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ALTERNATIVE PROOFS OF LINEAR RESPONSE FOR PIECEWISE EXPANDING UNIMODAL MAPS

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Abstract. We give two new proofs that the SRB measure \( t \mapsto \mu_t \) of a \( C^2 \) path \( f_t \) of unimodal piecewise expanding \( C^3 \) maps is differentiable at 0 if \( f_t \) is tangent to the topological class of \( f_0 \). The arguments are more conceptual than the original proof in [4], but require proving Hölder continuity of the infinitesimal conjugacy \( \alpha \) (a new result, of independent interest) and using spaces of bounded \( p \)-variation. The first new proof gives differentiability of higher order of \( f \psi \, d\mu_t \) if \( f_t \) is smooth enough and stays in the topological class of \( f_0 \) and if \( \psi \) is smooth enough (a new result). In addition, this proof does not require any information on the decomposition of the SRB measure into regular and singular terms, making it potentially amenable to extensions to higher dimensions. The second new proof allows us to recover the linear response formula (i.e., the formula for the derivative at 0) obtained in [4], and gives additional information on this formula.

1. Introduction

Many chaotic dynamical systems \( f : M \to M \) on a Riemannian manifold \( M \) admit an SRB measure \( \mu \) (see e.g. [18]) which describes the statistical properties of a “large” set of initial conditions in the sense of Lebesgue measure. (In dimension one, an SRB measure is simply an absolutely continuous ergodic invariant probability measure \( \mu_t = \rho_t \, dx \), with a positive Lyapunov exponent.) It is of interest (in particular in view of applications to statistical mechanics, see e.g. [14, 15]) to study the smoothness of \( t \mapsto \mu_t \), when \( f_t \) is a smooth family of dynamical systems, each having an SRB measure \( \mu_t \). If \( t \mapsto \mu_t \) is differentiable, one says that linear response holds. Ruelle [14] obtained not only differentiability, but also a formula for the derivative (the linear response formula), in the case of smooth uniformly hyperbolic dynamical systems.

In [4], we proved that the SRB measure \( t \mapsto \mu_t \) of a \( C^2 \) family of piecewise \( C^3 \) and piecewise expanding unimodal maps \( f_t \), with \( f_0 \) mixing (see §2.1 for formal definitions), is differentiable at \( t = 0 \) (as a Radon measure) if and only if \( f_t \) is tangent to the topological class of \( f_0 \) at \( t = 0 \) (for the necessity of the tangency condition, we require an additional mild technical condition). Keller [10] proved a long time ago that \( \rho_t \) has a \( |t| \ln |t| \) modulus of continuity, as an element of \( L^1(dx) \), and examples in [1, 4] show that this can be attained as a lower bound for non tangential families. We also obtained in [4] a linear response formula analogous to the one in [14], using a resummation introduced in [1].

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More recently, differentiability of the SRB measure (in the weak sense, that is, as an appropriate distribution) was obtained [16], [6] for smooth families $f_t$ of analytic and nonuniformly expanding unimodal maps which stay in the topological class of $f_0$. The cases of families of smooth nonuniformly expanding interval maps only tangent to the topological class (where Whitney differentiability is expected on suitable subsets of parameters), as well as higher-dimensional dynamical systems such as piecewise expanding/hyperbolic maps or Hénon-like maps, are still open, and much more difficult, see [2] for a discussion. In particular, the arguments in [4] and [16] used detailed information about the structure of the SRB measure, decomposing it into a regular and a singular term. This type of information may be far less accessible in higher dimensions.

In this article, we give two new proofs of the fact (Theorem 5.1 in [4]) that the SRB measure of a $C^2$ family of piecewise $C^3$ and piecewise expanding unimodal maps $f_t$, with $f_0$ mixing, is differentiable at $t = 0$ if $f_t$ is tangent to the topological class of $f_0$ at $t = 0$.

Section 3 contains our first new proof (see Corollary 3.2), more precisely, we obtain differentiability of $t \mapsto \int \psi d\mu_t$ for $\psi \in C^{1+\text{Lip}}$ if $f_t$ is a $C^2$ family of piecewise expanding $C^3$ unimodal maps tangent to the topological class of a mixing map $f_0$. The argument is based on thermodynamic formalism, using potentials $(s, t) \mapsto s(\psi \circ h_t) - \log |f_t' \circ h_t|$ (where $h_t$ conjugates $f_t$ with $f_0$ and $|f_t - f_0| = O(t^2)$) and does not require any knowledge about the structure of $\mu_t$. It may therefore be useful in more difficult situations (such as Hénon maps, see [2]). It requires proving Hölder differentiability of the infinitesimal conjugacy $\alpha$, a new result (Proposition 2.3), of independent interest. Also, this new proof gives that $t \mapsto \int \psi d\mu_t$ is a $C^j$ function, if $\psi \in C^{j+\text{Lip}}$ and $f_t$ is a $C^j$ family of piecewise expanding $C^{j+2}$ maps in the topological class of $f_0$, for any $j \geq 1$ (this is a new result, Theorem 3.1). Note also that we do not require the assumption from [4] that there is a function $X$ so that $\partial f_t|_{t=0} = X \circ f_0$.

The first new proof requires $\psi \in C^{1+\text{Lip}}$ (instead of $\psi \in C^0$ as in [4]) and does not furnish the linear response formula. Section 4 contains our second new proof (Theorem 4.1), which uses spectral perturbation theory for transfer operators associated to the dynamics $f_0$ and the weight $1/|f_t' \circ h_t|$. This other proof gives differentiability of $\int \psi d\mu_t$ for $\psi \in C^0$ and, using the assumption that $\partial f_t|_{t=0} = X \circ f_0$, allows us to recover the linear response formula from [4]. (This second proof also uses the Hölder regularity of $\alpha$ from Proposition 2.3.) Note however that this second proof requires information on the structure of $\mu_t$ from [1, Prop. 3.3].

Putting together Theorems 3.1 and 4.1 we get the following additional result (Corollary 4.4): If $f_t$ is a $C^2$ family of piecewise expanding $C^3$ unimodal maps in the topological class of $f_0$, then $t \mapsto \mu_t$ is $C^1$ from a neighbourhood of zero to Radon measures.

We emphasize that neither new proof gives that the condition to be tangent to the topological class is necessary, contrary to the argument in [4] (see Theorem 7.1 there). The proofs here are a bit shorter than the one given in [4], although the present account requires some results from our previous papers (such as [1, Prop. 3.3] [5, Prop. 3.2, Theorem 2] [4, Prop. 2.4, Lemma 2.6, Prop. 3.3]).
2. Definitions and notations – Hölder smoothness of the infinitesimal conjugacy

2.1. Formal definitions. Denote $I = [-1, 1]$. For an integer $k \geq 1$, we define $B^k$ to be the linear space of continuous functions $f : I \to \mathbb{R}$ such that $f$ is $C^k$ on the intervals $[-1, 0]$ and $[0, 1]$. Then $B^k$ is a Banach space for the norm $\max\{|f|_{C^k([-1, 0])}, |f|_{C^k([0, 1])}\}$. For an integer $k \geq 1$, we define the set $\mathcal{U}^k$ of piecewise expanding $C^k$ unimodal maps to be the set of $f \in B^k$ such that \footnote{A prime denotes differentiation with respect to $x \in I$, a priori in the sense of distributions.} $f(-1) = f(1) = -1$, $\inf_{x \neq 0} |f'(x)| > 1$, and $f(0) \leq 1$ (so that $f(I) \subset I$). The point $c = 0$ is called the critical point of $f$.

A piecewise expanding $C^k$ unimodal map $f$ is good if either $c$ is not periodic under $f$ or $|(f^{-1})'(f(c))\min\{|f'_+ (c)|, |f'_- (c)|\} > 2$, where $q \geq 2$ is the minimal period of $c$; it is mixing if $f$ is topologically mixing on $[c_2, c_1]$, where $c_k = f^k(c)$.

For $1 \leq j \leq k$, a $C^j$ family of piecewise expanding $C^k$ unimodal maps is a $C^j$ family of piecewise expanding maps $f_t$ from $t \in (-\epsilon, \epsilon)$ to $\mathcal{U}^k$ for some $\epsilon > 0$. (In this paper, $k \geq 1$ is an integer and $j$ is either an integer or $j = k - 1 + \text{Lip}$ for $k \geq 2$, the notations $B^{k+\text{Lip}}$ and $\mathcal{U}^{k+\text{Lip}}$ for integers $k \geq 1$ being self-explanatory. See also Remark 2.2.)

Remark 2.1. A $C^j$ family $f_t$ of piecewise expanding $C^k$ unimodal maps is a $C^{j,k}$ perturbation of $f_0$ in the sense of [4] if $j = k \geq 2$.

Remark 2.2. Considering $B^{k+\beta}$ and $\mathcal{U}^{k+\beta}$ for $k \geq 1$ integer and a Hölder exponent $0 < \beta < 1$ will perhaps allow to avoid the loss of regularity from $C^{k+1}$ to $C^{k+\text{Lip}}$ e.g. in [4, Prop. 2.4] (this question was asked by J.-C. Yoccoz). However, since the spectral result of Wong [17] only holds on the space $BV_p$ of functions of bounded $p$-variation if $1 \leq p < p_0$, for some $p_0 > 1$ depending on the dynamics, it may be necessary in this case to replace $BV_p$ by spaces of generalised $p$-variation, as introduced by Keller [11]. (See also Remark 2.5.)

Assume that $f_t$ is a $C^j$ family of piecewise expanding $C^k$ unimodal maps for $k \geq j > 1$. By classical results of Lasota–Yorke, each $f_t$ has a unique absolutely continuous invariant probability measure $\mu_t = \rho_t \, dx$. This measure is ergodic and it is called the SRB measure of $f_t$. If $f_t$ is mixing, then $\mu_t$ is mixing. If $f_0$ is good and mixing, then $f_t$ is mixing for all small enough $t$ (see [10] and references therein).

