FEYNMAN AMPLITUDES AND LIMITS OF HEIGHTS

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Abstract. We investigate from a mathematical perspective how Feynman amplitudes appear in the low-energy limit of string amplitudes. In this paper, we prove the convergence of the integrands. We derive this from results describing the asymptotic behavior of the height pairing between degree-zero divisors, as a family of curves degenerates. These are obtained by means of the nilpotent orbit theorem in Hodge theory.

À Jean-Pierre Serre, en témoignagne d'admiration

1. Introduction

This paper grew out of an attempt to understand from a mathematical perspective the idea we learned from physicists that Feynman amplitudes should arise in the low-energy limit $\alpha' \to 0$ of string theory amplitudes, cf. [33] and the references therein. Throughout we work in space-time $\mathbb{R}^D$ with a given Minkowski bilinear form $\langle \cdot\,, \cdot \rangle$.

String amplitudes are integrals over the moduli space $\mathcal{M}_{g,n}$ of genus $g \geq 1$ curves with n marked points. They are associated to a fixed collection of external momenta $\underline{\mathbf{p}} = (\mathbf{p}_1, \dots, \mathbf{p}_n)$, which are vectors in $\mathbb{R}^D$ satisfying the conservation law $\sum_{i=1}^n \mathbf{p}_i = 0$. Up to some factors carrying information about the physical process being studied, the string amplitude can be written as (see e.g. [35, p.182])

$$A_{\alpha'}(g, \underline{\mathbf{p}}) = \int_{\mathcal{M}_{g,n}} \exp(-i\,\alpha' \mathcal{F})\; d\nu_{g,n}. \tag{1.1}$$

In this expression, $d\nu_{g,n}$ is a volume form on $\mathcal{M}_{g,n}$, independent of the momenta, α' is a positive real number, which one thinks of as the square of the string length, and $\mathcal{F}\colon \mathcal{M}_{g,n} \to \mathbb{R}$ is the continuous function defined at the point $[C, \sigma_1, \dots, \sigma_n]$ of $\mathcal{M}_{g,n}$ by

$$\mathcal{F}([C, \sigma_1, \dots, \sigma_n]) = \sum_{1\leq i,j\leq n} \langle \mathbf{p}_i, \mathbf{p}_j \rangle\, \mathfrak{g}'_{\mathrm{Ar},C}(\sigma_i, \sigma_j),$$

where $\mathfrak{g}'_{\mathrm{Ar},C}$ denotes a regularized version of the canonical Green function on C so that it takes finite values on the diagonal.

On the other hand, correlation functions in quantum field theory are calculated using Feynman amplitudes, which are certain finite dimensional integrals associated to graphs. Recall that a (massless) Feynman graph $(G, \mathbf{p})$ consists of a finite graph $G = (V, E)$, with vertex and edge sets V and E, respectively, together with a collection of external momenta $\underline{\mathbf{p}} = (\mathbf{p}_v)_{v\in V}$, $\mathbf{p}_v \in \mathbb{R}^D$, such that $\sum_{v\in V} \mathbf{p}_v = 0$. To the Feynman graph $(G, \underline{\mathbf{p}})$ one associates two polynomials in the variables $\underline{Y} = (Y_e)_{e\in E}$. The first Symanzik ψ_G, which depends only on the graph G, is given by the following sum over the spanning trees of G:

$$\psi_G(\underline{Y}) = \sum_{T\subseteq G} \prod_{e\notin T} Y_e.$$

The second Symanzik polynomial ϕ_G, depending on the external momenta as well, admits the expression

$$\phi_G(\underline{\mathbf{p}}, \underline{Y}) = \sum_{F\subseteq G} q(F) \prod_{e\notin F} Y_e.$$

Here F runs through the spanning 2-forests of G, and $q(F)$ is the real number $-\langle \mathbf{p}_{F_1}, \mathbf{p}_{F_2}\rangle$, where $\mathbf{p}_{F_1}$ and $\mathbf{p}_{F_2}$ denote the total momentum entering the two connected components F_1 and F_2 of F. The polynomial ϕ_G is quadratic in $\underline{\mathbf{p}}$ and it will be also convenient to consider the corresponding bilinear form, which we denote by $\phi_G(\underline{\mathbf{p}}, \underline{\mathbf{p}}', \underline{Y})$.

One of the various representations of the Feynman amplitude associated to $(G, \underline{\mathbf{p}})$ is, up to some elementary factors which we omit,

$$I_G(\underline{\mathbf{p}}) = \int_{[0,\infty]^E} \exp(-i\, \phi_G/\psi_G)\; d\pi_G, \tag{1.2}$$

where $d\pi_G$ denotes the volume form $\psi_G^{-D/2} \prod_E dY_e$ on $[0, \infty]^E$. This can be found e.g. in formula (6-89) of [17]. If one interprets the locus of integration as the space of metrics (i.e. lengths of edges) on G, then (1.2) looks like a path integral with the action ϕ_G/ψ_G.

Although both amplitudes diverge in general, one may still ask if $I_G(\underline{\mathbf{p}})$ is related to the asymptotic of $A_{\alpha'}(g, \underline{\mathbf{p}})$ when α' goes to zero, as physics suggests. The graph G appears as the dual graph of a stable curve C_0 with n marked points lying on the boundary of the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ (recall that the irreducible components X_v of C_0 are indexed by the vertices of G, whereas the singular points correspond to the edges). The question can be then split into two different problems, namely (i) the convergence of the integrands, and (ii) the convergence of the measure $\nu_{g,n}$, in an appropriate sense, and along the boundary of $\overline{\mathcal{M}}_{g,n}$, to a linear combination of the measures π_G for the (marked) dual graphs G associated to the strata. Our main result in this paper answers question (i) in the affirmative when the external momenta satisfy the "on shell" condition: the integrand in string theory converges indeed to the integrand appearing in the Feynman amplitude.

To make this more precise, let us consider the versal analytic deformation $\pi\colon \mathcal{C}' \to S'$ of the marked curve C_0, which we think of as a smooth neighborhood of C_0 in the analytic stack $\overline{\mathcal{M}}_{g,n}$. Here S' is a polydisc of dimension $3g-3+n$, the total space $\mathcal{C}'$ is regular and $\mathcal{C}'_0$, the fibre of π at 0, is isomorphic to C_0. For each edge $e \in E$, let $D_e \subset S'$ denote the divisor parametrizing those deformations in which the point associated to e remains singular. Then $D = \bigcup_{e\in E} D_e$ is a normal crossings divisor whose complement $U' = S' \setminus D$ can be identified with $(\Delta^*)^E \times \Delta^{3g-3-|E|+n}$. Over U', the fibres $\mathcal{C}'_s$ are smooth curves of genus g. Moreover, the versal family comes together with n disjoint sections $\sigma_i\colon S' \to \mathcal{C}'$ which do not meet the double points of C_0. We denote by $\underline{\mathbf{p}}^G = (\mathbf{p}^G_v)_{v\in V}$ the *restriction* of $\underline{\mathbf{p}}$ to G. By this we mean that, for each vertex $v \in V$, the external momentum $\mathbf{p}^G_v$ is obtained by summing those $\mathbf{p}_i$ associated to the sections σ_i which meet C_0 on the irreducible component X_v.

An *admissible segment* is a continuous maps $\underline{t}\colon [0,\varepsilon] \to S'$ from an interval of length $\varepsilon > 0$ such that $\underline{t}((0,\varepsilon]) \in U'$ and, letting t_e denote the coordinate corresponding to $e \in E$ in the factor $(\Delta^*)^E$ of U', the limit $\lim_{\alpha'\to 0} |t_e(\alpha')|^{\alpha'}$ exists and belongs to $(0,1)$. To any admissible segment we attach a collection $\underline{Y} = (Y_e)_{e\in E}$ of positive real numbers (the edge lengths) as follows:

$$Y_e = -\lim_{\alpha'\to 0} \log |t_e(\alpha')|^{\alpha'}.$$

Theorem 1.1 (cf. Theorem 6.8)**.** *Let C_0 be a stable curve of genus $g \geq 1$ with n marked points $\sigma_1, \dots, \sigma_n$ and dual graph $G = (V, E)$, and let $\underline{\mathbf{p}} = (\mathbf{p}_1, \dots, \mathbf{p}_n)$ be a collection of external momenta satisfying the conservation law $\sum_{i=1}^n \mathbf{p}_i = 0$ and the "on shell" condition $\langle \mathbf{p}_i, \mathbf{p}_i\rangle = 0$ for all i. Then, for any admissible segment $\underline{t}\colon I \to \overline{\mathcal{M}}_{g,n}$ such that $\underline{t}(0) = [C_0, \sigma_1, \dots, \sigma_n]$, we have*

$$\lim_{\alpha'\to 0} \alpha' \mathcal{F}(\underline{t}(\alpha')) = \frac{\phi_G(\underline{\mathbf{p}}^G, \underline{Y})}{\psi_G(\underline{Y})},$$

where $\underline{Y} = (Y_e)_{e\in E}$ denotes the edge lengths determined by $\underline{t}$.

We derive the above theorem from results describing the asymptotic behaviour of the archimedean height pairing. We start with the case of disjoint divisors. We consider the versal analytic deformation $\pi\colon \mathcal{C} \to S$ of C_0 (without the marked points), which we think of as a smooth neighborhood of C_0 in the analytic stack $\overline{\mathcal{M}}_g$. Now S is a polydisc of dimension $3g-3$. Again the total space $\mathcal{C}$ is regular and we repeat the construction above letting $D_e \subset S$ denote the divisor parametrizing those deformations in which the point associated to e remains singular. Then $D = \bigcup_{e\in E} D_e$ is a normal crossings divisor whose complement $U = S \setminus D$ can be identified with $(\Delta^*)^E \times \Delta^{3g-3-|E|}$. To accommodate external momenta, we assume that we are given two collections of sections of π, which we denote by $\sigma_1 = (\sigma_{\ell,1})_{\ell=1,\dots,n}$ and $\sigma_2 = (\sigma_{\ell,2})_{\ell=1,\dots,n}$. Since $\mathcal{C}$ is regular, the points $\sigma_{l,i}(0)$ lie on the smooth locus of C_0. We label the markings with two vectors $\underline{\mathbf{p}}_1 = (\mathbf{p}_{l,1})_{l=1}^n$ and $\underline{\mathbf{p}}_2 = (\mathbf{p}_{l,2})_{l=1}^n$ with $\mathbf{p}_{l,i} \in \mathbb{R}^D$ subject to the conservation of momentum, thus obtaining a pair of relative degree zero $\mathbb{R}^D$-valued divisors

$$\mathfrak{A}_s = \sum_{l=1}^n \mathbf{p}_{l,1}\sigma_{l,1}, \qquad \mathfrak{B}_s = \sum_{l=1}^n \mathbf{p}_{l,2}\sigma_{l,2}.$$

We first assume that σ_1 and σ_2 are *disjoint* on each fiber of π. Recall that to any pair $\mathfrak{A}$, $\mathfrak{B}$ of degree zero (integer-valued) divisors with disjoint support on a smooth projective complex curve C, one associates a real number, the *archimedean height*

$$\langle \mathfrak{A}, \mathfrak{B} \rangle = \mathrm{Re}\Big(\int_{\gamma_{\mathfrak{B}}} \omega_{\mathfrak{A}}\Big),$$

by integrating a canonical logarithmic differential $\omega_{\mathfrak{A}}$ with residue $\mathfrak{A}$ along any 1-chain $\gamma_{\mathfrak{B}}$ supported on $C \setminus |\mathfrak{A}|$ and having boundary $\mathfrak{B}$. Coupling with the Minkowski bilinear form on $\mathbb{R}^D$, the definition extends to $\mathbb{R}^D$-valued divisors. We thus get a real-valued function

$$s \mapsto \langle \mathfrak{A}_s, \mathfrak{B}_s \rangle. \tag{1.3}$$

For each $e \in E$, denote by s_e the coordinate in the factor corresponding to e in $U = (\Delta^*)^E \times \Delta^{3g-3-|E|}$, write $y_e = \frac{-1}{2\pi}\log|s_e|$ and put $\underline{y} = (y_e)_{e\in E}$. After shrinking U if necessary, the asymptotic of the height pairing is given by the following result:

Theorem 1.2 (cf. Corollary 5.10)**.** *Assume, as above, that σ_1 and σ_2 are disjoint on each fibre. Then there exists a bounded function*

$h\colon U \to \mathbb{R}$ such that

$$\langle \mathfrak{A}_s, \mathfrak{B}_s \rangle = 2\pi \frac{\phi_G(\underline{\mathbf{p}}_1^G, \underline{\mathbf{p}}_2^G, \underline{y})}{\psi_G(\underline{y})} + h(s). \tag{1.4}$$

Theorem 1.2 only deals with disjoint sections. In order to derive Theorem 1.1 we need to allow the supports of the divisors to intersect, which requires a regularization of the height pairing. Pointwise, the regularization depends on the choice of a metric on the tangent space of the given curve. To regularize the height pairing globally we choose a smooth $(1,1)$-form μ on $\pi^{-1}(U)$ such that the restriction to each fibre $\mathcal{C}_s$ is positive. Using μ we define a regularized height pairing $\langle \mathfrak{A}_s, \mathfrak{A}_s \rangle'_\mu$ (see Section 6). In the case where $g \geq 1$ and μ is the Arakelov metric μ_{Ar} we recover the function $\mathcal{F}$ from the string amplitude:

$$\mathcal{F}([\mathcal{C}_s, \sigma_1(s), \ldots, \sigma_n(s)]) = \langle \mathfrak{A}_s, \mathfrak{A}_s \rangle'_{\mu_{\mathrm{Ar}}}.$$

The asymptotic of the regularized height pairing is described in the following result:

Theorem 1.3 (cf. Theorem 6.5 and Corollary 6.6)**.** *Let $\underline{\mathbf{p}} = (\mathbf{p}_i)_{i=1,\ldots,n}$ be external momenta satisfying the conservation law and $(\sigma_i)_{i=1,\ldots,n}$ a collection of sections $\sigma_i\colon S \to \mathcal{C}$. Put $\mathfrak{A}_s = \sum \mathbf{p}_i \sigma_i(s)$, and let μ be a smooth $(1,1)$-form on $\pi^{-1}(U)$ whose restriction to each curve $\mathcal{C}_s$ is positive. Assume that one of the following conditions hold:*

1. *μ extends to a continuous $(1,1)$-form on $\mathcal{C}$, or*
2. *the $\mathbf{p}_i$ satisfy the "on shell" condition $\langle \mathbf{p}_i, \mathbf{p}_i \rangle = 0$.*

Then there exists a bounded function $h\colon U \to \mathbb{R}$ such that

$$\langle \mathfrak{A}_s, \mathfrak{A}_s \rangle'_\mu = 2\pi \frac{\phi_G(\underline{\mathbf{p}}^G, \underline{y})}{\psi_G(\underline{y})} + h(s).$$

To get Theorem 1.1 from Theorem 1.3 we first observe that the latter can be easily extended to the versal family $\mathcal{C}' \to S'$ of genus g curves with n marked points. For any admissible segment $\underline{t}\colon I \to U'$, we have

$$\begin{aligned}\lim_{\alpha' \to 0} \alpha' \mathcal{F}(\underline{t}(\alpha')) &= \lim_{\alpha' \to 0} \left[\frac{\phi_G(\underline{\mathbf{p}}^G, (-\log |t_e(\alpha')|^{\alpha'})_e)}{\psi_G((-\log |t_e(\alpha')|^{\alpha'})_e)} + \alpha' h(s) \right] \\ &= \frac{\phi_G(\underline{\mathbf{p}}^G, \underline{Y})}{\psi_G(\underline{Y})},\end{aligned}$$

where we have used that ϕ_G/ψ_G is homogeneous of degree one, as well as the boundedness of h.

Remark 1.4. If one wants to compute the quantum field theory amplitude (1.2) for "off shell" momenta as a limit of heights in the spirit of this paper, the surprising "on shell" condition in Theorem 1.1 can be avoided by simply taking momenta $\underline{\mathbf{p}}^G = \underline{\mathbf{p}}_1^G = \underline{\mathbf{p}}_2^G$ and disjoint multisections σ_1, σ_2 which have the same intersection data with components of the curve at infinity. One can combine equation (1.4) and the above limit calculation, noting that $\phi_G(\underline{\mathbf{p}}^G, \underline{Y}) = \phi_G(\underline{\mathbf{p}}_1^G, \underline{\mathbf{p}}_2^G, \underline{Y})$.

The proofs of Theorems 1.2 and 1.3 are based on the Hodge theoretic interpretation of the archimedean height. Since both sides of the equality (1.4) are bilinear in the momenta, we can reduce to the case of integer-valued divisors. Then $\mathfrak{A}_s$ and $\mathfrak{B}_s$ define a *biextension* mixed Hodge structure $H_{\mathfrak{B}_s,\mathfrak{A}_s}$ with graded pieces $\mathbb{Z}(1), H^1(\mathcal{C}_s, \mathbb{Z}(1)), \mathbb{Z}(0)$. The moduli space of such biextensions is the $\mathbb{C}^\times$-bundle associated to the Poincaré line bundle over $J(\mathcal{C}_s) \times \widehat{J(\mathcal{C}_s)}$, and one recovers the archimedean height by evaluating its canonical metric at $H_{\mathfrak{B}_s,\mathfrak{A}_s}$. As s varies, the biextensions $H_{\mathfrak{B}_s,\mathfrak{A}_s}$ fit together into an admissible variation of mixed Hodge structures over U. We shall write the period map

$$\widetilde{\Phi}\colon \widetilde{U} \longrightarrow \mathbb{H}_g \times \mathrm{Row}_g(\mathbb{C}) \times \mathrm{Col}_g(\mathbb{C}) \times \mathbb{C},$$

where $\widetilde{U}$ is the universal cover of U and $\mathbb{H}_g$ the Siegel upper half-space. If $\mathcal{P}^\times$ denotes the Poincaré bundle over the universal family of abelian varieties and their duals, $\widetilde{\Phi}$ descends to the map $\Phi\colon U \to \mathcal{P}^\times$ which sends s to $H_{\mathfrak{B}_s,\mathfrak{A}_s}$. Then (a weak version of) the nilpotent orbit theorem for variations of mixed Hodge structures allows us to describe the asymptotic of the height pairing.

