a subsequence, \((\tilde{\mu}(\rho_n))_{n\in\mathbb{N}}\) converges strongly to some \(\overline{\mu}\) in \(L^2_{loc}((0,T)\times\Omega)\). In view of the \(L^\infty(0,T;L^{3/2}(\Omega))\) bound on \((\nabla(\phi \tilde{\mu}(\rho_n)))_{n\in\mathbb{N}}\), \((\tilde{\mu}(\rho_n))_{n\in\mathbb{N}}\) also converges to \(\overline{\mu}\) in \(L^2(0,T;W^{s,3/2}(B))\) for all bounded subset \(B\) in \(\Omega\) and \(s < 1\).

Finally using the fact that \(\tilde{\mu}\) is an increasing function of \(\rho\), we conclude that \((\rho_n)_{n\in\mathbb{N}}\) converges strongly to \(\tilde{\mu}^{-1}(\overline{\mu})\) in \(C([0,T];L^q_{loc}(\Omega))\) for all \(q < 6m - 3\), so that \(\rho = \tilde{\mu}^{-1}(\overline{\mu})\). On the other hand, we already know that the sequence \((u_n)_{n\in\mathbb{N}}\) is uniformly bounded in \(L^q_{loc}(0,T;W^{1,q_3}(\Omega)^d)\) for some \(q_1\) and \(q_3\) that satisfies in view of (75)

\[
q_1 > \frac{5m - 3}{3m - 2} > \frac{5}{3} \quad \text{and} \quad q_3 > \frac{3(5m - 3)}{8m - 5} > \frac{15}{8}.
\]

It follows that up to the extraction of a subsequence, \((u_n)_{n\in\mathbb{N}}\) converges weakly in \(L^q_{loc}(0,T;W^{1,q_3}(\Omega)^d)\) to some \(u \in L^q_{loc}(0,T;W^{1,q_3}(\Omega)^d)\). In view
of Sobolev embeddings, we also know that \((u_n)_{n \in \mathbb{N}}\) is uniformly bounded in \(L^{5/3}(0, T; L^{\infty}_{\text{loc}}(\Omega)^d)\). As a consequence, we are able to pass to the limit in the mass conservation equation

\[
\partial_t \rho + \text{div}(\rho u) = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}_+ \times \Omega),
\]

Concerning the zero isothermal part of the equation of state, we shall have to pass to the limit in sequences such as \((\rho_n e_c(\rho_n))_{n \in \mathbb{N}}, \rho_n(\rho_n)_{n \in \mathbb{N}}, (\rho_n e_c(\rho_n) u_n)_{n \in \mathbb{N}}, \text{ and } (p_c(\rho_n) u_n)_{n \in \mathbb{N}}\). Using assumption (23), we deduce that the first two sequences are bounded in \(L^\infty(0, T; L^{(6m-3)/k}(\Omega))\), where \((6m - 3)/k > q_2/(q_2 - 1)\), so that combined with the strong convergence of \((\rho_n)_{n \in \mathbb{N}}\) allows to conclude that \((\rho_n e_c(\rho_n))_{n \in \mathbb{N}}\) and \((p_c(\rho_n))_{n \in \mathbb{N}}\) converge in \(C([0, T]; L^{q_2/(q_2 - 1)}_{\text{loc}}(\Omega))\) respectively to \(\rho e_c(\rho)\) and \(p_c(\rho)\), which will be enough to conclude as soon as \((u_n)_{n \in \mathbb{N}}\) is weakly compact in \(L^q(0, T; L^{q_2}_{\text{loc}}(\Omega))\).

### 7.2 Compactness of \((\rho_n u_n)_{n \in \mathbb{N}}\)

Let us now prove compactness results on the momentum \((\rho_n u_n)_{n \in \mathbb{N}}\). We already know from the preceding step that \((\rho_n u_n)_{n \in \mathbb{N}}\) converges weakly to \(\rho u\) in \(L^2(0, T; L^{3/2}_{\text{loc}}(\Omega))\).

In order to apply classical compactness lemma, we shall prove that the sequence \((\partial_t(\rho_n u_n))_{n \in \mathbb{N}}\) is uniformly bounded in \(L^p_{\text{loc}}(\mathbb{R}_+; H^{-\sigma}(\Omega))\) for some \(p > 1\) and \(\sigma\) large enough. Rewriting the momentum conservation equation (2), we obtain for all compactly supported \(\phi \in C^\infty(\Omega)\)

\[
\partial_t(\rho_n u_n \phi) = -\phi \text{div}(\rho_n u_n \otimes u_n) - \phi \nabla p_n + \phi \text{div}(2\mu(\rho_n) D(u_n)) + \phi \nabla(\lambda(\rho_n) \text{div} u_n).
\]

In the first term of the right hand side, \((\rho_n u_n \otimes u_n)_{n \in \mathbb{N}}\) is bounded uniformly in \(L^{3/2}(0, T; L^{9/5}_{\text{loc}}(\Omega))\) as the product of \(|\rho_n^{1/3} u_n|^2\), bounded in \(L^{3/2}(0, T; L^{3/2}_{\text{loc}}(\Omega))\) and \(\rho_n^{1/3}\) bounded in \(L^\infty(0, T; L^{3(6m-3)/k}_{\text{loc}}(\Omega))\) (recall that \(6m - 3 \geq 3\)). The second term is bounded uniformly in \(L^2_{\text{loc}}(\mathbb{R}_+; H^{-1}(\Omega))\), since \((\rho_n)_{n \in \mathbb{N}}\) is uniformly bounded in \(L^2_{\text{loc}}(\mathbb{R}_+; L^\infty_{\text{loc}}(\Omega))\) (the sequences \((\rho_n)_{n \in \mathbb{N}}\) and \((\theta_n)_{n \in \mathbb{N}}\) are respectively bounded in \(L^\infty(0, T; L^3_{\text{loc}}(\Omega))\) and \(L^2(0, T; L^3_{\text{loc}}(\Omega))\) and the cold pressure component \((p_c(\rho_n))_{n \in \mathbb{N}}\) is bounded in \(L^\infty(0, T; L^\delta(\Omega))\) for some \(\delta > 1\) in view of the preceding subsection). For the last two terms, we use the fact that \((\sqrt{\mu(\rho_n)} D(u_n))_{n \in \mathbb{N}}\) and \((\sqrt{\lambda(\rho_n)} \text{div} u_n)_{n \in \mathbb{N}}\) are bounded in \(L^2((0, T) \times \Omega)\), and that \(\sqrt{\mu(\rho_n)}\) as well as \(\sqrt{\lambda(\rho_n)}\) are uniformly bounded.
in $L^\infty(0,T;L^6_{\text{loc}}(\Omega))$. Therefore, the sequence $(\partial_t(\rho_n u_n))_{n \in \mathbb{N}}$ is uniformly bounded in $L^p_{\text{loc}}(\mathbb{R}^+;H^{\kappa_0}(\Omega))$ for some large enough $\sigma_0 > 0$ and for $p > 1$.