We say that a $C^j$ family $f_t$ of piecewise expanding $C^k$ unimodal maps is in the topological class of $f_0$ if there exist homeomorphisms $h_t : I \to I$ such that

\begin{equation}
    h_0 = \text{id and } h_t \circ f_0 = f_t \circ h_t \text{, } \forall |t| < \epsilon .
\end{equation}

We say that a $C^j$ family $f_t$ of piecewise expanding $C^k$ unimodal maps $(k \geq j \geq 2)$ is tangent to the topological class of $f_0$ if there exists a $C^j$ family $\tilde{f}_t$ of piecewise expanding $C^k$ unimodal maps in the topological class of $f_0$ so that $\tilde{f}_t = f_0$ and $\partial_t f_t|_{t=0} = \partial_t \tilde{f}_t|_{t=0}$. (Note that there is a typographical mistake in [4, p. 682, line 6], where “$C^{2,2}$ perturbation” should be replaced by “$C^{r_{0,r}}$ perturbation.”)

We say that a bounded function $v : I \to \mathbb{R}$ is horizontal for $f$, if $v(-1) = v(1) = 0$, and setting $M_f = q$ if $c$ is periodic of minimal period $q$, and $M_f = +\infty$ otherwise,

\begin{equation}
    J(f, v) = \sum_{j=0}^{M_f-1} \frac{v(c_j)}{(f^j)'(c_1)} = 0 .
\end{equation}
In [4, Cor. 2.6] we proved that if \( f_t \) is a \( C^2 \) family of piecewise expanding \( C^2 \) unimodal maps tangent to the topological class of \( f_0 \), then \( v = \partial f_t \rvert_{t=0} \) is horizontal for \( f_0 \). By [4, Theorem 2], if \( f_t \) is a \( C^2 \) family of piecewise expanding \( C^2 \) unimodal maps with \( f_0 \) good and \( v = \partial f_t \rvert_{t=0} \) is \( C^2 \) and horizontal for \( f_0 \), then there exists a \( C^{1+\text{Lip}} \) family \( \tilde{f}_t \) of piecewise expanding \( C^2 \) unimodal maps in the topological class of \( f_0 \) so that \( \tilde{f}_t = f_0 \) and \( \partial_t \tilde{f}_t \rvert_{t=0} = \partial_t f_t \rvert_{t=0} \).

We proved in [4, Lemma 2.2] that if \( v : I \to \mathbb{R} \) is bounded then the twisted cohomological equation (TCE)

\[
v(x) = \alpha(f(x)) - f'(x) \alpha(x), \quad \forall x \in I, \quad x \neq c,
\]

admits a unique bounded solution \( \alpha \) satisfying \( \alpha(c) = 0 \). This solution is obtained as follows: If \( c \) is not in the forward orbit of \( x \), set \( M(x) = \infty \) and otherwise let \( M(x) \) be the smallest integer \( j \geq 0 \) satisfying \( f^j(x) = c \), then put

\[
\alpha(x) = \sum_{i=0}^{M(x)-1} \frac{v(f^i(x))}{(f^i)'(x)}.
\]

The function \( \alpha \) is called the \emph{infinitesimal conjugacy}.

If \( u : I \to \mathbb{R} \) is Hölder we denote its Hölder norm by \( |u|_\beta \). Slightly abusing notation, we shall sometimes write \( \partial_t f_t \) for \( \partial_t f_t \rvert_{t=t} \), and similarly for other functions depending on \( t \).

### 2.2. Hölder smoothness of the infinitesimal conjugacy \( \alpha \)

A new result that we shall require throughout (see Lemmas 3.3 and 4.2) is:

**Proposition 2.3** (Smoothness of the infinitesimal conjugacy). Let \( f \in \mathcal{U}_2 \) be such that \( c \) is not periodic. For any \( \beta \in (0,1) \) there exist \( C_\beta > 0 \) and \( \mathcal{V}_\beta \) a neighbourhood of \( f \) in \( \mathcal{U}_2 \) so that, for any \( g \in \mathcal{V}_\beta \) and every \( \beta \)-Hölder \( v : I \to \mathbb{R} \), with \( v(-1) = v(1) = 0 \) and \( J(g,v) = 0 \), the unique bounded function \( \alpha \) (4) satisfying \( \alpha(c) = 0 \) and \( v(x) = \alpha(g(x)) - g'(x) \alpha(x) \) for all \( x \neq c \) is \( \beta \)-Hölder, with

\[
|\alpha|_\beta \leq C_\beta |v|_\beta.
\]

If the critical point of \( f \in \mathcal{U}_2 \) is periodic, the statement holds up to taking (for appropriate \( \xi(\beta) > 0 \))

\[
\mathcal{V}_\beta = \{ g \mid \|g - f\|_{\mathcal{U}_2} < \xi(\beta), \exists \text{ homeomorphism } h : I \to I \text{ s.t. } g \circ h = h \circ f \}.
\]

In particular, if \( f \in \mathcal{U}_2 \) and \( J(f,v) = 0 \) for some Lipschitz \( v \) with \( v(-1) = v(1) = 0 \), the function \( \alpha \) solving (3) is \( \beta \)-Hölder for any \( \beta < 1 \).

**Remark 2.4.** Jérôme Buzzi [8] showed us a simple proof that if \( h \) is a homeomorphism so that \( h \circ f = g \circ h \), for two piecewise expanding \( C^1 \) unimodal maps \( f \) and \( g \), then \( h \) is \( \beta \)-Hölder, for any \( \beta < \log(\inf |g'|/2)/\log(2 \sup |f'|) \). This fact neither implies nor is implied by Proposition 2.3.

**Proof.** **Step I.** For any \( \beta < 1 \), there exist a neighbourhood \( \mathcal{V}_\beta \) of \( f \) in \( \mathcal{U}_2 \), \( \ell \geq 1 \), and \( \eta > 0 \) such that \( \lambda = (\inf_{g \in \mathcal{V}_\beta} \inf_{x \neq c} |g'(x)|)^{-1} < 1 \) and, for any \( g \in \mathcal{V}_\beta \), letting \( d_1 < d_2 < \cdots < d_p \) be the critical points of \( g' \), putting \( d_0 = -1, d_{p+1} = 1 \), and setting

\[
\theta = \max_{0 \leq i \leq p} \sup_{x,y \in (d_i,d_{i+1})} \frac{|(g')'(x)|^\beta}{|x-y|^\eta},
\]
we have $2\theta < 1$.

Put $\Delta_g = \min_{0 \leq i \leq p} \{d_{i+1} - d_i\}$. Then $\inf_{g \in \mathcal{V}_\beta} \Delta_g > 0$ if the critical point of $f$ is not periodic. Otherwise we have $\inf_{g \in \mathcal{V}_\beta} \Delta_g > 0$, up to replacing $\mathcal{V}_\beta$ by a $B^2$-neighbourhood of $f$ in its topological class. In particular, we can assume that $\eta < \inf_{g \in \mathcal{V}_\beta} \Delta_g$. From now on, we fix $\mathcal{V}_\beta, \ell \geq 1$, and $\eta > 0$ as above.

**Step II.** We claim it suffices to show the lemma for $g \in \mathcal{V}_\beta$ with a periodic critical point: Indeed, if $g$ has a nonperiodic critical point, then we consider $g_t = g + tw$ with $g_t \in \mathcal{U}^2$, $w \in B^2$, $w(-1) = w(1) = 0$, and $J(g, w) \neq 0$. By [5, Corollary 4.1], there exists a sequence $t_n \to 0$ such that each $g_n = g_{t_n}$ has a periodic critical point. In particular, $g_n$ converges to $g$ in the $\mathcal{U}^2$ topology. Then, by [5, Proposition 3.2] we have $\lim_{n \to \infty} J(g_n, v) = 0$. Let $w_n$ be a $\beta$-Hölder function, with $w_n(-1) = w_n(1) = 0$ and $|w_n|_\beta \leq 1$, such that $J(g_n, w_n) = 1$. Set

$$v_n = v - J(g_n, v)w_n.$$  

Then we have $J(g_n, v_n) = 0$ and $\lim_{n \to \infty} |v_n - v|_\beta = 0$. If the proposition holds for maps in $\mathcal{V}_\beta$ with a periodic turning point, the unique function $\alpha_n = \alpha_n(c) = 0$ and $v_n(x) = \alpha_n(g_n(x)) - g_n'(c)\alpha_n(x)$ for all $x \neq c$, satisfies $|\alpha_n|_\beta \leq C_\beta|v_n|_\beta$. We can choose a subsequence $\alpha_n$ converging in the sup norm to a function $\alpha$. If follows from the uniform convergence of $\alpha_n$ that $\alpha$ satisfies the TCE (3) for $g$ and $v$, and that $|\alpha|_\beta \leq C_\beta|v|_\beta$.

**Step III.** We assume from now on that $g \in \mathcal{V}_\beta$ has a periodic turning point. The proof will be via an “infinitesimal pull-back” argument.

First, since $J(g, v) = 0$, it is easy to see that there exists a $\beta$-Hölder function $\alpha_0: I \to \mathbb{R}$ with $\alpha_0(-1) = \alpha_0(1) = \alpha_0(c) = 0$, $\alpha_0(g(c)) = v(c)$, and

$$v(x) = \alpha_0(g(x)) - g'(c)\alpha_0(x)$$

for every $x \neq c$ in the (finite) forward orbit of $c$.