In order to prove Theorem 1.3, we need to study the effect of the regularization process. To this end, we consider an analytic family of sections σ_j^u parametrized by u in a small disc such that $\sigma_j^0 = \sigma_j$ and, for each $u \neq 0$, the sections σ_i^u and σ_j are disjoint for all i, j. We also choose a suitable rationally equivalent divisor $\mathfrak{A} + \mathrm{div}(f)$. When the metric extends continuously to $\mathcal{C}$, the difference between the regularizations obtained using the metric μ and changing the section through the function f is bounded, thus proving the result. On the other hand, if the external momenta satisfy the "on shell" condition, then all divergent terms vanish, which makes the regularization process independent of the metric. In this way we deduce the result in the "on shell" case for non-continuous metrics from the result for continuous metrics.

We would like to end this introduction by mentioning that the asymptotic of the height pairing has been previously studied from a mathematical perspective in [11, 14, 16, 23, 26]. In particular, an analogue of theorems 1.2 and 1.3 for families of curves over a one-dimensional base was established by Holmes and de Jong in [16]. Some Hodge theoretic aspects of stable curves appear already in the work of Hoffman [15]. We are less familiar with the physics literature, but many of the ideas in this paper are discussed from a physics viewpoint in [34].

The paper is organized as follows. In Section 2, we discuss Symanzik polynomials in an abstract setting, convenient for the sequel. Section 3 is devoted to the study of the local monodromy of the analytic versal deformation. In Section 4, we recall the definition of the archimedean height pairing, as well as its interpretation in terms of biextensions and the Poincaré bundle. Section 5 contains the proof of Theorem 1.2, for which we need to write the period map and use part of the nilpotent orbit theorem. Finally, in Section 6 we put everything together to get the convergence of the integrands.

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Contents

2. Symanzik Polynomials

In this section, we define the first and the second Symanzik polynomials in an abstract setting. We then show how to recover the usual formulas for ψ_G and ϕ_G in the case of graphs [10]. Throughout, if $\mathbb{K}$ is a field and E a finite set, we write $\mathbb{K}^E = \left\{ \sum_{e\in E} \kappa_e e \mid \kappa_e \in \mathbb{K} \right\}$. For each $e \in E$, we denote by $e^\vee : \mathbb{K}^E \to \mathbb{K}$ the functional which takes the e-th coordinate of a vector.

2.1. **Abstract setting.** Let H be a vector space of finite dimension h over a field $\mathbb{K}$, and suppose we are given a finite set E of cardinality at least $h+1$, and an embedding $\iota : H \hookrightarrow \mathbb{K}^E$. Abusing notation, we write $e^\vee$ as well for the composition $H \hookrightarrow \mathbb{K}^E \to \mathbb{K}$. The function which sends $x \in H$ to $e^\vee(x)^2$ defines a rank one quadratic form $e^{\vee,2}$ on H. When needed, we denote by $\langle \cdot , \cdot \rangle_e$ the corresponding bilinear form.

If we fix a basis $\gamma_1, \dots, \gamma_h$ of H, we can identify the quadratic form $e^{\vee,2}$ with an $h \times h$ symmetric matrix M_e of rank one so that, thinking of elements of H as column vectors, we have

$$e^{\vee,2}(x) = {}^{\mathrm{t}}x M_e x. \tag{2.1}$$

Let $\underline{Y} = \{Y_e\}_{e\in E}$ be a collection of variables indexed by E, and consider the matrix $M = \sum_{e\in E} Y_e M_e$ (associated to the quadratic form $\sum_{e\in E} Y_e e^{\vee,2}$). It is an $h \times h$ symmetric matrix whose entries are linear forms in the Y_e. The linear map $M : H \to H^\vee$ is in fact canonical and independent of the choice of a basis of H.

Definition 2.1. The first Symanzik polynomial $\psi(H, \underline{Y})$ associated to the configuration $H \hookrightarrow \mathbb{K}^E$ is defined as

$$\psi(H, \underline{Y}) = \det(M).$$

Remark 2.2. Note that this definition depends on the choice of a basis of H. For a different basis, M is replaced by ${}^{\mathrm{t}}PMP$, where P is the $h \times h$ invertible matrix transforming one basis into the other, so the determinant gets multiplied by an element of $\mathbb{K}^{\times,2}$. The same argument shows that when $H = L \otimes_{\mathbb{Z}} \mathbb{K}$ for a sublattice L of $\mathbb{Z}^E$ and we restrict to bases coming from L, the first Symanzik polynomial is well-defined.

Let $\mathcal{W} = \mathbb{K}^E/H$. For any nonzero $w \in \mathcal{W}$, we define $H_w \subseteq \mathbb{K}^E$ as the $(h+1)$-dimensional subspace of vectors in $\mathbb{K}^E$ whose images in $\mathcal{W}$ lie in the line spanned by w. Choosing a vector $\omega \in \mathbb{K}^E$ in the preimage of w, we can extend the basis $\{\gamma_1, \ldots, \gamma_h\}$ of H to a basis $\{\gamma_1, \ldots, \gamma_h, \omega\}$ of H_w. The first Symanzik polynomial $\psi(H_w, \underline{Y})$ with respect to this basis of H_w yields the second Symanzik.

Lemma 2.3. *$\psi(H_w, \underline{Y})$ does not depend on the choice of ω.*

Proof. Let $\langle\, .\, ,.\, \rangle = \sum_{e \in E} Y_e \langle\, .\, ,.\, \rangle_e$ be the bilinear form on H_w associated to the quadratic form $\sum Y_e e^{\vee,2}$. The polynomial $\psi(H_w, \underline{Y})$ is the determinant of $\langle\, .\, ,.\, \rangle$ with respect to the basis $\{\gamma_1, \ldots, \gamma_h, \omega\}$. Changing the basis of H_w by adding to ω a linear combination of $\gamma_1, \ldots, \gamma_h$ does not change the determinant of $\langle\, .\, ,.\, \rangle$, so $\psi(H_w, \underline{Y})$ only depends on w and the basis $\{\gamma_1, \ldots, \gamma_h\}$ of H. □

Definition 2.4. The second Symanzik polynomial associated to H, $w \in \mathcal{W}$, and the variables $\underline{Y} = \{Y_e\}_{e \in E}$, is the polynomial

$$\phi(H, w, \underline{Y}) = \psi(H_w, \underline{Y}).$$

Proposition 2.5. *The ratio $\phi(H, w, \underline{Y})/\psi(H, \underline{Y})$ between the first and the second Symanzik polynomials does not depend on the choice of a basis of H.*

Proof. Let $\langle\, .\, ,.\, \rangle$ still denote the bilinear form on H_w associated to the quadratic form $\sum Y_e e^{\vee,2}$. Then $\psi(H, \underline{Y})$ is the determinant, in the given basis, of the restriction of $\langle\, .\, ,.\, \rangle$ to H. Changing the basis multiplies both $\psi(H, \underline{Y})$ and $\phi(H, w, \underline{Y}) = \psi(H_w, \underline{Y})$ by the same factor in $\mathbb{K}^{\times,2}$, from which the claim follows. □

To give another formula for the second Symanzik polinomial which will be used later we introduce the following notation.

Definition 2.6. Let $w \in \mathcal{W} \setminus \{0\}$ and choose $\omega \in \mathbb{K}^E$ in the preimage of w. We denote by $W_e(\omega)$ the column vector with components $\langle \gamma_i, \omega \rangle_e$. If w' and ω' is another choice of such vectors we write

$$Q_e(\omega, \omega') = \langle \omega, \omega' \rangle_e \quad \text{and} \quad Q_e(\omega) = Q_e(\omega, \omega).$$

Proposition 2.7.

(1) *The first Symanzik polynomial $\psi(H, \underline{Y})$ is homogeneous of degree $h = \dim H$ in the variables Y_e.*

(2) *The second Symanzik is given by*

$$\phi(H, w, \underline{Y}) = \det \Big(\sum Y_e \begin{pmatrix} M_e & W_e(\omega) \\ {}^{\mathrm{t}}W_e(\omega) & Q_e(\omega) \end{pmatrix} \Big). \tag{2.2}$$

Moreover, it is homogeneous of degree $h+1$ in the variables Y_e and is quadratic in $w \in \mathcal{W} \setminus \{0\}$.

Proof. The first statement is clear from $\psi(H, \underline{Y}) = \det(\sum Y_e M_e)$. Equation (2.2) is just a reformulation of the definition of the second Symanzik polynomial, from which the last statement follows immediately. □

One can slightly generalize the definition of the second Symanzik polynomial. Let $\mathcal{V}$ be a vector space over $\mathbb{K}$ equipped with a quadratic form q associated to a symmetric bilinear form $\langle \cdot, \cdot \rangle_q$. Using it, one can make sense of the determinant on the right hand side of (2.2) and define $\phi(H, w, \underline{Y})$ for any nonzero element $w \in \mathcal{W} \otimes_{\mathbb{K}} \mathcal{V}$. Typically, for physics applications, $\mathbb{K} = \mathbb{R}$, $\mathcal{V} = \mathbb{R}^D$ is space-time, and q is the Minkowski metric. This works as follows.

One naturally extends $\langle \,.\,, . \,\rangle_e$ to a bilinear pairing

$$\langle \,.\,, . \,\rangle_e : H \times (\mathbb{K}^E \otimes_{\mathbb{K}} \mathcal{V}) \to \mathcal{V},$$

and $e^{\vee,2}$ to the quadratic form Q_e on $\mathbb{K}^E \otimes_{\mathbb{K}} \mathcal{V}$ given by

$$Q_e(\alpha \otimes \beta) = e^{\vee,2}(\alpha) q(\beta).$$

Let $w \in \mathcal{W} \otimes_{\mathbb{K}} \mathcal{V}$ and consider $\omega \in \mathbb{K}^E \otimes \mathcal{V}$ in the preimage of $w \in \mathcal{W} \otimes_{\mathbb{K}} \mathcal{V}$. Applying the bilinear pairing to (γ_i, ω) leads to the column vector $W_e = (v_{e,1}, \dots, v_{e,h})$ with entries $v_{e,i} \in \mathcal{V}$.

Using the above extension, one can now make sense of the determinant in (2.2) and define $\phi(H, w, \underline{Y})$ for $w \in \mathcal{W} \otimes \mathcal{V}$. To explain this, suppose we have a symmetric $(h+1) \times (h+1)$ matrix of the form

$$T = \begin{pmatrix} M & W \\ {}^{\mathrm{t}}W & S \end{pmatrix}$$

where M is an invertible $h \times h$ matrix, W is a (column) vector of dimension h, and S is a scalar. The formula $(\det M)M^{-1} = \operatorname{adj}(M)$, where $\operatorname{adj}(M)$ is the matrix of minors, gives

$$\frac{\det T}{\det M} = -\,{}^{\mathrm{t}}WM^{-1}W + S.$$

Taking $W = \sum_{e\in E} Y_e W_e$, where W_e has now entries in $\mathcal{V}$, the determinant in (2.2) can be written as

$$\frac{\phi(H, w, \underline{Y})}{\psi(H, \underline{Y})} = -\,{}^{\mathrm{t}}WM^{-1}W + Q(\omega), \tag{2.3}$$

where $Q(\omega) = \sum_{e\in E} Y_e Q_e(\omega)$, and the product ${}^{\mathrm{t}}WM^{-1}W$ is interpreted via the bilinear form (in the sense that, developing the product as the sum of the form $\sum v_{e,i} m_{i,j} v_{e,j}$, with $v_{e,i}, v_{e,j} \in \mathcal{V}$, $m_{i,j} \in \mathbb{K}[\underline{Y}]$, becomes $\sum m_{i,j}\langle v_{e,i}, v_{e,j}\rangle_q$).

The expression (2.3) will be later used to relate the second Symanzik to the archimedean height.

2.2. **Graphs.** In what follows, G is a connected graph with edge set $E = E(G)$ and vertex set $V = V(G)$. We will fix an orientation on the edges so we have a boundary map $\partial : \mathbb{Z}^E \to \mathbb{Z}^V,\ e \mapsto \partial^+(e) - \partial^-(e)$, where ∂^+ and ∂^- denote the head and the tail of e, respectively. The homology of G is defined via the exact sequence

$$0 \to H_1(G, \mathbb{Z}) \to \mathbb{Z}^E \xrightarrow{\partial} \mathbb{Z}^V \to \mathbb{Z} \to 0. \tag{2.4}$$

Homology with coefficients in any abelian group is defined similarly.

In order to apply the constructions of the previous section, we write $H = H_1(G, \mathbb{K})$. The exact sequence (2.4) yields an isomorphism

$$\mathcal{W} = \mathbb{K}^E/H \simeq \mathbb{K}^{V,0}, \tag{2.5}$$

where $\mathbb{K}^{V,0}$ consists of those $x \in \mathbb{K}^V$ whose coordinate sum to zero. We will use (2.5) to identify both spaces.

Definition 2.8. The first Symanzik polynomial of a connected graph G is the first Symanzkik polynomial, as in Definition 2.1, associated to the configuration $H = H_1(G, \mathbb{K}) \subset \mathbb{K}^E$. We will denote it by

$$\psi_G(\underline{Y}) = \psi(H, \underline{Y}).$$

Since $H = H_1(G, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{K}$, Remark 2.2 guarantees that ψ_G is independent of the choice of an integral basis. The link between the above definition and the expression for ψ_G given in the introduction is the content of Kirchhoff's matrix-tree theorem [21]. Recall that a subgraph T of G is called a spanning tree if it is connected and simply connected, and satisfies $V(T) = V(G)$.

Proposition 2.9. *The first Symanzik polynomial ψ_G is equal to*

$$\psi_G(\underline{Y}) = \sum_{T\subset G} \prod_{e\notin T} Y_e,$$

where T runs through all spanning trees of G.

The second Symanzik polynomial can be described explicitly via the *external momenta*, as we explain now. Note that, in the situations coming from physics, it also depends on the masses associated to the edges. However, in this paper we only consider the massless case.

Let $\mathcal{V}$ be a vector space over a field $\mathbb{K}$ with a symmetric bilinear form $\langle \cdot\,,\cdot\rangle_q$, and consider $\omega \in \mathbb{K}^E\otimes\mathcal{V}$ which reduces to the vector $w \in \mathcal{W}\otimes\mathcal{V}$, where $H = H_1(G,\mathbb{K})$ and $\mathcal{W} = \mathbb{K}^E/H$. Using the isomorphism (2.5), we have $\mathcal{W}\otimes\mathcal{V} \simeq \mathbb{K}^{V,0}\otimes\mathcal{V} \simeq \mathcal{V}^{V,0}$. In other words, the choice of $w \in \mathcal{W}\otimes\mathcal{V}$ is equivalent to the choice of external momenta $\mathbf{p}_v \in \mathcal{V}$ satisfying the conservation law $\sum_{v\in V}\mathbf{p}_v = 0$.

Definition 2.10. Let G be a connected graph and $\underline{\mathbf{p}} = \{\mathbf{p}_v\} \in \mathcal{V}^{V,0}$ be external momenta. The second Symanzik polynomial of $(G,\underline{\mathbf{p}})$ is

$$\phi_G(\underline{\mathbf{p}},\underline{Y}) = \phi(H,\omega,\underline{Y})$$

for the element $\omega \in \mathbb{K}^E\otimes\mathcal{V}$ with $\mathbf{p} = \partial(\omega)$.

To give an explicit description of the polynomial $\phi_G(\underline{\mathbf{p}},\underline{Y})$, we need to introduce some extra notation. Let G be a connected graph. A spanning 2-forest $F\subset G$ is a subgraph of G, with two connected components F_1 and F_2, satisfying $V(F) = V(G)$ and $H_1(F,\mathbb{Z}) = 0$ (so each F_i is a subtree of G). Given a collection of external momenta $\underline{\mathbf{p}} = (\mathbf{p}_v) \in \mathcal{V}^{V,0}$ and a spanning 2-forest, we define $\mathbf{p}(F_i) = \sum_{v\in V(F_i)}\mathbf{p}_v$, the total momentum entering F_i, and $q(F) = -\langle \mathbf{p}(F_1),\mathbf{p}(F_2)\rangle_q = q(\mathbf{p}(F_1))$, where the last equality follows from the conservation law. Then we have the following proposition, for which we refer the reader e.g. to [8]:

Proposition 2.11. *If G is a connected graph, then*

$$\phi_G(\underline{\mathbf{p}},\underline{Y}) = \sum_{F\subset G} q(F) \prod_{e\notin E(F)} Y_e\,,$$

where the sum runs over all spanning 2-forests F of G.

3. Degeneration of curves

The aim of this section is to interpret the rank one symmetric matrices M_e introduced in (2.1) in terms of the monodromy of a degenerating family of curves [5, 6]. For this, we fix a complex stable curve C_0 of arithmetic genus g and dual graph $G = (V, E)$. Throughout h denotes the first Betti number of G.