As a conclusion, the product $(\phi \rho_n |u_n|^2)_{n \in \mathbb{N}}$, as the scalar product of the two sequences $(\phi \rho_n u_n)_{n \in \mathbb{N}}$, bounded in $L^\infty(0,T;L^{3/2}(\Omega))$ and $(u_n)_{n \in \mathbb{N}}$, bounded in $L^{5/3}(0,T;L^5_{\text{loc}}(\Omega)) \cap L^{5/3}(0,T;W^{1,15/8}_{\text{loc}}(\Omega))$, converges strongly in $L^1((0,T) \times \Omega)$ to $\rho |u|^2$. Using the fact that $\rho_n^{1/3} u_n = \rho_n^{1/3} u_n 1_{\rho_n \leq \delta} + \rho_n^{1/2} u_n \rho_n^{-1/6} 1_{\rho_n > \varepsilon}$, is the sum of a uniformly small term in $L^1((0,T) \times B)$ and another term converging to $\rho_n^{1/3} u_n 1_{\rho_n > \varepsilon}$ in $L^1((0,T) \times B)$ for any given $\varepsilon > 0$, we deduce that $(\rho_n^{1/3} u_n)_{n \in \mathbb{N}}$ converges strongly in $L^1((0,T) \times B)$ to $\rho^{1/3} u$. Finally using the uniform bound of $\rho_n^{1/3} u_n$ in $L^2((0,T) \times B)$ for some $\delta > 3$, we conclude that $\rho_n^{1/3} u_n$ converges strongly in $L^2((0,T) \times B)$ to $\rho^{1/3} u$.

### 7.3 Compactness on the Temperature $(\theta_n)_{n \in \mathbb{N}}$.

We finally want to obtain strong compactness results for the internal energy $(e_n)_{n \in \mathbb{N}}$ and the temperature $(\theta_n)_{n \in \mathbb{N}}$. The first step is to derive uniform bounds in $L^p_{\text{loc}}(\mathbb{R}^+;H^{-\kappa}(\Omega))$ (for some $p > 1$ and large enough $\sigma_0$) for the sequence $(\partial_t(\rho_n E_n \phi))_{n \in \mathbb{N}}$, given $\phi \in C^{\infty}(\Omega)$ compactly supported:

$$
\partial_t(\phi \rho_n E_n) = -\phi \text{div}(\rho_n u_n H_n) + \phi \text{div}(2\mu(\rho_n) D(u_n) \cdot u_n) + \phi \text{div}(u_n \lambda(\rho_n) \text{div} u_n) + \phi \text{div}(\kappa(\rho_n, \theta_n) \nabla \theta_n),
$$

In order to deal with the first term of the right hand side, let us derive uniform bounds on $\rho_n u_n H_n$ in $L^q_{\text{loc}}(\mathbb{R}^+;L^q_{\text{loc}}(\Omega))$ for some $q > 1$. We already proved that $(\rho_n u_n |u_n|^2)_{n \in \mathbb{N}}$ is uniformly bounded in $L^q_{\text{loc}}(\mathbb{R}^+;L^q_{\text{loc}}(\Omega)^3)$ with $q > 1$ in Subsection 5.1. Similar uniform estimates in $L^q_{\text{loc}}(\mathbb{R}^+;L^q_{\text{loc}}(\Omega)^3)$ with $q > 1$ on $(\rho_n u_n e_n + p_n u_n)_{n \in \mathbb{N}}$ and $(\kappa(\rho_n, \theta_n) \nabla \theta_n)_{n \in \mathbb{N}}$ were similarly derived in Subsections 5.2 and 5.3.

The last two terms are the viscous fluxes and write as space derivatives of $u_n \cdot \mu(\rho_n) D(u_n)$ and $u_n \lambda(\rho_n) \text{div} u_n$, for which we use the following properties: $(\sqrt{\mu(\rho_n)} D(u_n))_{n \in \mathbb{N}}$ and $(\sqrt{|\lambda(\rho_n)|} \text{div} u_n)_{n \in \mathbb{N}}$ are bounded in $L^2((0,T) \times \Omega)$, $(\sqrt{\mu(\rho_n)} \rho_n^{-1/3})_{n \in \mathbb{N}}$ and $(\sqrt{|\lambda(\rho_n)|} \rho_n^{-1/3})_{n \in \mathbb{N}}$ are bounded uniformly in $L^\infty((0,T;L^{3(2m-1)/(3m-2)}_{\text{loc}}(\Omega)))$, hence in $L^\infty((0,T;L^{18}_{\text{loc}}(\Omega)))$, and $(\rho_n^{1/3} u_n)_{n \in \mathbb{N}}$ is bounded in $L^3((0,T) \times \Omega)$. It follows that the viscous fluxes are bounded in $L^{6/5}(0,T;L^{3/8}_{\text{loc}}(\Omega))$, hence in $L^q_{\text{loc}}(\mathbb{R}^+;L^q_{\text{loc}}(\Omega))$ for some $q > 1$.