Second, we define by induction continuous functions $\alpha_i: I \to \mathbb{R}$, for $i \geq 1$, such that $\alpha_i(-1) = \alpha_i(1) = \alpha_i(c) = 0$, $\alpha_i(g(c)) = v(g(c))$, that

(6) \quad $v(x) = \alpha_i(g(x)) - g'(c)\alpha_i(x)$

for every $x \neq c$ in the (finite) forward orbit of $c$, and, in addition,

(7) \quad $v(x) = \alpha_{i-1}(g(x)) - g'(c)\alpha_{i-1}(x)$,  $\forall x \neq c$.

Indeed, suppose we have defined $\alpha_i$, for $0 \leq i \leq n$. Set $\alpha_{n+1}(c) = 0$, and

$$\alpha_{n+1}(x) = \frac{\alpha_n(g(x)) - v(x)}{g'(x)}$$

$x \neq c$.

Clearly, $\alpha_{n+1}(-1) = \alpha_{n+1}(1) = 0$, and (7) holds for $i = n + 1$. Thus, since $v(x) = \alpha_n(g(x)) - g'(c)\alpha_n(x)$ for every $x \neq c$ in the forward orbit of $c$, we find $\alpha_n(x) = \alpha_{n+1}(x)$ for each $x \neq c$ in the forward orbit of $c$. Since $\alpha_n(c) = \alpha_{n+1}(c) = 0$, we conclude that (6) holds for $i = n + 1$, and $\alpha_{n+1}(g(x)) = v(x)$. Last, but not least, $\alpha_{n+1}$ is continuous on $I$ because $\alpha_n(g(x)) = v(x)$.

Thirdly, if $x$ is not a critical point of $g^j$, we set $v_0(x) = 0$, and

$$v_j(x) = \sum_{i=0}^{j-1} (g^{j-1-i})(g^{i+1}(x))v(g^i(x)),$$  \quad $j \geq 1$.

Recalling the notation $\ell, \{d_i\}$, from Step I, it is easy to see that

(8) \quad $v_\ell(x) = \alpha_{\ell}(g^\ell(x)) - (g^\ell)')(x)\alpha_{\ell+1}(x)$,  $\forall x \notin \{d_1, \ldots d_p\}$,  $\forall j \geq 0$. 
For \( \ell \geq 2 \) the function \( v_\ell \) may have jump discontinuities at the critical points \( d_i \) of \( g^\ell \), but it is \( \beta \)-Hölder in the connected components of \( I \setminus \{d_1, \ldots, d_p\} \).

Finally, we shall use the iterated twisted cohomological equation (8) to show that there exists \( C_\beta < \infty \) so that, for all \( g \in \mathcal{V}_\beta \) with a periodic turning point and all \( \beta \)-Hölder \( v \) with \( J(g,v) = 0 \) and \( v(-1) = v(1) = 0 \), there exists \( j_0 \geq 0 \) so that \( |\alpha_{j_0}|\beta \leq C_\beta |v|\beta \) for all \( j \geq j_0 \). In view of this, for \( j \geq 0 \), set \( K^\beta_j = sup_I |\alpha_{j\ell}| \), \( L^\beta = sup_I |v_\ell| \),

\[
K^\beta_j = \sup_{x \neq y} \frac{|\alpha_{j\ell}(x) - \alpha_{j\ell}(y)|}{|x - y|^\beta}, \quad \tilde{K}^\beta_j = \max_{0 \leq i \leq p} \sup_{x \neq y, x,y \in (d_i,d_{i+1})} \frac{|\alpha_{j\ell}(x) - \alpha_{j\ell}(y)|}{|x - y|^\beta},
\]

and

\[
L^\beta = \max_{0 \leq i \leq p} \sup_{x \neq y, x,y \in (d_i,d_{i+1})} \frac{|v_\ell(x) - v_\ell(y)|}{|x - y|^\beta}, \quad D = \max_{0 \leq i \leq p} \sup_{x \in (d_i,d_{i+1})} \frac{|(g^\ell)^\prime(y)|^2}{|x - y|^\beta}.
\]

Clearly, \( max(L^\beta, K^\beta_j) \leq \tilde{C}_\beta |v|\beta \) for all \( g \) and \( v \) under consideration, and we have

\[
|\alpha_{(j+1)\ell}(x) - \alpha_{(j+1)\ell}(y)| \leq 2\eta^{-\beta} K^\beta_{j+1} \quad \text{if} \quad |x - y| \geq \eta, \quad \forall j \geq 0.
\]

Therefore, recalling the definition of \( \lambda \) and (5) from Step I, it suffices to show that

\[
K^\beta_{j+1} \leq \lambda^\ell (K^\beta_j + L^\beta) \quad \text{and} \quad \tilde{K}^\beta_{j+1} \leq (L^\beta + K^\beta_j) D + \lambda^\ell \beta^\ell + \theta \beta^\beta, \quad \forall j \geq 0.
\]

Indeed, continuity of \( \alpha_{(j+1)\ell} \) together with (9) (recall also that \( \eta < \inf_\gamma \Delta_\gamma \), so that if \( |x - y| < \eta \) then \( [x,y] \) contains at most one point \( d_i \) implies

\[
K^\beta_{j+1} \leq \max(2\eta^{-\beta} K^\beta_{j+1}, 2\tilde{K}^\beta_{j+1}).
\]

The above bound together with (10) yield \( E_\beta < \infty \) so that, for all \( g \in \mathcal{V}_\beta \) with a periodic turning point, and all \( \beta \)-Hölder \( v \) with \( J(g,v) = 0 \), \( v(-1) = v(1) = 0 \), there exists \( j_0 \geq 0 \) so that

\[
K^\beta_{j+1} \leq E_\beta |v|\beta + 2 \theta K^\beta_j, \quad \forall j \geq j_0.
\]

Since \( 2\theta < 1 \), we conclude by a geometric series, taking larger \( j_0 \) if necessary.

It remains to show (10). We concentrate on the second bound (the first is easier and left to the reader). Let \( x, y \in (d_i, d_{i+1}) \) satisfy \( |x - y| < \eta \). Then (8) implies (since \( g \in \mathcal{U}^2 \), the function \( (g^\ell)^\prime \) is \( C^1 \) in the intervals of monotonicity of \( g^\ell \))

\[
|\alpha_{(j+1)\ell}(x) - \alpha_{(j+1)\ell}(y)| \leq |v_\ell(x) + \alpha_{j\ell}(g^\ell(x))| \left| \frac{1}{(g^\ell)^\prime(x)} - \frac{1}{(g^\ell)^\prime(y)} \right| + |v_\ell(x) - v_\ell(y)| + |\alpha_{j\ell}(g^\ell(x)) - \alpha_{j\ell}(g^\ell(y))| \left| \frac{(g^\ell)^\prime(y)}{(g^\ell)^\prime(x)} \right| \leq (L^\beta + K^\beta_j) \sup_{x,y} \left| \frac{(g^\ell)^\prime(x)}{(g^\ell)^\prime(y)} \right| |x - y| + \frac{L^\beta + \sup_{x,y} |(g^\ell)^\prime| |x - y|}{\inf_{x,y} |(g^\ell)^\prime|} \leq (L^\beta + K^\beta_j) D |x - y| + (\lambda^\ell \beta^\ell + \theta \beta^\beta) |x - y|^{\beta}.
\]

**Step IV.** Defining \( \tilde{\alpha}_n = \frac{1}{n} \sum_{j=0}^{n-1} \alpha_{j\ell} \), we can choose a subsequence \( \tilde{\alpha}_n \), converging uniformly on \( I \) to a function \( \tilde{\alpha} \) satisfying \( |\tilde{\alpha}|\beta \leq C_\beta |v|\beta \). By (8),

\[
v_\ell(x) = \tilde{\alpha}(g^\ell(x)) - (g^\ell)^\prime(x)\tilde{\alpha}(x), \quad \forall x \notin \{d_1, \ldots, d_p\}.
\]
Let $\alpha: I \to \mathbb{R}$ be the unique bounded solution vanishing at $c$ to the TCE (3) for $g$ and $v$, as in (4). Then
\begin{equation}
\psi_t(x) = \alpha(g'(x)) - (g')'(x)\alpha(x), \quad \forall x \notin \{d_1, \ldots, d_p\}.
\end{equation}
Since $\hat{\alpha}$ is continuous (11) and (12), imply $\alpha = \hat{\alpha}$. We proved $|\alpha|_p \leq C(\beta)v|_p$, for all $g \in V_\beta$ with a periodic turning point, and thus the proposition.

2.3. Banach spaces of functions of bounded variation. We shall consider the Banach space of functions of bounded variation
\[ BV = \{ \varphi: \mathbb{R} \to \mathbb{C} \mid \text{var}(\varphi) < \infty, \text{supp}(\varphi) \subset I \} / \sim, \]
endowed with the norm $\|\varphi\|_{BV} = \inf_{\varphi \sim \psi} \text{var}(\psi)$, where var denotes total variation, and $\varphi_1 \sim \varphi_2$ if the bounded functions $\varphi_1$, $\varphi_2$ differ on an at most countable set. In addition, for $1 \leq p < \infty$ we shall work with the Banach space of functions of bounded $p$-variation (used in interval dynamics by Wong [17])
\[ BV_p = \{ \varphi: \mathbb{R} \to \mathbb{C} \mid \text{var}_p(\varphi) < \infty, \text{supp}(\varphi) \subset I \} / \sim, \]
where
\[ \text{var}_p(\varphi) = \sup_{x_1 < x_2 < \ldots < x_n} \left( \sum_{i=1}^{n} |\varphi(x_{i+1}) - \varphi(x_i)|^p \right)^{1/p}, \]
the supremum ranging over all ordered finite subsets of $\mathbb{R}$. Note that $\text{var}_1 = \text{var}$ and $BV = BV_1$. Wong [17] does not quotient by the equivalence relation $\varphi_1 \sim \varphi_2$, but his results remain unchanged if we consider elements in $BV_p$ modulo $\sim$ (a function in $BV_p$ is continuous except on an at most countable set, see also [11, Lemma 1.4.a, Lemma 2.7] and [7]). Note that for each $p \geq 1$ there is $C \geq 1$ so that $|\varphi|_p \leq C\|\varphi\|_{BV_p}$ for all $\varphi$, and if $\varphi$ is $1/p$-Hölder, then $\|\varphi\|_{BV_p} \leq |\varphi|_1^{1/p}$. Also,
\begin{equation}
\|\varphi_1 \varphi_2\|_{BV_p} \leq 2\|\varphi_1\|_{BV_p}\|\varphi_2\|_{BV_p} \quad \forall p \geq 1,
\end{equation}
and $\|\varphi \circ h\|_{BV_p} = \|\varphi\|_{BV_p}$ for any homeomorphism $h: I \to I$ and all $p \geq 1$.