Concretely, C_0 is a projective connected nodal curve with smooth irreducible components X_v indexed by the vertices of G. It is obtained as a quotient of $\coprod_{v\in V} X_v$ by identifying a chosen point of X_v with a chosen point of X_w whenever there exists an edge connecting v and w. Stability means that the automorphism group of C_0 is finite: letting $g(X_v)$ denote the geometric genus of X_v and $\mathrm{val}(v)$ the valency of a vertex, this is equivalent to $2g(X_v) - 2 + \mathrm{val}(v) > 0$ for every $v \in V$.

Proposition 3.1. *The identification map $p : \coprod_{v\in V} X_v \to C_0$ induces a canonical isomorphism*

$$H^1(G, \mathbb{C}) \simeq \ker\Big(H^1(C_0, \mathcal{O}_{C_0}) \xrightarrow{p^*} \bigoplus_{v\in V} H^1(X_v, \mathcal{O}_{X_v}) \Big),$$

and the arithmetic genus of C_0 is equal to $h + \sum_{v\in V} g(X_v)$, where h is the first Betti number of G. Moreover,

$$H^1(G, \mathbb{Z}) \simeq \ker\Big(H^1(C_0, \mathbb{Z}) \xrightarrow{p^*} \bigoplus_{v\in V} H^1(X_v, \mathbb{Z})\Big).$$

Proof. Let us choose an orientation of the edges of G. Then we have an exact sequence of sheaves

$$0 \to \mathcal{O}_{C_0} \to p_*\mathcal{O}_{\coprod X_v} \xrightarrow{\varphi} \mathcal{S} \to 0, \tag{3.1}$$

where $\mathcal{S}$ is a skyscraper sheaf with stalk $\mathbb{C}$ over each singular point of C_0 and the map $\varphi = (\varphi_e)_{e\in E}$ is defined as follows: if f is a local section of $p_*\mathcal{O}_{\coprod X_v}$ near the singular point corresponding to e, then $\varphi_e(f) = f(P_v) - f(P_w)$ where v and w denote the head and the tail of e respectively, and $P_v \in X_v$ and $P_w \in X_w$ are the points identified to get C_0. Observe that, since p is finite, taking cohomology commutes with p_*, so we get the exact sequence

$$0 \to \mathbb{C} \to \mathbb{C}^V \xrightarrow{\delta} \mathbb{C}^E \to H^1(C_0, \mathcal{O}_{C_0}) \xrightarrow{p^*} \bigoplus_{v\in V} H^1(X_v, \mathcal{O}_{X_v}) \to 0,$$

where δ is dual to the boundary map (with respect to the same orientation of the edges) in the definition (2.4) of the graph homology. Thus, $H^1(G, \mathbb{C}) \simeq \mathrm{coker}(\delta)$ and the first isomorphism, as well as the expression for the arithmetic genus of C_0, follows.

The proof of the second assertion goes in the same way, up to replacing the exact sequence (3.1) by the analogous sequence of constructible sheaves calculating Betti cohomology. □

3.1. **Deformations.** We recall some basic facts about deformation theory of stable curves, for which we refer the reader to [9, 12, 13, 30].

Let C_0 be, as before, a complex stable curve of arithmetic genus g. Standard results in deformation theory provide a smooth formal scheme $\widehat{S} = \operatorname{Spf} \mathbb{C}[[t_1, \dots, t_N]]$ and a versal formal family of curves $\pi \colon \widehat{\mathcal{C}} \to \widehat{S}$ with a fibre $\mathcal{C}_0$ over $0 \in \widehat{S}$ isomorphic to C_0. In particular, the total space $\widehat{\mathcal{C}}$ is formally smooth over $\mathbb{C}$, and we get an identification of the tangent space T to $\widehat{S}$ at 0 with the Ext group $\operatorname{Ext}^1(\Omega^1_{C_0}, \mathcal{O}_{C_0})$. Locally (for the étale topology) at the singular points, we have

$$C_0 \simeq \operatorname{Spec} R, \quad R = \mathbb{C}[x,y]/(xy), \tag{3.2}$$

so $\Omega^1_{C_0} \simeq Rdx \oplus Rdy/(xdy + ydx)$. In particular, since $xdy \in \Omega^1_{C_0}$ is killed by both x and y, it follows that Ω^1 has a non-trivial torsion subsheaf supported at the singular points. By the vanishing of the higher degree terms [9], we get the following short exact sequence from the five term exact sequence of low degree terms in the local to global Ext-spectral sequence

$$0 \to H^1(C_0, \underline{\mathcal{H}om}(\Omega^1_{C_0}, \mathcal{O}_{C_0})) \to T \to \Gamma(C_0, \underline{\mathcal{E}xt}^1(\Omega^1_{C_0}, \mathcal{O}_{C_0})) \to 0. \tag{3.3}$$

The local $\mathcal{E}xt$ sheaf on the right can be easily calculated using the local presentation (3.2) at the singular points:

$$\begin{aligned} 0 \to R &\longrightarrow Rdx \oplus Rdy \to \Omega^1_R \to 0 \\ 1 &\longmapsto xdy + ydx \end{aligned}$$

One identifies in this way $\underline{\mathcal{E}xt}^1(\Omega^1_{C_0}, \mathcal{O}_{C_0})$ with the skyscraper sheaf consisting of one copy of $\mathbb{C}$ supported at each singular point, hence

$$\Gamma\big(C_0, \underline{\mathcal{E}xt}^1(\Omega^1_{C_0}, \mathcal{O}_{C_0})\big) \simeq \mathbb{C}^E. \tag{3.4}$$

In addition, $\Gamma\big(C_0, \underline{\mathcal{E}xt}^1(\Omega^1_{C_0}, \mathcal{O}_{C_0})\big)$ corresponds to smoothings of the double points [9], so the subspace $H^1(C_0, \underline{\mathcal{H}om}(\Omega^1_{C_0}, \mathcal{O}_{C_0})) \subset T$ corresponds to deformations such that only X_v with the points corresponding to incident edges to v in G move. Since r points on X_v of genus $g(X_v)$ have $3g(X_v) - 3 + r$ moduli, we get the dimensions

$$\dim H^1(C_0, \underline{\mathcal{H}om}(\Omega^1_{C_0}, \mathcal{O}_{C_0})) = \sum_{v \in V} (3g(X_v) - 3 + \operatorname{val}(v)),$$

$$\begin{aligned}\dim T &= \sum_{v\in V}(3(g(X_v)+\mathrm{val}(v)-3)+|E|\\ &= -3|V|+3|E|+\sum_{v\in V}3g(X_v)\\ &= 3g-3,\end{aligned}$$

where the last equality follows from Proposition 3.1. Note that the arithmetic genus of C_0 coincides with the usual genus of a smooth deformation of C_0, and that $3g-3$ is also the dimension of the moduli space of curves of genus g.

For each edge $e \in E$, let $p_e \in C_0$ be the corresponding singular point of C_0. Those deformations of C_0 which preserve the singularity at p_e are given by a divisor $\widehat{D}_e \subset \widehat{S}$. Suppose that $f_e \in \mathcal{O}_{\widehat{S}}$ defines $\widehat{D}_e$. Then the functional $T \to \mathbb{C}^E \xrightarrow{pr_e} \mathbb{C}$ is defined by df_e. Taking into account the identification (3.4), this yields the surjective map $T \to \mathbb{C}^E$ in (3.3). In the geometric picture, we have a collection of principal divisors $\widehat{D}_e \subset \widehat{S}$ indexed by the edges of G which meet transversally. The subvariety cut out by these divisors is precisely the locus of equisingular deformations of C_0 which are given by moving the singular points.

Similarly, if $(C_0, q_1, \dots, q_n)$ is a complex stable curve of arithmetic genus g with n marked points, there exists a formal disc $\widehat{S}'$ of dimension $3g-3+n$ and a versal formal deformation $\pi\colon \widehat{\mathcal{C}}' \to \widehat{S}'$ such that the tangent space to $0 \in \widehat{S}'$ is identified with $\mathrm{Ext}^1(\Omega^1_{C_0}, \mathcal{O}_{C_0}(-\sum_{i=1}^n q_i))$. The fibre at 0 is isomorphic to C_0, and the family comes together with sections $\sigma_i\colon \widetilde{S}' \to \widetilde{\mathcal{C}}'$ such that $\sigma_i(0) = q_i$.

3.2. **Monodromy.** The formal schemes given by the deformation theory can be spread out to yield an analytic deformation $\mathcal{C} \to S$, where S is a polydisc of dimension $3g-3$. In this way, the divisors lift to analytic divisors $D_e \subset S$ which are defined by the equation $\{f_e = 0\}$.

We fix a basepoint $s_0 \in S \setminus \bigcup_{e\in E} D_e$. The goal is to study the monodromy action on $H_1(C_{s_0}, \mathbb{Z})$. For this, we choose, for each $e \in E$, a simple loop $\ell_e \subset S \setminus \bigcup_{e\in E} D_e$ based at s_0 which loops around the divisor D_e. We assume that ℓ_e is contractible in the space $S \setminus \bigcup_{\varepsilon\neq e} D_\varepsilon$.

The monodromy for the action of ℓ_e on $H_1(C_{s_0}, \mathbb{Z})$ is given by the *Picard-Lefschetz formula*:

$$\beta \mapsto \beta - \langle \beta, a_e\rangle a_e, \tag{3.5}$$

where $a_e \in H_1(C_{s_0}, \mathbb{Z})$ denotes the vanishing cycle associated to the double point which remains singular as one deforms along D_e.

By a basic result in differential topology, after possibly shrinking the polydisc S, the inclusion $C_0 \hookrightarrow \mathcal{C}$ admits a retraction $\mathcal{C} \to C_0$ in such a way that the composition $\mathcal{C} \to C_0 \to \mathcal{C}$ becomes homotopic to the identity. The inclusion $C_0 \hookrightarrow \mathcal{C}$ is thus a homotopy equivalence, and from this, one gets the specialization map

$$\mathrm{sp}\colon H_1(C_{s_0}, \mathbb{Z}) \to H_1(\mathcal{C}, \mathbb{Z}) \simeq H_1(C_0, \mathbb{Z}).$$

Lemma 3.2. *The specialization map* sp *above is surjective.*

One can give a formal proof based on the Clemens-Schmid exact sequence, see e.g. [24]. Intuitively, a loop in $H_1(C_0, \mathbb{Z})$ can be broken up into segments which connect double points of the curve. Since these double points arise by shrinking (vanishing) cycles on C_{s_0}, we can model the segments by segments in C_{s_0} which connect the vanishing cycles. Connecting all these segments together yield a loop in C_{s_0} which specializes to the given loop in C_0.

Let $A \subset H_1(C_{s_0}, \mathbb{Z})$ denote the subspace spanned by the vanishing cycles a_e. Observe that we have an exact sequence

$$0 \to A \to H_1(C_{s_0}, \mathbb{Z}) \xrightarrow{\mathrm{sp}} H_1(C_0, \mathbb{Z}) \to 0.$$

Define $A' = A + \mathrm{sp}^{-1}(\bigoplus_{v\in V} H_1(X_v, \mathbb{Z})) \subseteq H_1(C_{s_0}, \mathbb{Z})$. Using Proposition 3.1, we have

$$H_1(C_{s_0}, \mathbb{Z})/A' \simeq H_1(C_0, \mathbb{Z})/\bigoplus_{v\in V} H_1(X_v, \mathbb{Z}) \simeq H_1(G, \mathbb{Z}). \tag{3.6}$$

Lemma 3.3. *The subspace $A \subset H_1(C_{s_0}, \mathbb{Z})$ defined by the vanishing cycles is isotropic and has rank h. In particular, it is maximal isotropic if the genera of all the components X_v are zero.*

Proof. For s_0 very close to $0 \in S$, the vanishing cycles a_e become disjoint since they approach different singular points $p_e \in C_0$. Thus, $\langle a_e, a_{e'} \rangle = 0$. The pairing on H_1 being symplectic, we automatically have $\langle a_e, a_e \rangle = 0$, which shows that the subspace A is isotropic.

To prove the claim about the dimension, note that (3.6) implies

$$h = \mathrm{rk}\, H_1(C_0, \mathbb{Z}) - 2\sum_{v\in V} g(X_v) = \mathrm{rk}\, H_1(C_{s_0}, \mathbb{Z}) - \mathrm{rk}\, A - 2\sum_{v\in V} g(X_v)$$
$$= 2g(C_{s_0}) - \mathrm{rk}\, A - 2\sum_{v\in V} g(X_v) = 2h - \mathrm{rk}\, A.$$

It follows that $\mathrm{rk}\, A = h$. □

The same reasoning as above implies that $\langle A, A' \rangle = 0$, so the symplectic pairing reduces to a pairing $A \times \Big(H_1(C_{s_0}, \mathbb{Z})/A'\Big) \to \mathbb{Z}$.

Write $N_e = \ell_e - \mathrm{Id}$. Note that by (3.5), we have $N_e(\beta) = \langle \beta, a_e \rangle a_e$ for any $\beta \in H_1(C_{s_0}, \mathbb{Z})$. Thus the image of N_e is contained in A. By Lemma 3.3, we get that N_e vanishes in A. Thus $N_e^2 = 0$ which shows that $N_e = \log(\ell_e)$. Consider now the composition

$$H_1(C_0, \mathbb{Z}) \simeq H_1(C_{s_0}, \mathbb{Z})/A \xrightarrow{N_e} A \simeq (H_1(C_{s_0}, \mathbb{Z})/A')^\vee \simeq H_1(G, \mathbb{Z})^\vee.$$

Note that, by Picard-Lefschetz, all the elements in

$$\mathrm{sp}^{-1}(\bigoplus_v H_1(X_v, \mathbb{Z})) \subset H_1(C_{s_0}, \mathbb{Z})$$

are in the kernel of N_e, so in fact the above map passes to the quotient to give a map

$$H_1(G, \mathbb{Z}) \simeq H_1(C_{s_0}, \mathbb{Z})/A' \xrightarrow{N_e} H_1(G, \mathbb{Z})^\vee. \tag{3.7}$$

The following proposition provides the relation between the monodromy and the combinatorics of the graph polynomials.

Proposition 3.4. *The bilinear form on $H_1(G, \mathbb{Z})$ given by* (3.7) *coincides with the bilinear form $\langle \cdot \,, \cdot \rangle_e$.*

Proof. For any $b \in H_1(C_{s_0}, \mathbb{Z})$, the image of $\mathrm{sp}(b)$ in the quotient $H_1(C_{s_0}, \mathbb{Z})/A' \simeq H_1(G, \mathbb{Z})$ can be identified with a loop $\gamma = \sum_e n_e e$, where $n_e = \langle b, a_e \rangle$ denotes the multiplicity of intersection of b with the vanishing cycle a_e. The quadratic form on $H_1(C_{s_0}, \mathbb{Z})$ associated to N_e sends b to $\langle b, \langle b, a_e \rangle a_e \rangle = n_e^2$. On the other hand, the bilinear form $\langle \cdot \,, \cdot \rangle_e$ on $H_1(G, \mathbb{Z})$ corresponding to the edge e sends the loop γ to n_e^2, from which the proposition follows. □

Fix a symplectic basis $a_1, \ldots, a_g, b_1, \ldots, b_g$ for $H_1(C_{s_0}, \mathbb{Z})$ such that $a_1, \ldots, a_h$ form a basis of A. For any $e \in E$, we can write

$$a_e = \sum_{i=1}^{h} c_{e,i} a_i. \tag{3.8}$$

Let $B \subset H^1(C_{s_0}, \mathbb{Z})$ be the subspace generated by $b_1, \ldots, b_h$. Note that we have isomorphisms $B \simeq A^\vee \simeq H_1(G, \mathbb{Z})$. Thus, we can see the monodromy operators as maps

$$N_e \colon A \longrightarrow B.$$

The following proposition is straightforward.

Proposition 3.5. *In terms of the basis $b_1, \ldots, b_h$ for $B \simeq H_1(G, \mathbb{Z})$, we can write $M_e = \Big(c_{e,i} c_{e,j}\Big)_{1 \le i,j \le h}$.*

We will denote by A_0 (respectively B_0) the subspace of $H^1(C_{s_0}, \mathbb{Z})$ generated by $a_1, \dots, a_g$ (respectively $b_1, \dots, b_g$). The A_0 is a maximal isotropic subspace with $A \subset A_0 \subset A'$.

4. Archimedean heights and the Poincaré bundle

In this section, we recall the definition of the archimedean height pairing between degree zero divisors with disjoint support on a smooth projective curve, as well as its interpretation in terms of biextensions and the Poincaré bundle. Throughout, given a set Σ and a ring R, we denote by $(\bigoplus_\Sigma R)^0$ the set of elements $(r_s) \in \bigoplus_\Sigma R$ with $\sum_s r_s = 0$.

