Jacques Herbrand (1908 - 1931)

Principal writings in logic

Recherches sur la théorie de la démonstration
(thesis, completed April 1929, defended June 1930)

Sur le problème fondamental de la logique mathématique
(body of paper completed September 1929, appendix added April 1931)

Sur la non-contradiction de l'arithmétique
(completed July 1931)

Collected editions

Écrits logiques, avec une préface de Jean van Heijenoort
(Presses Universitaires de France, 1968)

Logical Writings, ed. with an introduction by Warren Goldfarb, and notes by Burton Dreben, Warren Goldfarb and Jean van Heijenoort (Reidel and Harvard U. Press, 1971)

First Readings

S. Zaremba, *La logique des mathématiques* (Paris, 1926)
A. Whitehead and B. Russell, *Principia Mathematica*, vol. 1 (Cambridge, England, 1910, 2nd edition 1925)
D. Hilbert and W. Ackermann, *Grundzüge der theoretischen Logik* (Berlin 1928)
D. Hilbert, Neubegründung der Mathematik (1922), Die logischen Grundlagen der Mathematik (1923), Über das Unendliche (1925), Die Grundlagen der Mathematik (1927)

Herbrand on metamathematics:
"La caracteristique de cette nouvelle doctrine, que son fondateur a appelée la "métamathematique", voulant exprimer par là que toutes les questions de principe conçernant les mathématiques devaient lui être soumises, c'est qu'elle a pour object d'étude non pas les objets dont s'occupent habituellement les mathematiciens, mais les phrases mêmes qu'ils peuvent prononcer sur ces objets. ... c'est en quelque sorte une mathématique du langage." (1930a)

Claude Chevalley in 1982, about Herbrand
"C'est le goût de l'aventure intellectuelle qui le porta vers la logique. Il était séduit par le caractère grandiose de l'oeuvre de Hilbert."

Russell's axiom in *9 of *Principia*

$$Fu \vee Fv \supset \exists xFx$$

Herbrand's Axiomatization of First-Order Logic

Axioms: truth-functionally valid formulas that contain no quantifiers

Rules of inference:

(1) Generalization: from Fvv infer ∃xFvx

from Fv infer ∀xFx

(2) Rules of passage: inside a formula, replace

~∀xFx	by	∃x~Fx	or vice versa
~∃xFx	by	∀x~F	"
G . ∀xFx	by	∀x(G . Fx)	" if G does not contain free x
G.∃xFx	by	∃x(G . Fx)	" "
G ∨ ∀xFx	by	∀x(G ∨ Fx)	" "
G ∨ ∃xFx	by	∃x(G ∨ Fx)	" "

(3) Simplification: from G ∨ G infer G

(later version: inside a formula, replace G ∨ G with G)

(4) Modus ponens: From F and F ⊃ G, infer G.

To derive: $\exists xFx \vee \exists xFx \supset \exists xFx$

Cannot get there from $Fv \vee Fv \supset Fv$

Start with $(Fu \vee Fv \supset Fu) \vee (Fu \vee Fv \supset Fv)$
Obtain

$$(Fu \vee Fv \supset \exists xFx) \vee (Fu \vee Fv \supset \exists xFx)$$

Use rule of simplification

$$Fu \vee Fv \supset \exists xFx$$

Two universal generalizations and rules of passage now suffice.

Question: what formulas G can be obtained from the following procedure: find a disjunction of quantifier-free instances of G that is truth-functionally valid, and such that the variables that are used in the instances allow the use of generalization, simplification, and rules of passage so as to get G back.

Answer: all prenex formulas that are logically valid.

Fundamental Theorem

(Prenex example) Let F be $\exists x \forall y \exists z \forall w R(x,y,z,w)$.

Herbrand functional form: $\exists x \exists z R(x,f(x),z,g(x,z))$

Herbrand domain D(F,p): an arbitrary item, and values for f and g (and any other function signs in R) iterated up to p times.

Herbrand (validity) expansion: disjunction of instances of Herbrand functional form over D(F,p):

$$\bigvee R(s,f(s),t,g(s,t))$$

In general, functional form obtained by replacing each *essentially universal* variable v with terms made up of an index function sign f_v and arguments those *essentially existential* variables that govern v.

Theorem.

(I) If F is provable in a standard axiomatic system of first-order logic, then from its derivation we can calculate a number p such that the E(F,p) is truth-functionally valid

(II) If E(F,p) is truth-functionally valid, then we can construct a derivation of F that starts with a quantifier-free tautology and uses only generalization rules, rules of passages, and the rule of simplification.

Herbrand's Restatement

(I) If F is theorem of pure logic then ~F is not true in any infinite domain.