Remark 2.5. The reason we consider spaces $BV_p$ for $p \neq 1$ is because we are concerned with differentiability in the $t$-parameter and we will have to deal with derivatives $\partial_t(\psi \circ h_1)|_{t=0} = \psi' \cdot \partial_t h_1|_{t=0}$ or $\partial_t(f'_t \circ h_1)|_{t=0} = f'_0 \partial_t h_1|_{t=0} + v'$, where $v' = \partial_t f'_t|_{t=0} = C^1$, but $\partial_t h_1|_{t=0}$ does not belong to $BV$ in general. We shall see, however, that Proposition 2.3 implies that $\alpha = \partial_t h_1|_{t=0}$ lies in $BV_p$ for all $p > 1$.

3. Weak differentiability of the SRB via the pressure
The main result of this section (Theorem 3.1) says that for any $j \geq 1$, if $f_t$ is a $C^{J+j+1}$ family of piecewise expanding $C^{J+j+2}$ unimodal maps in the topological class of $f_0$, then $R(t) = \int \psi \text{d}\mu_t$ is $C^j$ if $\psi$ is $C^{J+j+1}$-Lipschitz. Even if $j = 1$, this is a new result (Theorem 5.1 in [4] only gives differentiability at $t = 0$). The argument is based on the topological pressure of the potential $(s, t) \mapsto -\log |f'_t \circ h_t| + s(\psi \circ h_t)$ for the map $f_0$. It is simple, but does not give the formula for $\partial_t R(t)|_{t=0}$ (or higher order derivatives). Using the linear response formula from [4, Theorem 5.1] or Theorem 4.1 below, Theorem 3.1 will imply Corollary 4.4.

Theorem 3.1. For any integer $j \geq 1$, if $f_t$ is a $C^{j+1}$ family of piecewise expanding $C^{j+2}$ unimodal maps in the topological class of a mixing map $f_0$, then there is $\hat{\epsilon} > 0$ so that for any $C^{J+j+1}$-Lipschitz function $\psi$ the map $R(t) = \int \psi \text{d}x$ is $C^j$ in $(-\hat{\epsilon}, \hat{\epsilon})$. 
As an immediate corollary of Theorem 3.1 and Proposition A.1, we recover the first claim of [4, Theorem 5.1] if $\psi$ is $C^{1+\text{Lip}}$ (we do not need the assumption $\partial f|_{l=0} = X \circ f_0$ used in [4, Theorem 5.1]):

**Corollary 3.2.** Assume that $f_t$ is a $C^2$ family of piecewise expanding $C^3$ unimodal maps, where $f_0$ is a good mixing map. If $f_t$ is tangent to the topological class of $f_0$ then for any $C^{1+\text{Lip}}$ function $\psi : I \to \mathbb{C}$, the map $R(t) = \int \psi \, d\mu_t$ is differentiable at $t = 0$.

**Proof of Theorem 3.1.** Fix $\psi \in C^{1+\text{Lip}}$, recall the notation $h_t$ from (1), put

$$g_{s,t}(y) = \frac{\exp(s\psi(h_t(y)))}{|f'_t(h_t(y))|}, \quad y \in I \setminus \{c\},$$

and consider the transfer operator

$$\widetilde{L}_{s,t}\phi(x) = \sum_{f(y)=x} g_{s,t}(y)\phi(y).$$

Note that $L_0 = \widetilde{L}_{0,0}$ is the usual transfer operator for $f_0$. We let $L_0$ act on $BV_p$ for any fixed $p \in [1, p_0)$, where $p_0$, depending on $f_0$ through $\inf|f'_0|$ and $\sup|f''_0|$, is given by the main Theorem of [17], which says that $L_0$ on $BV_p$ has spectral radius 1, essential spectral radius < 1, and 1 is the only eigenvalue of modulus 1. Furthermore, 1 is a simple eigenvalue with an eigenvector $\rho_0$ which is strictly positive on $\{c_2, c_1\}$. The corresponding fixed vector $\nu_0$ of $L_0$ is just Lebesgue measure $dx$. We normalise so that $\int \rho_0 \, d\nu_0 = 1$ and $\nu_0(I) = 1$. (Of course, $\mu_0 = \rho_0 \, dx$.) We shall view $\widetilde{L}_{s,t}$ as a perturbation of $L_0$, more precisely we write

$$\widetilde{L}_{s,t}(\phi) = L_0\left(\frac{g_{s,t}}{\rho_0^t}\phi\right).$$

Since $g_{0,0}^{-1} = |f'_0| \in BV_p$ the bound (13) implies that $P_{s,t}(\phi) = \frac{\partial \psi}{\rho_0^t} \phi$ is a bounded operator on $BV_p$. Clearly, $P_{0,0} = \text{id}$. Since $\|g_{s,t} - g_{0,0}\|_{BV_p} \to 0$ as $(s, t) \to (0, 0)$, the operators $\widetilde{L}_{s,t}$ on $BV_p$ have a real positive simple maximal eigenvalue with a spectral gap, uniformly in $(s, t)$ close enough to $(0, 0)$, by classical perturbation theory [9]. In particular, the operator $\widetilde{L}_t = \widetilde{L}_{0,t}$ on $BV_p$ has a simple eigenvalue at 1, for the fixed point $\rho_t = \rho_0 \circ h_t$, where $\mu_t = \rho_t \, dx$ is the SRB measure of $f_t$, and the rest of its spectrum lies in a disc of strictly smaller radius. Note that the fixed point of $\widetilde{L}_t$ is the measure $\nu_t$ defined by

$$\int \phi \, dx = \int \phi \, d\mu_t.$$

(By definition $\nu_t$ is a probability measure and $\int \rho_t \, d\nu_t = 1$.)

Consider first the case $j = 1$. Lemma 3.3 below implies that the map $s \mapsto P_{s,t}$ is $C^1$ from $\mathbb{R}$ to the Banach space of $C^1$ maps from $\{|t| < \epsilon\}$ to bounded operators on $BV_p$, and

$$\partial_s P_{s,t}|_{s=u} = (\psi \circ h_t) P_{u,t}, \quad \forall u \in \mathbb{R}.$$  

Therefore, $s \mapsto L_{s,t}$ is $C^1$ from $\mathbb{R}$ to the Banach space of $C^1$ maps from $\{|t| < \epsilon\}$ to bounded operators on $BV_p$, and

$$\partial_s \widetilde{L}_{s,t}|_{s=u}(\phi) = \widetilde{L}_{u,t}(\psi \circ h_t)(\phi), \quad \forall u \in \mathbb{R}. $$
We are thus in a position to apply classical perturbation theory of an isolated simple eigenvalue (see [9, Ch. VII.1.3] for the analytic case, see e.g. [3, Lemma 3.2] for the differentiable setting). It follows on the one hand that, in a neighbourhood of \((0,0)\), the maximal 2 eigenvalue \(\lambda_{s,t} > 0\) of \(\widetilde{\mathcal{L}}_{s,t}\) acting on \(BV_p\) is a \(C^1\) function of \(s\) to the space of \(C^1\) maps from \(\{\|t\| < \epsilon\}\) to \(\mathbb{R}\). On the other hand, by “tedious but straightforward calculations” and [9, Ch. VII.1.5, Ch. II.2.2], (to quote [13, (5.2)]), we have

\[
\partial_s (\log \lambda_{s,t})|_{s=0} = \int \psi \circ h_t \tilde{\rho}_t \, d\nu_t = \int \psi \, d\mu_t
\]

(use that \(\tilde{\rho}_t\) and \(\nu_t\) are the fixed eigenvectors of \(\widetilde{\mathcal{L}}_{0,t}\) and its dual). Since \(t \mapsto \partial_s (\log \lambda_{s,t})|_{s=0}\) is a \(C^1\) function in a neighbourhood of zero, we have proved Theorem 3.1 in the case \(j = 1\). If \(j \geq 2\), apply Lemma 3.4 instead of Lemma 3.3. \(\square\)

The following result is the key ingredient in the proof of Theorem 3.1, its proof hinges on Proposition 2.3 and [4, Prop. 2.4]:

**Lemma 3.3.** Let \(f_t\) be a \(C^2\) family of piecewise expanding \(C^3\) unimodal maps in the topological class of \(f_0\). For any \(p > 1\) there exists \(\epsilon_p > 0\) so that for any \(\psi : I \to \mathbb{R}\) which is \(C^1+\text{Lip}_p\), the map \(s \mapsto g_{s,t}\) defined by (14) is \(C^1\) from \(\mathbb{R}\) to the Banach space of \(C^1\) maps from \(\{\|t\| < \epsilon_p\}\) to \(BV_p\). In addition, recalling the notation (1),

\[
\partial_s g_{s,t}|_{s=0} = (\psi \circ h_t) g_{u,t}, \quad \forall u \in \mathbb{R}.
\]

In fact, \(s\)-analyticity holds in Lemma 3.3, but we shall not need this.