4.1. **Archimedean heights.** Let C be a smooth projective curve over the field of complex numbers and $\Sigma \subset C$ a finite set of points in C, which we also think of as a reduced effective divisor. The inclusion $j \colon C \setminus \Sigma \hookrightarrow C$ yields an exact sequence of mixed Hodge structures:

$$0 \to H^1(C, \mathbb{Z}(1)) \xrightarrow{j^*} H^1(C \setminus \Sigma, \mathbb{Z}(1)) \longrightarrow (\bigoplus_{\Sigma} \mathbb{Z})^0 \to 0. \tag{4.1}$$

Lemma 4.1. *The exact sequence of real mixed Hodge structures obtained from* (4.1) *by tensoring with* $\mathbb{R}$ *is canonically split.*

Proof. It suffices to show that an extension of real mixed Hodge structures of the form $0 \to H \to E \xrightarrow{\alpha} \mathbb{R}(0) \to 0$, where H is pure of weight -1, is canonically split. For this, we consider the subspace $M = F^0 E_{\mathbb{C}} \cap \overline{F^0 E_{\mathbb{C}}}$ of E. The map α induces a surjection $M \to \mathbb{R}$ with kernel $M \cap H$. Since H has weight -1, this intersection is empty. We thus get an isomorphism whose inverse map provides the splitting. □

Recall that the Hodge filtration on $H^1(C \setminus \Sigma, \mathbb{C})$ comes from the exact sequence of sheaves

$$0 \to \Omega^1_C \to \Omega^1_C(\log \Sigma) \xrightarrow{\mathrm{Res}_\Sigma} \bigoplus_{\Sigma} \underline{\mathbb{C}} \to 0, \tag{4.2}$$

were $\mathrm{Res}_\Sigma = \sum_{p \in \Sigma} \mathrm{Res}_p$, and Res_Σ is defined, for a local section ω, by

$$\mathrm{Res}_p(\omega) = \frac{1}{2\pi i} \int_{\gamma_{p,\varepsilon}} \omega,$$

for a small negatively oriented circle $\gamma_{p,\varepsilon}$ around the point p. We take $\gamma_{p,\varepsilon}$ negatively oriented because we want it to be part of the bounday of the complement of a small disk around p and not of the disc itself.

From the exact sequence (4.2) we deduce that $F^0 H^1(C \setminus \Sigma, \mathbb{C}(1)) = H^0(C, \Omega^1_C(\log \Sigma))$. Combining this information with Lemma 4.1, we get a canonical map

$$(\bigoplus_{\Sigma}\mathbb{R})^0 \longrightarrow H^0(C,\Omega^1_C(\log\Sigma))\cap H^1(C\setminus\Sigma,\mathbb{R}(1))$$
$$\mathfrak{D}\longmapsto \omega_{\mathfrak{D},\mathbb{R}}.$$

Remark 4.2. Concretely, $\omega_{\mathfrak{D},\mathbb{R}}$ can be understood as follows. By (4.2), the condition $\mathrm{Res}_\Sigma\omega_{\mathfrak{D}}=\mathfrak{D}$ determines a logarithmic differential $\omega_{\mathfrak{D}}$ only up to addition of elements in $H^0(C,\Omega^1_C)$. To fix it uniquely, we require that $\int_\gamma \omega_{\mathfrak{D},\mathbb{R}}\in\mathbb{R}(1)$ for every real valued cycle γ in $C\setminus\Sigma$. Note that $\omega_{\mathfrak{D},\mathbb{R}}$ is an "admissible integral" in the sense of [11, Def. 3.3.5].

Let $\mathfrak{A}$ be a degree zero $\mathbb{R}$-divisor on C with support Σ and let $\omega_{\mathfrak{A},\mathbb{R}}$ the form just defined. Given another degree zero $\mathbb{R}$-divisor $\mathfrak{B}$ with disjoint support, we can find a real-valued 1-chain $\gamma_{\mathfrak{B}}$ on $C\setminus\Sigma$ such that $\mathfrak{B}=\partial\gamma_{\mathfrak{B}}$.

Definition 4.3. The archimedean height pairing between $\mathfrak{A}$ and $\mathfrak{B}$ is the real number

$$\langle\mathfrak{A},\mathfrak{B}\rangle=\mathrm{Re}\left(\int_{\gamma_{\mathfrak{B}}}\omega_{\mathfrak{A},\mathbb{R}}\right). \tag{4.3}$$

Note that, since $\omega_{\mathfrak{A},\mathbb{R}}$ is an $\mathbb{R}(1)$-class, modifying $\gamma_{\mathfrak{B}}$ by an element of $H_1(C\setminus\Sigma,\mathbb{R})$ does not change the real part of the integral. Therefore the above definition is independent of the choice of $\gamma_{\mathfrak{B}}$. Though not apparent from (4.3), the archimedean height pairing is symmetric.

Example 4.4. When the divisor $\mathfrak{A}$ is of the form $\mathrm{div}(f)$ for a rational function f on C, the differential $\omega_{\mathfrak{A},\mathbb{R}}$ is nothing else than $-\frac{df}{f}$, hence

$$\langle\mathfrak{A},\mathrm{div}(f)\rangle=\mathrm{Re}\left(\int_{\gamma_{\mathfrak{B}}}-d\log|f|\right)=-\log|f(\mathfrak{B})|. \tag{4.4}$$

Finally, consider the case of divisors with values in space-time $\mathbb{R}^D$ with a given Minkowski metric. Tensoring with $\mathbb{R}^D$ and using the Minkowski metric, the archimedean height pairing extends to a pairing between degree zero $\mathbb{R}^D$-valued divisors with disjoint support.

4.2. **Biextensions.** Let $\mathfrak{A}$ and $\mathfrak{B}$ be integer-valued degree zero divisors with disjoint supports $|\mathfrak{A}|$ and $|\mathfrak{B}|$ on C. In this paragraph, we recall how to attach to $\mathfrak{A}$ and $\mathfrak{B}$ a mixed Hodge structure M with weights $-2,-1,0$ and graded pieces

$$\mathrm{gr}^W_{-2}M=\mathbb{Z}(1),\quad \mathrm{gr}^W_{-1}M=H^1(C,\mathbb{Z}(1)),\quad \mathrm{gr}^W_0M=\mathbb{Z}(0).$$

Such mixed Hodge structures are called *biextensions*. The standard reference is Section 3 of Hain's paper [11].

We first observe that pulling back the exact sequence (4.1) by the map $\mathbb{Z} \to \left(\bigoplus_{|\mathfrak{A}|} \mathbb{Z}\right)^0$ which sends 1 to the divisor $\mathfrak{A}$, we get an extension

$$H_{\mathfrak{A}} \in \operatorname{Ext}^1_{\mathrm{MHS}}\Big(\mathbb{Z}(0), H^1(C, \mathbb{Z}(1))\Big)$$

which fits into a diagram
(4.5)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(C,\mathbb{Z}(1)) & \longrightarrow & H^1(C \setminus |\mathfrak{A}|, \mathbb{Z}(1)) & \longrightarrow & \left(\bigoplus_{|\mathfrak{A}|} \mathbb{Z}\right)^0 & \longrightarrow & 0 \\ & & \| & & \uparrow{\scriptstyle \gamma} & & \uparrow & & \\ 0 & \longrightarrow & H^1(C,\mathbb{Z}(1)) & \longrightarrow & H_{\mathfrak{A}} & \longrightarrow & \mathbb{Z}(0) & \longrightarrow & 0. \end{array}$$

Abusing notation, $H_{\mathfrak{A}}$ is also denoted by $H^1(C \setminus \mathfrak{A}, \mathbb{Z}(1))$.

Similarly, from the cohomology of C relative to $|\mathfrak{B}|$ we obtain an exact sequence of mixed Hodge structures

$$0 \to \operatorname{coker}\Big(\mathbb{Z} \to \bigoplus_{|\mathfrak{B}|} \mathbb{Z}\Big) \to H^1(C, |\mathfrak{B}|; \mathbb{Z}) \to H^1(C, \mathbb{Z}) \to 0. \tag{4.6}$$

Pushing forward (4.6) by the map $\operatorname{coker}\Big(\mathbb{Z} \to \bigoplus_{|\mathfrak{B}|} \mathbb{Z}\Big) \to \mathbb{Z}$ given by the coefficients of $\mathfrak{B}$ and tensoring by $\mathbb{Z}(1)$, we get an element

$$H^1(C, \mathfrak{B}; \mathbb{Z}(1)) \in \operatorname{Ext}^1_{MHS}\Big(H^1(C, \mathbb{Z}(1))\,,\, \mathbb{Z}(1)\Big).$$

Remark 4.5. Applying $\operatorname{Hom}_{MHS}(-, \mathbb{Z}(1))$ to this extension and using Poincaré duality we get $H_{\mathfrak{B}}$.

Since $\mathfrak{A}$ and $\mathfrak{B}$ have disjoint support, replacing C by $C \setminus |\mathfrak{A}|$ in (4.6) and proceeding as before yields another extension

$$0 \to \mathbb{Z}(1) \to E \to H^1(C \setminus |\mathfrak{A}|, \mathbb{Z}(1)) \to 0. \tag{4.7}$$

Definition 4.6. The biextension mixed Hodge structure associated to $\mathfrak{A}$ and $\mathfrak{B}$ is the pullback of the extension (4.7) by the map γ in (4.5). It will be denoted either by $H_{\mathfrak{B},\mathfrak{A}}$ or by $H^1(C \setminus \mathfrak{A}, \mathfrak{B}; \mathbb{Z}(1))$.

By construction, $H_{\mathfrak{B},\mathfrak{A}}$ fits into the diagram

(4.8)
$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}(1) & \longrightarrow & H^1(C,\mathfrak{B};\mathbb{Z}(1)) & \longrightarrow & H^1(C,\mathbb{Z}(1)) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}(1) & \longrightarrow & H_{\mathfrak{B},\mathfrak{A}} & \longrightarrow & H_{\mathfrak{A}} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbb{Z}(0) & = & \mathbb{Z}(0) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

In particular, the weight filtration is given by

$$0 = W_{-3} \subset W_{-2} = \mathbb{Z}(1) \subset W_{-1} = H^1(C,\mathfrak{B};\mathbb{Z}(1)) \subset W_0 = H_{\mathfrak{B},\mathfrak{A}}$$

and hence satisfies

$$\mathrm{gr}^W_{-2} H_{\mathfrak{B},\mathfrak{A}} = \mathbb{Z}(1), \quad \mathrm{gr}^W_{-1} H_{\mathfrak{B},\mathfrak{A}} = H^1(C,\mathbb{Z}(1)), \quad \mathrm{gr}^W_0 H_{\mathfrak{B},\mathfrak{A}} = \mathbb{Z}(0).$$

Remark 4.7.

(1) By Poincaré duality, the biextension $H^1(C \setminus \mathfrak{A},\mathfrak{B};\mathbb{Z}(1))$ is isomomorphic to the biextension $H_1(C \setminus \mathfrak{B},\mathfrak{A};\mathbb{Z})$ which is constructed in the same way, but using homology.

(2) Going from integral to real coefficients, the same construction yields a *real biextension* which will be denoted by

$$H_1(C \setminus \mathfrak{B},\mathfrak{A};\mathbb{R}).$$

It has graded quotients $\mathbb{R}(1)$, $H_1(C,\mathbb{R})$ and $\mathbb{R}(0)$.

Lemma 4.8. *The set of isomorphism classes of real biextensions with graded quotients $\mathbb{R}(0), H_1(C,\mathbb{R})$ and $\mathbb{R}(1)$ is canonically isomorphic to $\mathbb{R} = \mathbb{C}/\mathbb{R}(1)$. Moreover, if we denote by η the composition of the change of coefficients from $\mathbb{Z}$ to $\mathbb{R}$ with this isomorphism, then for every pair $\mathfrak{A}$, $\mathfrak{B}$ of integer-valued degree zero divisors on C with disjoint support the following equality holds*

$$\langle \mathfrak{A},\mathfrak{B}\rangle = \eta(H_{\mathfrak{B},\mathfrak{A}}).$$

Proof. The first statement is [11, Cor. 3.2.9] and the second is [11, Prop. 3.3.7]. Note that, in this reference, the height pairing is defined as the class of the biextension while we have defined it as an integral. The content of the [11, Prop. 3.3.7] is that both definitions agree. □

4.3. **The Poincaré bundle.** For what follows, it will be more convenient to reformulate the height pairing in terms of Poincaré bundles. We first recall the construction for a single compact complex torus $T = V/\Lambda$, where V is a finite dimensional $\mathbb{C}$-vector space and $\Lambda \subset V$ a cocompact lattice. By definition, the dual torus $\widehat{T}$ is the quotient $\widehat{T} = \widehat{V}/\widehat{\Lambda}$ of the $\mathbb{C}$-vector space $\widehat{V} = \mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C})$ of $\mathbb{C}$-antilinear functionals on V by the dual lattice $\widehat{\Lambda} = \{\phi \in \widehat{V} \mid \mathrm{Im}(\phi(\Lambda)) \subset \mathbb{Z}\}$. Observe that a functional $\phi \in \widehat{V}$ is uniquely determined by its imaginary part $\eta = \mathrm{Im}(\phi)\colon V \to \mathbb{R}$, thanks to the formula $\phi(v) = \eta(-iv) + i\eta(v)$.

For $\phi \in \widehat{T}$, denote by L_ϕ the $\mathbb{C}^\times$-bundle on T associated to the representation of the fundamental group

$$\pi_1(T) = \Lambda \subset V \xrightarrow{\mathrm{Im}(\phi)} \mathbb{C} \xrightarrow{\exp(2\pi i \cdot)} \mathbb{C}^\times. \tag{4.9}$$

A Poincaré bundle $\mathcal{P}^\times$ is a $\mathbb{C}^\times$-bundle on $T \times \widehat{T}$, which is uniquely characterized up to isomorphisms by the following two properties:

(*i*) The restriction $\mathcal{P}^\times|_{\{0\}\times\widehat{T}}$ is trivial.

(*ii*) The restriction $\mathcal{P}^\times|_{T\times\{\phi\}}$ is L_ϕ.

Moreover, if $\mathcal{P}_1^\times$ and $\mathcal{P}_2^\times$ are two $\mathbb{C}^\times$-bundles satisfying conditions (i) and (ii) and we choose trivializations $\mathcal{P}_i^\times|_{\{(0,0)\}} \simeq \mathbb{C}^\times$, then there is a unique isomorphism $\mathcal{P}_1^\times \simeq \mathcal{P}_2^\times$ compatible with the trivializations. A Poincaré bundle $\mathcal{P}^\times$ together with a trivialization $\mathcal{P}^\times|_{\{(0,0)\}} \simeq \mathbb{C}^\times$ is called a *rigidified* Poincaré bundle.

More generally, if $T \to X$ is a holomorphic family of principally polarized abelian varieties, the dual abelian varieties fit together into a holomorphic family $\widehat{T} \to X$. A Poincaré bundle on the product $\pi\colon T \times_X \widehat{T} \to X$ is a $\mathbb{C}^\times$-bundle $\mathcal{P}^\times$ such that

(i) The restriction of $\mathcal{P}^\times$ to each fibre is a Poincaré bundle.

(ii) The restriction to the zero section $s_0\colon X \to T \times_X \widehat{T}$ is trivial.

A rigidification of $\mathcal{P}^\times$ is an isomorphism

$$s_0^*\mathcal{P}^\times \simeq \mathcal{O}_X^\times.$$

To extend the Poincaré bundle to the space $\mathcal{A}_g$ of all principally polarized abelian varieties of dimension g, first recall the construction of $\mathcal{A}_g$. The Siegel domain is by definition

$$\mathbb{H}_g = \{\ g \times g \text{ complex symmetric matrix } \Omega \mid \mathrm{Im}(\Omega) > 0\,\}.$$

The group $\mathrm{Sp}_{2g}(\mathbb{R})$ acts on $\mathbb{H}_g$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \Omega = (A\Omega + B)(C\Omega + D)^{-1}.$$

The quotient $\mathcal{A}_g = \mathrm{Sp}_{2g}(\mathbb{Z}) \setminus \mathbb{H}_g$ is the Siegel moduli space parametrizing principally polarized abelian varieties of dimension g. As a complex manifold, this quotient is not smooth due to the existence of elliptic fixed points, but it is a smooth Deligne-Mumford stack and as such is the fine moduli space of principally polarized abelian varieties.

Denote by $\mathrm{Row}_g(\mathbb{C}) \simeq \mathbb{C}^g$ and $\mathrm{Col}_g(\mathbb{C}) \simeq \mathbb{C}^g$ the g-dimensional vector space of row and column matrices, and let

$$\widetilde{X} = \mathbb{H}_g \times \mathrm{Row}_g(\mathbb{C}) \times \mathrm{Col}_g(\mathbb{C}) \times \mathbb{C}.$$

Define the group $\widetilde{G}$ by

$$\widetilde{G} = \left\{ \begin{pmatrix} 1 & \lambda_1 & \lambda_2 & \alpha \\ 0 & A & B & \mu_1 \\ 0 & C & D & \mu_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; \lambda_i \in \mathrm{Row}_g(\mathbb{R}),\ \mu_j \in \mathrm{Col}_g(\mathbb{R}), \alpha \in \mathbb{C}, \right.$$
$$\left. \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{R}) \right\}.$$

The space $\widetilde{X}$ is a homogeneous space for the group $\widetilde{G}$ with respect to the action given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A & B & 0 \\ 0 & C & D & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (\Omega, W, Z, \rho) = ((A\Omega + B)(C\Omega + D)^{-1},$$
$$W(C\Omega + D)^{-1}, {}^{\mathrm{t}}(C\Omega + D)^{-1} Z, \rho - W\,{}^{\mathrm{t}}C\,{}^{\mathrm{t}}(C\Omega + D)^{-1} Z), \tag{4.10}$$

$$\begin{pmatrix} 1 & \lambda_1 & \lambda_2 & 0 \\ 0 & \mathrm{Id}_g & 0 & 0 \\ 0 & 0 & \mathrm{Id}_g & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (\Omega, W, Z, \rho) = (\Omega, W + \lambda_1 \Omega + \lambda_2, Z, \rho + \lambda_1 Z), \tag{4.11}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{Id}_g & 0 & \mu_1 \\ 0 & 0 & \mathrm{Id}_g & \mu_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} (\Omega, W, Z, \rho) = (\Omega, W, Z + \mu_1 - \Omega \mu_2, \rho - W \mu_2), \tag{4.12}$$

$$\begin{pmatrix} 1 & 0 & 0 & \alpha \\ 0 & \mathrm{Id}_g & 0 & 0 \\ 0 & 0 & \mathrm{Id}_g & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (\Omega, W, Z, \rho) = (\Omega, W, Z, \rho + \alpha). \tag{4.13}$$

Denote by $\widetilde{G}(\mathbb{Z}) \subset \widetilde{G}$ the subgroup consisting of those matrices with entries in $\mathbb{Z}$. The matrices in (4.13) form a normal subgroup N of $\widetilde{G}$, so we can take the quotient $G = \widetilde{G}/N$ and consider $G(\mathbb{Z}) = \widetilde{G}(\mathbb{Z})/N(\mathbb{Z})$. The following result gives a characterization of the Poincaré bundle.