The second step consists in deriving enough space compactness: since
\[ E_n = C_n \theta_n + \rho_n e_c(\rho_n) + |u_n|^2 / 2, \]

one may take advantage of the strong convergence of \((\sqrt{\rho_n} u_n)_{n \in \mathbb{N}}\) to \(\sqrt{\rho u}\) in \(L^r_{\text{loc}}(\mathbb{R}^+; L^2_{\text{loc}}(\Omega))\) and the strong convergence of \((\rho_n e_c(\rho_n))_{n \in \mathbb{N}}\) to \(\rho e_c(\rho)\) in \(L^r_{\text{loc}}(\mathbb{R}^+; L^1_{\text{loc}}(\Omega))\) for all \(r \in (1, +\infty)\). Indeed, let us introduce \(K = \{ f \in L^1_{\text{loc}}(\Omega) / \| \nabla f \|_{L^2(\Omega)} = 1 \}\) and a sequence \((T_k)_{k \in \mathbb{N}}\) of regularizing kernels given for instance by convolution operators such that the following basic properties hold

\[
\sup_{f \in K} \| f - T_k f \| \leq \frac{C}{k}, \tag{89}
\]

and for all ball \(B \subset \Omega\), there exists \(C_{k,B}\) such that for all \(f \in K\)

\[
\| T_k f \|_{L^s(\Omega)} \leq C_{k,B}, \quad \text{and} \quad T_k f \in H^s(\Omega) \quad \text{for all} \quad s > 0. \tag{90}
\]

We deduce from (89) that for any ball \(B \subset \Omega\), one has

\[
\left| \int_{(0,T) \times B} (\rho_n \theta_n^2 - \rho \theta^2) \right| \leq \frac{C}{k} \left( \| \rho_n \theta_n \|_{L^2((0,T) \times B)} + \| \rho \theta \|_{L^2((0,T) \times B)} \right) + \int_{(0,T) \times B} \rho \theta (\theta_n - \theta) \hspace{1cm}
\]

\[
+ \frac{1}{2} \| (\rho_n |u_n|^2 - \rho |u|^2) T_k \theta_n \|_{L^1((0,T) \times B)} \hspace{1cm}
\]

\[
+ \| (\rho_n e_c(\rho_n) - \rho e_c(\rho)) T_k \theta_n \|_{L^1((0,T) \times B)} + \int_{(0,T) \times B} (\rho_n E_n - \rho E) T_k \theta_n \]. \tag{91}
\]

Let us observe that the first term of the above r.h.s. is estimated by

\[
\frac{C}{k} \left( \| \rho_n \|_{L^\infty(0,T;L^3(B))} \| \theta_n \|_{L^2(0,T;L^6(B))} + \| \rho \|_{L^\infty(0,T;L^3(B))} \| \theta \|_{L^2(0,T;L^6(B))} \right). \tag{92}
\]

Therefore, given \(\varepsilon > 0\), there exists \(k_\varepsilon \in \mathbb{N}\) such that the preceding term (92) is less than \(\varepsilon / 4\) uniformly in \(n \in \mathbb{N}\). Dealing with the second term is an easy task since \(\theta_n\) converges weakly to \(\theta\) in \(L^2(0,T;L^6(B))\), so that for \(n \geq n_1\), the second term is estimated by \(\varepsilon / 4\). Using (90) \(\| T_{k_\varepsilon} \theta_n \|_{L^\infty(\Omega)}\) is uniformly bounded in \(n\), whereas \((\rho_n |u_n|^2)_{n \in \mathbb{N}}\) and \((\rho_n e_c(\rho_n))_{n \in \mathbb{N}}\) converge strongly respectively to \(\rho |u|^2\) and \(\rho e_c(\rho)\) in \(L^1((0,T) \times B)\), so that the sum of the third and the fourth term is estimated by \(\varepsilon / 4\) for \(n \geq n_2\). For the last term with \(k = k_\varepsilon\), the uniform bound of \((\partial_t (\rho_n E_n))_{n \in \mathbb{N}}\) in \(L^p(0,T;H^{-s_{\varepsilon}}(\Omega))\)
where \( p > 1 \) implies that up to the extraction of a subsequence, \((\rho_n E_n)_{n \in \mathbb{N}}\) converges strongly to \(\rho E\) in \(C([0, T]; H^{-a}(\Omega))\), so that for \( n \geq n_3 \), the right hand side of (91) is less than \( \varepsilon \). It follows that up to the extraction of a subsequence, \((\sqrt{\rho_n} u_n)_{n \in \mathbb{N}}\) converges strongly in \(L^2_{\text{loc}}(\mathbb{R}^+ \times \Omega)\).

Using the strong compactness of \((\rho_n^{-1/2})_{n \in \mathbb{N}}\) to \(\rho^{-1/2}\) in \(C([0, T]; L^p(\Omega))\) for \( p < 6 \), we deduce that \((\theta_n)_{n \in \mathbb{N}}\) converges to \(\theta\) in \(L^2_{\text{loc}}(\mathbb{R}^+; L^p(\Omega))\) for all \( p < 3/2 \). Recalling the uniform \(L^2(0, T; L^6_{\text{loc}}(\Omega))\) bound on \((\theta_n^{a/2})_{n \in \mathbb{N}}\), we deduce that \((\theta_n)_{n \in \mathbb{N}}\) converges strongly to \(\theta\) in \(L^2_{\text{loc}}(\mathbb{R}^+; L^q(\Omega))\) for \( p < a \) if \( a > 2, p = 2 \) if \( a = 2 \), and \( q < 3a \).

### 7.4 Conclusion

It remains to check that one can pass to the limit in all the nonlinear terms, in order to obtain weak solutions in the sense of distributions, as defined in the introduction.

The limit mass conservation equation holds because of the strong convergence of \((\rho_n)_{n \in \mathbb{N}}\) to \(\rho\) in \(C([0, T]; L^2(\Omega))\) and the strong convergence of \((\sqrt{\rho_n} u_n)_{n \in \mathbb{N}}\) to \(\sqrt{\rho} u\) in \(L^2_{\text{loc}}(\mathbb{R}^+; L^2_{\text{loc}}(\Omega))\).