(II) If F is not an theorem of pure logic, we can construct an infinite domain in which ~F is true.

Herbrand's definition: ~F is true in an infinite domain = no expansion E(F,p) is valid.

Nonconstructive step.

If no expansion E(F,p) is valid, then there are fixed values that make every E(F,p) false. In this case there is a model for ~F.

Nonconstructive Consequence of Herbrand's Theorem (II)

If some E(F,p) is valid, then F is derivable (without modus ponens).

If no E(F,p) is valid, then ~F has a model.

This yields the completeness of first-order logic.

Dual Notion (“Satisfiability Expansion”)

$E^S(F,p)$: equivalent to the negation of $E(\sim F,p)$.

conjunction of instances of Skolem functional form of F (functional terms replace essentially existential quantifiers)

Theory T is consistent iff for each conjunction F of its axioms and each p, $E^S(F,p)$ is truth-functionally satisfiable.

Herbrand's Quick Consistency Proof

Axioms

Any universally true computational arithmetical truths

Restricted mathematical induction:

$$F(0) \,.\, \forall x(F(x) \supset F(x+1)) \supset \forall x F(x)$$

for all F(x) without quantifiers

Inference rules: full use of first-order logic.

Hilbert-Bernays, Grundlagen der Mathematik II
First ∈-Theorem
Any derivation in first order logic from premises that lack quantifiers to a conclusion that lacks quantifiers can be effectively transformed into a derivation entirely without quantifiers.

Gödel, *1938 (published 1995)
If we take a theory which is constructive in the sense that each existence assertion made in the axioms is covered by a construction, and if we add to this theory the non-constructive notion of existence and all the logical rules concerning it, e.g., the law of excluded middle, we shall never get into any contradiction.

No-counterexample Interpretation (Gödel 1938, Kreisel 1951) Suppose $\exists x \forall y \exists z \forall w R(x,y,z,w)$ is provable in a formalized arithmetic. Recall that the Herbrand functional form is $\exists x \exists z R(x,f(x),z,g(x,z))$. From the formal proof we can find computational instructions that, given any functions f and g yield numbers m and n such that $R(m,f(m),n,g(m,n))$ is true (thus defeating the claim that f and g always provide counterexamples).

Fallacy in Herbrand's Proof

(I) If F is provable in a standard axiomatic system of first-order logic, then from its derivation we can calculate a number p such that the E(F,p) is truth-functionally valid

Claims

(1) If E(F,p) and E(F ⊃ G),q) are valid, then so is E(G,n) for n = max(p,q).

(2) If E(F,p) is valid, and J comes from F by a rule of passage, then E(J,n) is valid for n = p.

Herbrand shows (2) implies (1).

False Lemma

Lemma 3.3. Suppose formula J comes from formula H by replacing a positive subformula G . ∃xFx by ∃x(G . Fx). If E(H,p) is valid then so is E(J,n) for n=p.

Gödel (c. 1941), Dreben and Denton (1966): this is true for n computable from p and syntactic featues of H.

Goldfarb 1993: n is no less than double exponential in p, single exponential in number of essentially universal quantifiers in H.

A Form of Incompleteness

A is conjunction of axioms of a theory containing arithmetic. Let $\Phi(p,q,r)$ formalize: r encodes a numerical interpretation of $E^S(A,p)$ that makes the expansion true and assigns the numerical value q to the constant c.

$\forall x \forall y \exists z \Phi(x,y,z)$ expresses the existence, for any p and q, of interpretations that make $E^S(A,p)$ true, and give the constant c the value q.

If the theory is a true theory of arithmetic, then $\forall x \forall y \exists z \Phi(x,y,z)$ is true.

Claim. $\forall x \forall y \exists z \Phi(x,y,z)$ is not derivable in the theory.
Proof. If so, so is $\forall x \exists z \Phi(x,x,z)$. By Herbrand's Theorem, there is an expansion of $A \supset \forall x \exists z \Phi(x,x,z)$ that is valid, say of order p. This has the form $E^S(A,p) \supset \bigvee \Phi(c,c,t)$, where the disjunction is over t in the domain of order p. Now take an interpretation I of $E^S(A,p)$ that makes it true, that assigns c the value p, and that minimizes the largest number assigned to any term in the domain.

Since I makes $E^S(A,p)$ true, it must make $\bigvee \Phi(c,c,t)$ true; since c is given the numerical value p, I must make $\Phi(p,p,n)$ true for some n among the numerical values that I assigns to terms. But then $\Phi(p,p,n)$ is true "in the real world", so n encodes an interpretation that makes $E^S(A,p)$ true, assigns c the value p, and has a smaller maximal number assigned to a term. Contradiction.