Theorem 4.9.

(1) *The quotient*

$$\mathcal{E}_g = G(\mathbb{Z}) \setminus \Big(\mathbb{H}_g \times \mathrm{Row}_g(\mathbb{C}) \times \mathrm{Col}_g(\mathbb{C}) \Big)$$

is isomorphic to the universal family of abelian varieties and their duals over the fine moduli stack $\mathcal{A}_g$.

(2) *Under the previous isomorphism, the quotient*

$$\mathcal{P}_g^{\times} = \widetilde{G}(\mathbb{Z}) \setminus \Big(\mathbb{H}_g \times \mathrm{Row}_g(\mathbb{C}) \times \mathrm{Col}_g(\mathbb{C}) \times \mathbb{C} \Big) \tag{4.14}$$

is a Poincaré bundle over $\mathcal{E}_g$. Moreover, there is a canonical isomorphism

$$(\mathrm{Sp}_{2g}(\mathbb{Z}) \times N(\mathbb{Z})) \setminus \Big(\mathbb{H}_g \times \{(0,0)\} \times \mathbb{C} \Big) = \mathcal{A}_g \times \mathbb{C}^{\times}.$$

that rigidifies $\mathcal{P}_g^{\times}$.

Proof. This result is classical. See for instance [3, §8.7] for the construction of the universal family of abelian varieties. We start by sketching the construction of the isomorphism claimed in the first statement. Let T be an element of $\mathcal{A}_g$ which is the image of the element $\Omega \in \mathbb{H}_g$. Denote by $\omega_1, \dots, \omega_g$ the rows of Ω, and let $e_1, \dots, e_g$ be the standard basis of $\mathbb{C}^g$, so that the lattice Λ_Ω is generated by ω_k, e_j. Then the corresponding abelian variety is $T = \mathbb{C}^g/\Lambda_\Omega$. We identify $\mathbb{C}^g$ with $\mathrm{Row}_g(\mathbb{C})$ and Λ with $\mathrm{Row}(\mathbb{Z}^g) \oplus \mathrm{Row}(\mathbb{Z}^g)$ using the above basis, so the inclusion $\Lambda \hookrightarrow \mathbb{C}^g$ is given by $(\lambda_1, \lambda_2) \mapsto \lambda_1\Omega + \lambda_2$. When we want to distinguish between an abstract vector $v \in \mathbb{C}^g$ and its image in $\mathrm{Row}_g(\mathbb{C})$ we will denote the latter by W_v.

By (4.11), the action of $\widetilde{G}$ identifies $W \in \mathrm{Row}_g(\mathbb{C})$ with $W + \lambda_1\Omega + \lambda_2$, for $\lambda_1, \lambda_2 \in \mathrm{Row}(\mathbb{Z}^g)$. Thus, the image of W in the quotient $\widetilde{G}\backslash\widetilde{X}$ varies in $T = \mathbb{C}^g/\Lambda$.

The action of $\widetilde{G}$ identifies $Z \in \mathrm{Col}(\mathbb{C}^g)$ with $Z + \mu_1 - \Omega\mu_2$. We verify as follows that the class of Z in $\widetilde{G}\backslash\widetilde{X}$ varies in the dual $\widehat{T}$ of T. If η denotes, as before, the imaginary part of $\phi \in \widehat{\mathbb{C}^g}$, we have $\phi(v) = \eta(-iv) + i\eta(v)$ for all v. First, we identify $\widehat{\mathbb{C}^g} = \mathrm{Hom}_{\overline{\mathbb{C}}}(\mathbb{C}^g, \mathbb{C})$ with $\mathrm{Col}(\mathbb{C}^g)$, via the identification $\phi \in \widehat{\mathbb{C}^g} \mapsto Z_\phi = \mu_1 - \Omega\mu_2$, where

$\mu_1 = \mathrm{Col}\big(\eta(\omega_1), \dots, \eta(\omega_g)\big)$ and $\mu_2 = \mathrm{Col}\big(\eta(e_1), \dots, \eta(e_g)\big)$. Under this identification, the pairing between ϕ and v is given by

$$\phi(v) = -\overline{W_v}\mathrm{Im}(\Omega)^{-1} Z_\phi,$$

while its imaginary part is

$$\eta(v) = \mathrm{Im}(\phi(v)) = \lambda_1\mu_1 + \lambda_2\mu_2.$$

Therefore, an element $\phi_0 \in \widehat{\mathbb{C}^g}$ belongs to $\widehat{\Lambda}$ if and only if $\eta_0 = \mathrm{Im}(\phi_0) \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$, which amounts to ask that the two associated vectors μ_1 and μ_2 have integer coordinates. We get $\widehat{\mathbb{C}^g}/\widehat{\Lambda} \simeq \mathbb{C}^g/\{\mu_1 - \Omega\mu_2 \,|\, \mu_1, \mu_2 \in \mathbb{Z}^g\}$. In this way, we have verified that the fibre of $\mathcal{E}_g \to \mathcal{A}_g$ over a point $T \in \mathcal{A}_g$ is identified with $T \times \widehat{T}$. This identification can be extended to an isomorphism of $\mathcal{E}_g$ with the universal family of abelian varieties and their duals.

Denote by $\mathcal{L}$ the $\mathbb{C}^\times$ bundle obtained as the quotient (4.14). If we restrict the actions (4.10) and (4.12) to the points of the form $(\Omega, 0, Z, \rho)$ we see that the restriction of $\mathcal{L}$ to the set $W = 0$ (which is the zero section in the abelian variety) is trivial. Thus we obtain the first condition that characterizes the Poincaré bundle. We now fix $\Omega_0 \in \mathbb{H}_g$ and $Z_0 \in \mathrm{Col}(\mathbb{C}^g)$. Denote X_0 the subvariety of $\widetilde{X}$ of equations $\Omega = \Omega_0$, $Z = Z_0$ and $\phi_0 \in \widehat{\mathbb{C}^g}$ corresponding to Z_0. That is,

$$\phi_0(v) = -\overline{W_v}\mathrm{Im}(\Omega_0)^{-1} Z_0.$$

The restriction of the action (4.11) to X_0 reads

$$(\lambda_1, \lambda_2)(\Omega_0, W, Z_0, \rho) = (\Omega_0, W + \lambda_1\Omega_0 + \lambda_2, \rho + \lambda_1 Z_0).$$

Hence the restriction of $\mathcal{L}$ to the abelian variety covered by X_0 is the $\mathbb{C}^\times$ bundle determined by the cocycle

$$a((\lambda_1, \lambda_2), W) = \exp(2\pi i \lambda_1 Z_0)$$

Consider the holomorphic function $\psi\colon X_0 \to \mathbb{C}$ given by $\psi(W) = \exp(2\pi i W \mathrm{Im}(\Omega_0)^{-1}\mathrm{Im}(Z_0))$. The cocycle

$$b((\lambda_1, \lambda_2), W) = a((\lambda_1, \lambda_2), W)\psi(W + \lambda_1\Omega_0 + \lambda_2)^{-1}\psi(W)$$

is equivalent to a and hence defines an isomorphic bundle. Computing this cocycle we obtain

$$b((\lambda_1, \lambda_2), W) = \exp(2\pi i \mathrm{Im}(\phi_0(\lambda_1\Omega + \lambda_2)).$$

By (4.9) this cocycle determines the line bundle L_{ϕ_0}, so $\mathcal{L}$ satisfies also the second condition that determines the Poincaré bundle. In consequence, we have seen that the restriction of $\mathcal{P}_g^\times$ to each fibre of $\mathcal{E}_g \to \mathcal{A}_g$ is a Poincaré bundle. The stated rigidification shows in particular that the restriction to the zero section is trivial implying the statement. □

Theorem 4.10. *Let $\mathcal{P}^\times$ be a rigidified Poincaré bundle. Then there is a unique metric on $\mathcal{P}^\times$ whose curvature is translation invariant and that, under the rigidification, satisfies $\|1\| = 1$. Moreover, for the Poincaré bundle $\mathcal{P}_g$ over the universal family $\mathcal{E}_p$ this metric is given, for an element (Ω, W, Z, ρ) in $\widetilde{X}$, by*

$$\log ||(\Omega, W, Z, \rho)|| = \Big(- 2\pi \mathrm{Im}(\rho) + 2\pi \mathrm{Im}(W)\big(\mathrm{Im}(\Omega)\big)^{-1}\mathrm{Im}(Z)\Big). \tag{4.15}$$

Proof. It is well known that the invariance of the curvature form fixes the metric up to a multiplicative constant on each fibre but the compatibility with the rigidification fixes this constant.

Consider the space $\widetilde{X}' = \mathbb{H}_g \times \mathrm{Row}_g(\mathbb{C}) \times \mathrm{Col}_g(\mathbb{C}) \times \mathbb{C}^\times$ and the map $\widetilde{X} \to \widetilde{X}'$ that sends ρ to $s = \exp(2\pi i \rho)$. Then $\widetilde{X}'$ is a trivial $\mathbb{C}^\times$-bundle over $\mathbb{H}_g \times \mathrm{Row}_g(\mathbb{C}) \times \mathrm{Col}_g(\mathbb{C})$. The formula (4.15) determines a hermitian metric on this trivial bundle given by

$$\|(\Omega, W, Z, s)\|^2 = |s|^2 \exp(4\pi \mathrm{Im}(W)\big(\mathrm{Im}(\Omega)\big)^{-1}\mathrm{Im}(Z)). \tag{4.16}$$

Let $\widetilde{G}_{\mathbb{R}} \subset \widetilde{G}$ be the subgroup consisting of matrices with $\alpha \in \mathbb{R}$. The fact that (4.16) induces a metric in the Poincaré bundle whose curvature is invariant under translation follows from the invariance of the function (4.15) under the action of $\widetilde{G}_{\mathbb{R}}$, which is a straightforward verification. □

4.4. **The Poincaré bundle and the archimedean height.** The interest for us on the Poincaré bundle is consequence of the relation between biextensions and the Poincaré bundle due to Hain [11].

Let H be a principally polarized pure Hodge structure of weight -1 and type $\{(-1, 0), (0, -1)\}$ and T_H the corresponding principally polarized abelian variety. Let $\mathcal{B}(H, \mathbb{Z})$ be the set of isomorphism classes of biextensions of H. That is, the isomorphism classes of mixed Hodge structures E of weights -2, -1 and 0 with

$$\mathrm{Gr}^W_{-2}(E) = \mathbb{Z}(1), \quad \mathrm{Gr}^W_{-1}(E) = H, \quad \mathrm{Gr}^W_0(E) = \mathbb{Z}(0).$$

There are natural maps

$$\begin{array}{ccc} \mathcal{B}(H, \mathbb{Z}) & \longrightarrow & \mathrm{Ext}^1(\mathbb{Z}(0), H) = T_H \\ E & \longmapsto & E/W_{-2}E. \end{array}$$

and

$$\begin{array}{ccc} \mathcal{B}(H, \mathbb{Z}) & \longrightarrow & \mathrm{Ext}^1(H, \mathbb{Z}(1)) = \widehat{T}_H \\ E & \longmapsto & W_{-1}E. \end{array}$$

Thus we obtain a map $\mathcal{B}(H,\mathbb{Z}) \to T_H \times \widehat{T}_H$. The canonical isomorphism $\mathrm{Ext}^1(\mathbb{Z}(0),\mathbb{Z}(1)) = \mathbb{C}^\times$ induces a structure of $\mathbb{C}^\times$-bundle on $\mathcal{B}(H,\mathbb{Z})$ and a rigidification of $\mathcal{B}(H,\mathbb{Z})$.

Let $\mathcal{B}(H,\mathbb{R})$ be the set of isomorphism classes of real biextensions. By Lemma 4.8 we can identify $\mathcal{B}(H,\mathbb{R})$ with $\mathbb{R}$. We have already denoted by $\eta\colon \mathcal{B}(H,\mathbb{Z}) \to \mathbb{R}$.

Theorem 4.11 (Hain [11])**.** *The bundle $\mathcal{B}(H,\mathbb{Z})$ is a rigidified Poincaré bundle and the invariant metric is given by*

$$\log \|E\| = \eta(E).$$

Using now the relation between the height pairing and the biextensions we can relate the height pairing and the Poincaré bundle. Summing up Theorem 4.11 and Lemma 4.8 we deduce:

Proposition 4.12. *Let C be a smooth projective curve over $\mathbb{C}$ and $\mathfrak{A}$, $\mathfrak{B}$ integer-valued degree zero divisors on C with disjoint support. The following three quantities coincide:*

(a) $\langle \mathfrak{A}, \mathfrak{B} \rangle$,
(b) $\log ||H_{\mathfrak{B},\mathfrak{A}}||$,
(c) $\eta(H_{\mathfrak{B},\mathfrak{A}})$.

Example 4.13. Let $C = \mathbb{P}^1$ and consider the divisors $\mathfrak{A} = z_1 - z_2$ and $\mathfrak{B} = z_3 - z_4$, where z_i are four distinct points of $\mathbb{P}^1$. Then:

$$\langle \mathfrak{A}, \mathfrak{B} \rangle = \mathrm{Re} \int_{z_4}^{z_3} \left(\frac{1}{z - z_1} - \frac{1}{z - z_2} \right) dz = \log \left| \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)} \right|.$$

Observe that the term inside the absolute value is nothing else than the cross-ratio of the points z_i. Since $H^1(\mathbb{P}^1,\mathbb{Z}(1)) = 0$, the biextension associated to $\mathfrak{A}$ and $\mathfrak{B}$ is in this case simply the extension

$$0 \to \mathbb{Z}(1) \to H^1(\mathbb{P}^1 \setminus \{z_1, z_2\}, \{z_3, z_4\}; \mathbb{Z}(1)) \to \mathbb{Z}(0) \to 0.$$

5. The asymptotic of the height pairing

The goal of this section is to prove Theorem 1.2 from the introduction, which relates the asymptotic of the height pairing between degree zero divisors with disjoint support, as a family of smooth curves degenerates to a stable curve C_0, to the ratio of the first and the second Symanzik polynomials of the dual graph of C_0.

5.1. **Asymptotic of the height pairing.** Let Δ be a small open disc around $0 \in \mathbb{C}$, and write $\Delta^* = \Delta \setminus \{0\}$ and $S = \Delta^{3g-3}$. Consider the versal analytic deformation $\pi\colon \mathcal{C} \to S$ from Section 3 of the stable curve C_0. The stability of C_0 implies in particular that we deal with curves of genus $g \geq 2$ and that the dual graph C_0 has $\leq 3g-3$ edges. These assumptions can be removed if one works systematically with moduli of stable marked curves.

Recall that the fibres are smooth outside a normal crossing divisor $D = \bigcup_{e\in E} D_e \subset S$, with irreducible components indexed by the set of singular points of C_0. We denote by U the complement of D in S and we identify it with $U = (\Delta^*)^E \times \Delta^{3g-3-|E|}$. The universal cover is then

$$\widetilde{U} = \mathbb{H}^E \times \Delta^{3g-3-|E|} \longrightarrow U, \tag{5.1}$$

where the map is induced by $z_e \mapsto \exp(2\pi i z_e)$ in the first factors and identity on the second factors.

We assume moreover that we are given two collections

$$\sigma_1 = \{\sigma_{l,1}\}_{l=1,\dots,n}, \quad \sigma_2 = \{\sigma_{l,2}\}_{l=1,\dots,n}$$

of sections $\sigma_{l,i}\colon S \to \mathcal{C}$ of π. Since $\mathcal{C}$ is regular over $\mathbb{C}$, the sections cannot pass through double points of C_0. Thus, for each l, $\sigma_{l,i}(S) \cap C_0$ lies in a unique irreducible component X_{v_l} of C_0, corresponding to a vertex v_l of G. We assume further that the sections $\sigma_{l,1}$ and $\sigma_{l,2}$ are distinct on C_0. It follows, possibly after shrinking S, that σ_1 and σ_2 are disjoint as well.

Let $\underline{\mathbf{p}}_1 = \{\mathbf{p}_{l,1}\}_{l=1}^n \in (\mathbb{R}^D)^{n,0}$ and $\underline{\mathbf{p}}_2 = \{\mathbf{p}_{l,2}\}_{l=1}^n \in (\mathbb{R}^D)^{n,0}$ be two collections of external momenta satisfying the conservation law. We label the marked points $\sigma_{l,i}$ with $\mathbf{p}_{l,i} \in \mathbb{R}^D$, and we write $\underline{\mathbf{p}}_1^G = (\mathbf{p}_{v,1}^G)$ and $\mathbf{p}_2^G = (\mathbf{p}_{v,2}^G)$ for the *restriction* of $\underline{\mathbf{p}}_1$ and $\underline{\mathbf{p}}_2$ to G. By definition, for each vertex v of G, the vector $\mathbf{p}_{v,i}^G$ is the sum of all $\mathbf{p}_{l,i}$ with $v_l = v$.