In the momentum equation, the strong convergence in \(L^1_{\text{loc}}(\mathbb{R}^+ \times \Omega)\) of \((\rho_n u_n)_{n \in \mathbb{N}}\) and \((\rho_n u_n \otimes u_n)_{n \in \mathbb{N}}\) to \(\rho u\) and \(\rho u \otimes u\) allows to pass to the limit in the sense of distributions in the first two terms. As a product of \((\rho_n)_{n \in \mathbb{N}}\) and \((\theta_n)_{n \in \mathbb{N}}\), which respectively converge strongly in \(C([0, T]; L^2_{\text{loc}}(\Omega))\) and strongly in \(L^2_{\text{loc}}(\mathbb{R}^+ \times \Omega)\), the perfect gas pressure term \(\nabla(\rho_n \theta_n)\) converges to the limit pressure \(\nabla(\rho \theta)\). The cold component of the pressure term has actually already been treated. It remains to consider the viscous fluxes: writing

\[
\mu(\rho_n) D(u_n) = D(\mu(\rho_n) u_n) - \frac{1}{2} \left( \sqrt{\rho_n} u_n \otimes \frac{\nabla \mu(\rho_n)}{\sqrt{\rho_n}} + \frac{\nabla \mu(\rho_n)}{\sqrt{\rho_n}} \otimes \sqrt{\rho_n} u_n \right),
\]

we conclude for the first term of the right hand side of (93) using the strong convergence in \(L^2(0, T; L^2_{\text{loc}}(\Omega))\) of \((\mu(\rho_n)/\sqrt{\rho_n})_{n \in \mathbb{N}}\) to \(\mu(\rho)/\sqrt{\rho}\) and the strong convergence in \(L^2(0, T; L^2_{\text{loc}}(\Omega))\) of \((\sqrt{\rho_n} u_n)_{n \in \mathbb{N}}\) to \(\sqrt{\rho} u\). For the last term of the right hand side of (93), we once again use the strong convergence of \((\sqrt{\rho_n} u_n)_{n \in \mathbb{N}}\) to \(\sqrt{\rho} u\) in \(L^2(0, T; L^2_{\text{loc}}(\Omega))\), and the weak convergence of \((\rho_n^{-1/2} \nabla \mu(\rho_n))_{n \in \mathbb{N}}\) in \(L^2(0, T; L^2_{\text{loc}}(\Omega))\) to \(\rho^{-1/2} \nabla \mu(\rho)\). Indeed the weak limit may be identified by writing \(\rho_n^{-1/2} \nabla \mu(\rho_n)\) as the gradient of a function.
of the density and by using the convergence properties of \((\rho_n)_{n \in \mathbb{N}}\). Finally, the bulk viscous term \(\lambda(\rho_n) \nabla u_n\) may be rewritten as follows

\[
\lambda(\rho_n) \nabla u_n = -2 (\partial_t \mu(\rho_n) + \text{div} (\mu(\rho_n) u_n)),
\]

which makes the conclusion straightforward, and concludes for the momentum conservation equation.

In the total energy conservation equation, the only difficulty is to pass to the limit in the energy flux \(\rho_n u_n (e_n + |u_n|^2/2)\), the heat flux \(\kappa(\rho_n, \theta_n) \nabla \theta_n\), and the stress flux \(2\mu(\rho_n) D(u_n) \cdot u_n\), \(\lambda(\rho_n) u_n \text{div} u_n\). For the energy flux, \(\rho_n u_n \theta_n\) converges strongly to \(\rho \theta\) to \(L^1_{\text{loc}}(\mathbb{R}^+ \times \Omega)\) as a product of \(\sqrt{\rho_n} u_n\) and \(\sqrt{\rho_n} \theta_n\) which converge strongly in \(L^2_{\text{loc}}(\mathbb{R}^+ \times \Omega)\). Moreover, \(\rho_n^{1/3} u_n = \rho_n^{-1/6} \sqrt{\rho_n} u_n\) converges strongly to \(\rho^{1/3} u\) in \(L^1_{\text{loc}}(\mathbb{R}^+ \times \Omega)\) as a product of two strongly converging sequences in \(L^2_{\text{loc}}(\mathbb{R}^+ \times \Omega)\). Recalling that \(\rho_n^{1/3} u_n\) is also uniformly bounded in \(L^q_{\text{loc}}(\mathbb{R}^+ \times \Omega)\) for some \(q > 3\), we deduce that \(\rho_n^{1/3} u_n\) converges to \(\rho^{1/3} u\) in \(L^3_{\text{loc}}(\mathbb{R}^+ \times \Omega)\).

For the heat flux, we first observe that \(\kappa(\rho_n, \theta_n)^{1/2}\) converges strongly in \(L^2_{\text{loc}}(\mathbb{R}^+ \times \Omega)\) to \(\kappa(\rho, \theta)^{1/2}\) (since it basically behaves like \((1 + \rho_n)^{1/2}(1 + \theta_n^{n/2})\), and \(\kappa(\rho_n, \theta_n)^{1/2} \nabla \theta_n\) converges weakly to \(\kappa(\rho, \theta)^{1/2} \nabla \theta\) in \(L^2_{\text{loc}}(\mathbb{R}^+ \times \Omega)\) (the identification of weak limits is deduced from the preceding strong convergence of \(\theta_n\) and \(\rho_n\)). As a consequence, \(\kappa(\rho_n, \theta_n) \nabla \theta_n\) converges weakly to \(\kappa(\rho, \theta) \nabla \theta\) in \(L^1_{\text{loc}}(\mathbb{R}^+ \times \Omega)\).

Finally the stress flux \(2\mu(\rho_n) D(u_n) \cdot u_n\) converges weakly to \(2\mu(\rho) D(u) \cdot u\). Indeed, \(\sqrt{\mu(\rho_n)} D(u_n)\) and similarly \(\sqrt{\mu(\rho)} D(u)\) converge weakly in \(L^2_{\text{loc}}(\mathbb{R}^+ \times \Omega)\) to \(\sqrt{\mu(\rho)} D(u)\) and \(\sqrt{\lambda(\rho)} \text{div} u\). On the other hand, \((\rho_n^{1/3} u_n)_{n \in \mathbb{N}}\) converges strongly to \(\rho^{1/3} u\) in \(L^3_{\text{loc}}(\mathbb{R}^+ ; L^3_{\text{loc}}(\Omega))\). Finally, the sequence \((\rho_n^{-2/3} \mu(\rho_n))_{n \in \mathbb{N}}\) as well as \((\rho_n^{-2/3} |\lambda(\rho_n)|)_{n \in \mathbb{N}}\) converge strongly to \((\rho^{-2/3} \mu(\rho))^{1/2}\) and \((\rho^{-2/3} |\lambda(\rho)|)^{1/2}\) in \(C([0, T]; L^3_{\text{loc}}(\Omega))\) since \(6m - 3 > 3\).