**Proof of Lemma 3.3.** Fix \(p > 1\). For every \(x \neq c\), all small \(t\), and all \(s_1 < s_2 \leq 1\), there exists \(s_3 \in [s_1, s_2]\) so that

\[
\frac{g_{s_1,t}(x) - g_{s_2,t}(x)}{(s_1 - s_2)} = \psi(h_t(x)) g_{s_2,t}(x) - (s_1 - s_2)^2 (\psi(h_t(x)))^2 g_{s_2,t}(x).
\]

(Just use the Taylor formula for \(s \mapsto g_{s,t}(x)\) and the intermediate value theorem.)

So, to prove both differentiability and (19), it suffices to see that the three maps

\[
t \mapsto g_{s,t}, \quad t \mapsto (\psi \circ h_t) g_{s,t}, \quad t \mapsto (\psi \circ h_t)^2 g_{s,t},
\]

are \(C^1\) from a neighbourhood of 0 to \(BV_p\), uniformly in \(s\) in any compact set \(K \subset \mathbb{R}\).

In view of this, we first study the maps \(t \mapsto h_t(x)\). By [4, Proposition 2.4], there exists \(\tilde{\epsilon} > 0\) so that the set of maps \(\{t \mapsto h_t(x), \ x \in I\}\) is bounded in \(C^1+\text{Lip}_{[\tilde{\epsilon}, \epsilon]}(I)\). Differentiating with respect to \(t\) the equation \(h_t \circ f_0 = f_t \circ h_t\), and setting \(\alpha_t = \partial_t h_t \circ h_t^{-1}\), we get

\[
\alpha_t(f_t(c)) = \partial_t f_t(c), \quad \partial_t f_t(x) = \alpha_t(f_t(x)) - f'_t(x) \alpha_t(x), \forall x \neq c, |x| < \tilde{\epsilon}.
\]

Since \(\alpha_t(c) = 0\) this implies \(J(f_t, \partial_t f_t) = 0\) for \(|t| < \tilde{\epsilon}\) (recall (4)), so, for any fixed

\[
\beta \in (1/p, 1) \quad \text{(we may and shall assume also that } \beta < 1/\sqrt{p}, \text{ e.g.)},
\]

Proposition 2.3 gives \(C\) and \(\epsilon_p > 0\) so that

\[
|\alpha_t|_{\beta} \leq C, \quad \forall |t| < \epsilon_p.
\]

Let \(\alpha_t^\eta\) be the \(\eta\)-regularisation (in the variable \(x\)) of \(\alpha_t\), that is the convolution \(\alpha_t^\eta(x) = \int \alpha_t(y) \kappa_\eta(x-y) \, dy\) of \(\alpha_t\) with a convolution kernel \(\kappa_\eta(x) = \eta^{-1} \kappa(x/\eta)\),

\[\text{Of course, } \log \lambda_{s,t} \text{ is the topological pressure of } \log g_{s,t}.\]
where the $C^\infty$ function $\kappa : \mathbb{R} \to \mathbb{R}_+$ is supported in $[-1,1]$, and $\int \kappa(x) \, dx = 1$.

Note for further use that (21) gives $C$ so that, for all $|t| < \epsilon_p$,

$$\|a^n_t\|_{C^{1+1/p}} \leq \frac{\tilde{C}}{\eta^{1+1/p} - \eta} \cdot \|a^n_t\|_{C^{\beta}} \cdot |a^n_t - \alpha_t|_{1/p} \leq \tilde{C} \eta^{\beta-1/p} \cdot \forall \eta \in (0,1).$$

We now consider $t \mapsto g_{s,t}$. For $x \neq c$, we have

$$\partial_t g_{s,t}(x) = e^{s\psi(h_t(x))} \left[ \frac{\psi'(h_t(x))a_t(h_t(x))}{|f'_t(h_t(x))|^2} - \frac{\partial_t(|f'_t(h_t(x))|^2)}{|f'_t(h_t(x))|^2} \right],$$

where

$$\partial_t(|f'_t(h_t(x))|^2) = -\text{sgn}(x)(f''_t(h_t(x))a_t(h_t(x)) + \partial_t f'_t(h_t(x))).$$

We claim that the function $x \mapsto \partial_t g_{s,t}(x)$ has bounded $BV_{1/\beta}$ norm, uniformly in $s \in K$ and $|t| < \epsilon_p$. Indeed, decomposing

$$\partial_t g_{s,t} = b_{s,t} \circ h_t,$$

note that each $h_t : I \to I$ is a homeomorphism leaving both $[-1,c]$ and $[c,1]$ invariant, while $b_{s,t}$ is $\beta$-H"older on $[-1,c)$ and $(c,1]$, uniformly in $s \in K$ and $|t| < \epsilon_p$ (because $\psi'$ is $C^\beta$, $f_t$ is a $C^2$ family of $C^3$ maps$^3$, and $a_t$ is $\beta$-H"older, uniformly in $|t| < \epsilon_p$), and $\sup_{s \in K,|t| < \epsilon_p} |b_{s,t}| < \infty$.

To conclude, it suffices to prove that our candidate $b_{s,t} \circ h_t \in BV_p$ is really the $t$-derivative of $g_{s,t}$ (uniformly in $s$), that is,

$$\lim_{t \to t_1} \sup_{s \in K} \frac{\|g_{s,t_2} - g_{s,t_1} - b_{s,t_1} \circ h_{t_1}\|_{BV_p}}{|t_2 - t_1|} = 0, \forall |t_1| < \epsilon_p,$$

and that this derivative is continuous in $t$ (uniformly in $s$), that is,

$$\lim_{t \to t_1} \sup_{s \in K} \sup_{t_1, t_2 \in [t,\bar{t}]} \|b_{s,t_2} \circ h_{t_2} - b_{s,t_1} \circ h_{t_1}\|_{BV_p} = 0, \forall |t_1| < \epsilon_p.$$
Hölder (or Jensen) inequality, there exists $\bar{C} > 0$ so that for all $s \in K$, all $|t_1| < \epsilon_p$, $|t_2| < \epsilon_p$, all $\eta \in (0,1)$, and any $x_0 < x_1 < \cdots < x_N \leq c$,

$$\sum_{i=0}^{N-1} \left| b^\eta_{s,t_1} (h_{t_2} (x_i)) - b^\eta_{s,t_1} (h_{t_1} (x_i)) - b^\eta_{s,t_1} (h_{t_2} (x_{i+1})) + b^\eta_{s,t_1} (h_{t_1} (x_{i+1})) \right|^p$$

$$= \sum_i \left| \int_{t_1}^{t_2} \partial_t (b^\eta_{s,t_1} (h_t (x_i))) \, dt - \int_{t_1}^{t_2} \partial_t (b^\eta_{s,t_1} (h_t (x_{i+1}))) \, dt \right|^p$$

$$\leq \sum_{t_1} \int_{t_1}^{t_2} \left| (b^\eta_{s,t_1})' (h_t (x_i)) \alpha_t (h_t (x_i)) - (b^\eta_{s,t_1})' (h_t (x_{i+1})) \alpha_t (h_t (x_{i+1})) \right|^p \, dt$$

$$= \int_{t_1}^{t_2} \sum_{t_1} \left| (b^\eta_{s,t_1})' (h_t (x_i)) \alpha_t (h_t (x_i)) - (b^\eta_{s,t_1})' (h_t (x_{i+1})) \alpha_t (h_t (x_{i+1})) \right|^p \, dt$$

$$\leq |t_2 - t_1| \left( \sup_{x \in (-\infty, c)} \left| (b^\eta_{s,t_1})' (x) \right| \sup_t \| \alpha_t \circ h_t \|_{BV_p} + \sup_{x,t} \| \alpha_t \| \left| (b^\eta_{s,t_1})' (-\infty, c) \right|^{1/p} \right)$$

$$\leq \bar{C} \frac{|t_2 - t_1|}{y^{1+1/p} - \beta}.$$  

(28)

We used (21) in the last inequality. The same bounds hold for $c \leq x_0 < x_1 < \cdots < x_N$, and it is easy to estimate the jump of $b^\eta_{s,t_1} \circ h_{t_2} - b^\eta_{s,t_1} \circ h_{t_1}$ at $x = c$ uniformly in $s$ and $t_1, t_2$.

We next analyse the contribution of $b_{s,t_1} - b_{s,t_2}$ to the second term of (27). For this, observe that if $h$ is an orientation preserving homeomorphism fixing $c$ and $b$ is $\beta$-Hölder on $[-\infty, c]$ and $[c, \infty]$, then $\| b \circ h \|_{BV_p} \leq \| b \|_{(-\infty,c)} + \| b \|_{[c,\infty)} + \| b(c) - b(c^-) \|$. Then, the last bound of (22) and its analogue for the $\eta$-regularisation of $\psi'$, $f''$ and $\partial_t f''$ give a constant $C'$ so that for all $|t_1| < \epsilon_p$, $|t_2| < \epsilon_p$ and $\eta \in (0,1)$

$$\sup_{s \in K} \| (b_{s,t_1} - b^\eta_{s,t_1}) \circ h_{t_2} - (b_{s,t_1} - b^\eta_{s,t_1}) \circ h_{t_1} \|_{BV_p} \leq 2 \sup_{s,t} \| (b_{s,t_1} - b^\eta_{s,t_1}) \circ h_{t} \|_{BV_p}$$

$$\leq C' \eta^{\beta - 1/p}. \tag{29}$$

Taking $\xi \in (0,1)$ and setting $\eta = \left( t_2 - t_1 \right) \frac{\xi}{t_2 - t_1}$, we get from (28–29) that

$$\lim_{t_2 \to t_1} \sup_{t_2 \in [t_1, \bar{t}]} \sup_{s \in K} \| b_{s,t_1} \circ h_{t_2} - b^\eta_{s,t_1} \circ h_{t_1} \|_{BV_p} = 0.$$