For any $s \in S$, let $\mathfrak{A}_s$ and $\mathfrak{B}_s$ denote the $\mathbb{R}^D$-valued degree zero divisors on C_s

$$\mathfrak{A}_s = \sum_{l=1}^n \mathbf{p}_{l,1}\sigma_{l,1}(s), \qquad \mathfrak{B}_s = \sum_{l=1}^n \mathbf{p}_{l,2}\sigma_{l,2}(s).$$

Recall that in Section 4.1 we have extended the usual archimedean height pairing to $\mathbb{R}^D$-valued degree zero divisors by means of the given Minkowski bilinear form. We thus get a function

$$U \longrightarrow \mathbb{R}, \quad s \longmapsto \langle \mathfrak{A}_s, \mathfrak{B}_s \rangle.$$

Definition 5.1. An *admissible segment in S* is a continuous map

$$\underline{t} : I = (0, \varepsilon) \to U = (\Delta^*)^E \times \Delta^{3g-3-|E|}$$

from an open interval of positive length ε, which satisfies the following:

(i) letting $(t_e)_{e \in E}$ denote the coordinates in the factor $(\Delta^*)^E$, the limit $\lim_{\alpha' \to 0} |t_e(\alpha')|^{\alpha'}$ exists and belongs to $(0, 1)$ for all $e \in E$;
(ii) the segment can be extended to a continuous map $\underline{t} : [0, \varepsilon) \to S$.

Note that it follows from property (i) that $t_e(0) = 0$ for all $e \in E$.

Example 5.2.

(1) Given tuples of real numbers $(x_e)_{e \in E}$ and $(Y_e)_{e \in E}$ with $Y_e > 0$ for all $e \in E$, we define a map $\underline{z}\colon (0, 1) \to \mathbb{H}^E \times \{0\} \subset \widetilde{U}$ by $z_e(\alpha') = x_e + i\frac{Y_e}{2\pi\alpha'}$. Projecting to U by the universal cover (5.1), we get an admissible segment $\underline{t}\colon (0, 1) \to U$ for which $|t_e(\alpha')|^{\alpha'} = \exp(-Y_e)$.

(2) Let Δ^*_δ be a punctured disc of radius δ centered at the origin, and $\gamma : \Delta^*_\delta \to U$ be an analytic map which can be analytically extended to $\widetilde{\gamma} : \Delta_\delta \to S$. Let $\epsilon = -\frac{1}{\log \delta}$ and let $\pi : \Delta^*_\delta \to (0, \epsilon)$ be the map $\pi(t) = -\frac{1}{\log |t|}$. Then for any continuous section η of π, the composition $\gamma \circ \eta : (0, \epsilon) \to U$ is an admissible segment.

Our first result describes the asymptotic behavior of the height pairing $\langle \mathfrak{A}_s, \mathfrak{B}_s \rangle$ as the smooth curves C_s degenerate to C_0 through an admissible segment.

Theorem 5.3. *For any admissible segment $\underline{t}\colon I \to U$ the following asymptotic estimate holds*

$$\lim_{\alpha' \to 0} \alpha' \langle \mathfrak{A}_{\underline{t}(\alpha')}, \mathfrak{B}_{\underline{t}(\alpha')} \rangle = \frac{\phi_G(\underline{\mathbf{p}}^G_1, \underline{\mathbf{p}}^G_2, \underline{Y})}{\psi_G(\underline{Y})}, \tag{5.2}$$

where, for each edge $e \in E$, we define

$$Y_e = -\lim_{\alpha' \to 0} \log |t_e(\alpha')|^{\alpha'} > 0,$$

and ψ_G and ϕ_G denote the first and second Symanzik polynomials of G.

Before proving the theorem, we need to consider the period map obtained from the variation of the biextension mixed Hodge structures given by the divisors $\mathfrak{A}_s$ and $\mathfrak{B}_s$. This is what we do next.

5.2. **The period map and its monodromy.** Throughout this section we assume that the divisors $\mathfrak{A}_s$ and $\mathfrak{B}_s$ are integer-valued, that is, $\mathbf{p}_{l,i} \in \mathbb{Z}$ for all l, i. We see the $\underline{\mathbf{p}}_i$ as row vectors.

The family of mixed Hodge structures $H_{\mathfrak{B}_s,\mathfrak{A}_s}$ fit together into an admissible variation of mixed Hodge structures (see [32] for the definition). This can be seen as follows. Using the theory of mixed Hodge modules [28], [29] one can form a mixed Hodge module $H_{\mathfrak{B}_s,\mathfrak{A}_s}$. Since the relative homology $H_1(C_s \setminus \mathfrak{A}_s, \mathfrak{B}_s; \mathbb{Z})$ is a local system, then $H_{\mathfrak{B}_s,\mathfrak{A}_s}$ is an admissible variation of mixed Hodge structures. See [2] for a survey of mixed Hodge modules with all the needed properties.

In what follows we give a description of the period map of the variation of $H_{\mathfrak{B}_s,\mathfrak{A}_s}$ and its monodromy. Since the period map is only well defined up to the action of the group $\widetilde{G}(\mathbb{Z})$ we have to make some choices. We start by stating explicitely all the choices. We fix base-points $s_0 \in U$ and $\tilde{s}_0 \in \widetilde{U}$ lying above s_0, and a symplectic basis

$$a_1, \dots, a_g, b_1, \dots, b_g \in H_1(C_{s_0}, \mathbb{Z}) = A_0 \oplus B_0.$$

such that the space of vanishing cycles A is generated by $a_1, \dots, a_h \in A$, and $b_1, \dots, b_h$ generate $H_1(C_{s_0}, \mathbb{Z})/A' \simeq H_1(G, \mathbb{Z})$ as in (3.6).

For $i = 1, 2$, we write $\Sigma_{i,s} = \{\sigma_{1,i}(s), \dots, \sigma_{n,i}(s)\}$, $\Sigma_s = \Sigma_{1,s} \cup \Sigma_{2,s}$ and $\Sigma_i = \bigcup_s \Sigma_{i,s}$. We lift the classes a_j and b_j, $j = 1, \dots, g$ to elements of $H_1(C_{s_0} \setminus \Sigma_{s_0}, \mathbb{Z})$ by choosing loops that do not meet the points in Σ_{s_0}. We will denote these new classes also by a_j and b_j.

Since the cohomology groups $H_1(C_s \setminus \Sigma_s, \mathbb{Z})$ form a local system, we can spread out this symplectic basis to a basis

$$a_{1,\tilde{s}}, \dots, a_{g,\tilde{s}}, b_{1,\tilde{s}}, \dots, b_{g,\tilde{s}}$$

of $H_1(C_s \setminus \Sigma_s, \mathbb{Z})$, for any $s \in U$ and $\tilde{s} \in \widetilde{U}$ over it. If there is no risk of confusion, we drop $\tilde{s}$, and simply use a_i and b_i for these elements. Note also that, since A_0 is isotropic and contains the subspace of vanishing cycles, the Picard-Lefschetz formula (3.5) implies that the elements $a_{i,\tilde{s}}$ only depend on s and not on $\tilde{s}$. Thus we will also denote them by $a_{i,s}$.

By the admissibility of $H_{\mathfrak{B},\mathfrak{A}}$ we know that $H_{\mathfrak{B},\mathfrak{A}} \otimes_{\mathbb{C}} \mathcal{O}_U$ can be extended to a holomorphic vector bundle over S and that $F^0 W_{-1} H_{\mathfrak{B},\mathfrak{A}}$ can be extended to a coherent subsheaf of it. From this we deduce the existence of a collection of 1-forms $\{\omega_i\}_{i=1,\dots,g}$ on $\pi^{-1}(U) \subset \mathcal{C}$ such that, for each $s \in U$, the forms $\{\omega_{i,s} \coloneqq \omega_i|_{C_s}\}_{i=1,\dots,g}$ are a basis of the holomorphic differentials on C_s and

$$\int_{a_{i,s}} \omega_{j,s} = \delta_{i,j}. \tag{5.3}$$

Then the classical period matrix for the family of curves $\mathcal{C}$ is $(\int_{b_{i,s}} \omega_{j,s})$.

We choose an integer valued 1-chain $\gamma_{\mathfrak{B}_{s_0}}$ on $C_{s_0} \setminus \Sigma_{1,s_0}$ having $\mathfrak{B}_{s_0}$ as boundary. By adding a linear combination of the b_j if needed, we can assume that

$$\langle a_i, \gamma_{\mathfrak{B}_{s_0}} \rangle = 0. \tag{5.4}$$

The chain $\gamma_{\mathfrak{B}_{s_0}}$ determines a class

$$[\gamma_{\mathfrak{B}_{s_0}}] \in H_1(C_{s_0} \setminus \Sigma_{1,s_0}, \Sigma_{2,s_0}, \mathbb{Z})$$

that we can spread to classes $\gamma_{\mathfrak{B}_{\tilde{s}}}$ as before.

Invoking again the admissibility of $H_{\mathfrak{B},\mathfrak{A}}$, we can find a 1-form $\omega_{\mathfrak{A}}$ on $\pi^{-1}(U) \setminus \Sigma_1$ such that each restriction $\omega_{\mathfrak{A},s} := \omega_{\mathfrak{A}}|_{C_s}$ is a holomorphic form of the third kind, with residue $\mathfrak{A}_s$ and normalized in such a way that

$$\int_{a_{i,s}} \omega_{\mathfrak{A},s} = 0, \quad i = 1, \dots, g. \tag{5.5}$$

Note that this last condition is easily achieved by adding to $\omega_{\mathfrak{A}}$ a suitable linear combination of the ω_i.

Proposition 5.4. *The period map of the variation of mixed Hodge structures $H_{\mathfrak{B}_s,\mathfrak{A}_s}$ is given by*

$$\widetilde{\Phi} \colon \widetilde{U} \longrightarrow \mathbb{H}_g \times \mathrm{Row}_g(\mathbb{C}) \times \mathrm{Col}_g(\mathbb{C}) \times \mathbb{C}$$

$$\tilde{s} \longmapsto \Big(\big(\int_{b_{i,\tilde{s}}} \omega_{j,s} \big)_{i,j}, \big(\int_{\gamma_{\mathfrak{B},\tilde{s}}} \omega_{j,s} \big)_j, \big(\int_{b_{i,\tilde{s}}} \omega_{\mathfrak{A},s} \big)_i, \int_{\gamma_{\mathfrak{B},\tilde{s}}} \omega_{\mathfrak{A},s} \Big). \tag{5.6}$$

Proof. We drop the index s and work pointwise. Recall the definition of the biextension mixed Hodge structure $H_{\mathfrak{B},\mathfrak{A}}$ from Section 4.2. The integral part $H_{\mathfrak{B},\mathfrak{A}}$ has a basis given by $\alpha_{\mathfrak{A}}, a_1, \dots, a_g, b_1, \dots, b_g, \gamma_{\mathfrak{B}}$, where $\alpha_{\mathfrak{A}}$ is the generator of $\mathbb{Q}(1) \subset W_{-2}H_{\mathfrak{B},\mathfrak{A},\mathbb{Q}}$ determined by the divisor $\mathfrak{A}$. This means that, if δ_l is a small negatively oriented disc centered at $\sigma_{l,1}$, then the image of $\partial\delta_l$ in $H_{\mathfrak{B},\mathfrak{A},\mathbb{Q}}$ is $\mathbf{p}_{l,1}\alpha_{\mathfrak{A}}$.

The quotient $H_{\mathfrak{B},\mathfrak{A},\mathbb{C}}/F^0$ has a basis given by the classes

$$[\alpha_{\mathfrak{A}}], [a_1], \dots, [a_g] \tag{5.7}$$

The class of the biextension $H_{\mathfrak{B},\mathfrak{A}}$ is given by the expression of the classes $[b_1], \dots, [b_g], [\gamma_{\mathfrak{B}}]$ is the basis (5.7):

$$\begin{pmatrix} W & \rho \\ \Omega & Z \end{pmatrix}$$

with $\Omega \in \mathbb{H}_g$, $W \in \mathrm{Row}_g(\mathbb{C})$, $Z \in \mathrm{Col}_g(\mathbb{C})$ and $\rho \in \mathbb{C}$. Given a path γ representing a class in $H_1(C \setminus \mathfrak{B}, \mathfrak{A}; \mathbb{Z})$, then, by the choice of the forms

ω_j and $\omega_{\mathfrak{A}}$, the expression of the class $[\gamma]$ is the basis (5.7) is given by

$$\Big(\int_\gamma \omega_{\mathfrak{A}}, \int_\gamma \omega_1, \dots, \int_\gamma \omega_g\Big),$$

which proves the proposition. □

We next describe the action of the logarithm of monodromy maps N_e on the entries in (5.6), for $e \in E$. To this end we observe first that, since the sections $\sigma_{l,i}$ do not meet the double points of C_0, the vanishing cycles $a_e \in H_1(C_{s_0}, \mathbb{Z})$ can be lifted canonically to cycles in $H_1(C_{s_0} \setminus \Sigma_{s_0}, \mathbb{Z})$. These cycles will also be denoted by a_e. In this group we can write

$$a_e = \sum_i c_{e,i} a_i + \sum_l d_{e,l,1} \gamma_{l,1} + \sum_l d_{e,l,2} \gamma_{l,2}, \tag{5.8}$$

where $\gamma_{l,i}$ is a small negatively oriented loop around $\sigma_{l,i}(s_0)$. By the choice of the basis $\{a_i, b_i\}$ the coefficients $c_{e,i}$ are zero for $i > h$.

By the Picard-Lefschetz formula (3.5), the assumption (5.4) and the formula (5.8), we deduce that

$$N_e(b_i) = -\langle b_i, a_e\rangle a_e = c_{e,i} a_e, \tag{5.9}$$

$$N_e(\gamma_{\mathfrak{B}_{s_0}}) = -\langle \gamma_{\mathfrak{B}_{s_0}}, a_e\rangle a_e = -a_e \sum_l \mathbf{p}_{l,2} d_{e,l,2}. \tag{5.10}$$

Since the forms ω_j and $\omega_{\mathfrak{A}}$ are defined globally, they are invariant under monodromy. The integral of these forms with respect to the vanishing cycles is computed using (5.8), (5.5) and (5.3):

$$\int_{a_e} \omega_j = c_{e,j}, \quad \int_{a_e} \omega_{\mathfrak{A}_{s_0}} = \sum_l \mathbf{p}_{l,1} d_{e,l,1}. \tag{5.11}$$

Applying (5.9), (5.10) and (5.11) we deduce

$$\begin{aligned}
N_e(\int_{b_i} \omega_{j,s_0}) &= -\langle b_i, a_e\rangle \int_{a_e} \omega_{j,s} = c_{e,i} c_{e,j} \\
N_e(\int_{\gamma_{\mathfrak{B}_{s_0}}} \omega_{j,s_0}) &= -\langle \gamma_{\mathfrak{B}_{s_0}}, a_e\rangle \int_{a_e} \omega_{j,s} = -c_{e,j} \sum_l \mathbf{p}_{l,2} d_{e,l,2} \\
N_e(\int_{b_i} \omega_{\mathfrak{A}_{s_0}}) &= -\langle b_i, a_e\rangle \int_{a_e} \omega_{\mathfrak{A}_{s_0}} = c_{e,i} \sum_l \mathbf{p}_{l,1} d_{e,l,1}, \\
N_e(\int_{\gamma_{\mathfrak{B}_{s_0}}} \omega_{\mathfrak{A}_{s_0}}) &= -\langle \gamma_{\mathfrak{B}_{s_0}}, a_e\rangle \int_{a_e} \omega_{\mathfrak{A}_{s_0}} = -\Big(\sum_l \mathbf{p}_{l,1} d_{e,l,1}\Big)\Big(\sum_k \mathbf{p}_{k,2} d_{e,k,2}\Big).
\end{aligned}$$

We introduce the matrices $\widetilde{M}_e$, $\widetilde{W}_e$, $\widetilde{Z}_e$ and Γ_e given by

$$(\widetilde{M}_e)_{i,j} = c_{e,i}c_{e,j}, \qquad (\widetilde{W}_e)_{l,j} = -c_{e,j}d_{e,l,2},$$
$$(\widetilde{Z}_e)_{i,l} = c_{e,i}d_{e,l,1}, \qquad (\Gamma_{k,l}) = -d_{e,k,2}d_{e,l,1}.$$

Then the logarithm of the monodromy is given by the element of the Lie algebra of $\widetilde{G}$

$$N_e = \begin{pmatrix} 0 & 0 & \underline{\mathbf{p}}_2\widetilde{W}_e & \underline{\mathbf{p}}_2\Gamma_e{}^t\underline{\mathbf{p}}_1 \\ 0 & 0 & \widetilde{M}_e & \widetilde{Z}_e{}^t\underline{\mathbf{p}}_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.12}$$

Note that all the entries of this matrix are integers.

By Proposition 3.5, the matrix $\widetilde{M}_e$ is the $h \times h$ matrix M_e from Section 2 filled with zeros to a $g \times g$ matrix. Similarly, the matrix $\widetilde{W}_e$ (resp. $\widetilde{Z}_e$) is the extension with zeros of a matrix W_e (resp. Z_e) that has only h columns (resp. rows).

The choice of the path $\gamma_{\mathfrak{B}}$ determines a preimage ω_2 of the vector $\mathbf{p}_2^G$ in $\mathbb{Z}^E$ by counting the number of times (with sign) that $\gamma_{\mathfrak{B}}$ crosses the vanishing cycle a_e. Similarly, the form $\omega_{\mathfrak{A}}$ determines a preimage ω_1 of $\mathbf{p}_1^G$ in $\mathbb{C}^E$ with e-th component given by

$$\int_{a_e} \omega_{\mathfrak{A}}.$$

Recall the definitions of $W_e(\omega)$ and $Q_e(\omega_1, \omega_2)$ given in Definition 2.6.