The proof is complete.

\[\Box\]

8 Velocity integrability with drag forces

As explained in Section 4, the \(L^4(0, T; L^4_{\text{loc}}(\Omega))\) integrability of \(\rho^{1/3} u\) may also be obtained by assuming the presence of drag forces as internal forces \(f_{\text{int}} = -|u|^{b-2} u\). The work in internal energy of the preceding force writes as

\[
\int_{\Omega} \rho |u|^b,
\]

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which provides an additional bound on $u$ at the left hand side of (54) provided the associated term in (55) is controlled. This term, denoted

$$I = \int_\Omega \rho |u|^{b-2} u \cdot \nabla \varphi (\rho),$$

may be integrated by parts

$$I = -\int_\Omega \mu (\rho) \left( |u|^{b-2} \text{div} u + (b - 2) |u|^{b-4} \sum_i u \cdot (u_i \nabla u_i) \right).$$

Thus, it can be estimated as follows for $(\alpha, \beta) \in [0, 1]^2$ to be chosen

$$|I| \leq C \int_\Omega \mu (\rho) |u|^{b-2} |\nabla u|$$

$$\leq C \left\| \sqrt{\mu (\rho)} \nabla u \right\|_{L^2 (\Omega)}^{1-\alpha} \left\| \rho |u|^b \right\|_{L^1 (\Omega)}^{(1-\beta)(b-2)/b} \left\| \sqrt{\mu (\rho)} \nabla u \right\|_{L^2 (\Omega)}^{\alpha}$$

$$\times \left\| \frac{\rho |u|^b}{\theta} \right\|_{L^1 (\Omega)}^{\beta(b-2)/b} \left\| \mu (\rho)^{1/2} \theta^{\alpha/2 + \beta(b-2)/b} \rho^{(2-b)/b} \right\|_{L^{2b/(4-b)} (\Omega)}, (94)$$

Since as a function of $\rho$, $\mu$ is bounded by $\rho^n$ with $2/3 < n < 1$ in the neighborhood of 0, we have to make sure that

$$bm + 4 - 2b \geq 0, \quad \text{i.e.} \quad b \leq \frac{4}{2-n}. \quad (95)$$

Then, we can write

$$|I| \leq \varepsilon \left\| \sqrt{\mu (\rho)} \nabla u \right\|_{L^2 (\Omega)}^2 + \varepsilon \left\| \rho |u|^b \right\|_{L^1 (\Omega)}$$

$$+ C_{\varepsilon} \left\| \sqrt{\mu (\rho)} \nabla u \right\|_{L^2 (\Omega)}^{q\alpha} \left\| \rho |u|^b \right\|_{L^1 (\Omega)}^{q\beta(b-2)/b} \left( \left\| \theta^{\alpha/2 + \beta(b-2)/b} \right\|_{L^{2b/(4-b)} (\Omega)}^q \right)$$

$$+ \left\| \theta \right\|_{L^{2b/(4-b)} (\Omega)}^{q\alpha/2 + \beta(b-2)/b} \left\| \mu (\rho)^{1/2} \rho^{(2-b)/b} \right\|_{L^{2b/(4-b)} (\Omega)} \left\| \mu (\rho)^{1/2} \rho \right\|_{L^1 (\Omega)} \left( \right). \quad (96)$$

Away from zero, $\mu$ is estimated by $\rho^m$ with $m > 1$, so that we have to make sure that (both in the periodic and in the whole space case)

$$bm + 4 - 2b \leq 0. \quad (97)$$
Moreover, \( \varepsilon > 0 \) will be chosen small enough, and \( q \) and \( r \) given in such a way that
\[
\frac{1}{q} = 1 - \frac{1 - \alpha}{2} - \frac{(1 - \beta)(b - 2)}{b} = \frac{4 - b}{2b} + \frac{\alpha}{2} + \frac{\beta(b - 2)}{b},
\]
and
\[
\frac{1}{r} = \frac{4 - b}{2b} - \left( \frac{\alpha}{2} + \frac{\beta(b - 2)}{b} \right) \frac{1}{3a} \geq \frac{mb + 2b - 4}{6b(2m - 1)}.
\]
The time integrability of the last product of the right hand side of (96) is suitable to apply Gronwall’s Lemma if
\[
q \left( \frac{\alpha}{2} + \frac{\beta(b - 2)}{b} \right) \left( 1 + \frac{1}{a} \right) \leq 1 \quad \text{and} \quad \frac{1}{q} = \frac{mb + 2b - 4}{2b(2m - 1)}.
\]
Such a value for \( q \) is permitted since it belongs to \([1, 2b/(4-b)]\) (using \( m \geq 1 \)).

9 Real equations of state

In order to deal with general equations of state (5), additional assumptions have to be considered, some of them being driven by physical considerations. First, the compatibility condition (see (105)) is assumed. It guarantees the existence of an entropy \( s \) as a function of the specific volume \( v = 1/\rho \) and temperature \( \theta \). We also require that the obtained entropy function \((v, e) \mapsto S(v, e)\) is a concave function over \( \mathbb{R}^2_+ \), which will allow to adapt the estimates derived from the entropy dissipation equation.