To analyse the first term of (27), we start by noticing that since $t \mapsto \partial_t h_t$ is Lipschitz, there exists a set $D_p \subset (-\epsilon_p, \epsilon_p)$ of full Lebesgue measure so that $\partial_t h_t$ is differentiable at all $t$ in $D_p$. Differentiating twice $f_t \circ h_t (x) = h_t \circ f (x)$ with respect to $t$ and $^4$ setting $\alpha_t^2 = \partial^2_{tt} h_t \circ h_t^{-1}$, we obtain for all $x \neq c$ and all $t \in D_p$ that

$$f''_t (x) \alpha_t (x)^2 + 2 \partial_t f'_t (x) \alpha_t (x) + \partial_{tt} f_t (x) = \alpha^2_t (f_t (x)) - f'_t (x) \alpha^2_t (x). \tag{30}$$

The left-hand-side of the above TCE is $\beta$-Hölder in $[-1, c]$ and $[c, 1]$ and continuous in $I$, since $\alpha_t (c) = 0$ for every small $t$, so it is $\beta$-Hölder continuous. Therefore, by Proposition 2.3, there exist $\epsilon_p > 0$ and a constant $C''$ so that

$$|\alpha^2_t|_\beta \leq C'', \ \forall t \in D_p. \tag{31}$$

$^4$This is similar the proof of [4, Proposition 2.4], but we will make a more careful analysis of what was called $F_t$ there.
The fundamental theorem of calculus holds for the Lipschitz (and therefore almost everywhere differentiable) function \( t \mapsto b_{s,t} \) and gives

\[
(b_{s,t_2} - b_{s,t_1})h_{t_2}(x) = \int_{t_1}^{t_2} \partial_h b_{s,t}(h_{t_2}(x)) \, dt, \quad \forall x \neq c.
\]

The first term of (27) may then be estimated via the Hölder inequality and the fundamental theorem of calculus (32), as in (28), but exploiting (31) instead of using \( \eta \)-regularisation. Details are left to the reader.

Finally, to show (25), start from

\[
g_{s,t}(x) - g_{s,t_1}(x) - (t - t_1)b_{s,t_1}(h_{t_1}(x)) = \int_{t_1}^{t} (b_{s,t_2}(h_{t_2}(x)) - b_{s,t_1}(h_{t_1}(x))) \, dt_2,
\]

for all \( x \neq c \), and use the Hölder inequality and (26) (details are left to the reader).

The analysis of the maps \( t \mapsto (\psi \circ h_t)g_{s,t} \) and \( t \mapsto (\psi \circ h_t)^2g_{s,t} \) goes along exactly the same lines.

For the higher regularity statement in Theorem 3.1, we use the following result (again, analyticity in \( s \) holds):

**Lemma 3.4.** Let \( j \geq 2 \). Let \( f_t \) be a \( C^{j+1} \) family of piecewise expanding \( C^{j+2} \) unimodal maps in the topological class of \( f_0 \). For any \( p > 1 \) there exists \( \epsilon_p > 0 \) so that for any \( \psi : I \to \mathbb{R} \) which is \( C^{j+1,Lip} \), the map \( s \mapsto g_{s,t} \) defined by (14) is \( C^1 \) from \( \mathbb{R} \) to the space of \( C^j \) maps from \( \{|t| < \epsilon_p\} \) to \( BV_p \), and, recalling (1), \( \partial_s g_{s,t}\big|_{s=u} = (\psi \circ h_t)g_{s,t} \).

**Proof.** Since the family \( f_t \) is \( C^{j+1} \), the set \( \{t \mapsto h_t(x), \ x \in I\} \) is bounded in \( C^{j+1,Lip} \) by [4, Proposition 2.4]. Let \( \beta \in (1/p,1) \) (with \( \beta < 1/\sqrt{p} \), say). Assume first \( j = 2 \). Then, by (30), the function \( \alpha_t^2 = \partial_{h_t}^2 h_t \circ h_t^{-1} \), is well-defined for all \( |t| < \epsilon_p \) and there exists \( C \) so that \( |\alpha_t^2|_{\beta} \leq C \) for every \( |t| < \epsilon_p \). For \( j \geq 3 \), a higher order TCE similar to (30) gives that \( \alpha_t^j = \partial_{h_t}^j h_t \circ h_t^{-1} \) is \( \beta \)-Hölder for all \( |t| < \epsilon_p \). We put \( \alpha_t^1 = \alpha_t \).

Then, computing \( \partial_{h_t}^j g_{s,t}(x) \) at \( x \neq c \) gives \( b_{s,t}^{(j)}(h_t(x)) \), where \( b_{s,t}^{(j)} \) is an expression involving derivatives of order at most \( j \) of \( \psi(x) \), functions \( \alpha_{t}^\ell \), for \( 1 \leq \ell \leq j \), and derivatives (in \( x \), \( t \), or mixed) of total order at most \( j \) of \( f_t'(x) \), in the numerator, and \( |f_t'(x)|^m \) for \( m \geq 1 \) in the denominator. Our differentiability assumptions on \( \psi \) and the family \( f_t \) then allow us to proceed as in the proof of Lemma 3.3 (using Taylor series of higher order).

\[\square\]

4. Recovering the linear response formula

Here we give a slightly different proof of the differentiability of \( R(t) = \int \psi \, d\mu_t \), where \( \mu_t \) is the SRB measure of \( f_t \), still relying heavily on Proposition 2.3 (via Lemma 4.2). The advantage with respect to Theorem 3.1 is that we recover the formula for \( \partial_s R(t)|_{s=0} \), and we need only assume that \( \psi \) is \( C^0 \). (In particular, this gives a new proof of [4, Theorem 5.1].) We also get new information in Corollary 4.4 by combining Theorems 3.1 and 4.1.

We need notation. By [1, Proposition 3.3], we may decompose the invariant density of a piecewise expanding \( C^3 \) unimodal mixing map \( f_t \) as \( \rho_t = \rho_{reg,t} + \rho_{sal,t} \).
where $\rho_{reg,t} \in BV \cap C^0$, $\rho_{reg,t}' \in BV$, and

$$\rho_{sal,t} = \sum_{k=1}^{M_f} s_{k,t} H_{c_{k,t}}.$$  

(Here, $H_u(x)$ denotes the Heaviside function $H_u(x) = -1$ if $x < u$, $H_u(x) = 0$ if $x > u$ and $H_u(u) = 1/2$.) If $M_f = \infty$ then it is not difficult to show that (see e.g. [1, 4], noting that if $c_{1,t}$ is preperiodic but not periodic our notation is slightly different than the notation there)

$$s_{k,t} = \frac{s_{1,t}^{(k-1)}}{(f^{k-1})'(c_{1,t})}, \quad \forall k \geq 1.$$  

We simply write $\rho_0 = \rho = \rho_{reg} + \rho_{sal}$.

To compute the formula for the derivative, we shall assume, as in [4], that $v = \partial_t f_t|_{t=0}$ is of the form $v = X \circ f_0$ for a $C^2$ function $X : I \to \mathbb{R}$.  

**Theorem 4.1.** Let $f_t$ be a $C^2$ family of piecewise expanding $C^3$ unimodal maps. Assume that $f_0$ is good and mixing, that $f_t$ is tangent to the topological class of $f_0$, and that $v = \partial_t f_t|_{t=0} = X \circ f_0$ for a $C^2$ function $X$. Then, as Radon measures,

$$\lim_{t \to 0} \frac{\mu_t - \mu_0}{t} = -\alpha \rho_{sal}' - (\text{id} - \mathcal{L}_0)^{-1}(X' \rho_{sal} + (X \rho_{reg})') \, dx,$$

where the function $\alpha$ is given by (4), and the operator $\mathcal{L}_0 = \tilde{\mathcal{L}}_{0,0}$ is defined by (15). In addition, $\alpha$ is $\beta$-Hölder for any $\beta < 1$.

**Proof.** Set $f = f_0$ for convenience. By Proposition A.1, we can assume that $f_t$ lies in the topological class of $f$, denoting the conjugacies by $h_t$ as usual. The transfer operator $\tilde{\mathcal{L}}_t = \tilde{\mathcal{L}}_{0,t}$ for $f$ and the weight $|f_t'| h_t$ (recall (15)) is conjugated to the transfer operator for $f_t$ and $|f_t'|^{-1}$ defined by

$$\mathcal{L}_t \varphi(x) = \sum_{f_t(y) = x} \frac{\varphi(y)}{|f_t'(y)|},$$

via

$$\tilde{\mathcal{L}}_t(\varphi \circ h_t) = \mathcal{L}_t(\varphi) \circ h_t.$$  

Since $h_t$ is a homeomorphism, it gives rise to an isometry of $BV_p$, and (35) together with the main Theorem of [17] applied to $\mathcal{L}_t$ imply that there exists $p_0 > 1$ so that for any $p \in (1, p_0)$ there exists $\epsilon_p > 0$ so that for all $|t| < \epsilon_p$ the operator $\tilde{\mathcal{L}}_t$ acting on $BV_p$ has a maximal eigenvalue equal to 1, which is simple, and the rest of the spectrum lies in a disc of strictly smaller radius (i.e., $\tilde{\mathcal{L}}_t$ has a spectral gap). The fixed points of $\tilde{\mathcal{L}}_t$ and its dual, $\tilde{\mathcal{L}}_t^* = \rho_t \circ h_t$ and $\nu_t$ from (17), were introduced in the proof of Theorem 3.1. We can alternatively see that $\tilde{\mathcal{L}}_t$ has a spectral gap on $BV_p$ by noting that it is a small multiplicative perturbation of $\mathcal{L}_0 = \tilde{\mathcal{L}}_0$ on $BV_p$; Recall that $\lim_{t \to 0} \|g_{0,t} - g_{0,0}\|_{BV_1} = 0$ and (16). Observe for further use that this implies $\lim_{t \to 0} \|\rho_t - \rho_0\|_{BV} = 0$.