Proposition 5.5. *The following equalities hold:*

$$Z_e{}^t\underline{\mathbf{p}}_1 = -W_e(\omega_1), \qquad \underline{\mathbf{p}}_2 W_e = {}^tW_e(\omega_2), \qquad \underline{\mathbf{p}}_2\Gamma_e{}^t\underline{\mathbf{p}}_1 = -\,Q_e(\omega_1,\omega_2).$$

Proof. The j-th component of $W_e(\omega_2)$ is given by

$$W_e(\omega_2)_j = \langle b_j, \gamma_{\mathfrak{B}}\rangle_e = \langle b_j, a_e\rangle\langle \gamma_{\mathfrak{B}}, a_e\rangle = -c_{e,j}\sum_l \mathbf{p}_{l,2}d_{e,l,2} = (\underline{\mathbf{p}}_2 W_e)_j.$$

The i-th component of $W_e(\omega_1)$ is given by

$$W_e(\omega_1)_i = \langle b_i, a_e\rangle \int_{a_e}\omega_{\mathfrak{A}} = -c_{e,i}\sum_l \mathbf{p}_{l,1}d_{e,l,1} = -(\widetilde{Z}_e{}^t\underline{\mathbf{p}}_1)_i.$$

Finally

$$Q_e(\omega_1,\omega_2) = \langle \gamma_{\mathfrak{B}}, a_e\rangle \int_{a_e}\omega_{\mathfrak{A}} = \sum_{k,l}\mathbf{p}_{k,2}d_{e,k,2}d_{e,l,1}\mathbf{p}_{l,1} = -\underline{\mathbf{p}}_2\Gamma_e{}^t\underline{\mathbf{p}}_1.$$

∎

Since the monodromy is given by an element of $\widetilde{G}(\mathbb{Z})$, the map $\widetilde{\Phi}$ descends to U, making the following diagram commutative:

$$\begin{array}{ccc} \widetilde{U} & \xrightarrow{\widetilde{\Phi}} & \mathbb{H}_g \times \mathrm{Row}_g(\mathbb{C}) \times \mathrm{Col}_g(\mathbb{C}) \times \mathbb{C} \\ \downarrow & & \downarrow \\ U & \xrightarrow{\Phi} & \widetilde{G}(\mathbb{Z})\backslash\Big(\mathbb{H}_g \times \mathrm{Row}_g(\mathbb{C}) \times \mathrm{Col}_g(\mathbb{C}) \times \mathbb{C}\Big). \end{array} \tag{5.13}$$

Clearly the definition of the map $\widetilde{\Phi}$ can be extended to the case when $D = 1$ and the divisors are real-valued. But in this case the monodromy will not be integral valued and the map $\widetilde{\Psi}$ will not descend to U. Thus there is no analogue to the diagram (5.13). Finally we extend to the case of $\mathbb{R}^D$-valued divisors simply working componentwise. That is, when the divisors $\underline{\mathbf{p}}_1$ and $\underline{\mathbf{p}}_2$ have values in $\mathbb{R}^D$ we define a period map

$$\widetilde{\Psi}\colon \widetilde{U} \longrightarrow (\mathbb{H}_g \times \mathrm{Row}_g(\mathbb{C}) \times \mathrm{Col}_g(\mathbb{C}) \times \mathbb{C})^D.$$

5.3. **Asymptotic of the period map and proof of Theorem 5.3.** The proof of Theorem 5.3 is based on the Nilpotent Orbit Theorem. We refer to the paper by Schmidt [31] and Cattani-Kaplan-Schmidt [7] for the the case of variations of polarized pure Hodge structures. We need the more general case of a variation of mixed Hodge structures [19, 25]. We actually only need a small part of the Nilpotent Orbit Theorem that can be found in [27, Section 6].

Back to the case of integral valued divisors, consider the diagram (5.13). The action of the fundamental group $\mathbb{Z}^E$ of U is unipotent, and we write N_e for the logarithm of the generator $1_e \in \mathbb{Z}^E$. These operators are given explicitly in (5.12). To ease notation we write

$$\widetilde{X} = \mathbb{H}_g \times \mathrm{Row}_g(\mathbb{C}) \times \mathrm{Col}_g(\mathbb{C}) \times \mathbb{C}.$$

The untwisted period map

$$\widetilde{\Psi}(z) = \exp(-\sum_E z_e N_e)\widetilde{\Phi}(z) \tag{5.14}$$

takes values in a "compact dual" $\check{\mathcal{M}}$ which is (essentially) a flag variety which parametrizes filtrations $F^*\mathbb{C}^{g+2}$ which satisfy the conditions to be the Hodge filtration on a biextension of genus g. The space $\check{\mathcal{M}}$ contains $\widetilde{X}$ as an open subset. It is called the compact dual by analogy with the theory of semisimple Lie groups although in general is not compact. Since the map $\widetilde{\Psi}$ is invariant under the transformation $z_e \mapsto z_e + 1$, it descends to a map $\Psi : U \to \check{\mathcal{M}}$.

We separate the variables corresponding to the edges as s_E, and write any point s of U as $s = s_E \times s_{E^c}$. The coordinates in the universal cover $\widetilde{U}$ will be denoted by z_e. The projection $\widetilde{U} \to U$ is given in these coordinates by

$$s_e = \begin{cases} \exp(2\pi i z_e), & \text{for } e \in E, \\ z_e, & \text{for } e \notin E. \end{cases} \tag{5.15}$$

The following result is the part of the Nilpotent Orbit Theorem that we need. A proof of it for admissible variations of mixed Hodge structures can be found in [27, Section 6]. Recall that Δ is a disk of small radius and we denote $S = \Delta^{3g-3}$.

Theorem 5.6. *After shrinking the radius of Δ if necessary, the map Ψ extends to a holomorphic map*

$$\Psi : S \longrightarrow \check{\mathcal{M}}.$$

Moreover, there exists a constant h_0 such that, if for all $e \in E$, $\mathrm{Im}(z_e) \geq h_0$, then

$$\exp(\sum_{e\in E} z_e N_e)\Psi(s) \in \widetilde{X}.$$

We now write $\Psi_0(s) = \exp(\sum_e ih_0 N_e)\Psi(s)$ so $\Psi_0 \colon S \to \widetilde{X}$ is holomorphic. We write

$$\Psi_0(s) = (\Omega_0(s), W_0(s), Z_0(s), \rho_0(s)), \tag{5.16}$$

$$y_e = \mathrm{Im}(z_e) = \frac{-1}{2\pi}\log|s_e|. \tag{5.17}$$

Gathering together all the computations we have made we obtain an expression for the height pairing function.

Proposition 5.7. *The height pairing is given by*

$$\begin{aligned} \langle \mathfrak{A}_s, \mathfrak{B}_s\rangle = -2\pi\,\mathrm{Im}(\rho_0) - \sum_{e\in E} 2\pi y'_e \underline{\mathbf{p}}_2 \Gamma_e {}^t\underline{\mathbf{p}}_1 + \\ 2\pi\Big(\mathrm{Im}(W_0) + \sum_{e\in E} y'_e \underline{\mathbf{p}}_2 \widetilde{W}_e\Big)\cdot\Big(\mathrm{Im}(\Omega_0) + \sum_{e\in E} y'_e \widetilde{M}_e\Big)^{-1} \\ \cdot\Big(\mathrm{Im}(Z_0) + \sum_{e\in E} y'_e \widetilde{Z}_e {}^t\underline{\mathbf{p}}_1\Big), \end{aligned} \tag{5.18}$$

where $y'_e = y_e - h_0$.

Proof. By Proposition 4.12 and equation (5.14) we know that

$$\langle \mathfrak{A}_s, \mathfrak{B}_s \rangle = \log \| H_{\mathfrak{B}_s, \mathfrak{A}_s} \| = \log \| \widetilde{\Phi}(z) \| = \log \| \exp(\sum_{e \in E} z_e \, N_e) \widetilde{\Psi}(z) \|.$$

The proposition follows from the explicit description of the operators N_e in (5.12), and the function $\log \| \cdot \|$ in Theorem 4.10, as well as equations (5.16) and (5.17). □

From the previous proposition we derive the following estimate.

Theorem 5.8. *After shrinking the radius of Δ if necessary, the height pairing can be written as*

$$\begin{aligned}(5.19)\quad \langle \mathfrak{A}_s, \mathfrak{B}_s \rangle = &- \sum_{e \in E} 2\pi y_e \underline{\mathbf{p}}_2 \Gamma_e {}^t\underline{\mathbf{p}}_1 \\ &+ 2\pi \Big(\sum_{e \in E} y_e \underline{\mathbf{p}}_2 W_e \Big) \Big(\sum_{e \in E} y_e M_e \Big)^{-1} \Big(\sum_{e \in E} y_e Z_e {}^t\underline{\mathbf{p}}_1 \Big) + h(s),\end{aligned}$$

where $h \colon U \to \mathbb{R}$ is a bounded function.

Proof. Since ρ_0 is a holomorphic function on $S = \Delta^{3g-3}$, after shrinking the radius of Δ we can assume that $\mathrm{Im}(\rho_0)$ is bounded. So we only need to prove that the third term in the right hand side of equation (5.18) is, up to a bounded function, equal to the second term in the right hand side of (5.19).

Using that, for any symmetric bilinear form $\langle \cdot, \cdot \rangle$ the equality

$$2\langle a, b \rangle = \langle a+b, a+b \rangle - \langle a, a \rangle - \langle b, b \rangle$$

holds, we may assume that $W_0 = {}^tZ_0$ and that $\underline{\mathbf{p}}_2 \widetilde{W}_e = \underline{\mathbf{p}}_1 {}^tZ_e$.

On the other hand, if we denote by c_e (resp. $d_{e,1}$) the column vector $(c_{e,i})_i$ (resp. $(d_{e,l,1})_l$), then

$$\widetilde{M}_e = c_e {}^tc_e, \qquad \widetilde{Z}_e = c_e {}^td_e.$$

Therefore we can choose a colum vector v such that

$$\widetilde{Z}_e {}^t\underline{\mathbf{p}}_1 = \widetilde{M}_e v.$$

For shorthand we write $a = \mathrm{Im}(Z_0)$ and $B = \mathrm{Im}(\Omega_0)$. Then

$$\Big({}^ta + \sum_{e \in E} y'_e \underline{\mathbf{p}}_1 {}^t\widetilde{Z}_e \Big) \cdot \Big(B + \sum_{e \in E} y'_e \widetilde{M}_e \Big)^{-1} \cdot \Big(a + \sum_{e \in E} y'_e \widetilde{Z}_e {}^t\underline{\mathbf{p}}_1 \Big),$$

is a normlike function in the terminology of [4, Section 3.1]. Taking this into account, the result follows from [4, Theorem 3.2 (1)]. □

Remark 5.9. A graph-theoretic proof of this theorem based on Equation 2.3, Proposition 5.7, and exchange properties between spanning trees and 2-forests in a graph is given in [1].

From Theorem 5.8, Proposition 5.5 and equation (2.3) we derive:

Corollary 5.10. *The height pairing can be written as*

$$\langle \mathfrak{A}_s, \mathfrak{B}_s \rangle = 2\pi \frac{\phi_G(\underline{\mathbf{p}}_1^G, \underline{\mathbf{p}}_2^G, \underline{y})}{\psi_G(\underline{y})} + h(s),$$

where $\underline{y} = (y_e)_{e \in E}$ *and* $h\colon U \to \mathbb{R}$ *is a bounded function.*

Remark 5.11. Let now $\pi\colon \mathcal{C}' \to S'$ denote the versal analytic deformation of the marked stable curve C_0, and let $\sigma_i\colon S \to \mathcal{C}$, for $i = 1, \ldots, n$, be the sections corresponding to the markings. Then S is a polydisc Δ^{3g-3+n} and the fibres of π are smooth over the open subset $U' = (\Delta^*)^E \times \Delta^{3g-3-|E|+n}$.

Assume that we are given another family of sections $\lambda_1, \ldots, \lambda_n$ disjoint between them and from the σ_i. Then, for any two collections of external momenta $\underline{\mathbf{p}}_1 = (\mathbf{p}_{1,i})$ and $\underline{\mathbf{p}}_2 = (\mathbf{p}_{2,j})$ satisfying the conservation law, we still have

$$\langle \sum_{i=1}^n \mathbf{p}_{1,i}\sigma_i, \sum_{j=1}^n \mathbf{p}_{2,j}\lambda_j \rangle = 2\pi \frac{\phi_G(\underline{\mathbf{p}}_1^G, \underline{\mathbf{p}}_2^G, \underline{y})}{\psi_G(\underline{y})} + h(s)$$

for a bounded function $h\colon U' \to \mathbb{R}$.

Remark 5.12. By considering a ramified covering of S', étale over U', the same results holds when the λ_i in Remark 5.11 are multi-valued sections which do not meet the double points of C_0.

Proof of Theorem 5.3. Let $\underline{t}\colon I \to U$ be an admissible segment, and consider a lift $\underline{z}\colon I \to \widetilde{U}$. Then $t_e(\alpha') = \exp(2\pi i z_e(\alpha'))$, so

$$\alpha' y_e(\alpha') = \alpha' \mathrm{Im}(z_e(\alpha')) = -\frac{1}{2\pi} \log |t_e(\alpha')|^{\alpha'}.$$

In particular, $\lim_{\alpha' \to 0} \alpha' y_e(\alpha) = \frac{1}{2\pi} Y_e$.

Using Corollary 5.10, together with the fact that the quotient of the Symanzik polynomials is homogeneous of degree one, we get

$$\alpha' \langle \mathfrak{A}_{\underline{t}(s)}, \mathfrak{B}_{\underline{t}(s)} \rangle = 2\pi \frac{\phi_G(\underline{\mathbf{p}}_1^G, \underline{\mathbf{p}}_2^G, (\alpha' y_e(\alpha'))_{e \in E})}{\psi_G((\alpha' y_e(\alpha'))_{e \in E})} + \alpha' h(\underline{t}(s)).$$

Since the function h is bounded, passing to the limit yields

$$\lim_{\alpha' \to 0} \alpha' \langle \mathfrak{A}_{\underline{t}(s)}, \mathfrak{B}_{\underline{t}(s)} \rangle = \frac{\phi_G(\underline{\mathbf{p}}_1^G, \underline{\mathbf{p}}_2^G, \underline{Y})}{\psi_G(\underline{Y})}. \qquad \square$$

6. Convergence of the integrands

In this final section, we prove the main result of the paper: the convergence of the integrand in string theory to the integrand of Feynman amplitudes in the low-energy limit $\alpha' \to 0$. For this, we first recall the definition of regularized Green functions.

6.1. **Green functions.** Let C be a smooth projective complex curve, together with a smooth positive $(1,1)$-form μ.

Example 6.1. If C has genus $g \geq 1$, a natural choice for μ is the Arakelov form

$$\mu_{\mathrm{Ar}} = \frac{i}{2g} \sum_{j=1}^{g} \omega_i \wedge \overline{\omega_i},$$

where $\omega_1, \ldots, \omega_g$ is any orthonormal basis of the holomorphic differentials $H^0(C, \Omega^1_C)$ for the Hermitian product $(\omega, \omega') = \frac{i}{2} \int_C \omega \wedge \overline{\omega'}$.

To μ one associates a Green function $\mathfrak{g}_\mu$ as follows. For a fixed point x of C, consider the differential equation

$$\partial\overline{\partial}\varphi = \pi i (\delta_x - \mu), \tag{6.1}$$

where δ_x is the Dirac delta distribution. It admits a unique solution

$$\mathfrak{g}_\mu(x, \cdot) \colon C \setminus \{x\} \longrightarrow \mathbb{R}$$

satisfying the following conditions:

- If we choose local coordinates in an analytic chart U, then, for fixed $x \in U$ there exists a smooth function α such that $\mathfrak{g}_\mu(x, y) = -\log|y - x| + \alpha(y)$ for any $y \in U \setminus \{x\}$.
- (Normalization) $\int_C \mathfrak{g}_\mu(x, y)\mu(y) = 0$.

Letting x vary, we can view $\mathfrak{g}_\mu$ as a function on $C \times C \setminus \Delta$. The chosen normalization implies that $\mathfrak{g}_\mu$ is symmetric.

The following lemma, proved in [22, Chap, II, Prop. 1.3], explains how the Green function varies when μ is changed.

Lemma 6.2. *If μ' is another positive $(1,1)$-form on C, then there exists a smooth function f on C such that*

$$\mathfrak{g}_{\mu'}(x, y) = \mathfrak{g}_\mu(x, y) + f(x) + f(y). \tag{6.2}$$

The archimedean height pairing between $\mathbb{R}^D$-valued divisors can be expressed in terms of Green functions as follows:

Lemma 6.3. *Let $\mathfrak{A} = \sum \mathbf{p}_{i,1}\sigma_{i,1}$ and $\mathfrak{B} = \sum \mathbf{p}_{j,2}\sigma_{j,2}$ be $\mathbb{R}^D$-valued degree zero divisors with disjoint support on C. Then*

$$\langle \mathfrak{A}, \mathfrak{B} \rangle = \sum_{i,j} \langle \mathbf{p}_{i,1}, \mathbf{p}_{j,2} \rangle \mathfrak{g}_\mu(\sigma_{i,1}, \sigma_{j,2}) \tag{6.3}$$

for any positive $(1,1)$-form μ on C. In particular, the right hand side of (6.3) is independent of μ.