9.1 Entropy estimates

We use here the property that the entropy \( s \) is a concave function of the specific volume \( v \) and the internal energy \( e \). It follows that every point \((v, e, s(v, e))\) lies below the tangent plane defined in \((v_0, e_0, s_0)\) (where \( v_0 = 1/\rho_0 \)), so that using (7) yields
\[
s(\rho, e) \leq s(\rho_0, e_0) + \frac{\rho_0}{\theta_0} \left( \frac{1}{\rho} - \frac{1}{\rho_0} \right) + \frac{1}{\theta_0} (e - e_0)
\]
It follows that
\[ \int_{\Omega} \rho s(t, \cdot) \leq \int_{\Omega} \frac{p_0}{\theta_0} (\rho_0 - \rho(t, \cdot)) + \int_{\Omega} \frac{1}{\theta_0} (\rho e(t, \cdot) - \rho(t, \cdot)e_0), \]
hence using the fact that for all \( t \in \mathbb{R}_+ \), \( \rho(t, \cdot) \geq 0 \), \( e(t, \cdot) \geq 0 \), and \( \theta_0(\cdot) \geq \theta > 0 \), we deduce that
\[ \int_{\Omega} \rho s(t, \cdot) \leq \int_{\Omega} \frac{p_0}{\theta_0} + \frac{1}{\theta} \int_{\Omega} \rho e(t, \cdot), \]
hence the physical energy inequality (35) can be used to conclude that under the supplementary assumption \( p_0/\rho_0 \in L^1(\Omega) \), \( \theta_0 \geq \theta > 0 \), the following \textit{a priori} estimates for some constant \( C \) depending on the initial data
\[ \int_0^t \int_{\Omega} \frac{1}{\theta}(2\mu D(u) : D(u) + \lambda |\nabla u|^2) + \int_0^t \int_{\Omega} \frac{\kappa}{\theta^2} |\nabla \theta|^2 \leq C + \int_{\Omega} \rho_0 |s_0|. \]

Concerning the control of \( \int_{\Omega} \rho \div u \) and \( \int_{\Omega} \nabla p \cdot \nabla \varphi \) necessary in Equalities (54) and (55), we follow the same lines than in the ideal case assuming still that \( \rho \theta \) belongs to \( L^\infty_{\text{loc}}(\mathbb{R}_+; L^1(\Omega)) \) and making the following assumptions on the conductivity \( \kappa \)
\[ \kappa(\rho, \theta) \geq C \rho \theta^2 \rho^2 C^2 r^2, \quad (98) \]
with \( r \) also satisfying Inequalities (18) (19). We assume in addition that, \( Z \) and \( r \) being given by relations (57),
\[ Zr = \frac{p_c(\rho)}{\rho \theta} + Y, \]
where \( p_c(\rho) = \rho^2 e'_c(\rho) \) and \( \frac{Y}{1 + \rho^{m/2}} \in L^\infty((0, T) \times \Omega) \),
\[ (99) \]
where \( e_c \) denotes a \( C^1 \) nonnegative and nonincreasing function of \( \rho \).

As a matter of fact, Assumption (98) was already justified in (62). For the \( Y \) part in Assumption (99), we make use of the properties (15) (16) (17) of the viscosity coefficient \( \lambda \) and \( \mu \), recalling that \( (1 + \rho^{m/2}) \div u \) may be used instead of \( \div u \) alone in (63). In order to deal with the part of (99) involving \( p_c \), we observe that
\[ \int_{\Omega} \rho e'_c(\rho) \div u = -\frac{d}{dt} \int_{\Omega} \rho e_c, \]
which is non negative and can be handled at the left hand side of Equality (54).
9.2 Notations and Thermodynamics

This appendix recalls some notations and useful differential identities between the thermodynamic variables: the density \( \rho \), pressure \( p \), temperature \( \theta \), specific internal energy \( e \), and specific entropy \( s \). As in Subsection 4.4.1, we introduce the zero Kelvin isothermal part of the equation of state indexed by ”c” and its complement indexed by ”h”, namely the cold pressure and internal energy \( p_c(\rho) = p(\rho, 0) \), \( e_c(\rho) = e(\rho, 0) \), so that \( p(\rho, \theta) = p_c(\rho) + p_h(\rho) \) and \( e(\rho, \theta) = e_c(\rho) + e_h(\rho, \theta) \). We also require that \( p_c(\rho) = \rho d e_c(\rho)/d\rho \). Let us now introduce thermodynamical coefficients, possibly depending on the temperature and density for general equations of state:

- the heat capacity at constant volume \( C_v \) and the heat capacity at constant pressure \( C_p \)

\[
C_v = \left. \frac{\partial e}{\partial \theta} \right|_{\rho}, \quad C_p = \left. \frac{\partial(e_h + p_h/\rho)}{\partial \theta} \right|_{p_h}, \quad (100)
\]

- the volume expansion coefficient at constant pressure

\[
\alpha_p = -\frac{\theta \left. \frac{\partial \rho}{\partial \theta} \right|_{p_h}}{\rho}, \quad (101)
\]

- the compressibility at constant temperature

\[
Z = \frac{p_h \left. \frac{\partial \rho}{\partial \rho} \right|_{\theta}}{\rho}, \quad (102)
\]

- the Grüneisen coefficient

\[
\Gamma = \frac{1}{\rho} \left. \frac{\partial p}{\partial e} \right|_{\rho}, \quad (103)
\]

- and

\[
\gamma = \frac{\rho \left. \frac{\partial p_h}{\partial \rho} \right|_{s}}{p_h \left. \frac{\partial \rho}{\partial \rho} \right|_{s}}, \quad (104)
\]

In order that the expression

\[
\theta ds = de - \frac{p}{\rho^2} d\rho = de_h - \frac{p_h}{\rho^2} d\rho
\]

defines the entropy \( s \) as a closed differential form, the compatibility condition – the so-called Maxwell equation – has to be satisfied between the
pressure law $P$ and the internal energy law $E$ as functions of the density and temperature

\begin{equation}
p = \theta \frac{\partial p}{\partial \theta} \bigg|_\rho + \rho^2 \frac{\partial e}{\partial \rho} \bigg|_\theta.
\end{equation}

Assuming that (105) is satisfied, the following expressions follow from tedious but straightforward computations

\begin{align}
C_p &= C_v (1 + \alpha_p \Gamma), \\
\gamma &= \frac{C_p}{C_v Z},
\end{align}

\begin{equation}
\frac{dp}{dr} = r (\alpha_p \rho \frac{d\theta}{d\rho} + \theta \frac{dp}{dr}), \quad \text{where} \quad r = \frac{C_v \Gamma}{\alpha_p},
\end{equation}

and

\begin{equation}
\frac{de}{dr} = r \left( \frac{\alpha_p \theta}{\Gamma} \frac{d\theta}{d\rho} + \frac{\theta}{\rho} (Z - \alpha_p) \frac{dp}{dr} \right).
\end{equation}

Let us also finally observe that the compressibility $Z$ also writes

\begin{equation}
Z = \frac{p_h}{\rho r \theta},
\end{equation}

where the coefficient $r$ previously defined is constant in the case of a perfect polytropic gas.