From now on, we fix $p \in (1, p_0)$.

We next show that $t \mapsto \tilde{\rho}(t) \in BV_p$ and $t \mapsto \nu_t \in BV^*_p$ are differentiable at $t = 0$. By [4, Prop. 2.4, Cor. 2.6] $v$ is horizontal for $f_0$, $t \mapsto h_t(x)$ is differentiable, uniformly in $x \in I$, and $\alpha = \partial_t h_t|_{t=0}$ is continuous, with $\alpha(c) = 0$, $\alpha(c_1) = \cdots$.
\(X(c)\), and \(\alpha\) is the unique bounded solution (4) to the TCE (3). In addition, Proposition 2.3 gives that \(\alpha\) is \(\beta\)-Hölder for arbitrary \(\beta < 1\) (we shall take \(\beta \in (1/p, 1/\sqrt{p})\)).

Our assumptions on \(f_t\) then imply that \(v'\) is \(C^1\) and the following operator is bounded on \(BV_p^r\):

\[
\mathcal{M}\varphi(x) = - \sum_{f(y) = x} \frac{f''(y)\alpha(y) + v'(y)}{|f'(y)|^p} \varphi(y).
\]

(Write \(\mathcal{M}\) as \(L_0\) composed with a multiplication operator, like in (16), and use (13).) Lemma 4.2 below easily implies that \(t \mapsto \tilde{\mathcal{L}}_t\) is differentiable as an operator on \(BV_p\), and that

\[
\partial_t \tilde{\mathcal{L}}_t|_{t=0} = \mathcal{M}.
\]

As in the proof of Theorem 3.1, perturbation theory then gives that \(t \mapsto \tilde{\rho}_t \in BV_p\) and \(t \mapsto \nu_t \in BV_p^*\) are differentiable at \(t = 0\).

We next show that \(t \mapsto \mu_t = \rho_t \, dx\) is differentiable as a Radon measure, exploiting the formula for \(\mathcal{M}\) to get the claimed formula for \(\partial_t \mu_t|_{t=0}\). Fix \(\psi : I \to \mathbb{C}\) continuous. Since \(\tilde{\rho}_0 = \rho_0\), we can decompose

\[
\int \psi \rho_t \, dx - \int \psi \rho_0 \, dx = \int \psi \tilde{\rho}_t \, dx - \int \psi \tilde{\rho}_0 \, dx + \int \psi \hat{\rho}_t \, dx - \int \psi \hat{\rho}_0 \, dx.
\]

We shall now see that

\[
\lim_{t \to 0} \frac{\int \psi(\rho_t - \rho_0 \circ \hat{h}_t) \, dx}{t} = - \int \psi \alpha \rho_0' \, dx.
\]

In view of (39), note first that \(s_{k,t} \to s_k\) as \(t \to 0\): Indeed, since \(\lim_{t \to 0} \|\tilde{\rho}_t - \rho_0\|_{BV} = 0\) we have (in \(BV\))

\[
\lim_{t \to 0} \tilde{\rho}_t = \lim_{t \to 0} \left( \rho_{reg,t} \circ \hat{h}_t + \sum_{k=1}^{M_f} s_{k,t} H_{c_k} \right) = \rho_0 = \rho_{reg} + \sum_{k=1}^{M_f} s_k H_{c_k}.
\]

(We gave another proof of \(\lim_{t \to 0} s_{k,t} = s_k\) in Step 1 of [4, Proof of Theorem 5.1].)

Decompose \(\rho_t - \rho_0 \circ \hat{h}_t\) in (39) into \(\rho_{sal,t} - \rho_{sal,t} \circ \hat{h}_t + \rho_{reg,t} - \rho_{reg,t} \circ \hat{h}_t\). For the singular term, we have in the sense of Radon measures:

\[
\lim_{t \to 0} \frac{\rho_{sal,t} - \rho_{sal,t} \circ \hat{h}_t}{t} = - \sum_{k=1}^{M_f} \alpha(c_k) s_{k,t} \text{Dirac}_{c_k} = - \alpha \rho_{sal}'.
\]

(Just use that \(s_{k,t} \to s_k\) and (33), which implies the \(s_{k,t}\) decay exponentially in \(k\) uniformly in \(t\).)

We claim that the contribution of the regular term \(\rho_{reg,t}\) in the decomposition of (39) is \(- \int \psi \alpha \rho_{reg}' \, dx\). Indeed, recall that \(\rho_{reg,t} \in BV\). In fact, we may decompose

\[
\rho_{reg,t} = \rho_{reg,t} + \rho_{reg,sal,t},
\]

with (see Step 2 of [4, Proof of Theorem 5.1]) \(\rho_{reg,sal,t}\) is \(C^1\) (uniformly in \(t\)) and

\[
\rho_{reg,sal,t}' = \sum_{k=1}^{M_f} s_{k,t}' H_{c_k,t}.
\]

\(^6\)Claim (41) was also proved in Step 1 of [4, Proof of Theorem 5.1].
where the \( s'_{k,t} \) decay exponentially uniformly in \( t \). In fact (see [4, (69), (71)]) \(^7\)

\[ s'_{k,t} = E'_{k,t} - E_{k,t}, \text{ with } E_{k,t}' = \frac{s'_{k-1,t}}{(f'_1(c_{k-1,t}))^2}, \ k \geq 2, \]

and

\[ E_{k,t} = \frac{s_{k-1,t}f''(c_{k-1,t})}{(f'_1(c_{k-1,t}))^3}, \ k \geq 2, \ E'_{1,t} = -\frac{\rho_{reg,t}')(c)}{(f'_1(c_-))^2} + \frac{\rho_{reg,t}')(c)}{(f'_1(c_+))^2} \]

\[ E_{1,t} = \left( \frac{\rho_{reg,t}(c)f''(c_-)}{(f'_1(c_-))^3} + \frac{\rho_{reg,t}(c)f''(c_+)}{(f'_1(c_+))^3} \right) \]

\[ + \sum_{k=2, c_{k-1,t} > c} s_{k-1,t} \left( \frac{f''(c_-)}{(f'_1(c_-))^3} - \frac{f''(c_+)}{(f'_1(c_+))^3} \right). \]

The argument above giving \( s_{k,t} \to s_k \) also yields \( s'_{k,t} \to s'_k \) (just differentiate once). Therefore, just like in (41), we have

\[ \lim_{t \to 0} \int \psi \left( \frac{\rho_{reg,t} - \rho_{reg,t}}{t} \right) \circ h_t \ dx = -\sum_{k=1}^{M_f} \int \alpha(x) s'_k \psi(x) H_{c_k}(x) \ dx \]

\[ = \int \psi \alpha \rho'_{reg,t} \ dx. \]

In view of handling the term \( \rho_{reg,t} \) from (42), observe that

\[ \lim_{t \to 0} \| \varphi - \varphi \circ h_t \|_{BV} = 0, \ \forall \varphi \in C^1. \]

Indeed, for \( \delta > 0 \) and any partition \( x_0 < x_i < x_{i+1} < \ldots < x_n \) let \( N \leq n \) be so that \( \min(x_N, \inf_t h_t(x_N)) > 1 - \delta, \) and since \( |h_t(y) - y| = O(t) \) uniformly in \( y, \) take \( t_0 \) so that \( |x_i - h_t(x_i)| < \delta/N \) for all \( i \leq N \) and \( |t| < t_0. \) Then use

\[ \sum_{i=0}^{n-1} |\varphi(x_i) - \varphi(x_{i+1}) - \varphi(h_t(x_i)) + \varphi(h_t(x_{i+1}))| \leq 2 \sum_{i=0}^{N} |\varphi(x_i) - \varphi(h_t(x_i))| \]

\[ + \sum_{i=N+1}^{n-1} |\varphi(x_i) - \varphi(x_{i+1})| + \sum_{i=N+1}^{n-1} |\varphi(h_t(x_i)) - \varphi(h_t(x_{i+1}))|. \]

Since the \( C^1 \) norm of \( \rho_{reg,t} \) is bounded uniformly in \( t, \) (44) easily implies that

\[ \lim_{t \to 0} \| \rho_{reg,t} - \rho_{reg} \|_{BV} = 0. \] (Note for the record that, since \( \| \rho_t - \rho_0 \|_{BV} = 0 \) and \( s_{k,t} \to s_k, \ s'_{k,t} \to s'_k, \) with \( t \)-uniformly exponentially decaying \( s_{k,t}, s'_{k,t}, \) this implies \( \lim_{t \to 0} \| \rho_{reg,t} - \rho_{reg} \|_{BV} = 0. \) \(^8\)) Then, by the mean value theorem and the \( x \)-uniform differentiability of

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\(^7\)If \( c \) is periodic then \( \rho_{reg,t}'(c) \) may be undefined, but \( \rho_{reg,t}'(c_{\pm}) \) are both defined.

\(^8\)\( \lim_{t \to 0} \| \rho_{reg,t} - \rho_{reg} \|_{BV} = 0 \) can also be proved from Keller-Liverani techniques on the spaces in [4], without using the fact that \( \rho_t \to \rho_0 \) in the \( BV \) norm. This implies \( s'_{1,t} \to s'_1, \) like we proved \( s_{1,t} \to s_1 \) in [4, Proof of Theorem 5.1].
\[
t \mapsto h_t(x)
\]
\[
\lim_{t \to 0} \int \psi \frac{\rho_{reg,t} - \rho_{reg,t} \circ h_t}{t} \, dx
= \lim_{t \to 0} \int \psi(x) \frac{\rho_{reg,t}(x) - \rho_{reg,t}(h_t(x))}{t} \, dx
= \lim_{t \to 0} \int \psi(x) \rho_{reg,t}'(x_t) \alpha(x) \, dx
\]
where \( x_t \) is in the interval between \( x \) and \( h_t(x) \), and we used in the last line that \( \rho_{reg,t}' \) is continuous on the compact interval \( I \), uniformly in \( t \), together with (45).