Proof. By bilinearity, it suffices to prove the result for divisors of the form $\mathfrak{A} = x_1 - x_2$ and $\mathfrak{B} = y_1 - y_2$. For this, consider the function

$$\mathfrak{g}_{\mathfrak{A},\mu}(\cdot) = \mathfrak{g}_\mu(x_1, \cdot) - \mathfrak{g}_\mu(x_2, \cdot).$$

We claim that $\omega_{\mathfrak{A}} = 2\partial \mathfrak{g}_{\mathfrak{A},\mu}$. By Remark 4.2, this amounts to say that $2\partial \mathfrak{g}_{\mathfrak{A},\mu}$ has residue 1 at x_1 and -1 at x_2, and $\int_\gamma \partial \mathfrak{g}_{\mathfrak{A},\mu} \in \mathbb{R}(1)$ for any real-valued cycle γ on $C \setminus |\mathfrak{A}|$. The first property follows from the local expression of $\mathfrak{g}_\mu(x_1, \cdot)$ and $\mathfrak{g}_\mu(x_2, \cdot)$ around the points x_1 and x_2, and the second one uses the fact that $\overline{\partial} \mathfrak{g}_{\mathfrak{A},\mu} = \overline{\partial \mathfrak{g}_{\mathfrak{A},\mu}}$ since $\mathfrak{g}_{\mathfrak{A},\mu}$ is a real function. Therefore,

$$\begin{aligned} \langle \mathfrak{A}, \mathfrak{B} \rangle &= \operatorname{Re}\left(\int_{\gamma_{\mathfrak{B}}} \omega_{\mathfrak{A}} \right) = \operatorname{Re}\left(\int_{\gamma_{\mathfrak{B}}} \partial \mathfrak{g}_{\mathfrak{A},\mu} + \overline{\partial} \mathfrak{g}_{\mathfrak{A},\mu} \right) = \operatorname{Re}\left(\int_{\gamma_{\mathfrak{B}}} d\mathfrak{g}_{\mathfrak{A},\mu} \right) \\ &= \mathfrak{g}_\mu(x_1, y_1) - \mathfrak{g}_\mu(x_1, y_2) - \mathfrak{g}_\mu(x_2, y_1) + \mathfrak{g}_\mu(x_2, y_2), \end{aligned}$$

as we wanted to show. □

To prove the convergence of the integrands, we need to extend the definition of the height pairing to divisors with non-disjoint supports. For this we introduce the *regularized Green function* $\mathfrak{g}'_\mu \colon C \times C \to \mathbb{R}$, which agrees with $\mathfrak{g}_\mu$ outside the diagonal, and is defined on Δ by

$$\mathfrak{g}'_\mu(x, x) = \lim_{x' \to x} \big(\mathfrak{g}_\mu(x', x) + \log d_\mu(x', x) \big),$$

where x' is a holomorphic coordinate in a small neighborhood of x and d_μ denotes the distance function associated to the metric μ.

Replacing the Green function by its regularization in (6.3), we can extend the definition of the height pairing to arbitrary $\mathbb{R}^D$-valued divisors and, in particular, define

$$\langle \mathfrak{A}, \mathfrak{A} \rangle'_\mu = \sum_{1 \le i,j \le n} \langle \mathbf{p}_i, \mathbf{p}_j \rangle \mathfrak{g}'_\mu(\sigma_i, \sigma_j).$$

Without further assumptions, the real number $\langle \mathfrak{A}, \mathfrak{A} \rangle'_\mu$ depends on the choice of μ. However, we have the following straightforward consequence of Lemma 6.3:

Corollary 6.4. *Assume that the external momenta $\mathbf{p}_i \in \mathbb{R}^D$ satisfy the conservation law $\sum_{i=1}^n \mathbf{p}_i = 0$ and the on shell condition $\langle \mathbf{p}_i, \mathbf{p}_i \rangle = 0$ for all i. Then $\langle \mathfrak{A}, \mathfrak{A} \rangle'_\mu$ is independent of the choice of μ.*

Proof. Let μ and μ' be two different metrics. Then, using the on shell condition, the conservation law and Lemma 6.3

$$\begin{aligned}\langle \mathfrak{A}, \mathfrak{A} \rangle'_\mu &= \sum_{1 \le i,j \le n} \langle \mathbf{p}_i, \mathbf{p}_j \rangle \mathfrak{g}'_\mu(\sigma_i, \sigma_j) = 2 \sum_{1 \le i < j \le n} \langle \mathbf{p}_i, \mathbf{p}_j \rangle \mathfrak{g}'_\mu(\sigma_i, \sigma_j) \\ &= 2 \sum_{1 \le i < j \le n} \langle \mathbf{p}_i, \mathbf{p}_j \rangle \mathfrak{g}'_{\mu'}(\sigma_i, \sigma_j) = \langle \mathfrak{A}, \mathfrak{A} \rangle'_{\mu'}. \end{aligned}$$ □

6.2. **Asymptotic of the regularized height pairing.** Let $\pi\colon \mathcal{C}' \to S'$ be the analytic versal deformation of the stable marked curve C_0 over a polydisc $S' = \Delta^{3g-3+n}$, and let $U' = (\Delta^*)^E \times \Delta^{3g-3-|E|+n} \subset S'$ denote the smooth locus. Consider the $\mathbb{R}^D$-valued relative divisor

$$\mathfrak{A} = \sum_{i=1}^n \mathbf{p}_i \sigma_i.$$

Given a smooth $(1,1)$-form μ on $\pi^{-1}(U')$ such that every restriction $\mu_s = \mu_{|C_s}$ is positive, we get a function $U' \to \mathbb{R}$ by

$$\langle \mathfrak{A}_s, \mathfrak{A}_s \rangle'_\mu = \sum_{1 \le i,j \le n} \langle \mathbf{p}_i, \mathbf{p}_j \rangle \mathfrak{g}'_{\mu_s}(\sigma_i(s), \sigma_j(s)).$$

To study the asymptotic of $\langle \mathfrak{A}_s, \mathfrak{A}_s \rangle'_\mu$ as s approaches the boundary, we introduce the following function:

$$h_{\mathbf{p},\mu}(s) = \langle \mathfrak{A}_s, \mathfrak{A}_s \rangle'_\mu - 2\pi \frac{\phi_G(\underline{\mathbf{p}}^G, \underline{y})}{\psi_G(\underline{y})}.$$

Theorem 6.5. *If μ extends to a continuous $(1,1)$-form on $\mathcal{C}'$, then the function $h_{\mathbf{p},\mu}$ is bounded.*

Proof. By bilinearity, it suffices to prove that the function is bounded for integer-valued divisors

$$\mathfrak{A} = \sum_{i=1}^n \mathbf{p}_i \sigma_i, \quad \mathbf{p}_i \in \mathbb{Z}, \quad \sum_{i=1}^n \mathbf{p}_i = 0.$$

Let $\Sigma \subset C_0$ denote the union of the set of singular points of C_0 and the marked points $\sigma_1(0), \dots, \sigma_n(0)$. Using the moving lemma, one can find a rational function f on $\mathcal{C}$ such that $\mathfrak{A} + \operatorname{div}(f)$ does not meet Σ. Possibly after shrinking Δ, we may assume that the divisor $\mathfrak{A} + \operatorname{div}(f)$ has support disjoint from $\mathfrak{A}$ and no vertical components.

Since $\operatorname{div}(f)$ does not meet the double points of C_0 and $\operatorname{div}(f)_{|X_v}$ has degree zero for each $v \in V$, the restrictions to G of the momenta of the divisors $\mathfrak{A}$ and $\mathfrak{A} + \operatorname{div}(f)$ coincide. Let $\underline{\mathbf{p}}^G$ denote their common value. Then Corollary 5.10 and remarks 5.11 and 5.12 yield

$$\langle \mathfrak{A}, \mathfrak{A} + \operatorname{div}(f) \rangle = 2\pi \frac{\phi_G(\underline{\mathbf{p}}^G, \underline{y})}{\psi_G(\underline{y})} + h_1(s), \tag{6.4}$$

where $\underline{y} = (y_e)_{e \in E}$ and $h_1 \colon U' \to \mathbb{R}$ is a bounded function.

Let π_i be a local equation of the divisor $\sigma_i \subset \mathcal{C}'$ around the point $\sigma_i(0)$, and consider the first order deformation σ_i^u given by $\{\pi_i = u\}$ for u in a small disc. Then the relative divisor

$$\mathfrak{A}^u = \sum_{i=1}^{n} \mathbf{p}_i \sigma_i^u$$

coincides with $\mathfrak{A}$ for $u = 0$ and is disjoint both from $\mathfrak{A}$ and $\mathfrak{A} + \operatorname{div}(f)$ for $u \neq 0$ sufficiently small. Moreover,

$$\langle \mathfrak{A}, \mathfrak{A} \rangle'_\mu = \lim_{u \to 0} \left(\langle \mathfrak{A}^u, \mathfrak{A} \rangle - \sum_{i=1}^{n} \mathbf{p}_i^2 \log d_\mu(\sigma_i^u, \sigma_i) \right).$$

By Example 4.4, this can be rewritten as

$$\langle \mathfrak{A}, \mathfrak{A} \rangle'_\mu = \lim_{u \to 0} \big(\langle \mathfrak{A}^u, \mathfrak{A} + \operatorname{div}(f) \rangle - \log |f(\mathfrak{A}^u)| - \sum_{i=1}^{n} \mathbf{p}_i \log d_\mu(\sigma_i^u, \sigma_i) \big).$$

Note that, since $\mathfrak{A}+\operatorname{div}(f)$ is disjoint from $\mathfrak{A}$, the function f is, locally around the point $\sigma_i(0)$, of the form $f = \pi_i^{-p_i} v_i$ with v_i invertible, hence

$$\log |f(\mathfrak{A}^u)| = -\sum \mathbf{p}_i^2 \log |u| + \mathbf{p}_i \log v_i(\sigma_i^u).$$

On the other hand, since the metric μ is continuous, there exists a continuous function η_μ such that $d_\mu(\sigma_i^u, \sigma_i) = \eta_\mu |u|$. It follows that the function

$$h_2(s) = \lim_{u \to 0} \big(\log |f(\mathfrak{A}_s^u)| + \sum_{i=1}^{n} \mathbf{p}_i^2 \log d_\mu(\sigma_i^u, \sigma_i) \big)$$

is bounded. Combining this with equation (6.4), we get $h_{\mathbf{p},\mu} = h_1 - h_2$, so it is a bounded function. □

Corollary 6.6. *Assume that the external momenta satisfy $\sum_{i=1}^n \mathbf{p}_i = 0$ and $\langle \mathbf{p}_i, \mathbf{p}_i \rangle = 0$ for all i. Then the function $\mathfrak{g}_{\mathbf{p},\mu}$ is independent of μ. In particular, when $g \geq 1$ and $\mu = \mu_{\mathrm{Ar}}$, the following holds*

$$\langle \mathfrak{A}, \mathfrak{A} \rangle'_{\mu_{Ar}} = 2\pi \frac{\phi_G(\mathbf{p}^G, \underline{y})}{\psi_G(\underline{y})} + \textit{bounded.} \tag{6.5}$$

Remark 6.7. By [18, Thm 1.2], when the base is one dimensional, the Arakelov metric has logarithmic singularities. This implies that equation (6.5) does not hold for μ_{Ar} without the "on shell" condition $\langle \mathbf{p}_i, \mathbf{p}_i \rangle = 0$. Since the asymptotic behaviour of the Arakelov metric is also determined by the combinatorics of the dual graph of C_0, one may ask what the asymptotic of $\langle \mathfrak{A}, \mathfrak{A} \rangle_{\mu_{Ar}}$ is in the general case. We have a formula in terms of the Green's function associated to the Zhang measure on the metric graph G with edge lengths $\underline{Y}$. We hope to return to this point in a future publication.

From Corollary 6.6 we immediately derive:

Theorem 6.8. *Assume that the external momenta satisfy* $\sum_{i=1}^{n} \mathbf{p}_i = 0$ *and* $\langle \mathbf{p}_i, \mathbf{p}_i \rangle = 0$ *for all* i*. Then, for any admissible segment* $\underline{t} \colon I \to U'$*, the following holds:*

$$\lim_{\alpha' \to 0} \alpha' \langle \mathfrak{A}_{\underline{t}(\alpha')}, \mathfrak{A}_{\underline{t}(\alpha')} \rangle'_{\mu_{Ar}} = \frac{\phi_G(\mathbf{p}^G, \underline{Y})}{\psi_G(\underline{Y})}.$$

References

[1] O. Amini, The exchange graph and variations of the ratio of the two Symanzik polynomials, preprint.

[2] M. Asakura, Motives and algebraic de Rham cohomology, in *The arithmetic and geometry of algebraic cycles (Banff, AB, 1998)*, 133–154, CRM Proc. Lecture Notes, **24**, 2000.

[3] C. Birkenhake, H. Lange, *Complex abelian varieties.* Second edition. Grundlehren der Math. Wiss. **302**. Springer-Verlag, Berlin, 2004.

[4] J. I. Burgos Gil, R. de Jong, D. Holmes, Singularity of the biextension metric for families of abelian varieties, preprint arxiv:1604.00686.

[5] J. A. Carlson, E. H. Cattani, A. G. Kaplan, Mixed Hodge structures and compactifications of Siegel's space (preliminary report), *Journées de Géometrie Algébrique d'Angiers*, Sijthoff Nordhoff, Alphen an den Rijn, the Netherlands (1979), 77-106.

[6] E. H. Cattani, Mixed Hodge structures, compactifications, and monodromy weight filtration, *Topics in Transcendental Algebraic Geometry*, 75–100, Ann. Math. Studies **106**, Princetonn Univ. Press, 1984.

[7] E. Cattani, A. Kaplan, and W. Schmidt, Degeneration of Hodge structures, *Annals of Math.* **123** (1986), 457–535.

[8] S. Chaiken, A combinatorial proof of the all minors matrix tree theorem, *SIAM J. Algebraic Discrete Methods* **3** (1982), no. 3, 319–329.

[9] P. Deligne and D. Mumford, The irreducibility of the space of curves of given genus, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 75–109.

[10] R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S-Matrix*, Cambridge University Press, London-New York-Ibadan, 1966.

[11] R. Hain, Biextensions and heights associated to curves of odd genus, Duke Math. J. **61** (1990), no. 3, 859–898.

[12] J. Harris and I. Morrison, *Moduli of curves*, Grad. Texts in Math. **187**, Springer-Verlag, New York, 1998.

[13] R. Hartshorne, *Deformation theory*, Grad. Texts Math. **257**, Springer, New York, 2010.

[14] T. Hayama and G. Pearlstein, Asymptotics of degenerations of mixed Hodge structures, *Adv. Math.* **273** (2015), 380–420.

[15] J. W. Hoffman, *The Hodge theory of stable curves.* Mem. Amer. Math. Soc. **51** (1984), no. 308.

[16] D. Holmes and R. de Jong, Asymptotics of the Néron height pairing, *Math. Res. Lett.*, arXiv:1304.4768, to appear.

[17] C. Itzykson and J.-B. Zuber, *Quantum Field Theory*, International Series in Pure and Applied Physics, McGraw-Hill International, New York, 1980.

[18] R. de Jong, Faltings delta-invariant and semistable degeneration, arxiv:1511:06567.

[19] K. Kato, C. Nakayama, and S. Usui, $SL(2)$-orbit theorem for degeneration of mixed Hodge structure, *J. Algebraic Geom.* **17** (2008), no. 3, 401–479.

[20] K. Kato, S. Usui, *Classifying Spaces of Degenerating Polarized Hodge Structures*, Ann. Math. Studies **169**, Princeton University Press, 2009.

[21] G. Kirchhoff, Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Vertheilung galvanischer Ströme geführt wird, *Annalen der Physik* **148** (1847), no. 12, 497–508.

[22] S. Lang, *Introduction to Arakelov theory*, Springer-Verlag, New York, 1988.

[23] D. Lear, *Extensions of normal functions and asymptotics of the height pairing*, PhD thesis, University of Washington, 1990.

[24] D. R. Morrison, The Clemens-Schmid exact sequence and applications, *Topics in transcendental algebraic geometry*, 101-199, Ann. of Math. Stud. **106**, Princeton Univ. Press, 1984.

[25] G. J. Pearlstein, Degenerations of Mixed Hodge Structures, *Duke Math. J.* **110**, no. 2, (2001), 217–251.

[26] G. J. Pearlstein, SL_2-orbits and degenerations of mixed Hodge structures, *J. Diff. Geometry* **74** (2006), 1–67.

[27] G. J. Pearlstein, Variations of mixed Hodge structure, Higgs fields, and quantum cohomology. *Manuscripta Math.* **102** (2000), no. 3, 269–310.

[28] M. Saito, Modules de Hodge polarisables, *Publ. Res. Inst. Math. Sci.* **24** (1988), 849–995.

[29] M. Saito, Mixed Hodge modules, *Publ. Res. Inst. Math. Sci.* **26** (1990), 221-333.

[30] M. Schlessinger, Functors of Artin rings, *Trans. Amer. Math. Soc.* **130** (1968), 208–222.

[31] W. Schmid, Variation of Hodge structure; the singularities of the period mapping, *Invent. Math.* **22** (1973), 211–319.

[32] J. Steenbrink and S. Zucker, Variation of mixed Hodge structure, *Invent. Math.* **80** (1985), 489–542.

[33] P. Tourkine, Tropical amplitudes, `http://arxiv.org/abs/1309.3551`.

[34] P. Vanhove, The physics and the mixed Hodge structure of Feynman integrals, `http://arxiv.org/abs/1401.6438v2`.

[35] A. A. Voronov, A unified approach to string scattering amplitudes, *Comm. Math. Phys.* **131** (1990), no. 1, 179–218.

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