Let us finally give some examples of equations of state commonly used in real applications.

- For polytropic ideal gases, one has

\begin{equation}
p_h(\rho, \theta) = \rho r \theta, \quad e_h(\rho, \theta) = C_v \theta,
\end{equation}

where $r$ and $C_v$ are constants, and $C_v = r/(\gamma - 1)$, where $\gamma > 1$ denotes the polytropic coefficient ($\gamma = 5/3$ (resp. $\gamma = 7/5$) in the case of monoatomic (resp. diatomic) gases). This model is quite relevant for moderate pressure and temperature regimes.

Notice that all the above coefficients $C_p, \alpha_p, Z, \Gamma$, are constant and write in terms of $r$ and $\gamma$

\begin{align}
C_p &= \frac{r \gamma}{\gamma - 1}, \\
\alpha_p &= 1, \\
Z &= 1, \\
\Gamma &= \gamma - 1,
\end{align}

which simplifies much the mathematical analysis of the flow model.

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- Two molecular vibrating gas: the $C_v$ coefficient depends on the temperature $\theta$

\[ p_h(\rho, \theta) = \rho r \theta, \quad e_h(\rho, \theta) = C^0_v \theta + \frac{r \Theta_0}{\exp(\Theta_0/\theta) - 1}, \] (111)

where $r$, $C^0_v$ and $\Theta_0$ denote positive constants.

In such cases, the $C_v$ coefficient is a function of $\theta$ only, which makes the analysis much easier. As a matter of fact, if $\partial C_v / \partial \rho|_{\theta} = 0$, then Maxwell equation (6) yields

\[ \frac{\partial^2 p}{\partial \theta^2}|_{\rho} = 0, \]

hence $p$ is an affine function of $\theta$. The extension of the main existence theorem to this family of equations of state is the purpose of the next subsection.

### 9.3 Assumptions and results

Extension of Theorem 3.1 to general equations of state requires to change drastically Lemma 4.3. This new development will be detailed in a forthcoming work [3]. However, Lemma 4.3 may be quite easily adapted in the case when the $C_v$ coefficient does not depend on the density. The pressure function then depends on the temperature in an affine way, and the specific entropy and internal energy express as the sum of separated contributions depending on $\theta$ and $\rho$

\[ p(\rho, \theta) = p_c(\rho) - \theta \rho^2 S'_\rho(\rho), \] (112)
\[ s(\rho, \theta) = S_\theta(\theta) + S_\rho(\rho), \] (113)
\[ e(\rho, \theta) = e_\theta(\theta) + e_c(\rho), \] (114)

where $p_c$ is a $C^1$ function corresponding to the "cold pressure" (pressure at zero temperature), $S_\rho$ and $S_\theta$ are concave functions $\mathbb{R}_+ \to \mathbb{R}$, and $S_\theta$, $e_\theta$, $e_c$ are defined up to an additive constant in terms of $C_v$ and $p_c$ by

\[ \theta S'_\rho(\theta) = C_v(\theta), \quad e'_\theta(\theta) = C_v(\theta), \quad e'_c(\rho) = \frac{p_c(\rho)}{\rho^2}. \]

Notice that the concavity assumption on the entropy requires the $C_v$ coefficient to be such that $\theta \mapsto C_v(\theta)/\theta$ to be nonincreasing.
As mentioned in Subsection 9.1, one may adapt (63)(64) by assuming in addition (98) and (99). Moreover, additional integrability is obtained in a similar way as in the perfect gas case: in Lemma 4.3, functions $f$ are assumed to be such that $\theta \mapsto f'(\theta)/C_v(\theta)$ is non negative and non increasing. Assuming for instance that the $C_v$ coefficient is bounded from above and bounded away from zero by a positive constant (as in the case of the two molecular vibrating gas), the proof in the case of a constant $C_v$ coefficient easily adapts to the present case (Notice that much more general laws may be considered; it will be detailed in [3]). In particular, $\rho \theta$ still belongs to $L^\infty(0,T;L^1(\Omega))$ since $e \geq C \theta$ for some positive constant $C$. Then, the same methodology leads to additional estimates locally in space and time that allow to pass to the limit in all the non linear terms.

10 The two-dimensional case

Theorem 3.1 can be extended to the two dimensional case. It requires some adaptations of technical issues. Let us give in this section corresponding Lemma of Section 6 for the two dimensional space dimension.

10.1 Auxiliary a priori estimates in 2D

Concerning Lemmas 6.1 and 6.2, we have to change, in the first Lemma, $L^6$ by $L^q$ for all $q \in [2, +\infty)$ with a constant coefficient depending on $q$.

For the second Lemma, some additional bounds on the initial density is needed in order to deal with the lack of compactness at infinity of the homogeneous $H^1$ space in $\mathbb{R}^2$. More precisely, the following assumption is made on $\rho_0$: there exists a positive constant $A_o < A$ and $p \in (0, m/2]$ such that

$$\rho^p 1_{\rho \geq A_o} \in L^2(\Omega). \quad (115)$$

Then, Lemma 6.2 is replaced by

**Lemma 10.1** For all $\rho$ satisfying (115) and $\rho^{-1/2} \nabla \mu(\rho) \in L^2(\Omega)$, one has

$$\|\rho^{m-1/2} 1_{\rho > A} \|_{L^q(\Omega)} \leq C_q \left\| \frac{\nabla \mu(\rho)}{\sqrt{\rho}} \right\|_{L^2(\Omega)}^{1-2/q} \|\rho^p 1_{\rho \geq A_o}\|_{L^2(\Omega)}^{2/q} \quad (116)$$