Putting (41-43-46) together, we find (39).

Next, we claim that we have (in \( BV_p \))
\[
-\alpha \rho_{reg} + (id - \mathcal{L}_0)^{-1}(id - \Pi_0)(\mathcal{M}\rho_0) = -(id - \mathcal{L}_0)^{-1}(id - \Pi_0)(X'\rho_0 + X\rho_{reg}) .
\]
(Recall from [1, Proof of Proposition 4.4] that \( \Pi_0(X'\rho_0 + X\rho_{reg}) = 0 \).) Since the TCE (3) implies, using \( \nu' = (X' \circ f) \cdot f' \),
\[
\mathcal{M}\rho_0(x) = (X(x) - \alpha(x)) \sum_{f(y) = x} \frac{f''(y)}{|f'(y)|^2} \rho_0(y) - X'(x)\rho_0(x) ,
\]
to prove (10), it suffices to show
\[
-\alpha \rho_{reg} + (id - \mathcal{L}_0)^{-1}(id - \Pi_0)(\mathcal{M}\rho_0) = -(id - \mathcal{L}_0)^{-1}(id - \Pi_0)(X\rho_{reg}) ,
\]
where \( \mathcal{M}\varphi(x) = \sum_{f(y) = x} \frac{f''(y)}{|f'(y)|^2} \varphi(y) \). Note that \( \varphi \) if \( x \in [-1, c_1] \) is not along the postcritical orbit we have, using \( (\rho_{reg})' = (\rho_0)' \) if \( y \) is not on the postcritical orbit,
\[
(\rho_{reg})'(x) = (\rho_0)'(x) = (\mathcal{L}_0(\rho_0))'(x) = \sum_{f(y) = x} \frac{(\rho_{reg})'(y)}{|f'(y)|^2} - \frac{\rho_0(y)f''(y)}{|f'(y)|^2} |f'(y)|^2 .
\]
Therefore, for any \( x \in I \) which is not on the postcritical orbit
\[
\mathcal{M}(\rho_0)(x) = \sum_{f(y) = x} \frac{\rho_{reg}'(y)}{|f'(y)|^2} - \rho_{reg}'(x) .
\]
\text{This is [4, (70)].}
In other words, we have (in $BV_p$)
\[ \tilde{M}(\rho_0) = \bar{M}(\rho_{\text{reg}}) - \rho_{\text{reg}}', \]
where $\bar{M}(x) = \sum_{f(y)=x} \frac{\varphi(y)}{|f'(y)|}$. So we have reduced the claim (48) to
\[ -\alpha \rho_{\text{reg}}' - (\id - \mathcal{L}_0)^{-1}(\id - \Pi_0)(\mathcal{L}_0 - \id)(\alpha \rho_{\text{reg}}') = 0, \]
that is, using $\Pi_0 \mathcal{L}_0 = \Pi_0$,
\[ (\mathcal{L}_0 - \id)(\alpha \rho_{\text{reg}}') = (\id - \Pi_0)(\mathcal{L}_0 - \id)(\alpha \rho_{\text{reg}}') = (\mathcal{L}_0 - \id)(\alpha \rho_{\text{reg}}'). \]

Finally, since $\rho_0 \in BV_p$, and since $t \mapsto \nu_t$ and $t \mapsto \tilde{\rho}_t$ are differentiable in $BV_p$ and $BV_p^*$, respectively, we have (in $BV_p$)
\[ \partial_t \nu_t(\rho_0) \tilde{\rho}_t |_{t=0} = \partial_t \nu_t(\rho_0) |_{t=0} = 0. \]
Take the Lebesgue average of both sides of (50). Since $\partial_t \int \tilde{\rho}_t \, dx = 0$ (because each $\tilde{\rho}_t \, dt$ is a probability), and since
\[ -\int (\id - \mathcal{L}_0)^{-1}(X' \rho_{\text{sat}} + (X \rho_{\text{reg}})' ) \, dx = 0 \]
(use again $\Pi_0 (X' \rho_0 + X \rho_{\text{reg}}' ) = 0$), we find that $\partial_t \nu_t(\rho_0) |_{t=0} \int \rho_0 \, dx = 0$. Therefore, $\partial_t \nu_t(\rho_0) |_{t=0}$, and putting together (38), (39), (47), (48), and (50), we have proved the theorem.

We have (a simplification of Lemma 3.3):

**Lemma 4.2.** Let $f_t$ be a $C^2$ family of piecewise expanding $C^3$ unimodal maps in the topological class of $f_0$. Set $v = \partial_t f_t |_{t=0}$. For any $p > 1$ the map $t \mapsto g_t = \frac{1}{|f'_t|_{t=0}} \in BV_p$ is $C^1$ in a neighborhood of 0, and $\partial_t g_t |_{t=0} = -\frac{\psi''(v')}{|f'_0|_{t=0}}$.

**Proof.** Differentiability follows from Lemma 3.3 applied to $\psi \equiv 0$. The value of the derivative is given by (23–24) in the proof of that lemma, since $\partial_t f_t |_{t=0} = v'$.

**Remark 4.3.** We have the following strengthening of Lemma 4.2 if $f_t$ is a $C^3$ family of piecewise expanding $C^4$ unimodal maps: For any $p > 1$ the map $t \mapsto g_t = \frac{1}{|f'_t|_{t=0}} \in BV_p$ is $C^2$ in a neighborhood of 0, and $|g_t - g_0| + \frac{\psi''(v')}{|f'_0|_{t=0}} = O(t)$.

Recalling (36–37), this implies that $\| \frac{\mathcal{L}_0}{t} - \mathcal{M} \|_{BV_p} = O(t)$.

We shall get the following (new) result as a corollary of Theorems 3.1 and 4.1:

**Corollary 4.4.** If $f_t$ is a $C^2$ family of piecewise expanding $C^3$ unimodal maps in the topological class of $f_0$, and if $\partial_t f_t |_{t=0} = X \circ f_0$ for a $C^2$ function $X$, then there exists $\epsilon > 0$ so that $t \mapsto \mu_t$ is $C^1$ from $(-\epsilon, \epsilon)$ to Radon measures.

In particular, under the assumptions of Corollary 4.4, the Radon measure
\[ -\alpha \rho_{\text{sat}}' - (\id - \mathcal{L}_t)^{-1}(X_t \rho_{\text{reg}}, t)' \, dx \]
(recall (34)) is continuous as a function of $t$. (Here, $\alpha_t$ solves (3) for $f_t$ and $v_t = \partial_t f_t |_{t=t}$, and $X_t \circ f_t = v_t$.) This fact is not clear a priori from the formula.

**Remark 4.5.** We expect that a careful analysis of the term (39) for $C^3$ functions $\psi$ would allow to bypass the reference to Theorem 3.1 in the proof of Corollary 4.4.
Proof of Corollary 4.4. We want to show that \( t \mapsto \partial_u \mu_u |_{u=t} = \tilde{\mu}_t \) is continuous: We know that \( \tilde{\mu}_t \) exists for all small \( t \) (as a Radon measure) by Theorem 4.1. Clearly, \( | \int \psi \, d\tilde{\mu}_u | \leq C \sup |\psi| \) for all continuous \( \psi \) and all small enough \( t \).

Assume for a contradiction that \( t \mapsto \tilde{\mu}_t \) is discontinuous at \( t_0 \). This means that there exist \( \psi \in C^0 \), with \( \sup |\psi| = 1, \delta > 0 \), and a sequence \( t_m \), with \( |t_m - t_0| < 1/m \), so that \( | \int \psi \, d\tilde{\mu}_{t_0} - \int \psi \, d\tilde{\mu}_{t_m} | > \delta \) for all \( m \). Take \( \psi \in C^1 \) so that \( \sup |\psi - \tilde{\psi}| < \delta/4 \). Then \( | \int \psi \, d\tilde{\mu}_{t_0} - \int \psi \, d\tilde{\mu}_{t_m} | > \delta/2 \) for all \( m \). But Theorem 3.1 implies \( | \int \psi \, d\tilde{\mu}_{t_0} - \int \psi \, d\tilde{\mu}_{t_m} | < \delta \) if \( m \) is large enough, a contradiction. \( \square \)

APPENDIX A. A CONSEQUENCE OF THE KELLER-LIVERANI BOUNDS FROM [4]

We state here for the record an immediate corollary of [4, Proposition 3.3] which was based on results in [12] (see Remark 2.1 and note that the assumptions below imply \( \sup t |f_t - \hat{f}_t| = O(t^2) \)):

**Proposition A.1.** Let \( f_t \) be a \( C^2 \) family of piecewise expanding \( C^2 \) unimodal maps. Assume that \( f_0 \) is mixing and good, and that \( f_t \) is tangent to the topological class of \( f_0 \), denoting by \( \hat{f}_t \) a family in the topological class of \( f_0 \) with \( \hat{f}_0 = f_0 \) and \( \partial f_{t=0} = \partial \hat{f}_{t=0} \).

Let \( \mu_t = \rho_t \, dx \) and \( \tilde{\mu}_t = \tilde{\rho}_t \, dx \) be the SRB measures of \( f_t \) and \( \hat{f}_t \), respectively. Then for any \( \xi < 2 \) there exists \( C > 0 \) so that for all small \( t \)

\[ \| \rho_t - \tilde{\rho}_t \|_{L^\xi(\text{Leb})} \leq C |t|^\xi. \]

**References**


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