where $q \in (2, +\infty)$ is arbitrary.
Proof. In two space dimension, Gagliardo – Nirenberg’s inequality leads to
\[
\|\beta(\rho)\|_{L^q(\Omega)} \leq C_q \left\| \frac{\nabla \mu(\rho)}{\sqrt{\rho}} \right\|_{L^2(\Omega)}^{1-2/q} \|\beta(\rho)\|_{L^2(\Omega)}^{2/q}
\]
for all \( q \in (2, +\infty) \). We now use the same \( \xi \) function than in the proof of Lemma 6.1 written in the three dimensional case. Using the fact that \( \xi(\rho) \) belongs to \( L^{2p}(\Omega) \) and that \( m > 1 \), one may interpolate as follows
\[
\|\xi(\rho)\|_{L^{2m-1}(\Omega)} \leq \|\xi(\rho)\|_{L^{2p}(\Omega)}^{\alpha} \|\xi(\rho)\|_{L^q(m\frac{m-1}{2})(\Omega)}^{1-\alpha},
\]
where \( \alpha \in (0, 1) \) is given by
\[
\frac{1}{2m-1} = \frac{\alpha}{2p} + \frac{1-\alpha}{q(m-1/2)}.
\]
Therefore, one gets
\[
\|\beta(\rho)\|_{L^q(\Omega)} \leq C_q \left\| \frac{\nabla \mu(\rho)}{\sqrt{\rho}} \right\|_{L^2(\Omega)}^{\frac{(q(m-1/2)-2p)/(q(m-1/2))}{(q(m-1/2)-2p)/(q(m-1/2))}} \|\xi(\rho)\|_{L^q(\Omega)}^{2/q}.
\]
(117)
This ends the proof.

Still in the whole space case, one may use the transport equation on \( \rho \) in a renormalized manner together with the space and time \( L^2 \) bound on \( \text{div} \, u \) to prove that
\[
\|\rho 1_{B \leq \rho \leq A}(t, \cdot) - \rho_\infty\|_{L^2(\Omega)}^2
\]
\[
\leq C \left( \int_0^t \|\text{div} \, u(s, \cdot)\|_{L^2(\Omega)}^2 ds + \|\rho_0 1_{B \leq \rho_0 \leq A} - \rho_\infty\|_{L^2(\Omega)}^2 \right),
\]
(118)
as soon as the initial data are taken such that the second term of the r.h.s. of (118) is finite for some constant density \( \rho_\infty \).

Concerning the bounds on the velocity as in Subsection 5.1 but in dimension 2, we just have to write local in space estimates instead of global estimates with \( L^6(\Omega) \) replaced by \( L^q_{\text{loc}}(\Omega) \) for all \( q \in (2, +\infty) \). The same changes hold in Lemma 6.3.

Let us remark that only local in space compactness is needed to get existence in the distribution sense.
11 Some extensions.

Bounded domains. In the spirit of the work done in [6], the case of domains with no curvature, such as \( \mathbb{T}^d \times (0, 1) \) with suitable boundary conditions may be considered without additional difficulties. In fact, we can consider a general smooth bounded domain with suitable boundary conditions (Navier boundary conditions), see [9]. It suffices to show that we can control the additional estimates in a good way, concluding with Gronwall’s estimate.

Low Mach number limit. The global existence result obtained by P.–L. Lions on the barotropic compressible Navier–Stokes equations has allowed a lot of works to be done such as low Mach number limit in general domains (see [20] and [21]); Inflow-Outflow compressible Navier–Stokes equations (see [51]), fluid-structure interactions studies (see [19]). Our result allow to study the same problems using now the compressible Navier–Stokes equations with the full conductivity equation. The low Mach number problem for weak solutions is for instance still in progress in [3].

The capillary case. The capillary compressible Navier–Stokes equations, in the spirit of Korteweg models for diffusive interfaces, can be written as follows

\[
\begin{align*}
\partial_t \rho + \text{div} (\rho u) &= 0, \quad (119) \\
\partial_t \rho u + \text{div} (\rho u \otimes u) &= \text{div} \sigma + \text{div} K + \rho f, \quad (120) \\
\partial_t (\rho E) + \text{div} (\rho u H) &= \text{div} (\sigma \cdot u) + \text{div} (K \cdot u) \\
&\quad + \text{div} (\kappa \nabla \theta) + \rho f \cdot u, \quad (121)
\end{align*}
\]

where \( \chi > 0 \) denotes the capillarity coefficient. Such model was first introduced by Korteweg a century ago, [39]. In [4], they give the formulation of such Korteweg system and formally prove that asymptotically, \( \chi \to 0 \) the diffuse-interface equations represent the classical, sharp-interface system. That means the classical model of fluid–fluid systems with a sharp interfacial endowed with properties such as surface tension. In an appendix, they formulate general diffuse-interface equations that account for thermal and viscous dissipation. Using the magic structure, we can prove the global well
posedness of weak solutions of such system for the appropriate viscosities and conductivity coefficients. Readers interested by local strong solution or global strong solution close to equilibrium for general capillarity, viscosities and conductivity coefficients are referred to [18] [46].

Physical energy bounds. As before, the physical energy equality

$$\frac{d}{dt} \int_{\Omega} \left[ \rho \left( e + \frac{|u|^2}{2} \right) + \frac{\chi}{2} |\nabla \mu(\rho)|^2 \right] (t, x) \, dx = 0$$

(122)
is obtained in a classical way by multiplying the momentum equation by $u$ and by using the energy equation. This gives, integrating in time from 0 to $T > 0$

$$\int_0^T \int_{\Omega} \left[ \rho \left( e + \frac{|u|^2}{2} \right) + \frac{\chi}{2} |\nabla \mu(\rho)|^2 \right](t, x) \leq \int_{\Omega} \left[ \rho_0 e_0 + \frac{|m_0|^2}{2\rho_0} + \frac{\chi}{2} |\nabla \mu(\rho_0)|^2 \right] (x) \, dx$$

(123)
The additional estimates comes from Lemma 4.1: (36) (37). Comparing to the standard compressible Navier–Stokes equations, we remark that, assuming $\nabla \rho_0 \in L^2(\Omega)$, we get the following additional information on the approximate solutions

$$\nabla \rho \in L^\infty(0, T; L^2(\Omega)), \quad \sqrt{\mu'(\rho)} \Delta \mu(\rho) \in L^2(0, T; L^2(\Omega)).$$

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