LOGARITHMIC TREE FACTORIALS

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Abstract. To any rooted tree, we associate a sequence of numbers that we call the logarithmic factorials of the tree. This provides a generalization of Bhargava's factorials to a natural combinatorial setting suitable for studying questions around generalized factorials.

We discuss several basic aspects of the framework in this paper. From an arithmetical point of view, we obtain an alternative direct proof of the well-definedness of Bhargava's factorials, show a realization theorem for certain sequences of integer numbers as factorials of subsets of local fields, and prove an equidistribution theorem with respect to the equilibrium measure for factorial-determining-sequences of subsets of local fields. From a probabilistic point of view, our results lead to a new characterization of transient trees as those trees whose sequence of normalized logarithmic factorials converge to a finite limit, while in such a case, we obtain an explicit way of sampling the harmonic measure on the tree.

Our treatment is based on a local weighting process in the tree which gives an effective way of constructing the factorial sequence.

1. Introduction

Let T be a rooted tree with root $\mathfrak{r}$, and let $\ell : E(T) \to \mathbb{R}_+$ be a length function on the edges of T. Denote by Γ the metric realization of the pair (T, ℓ), which is a rooted metric tree with root $\mathfrak{r}$. We call the unite length function $\ell \equiv 1$ which assigns value one to all the edges of a tree T the *standard* length function.

We orient T away from the root, and, by an abuse of the notation, denote by $E(T)$ the set of oriented edges of T. For any vertex of T, we denote by $[\mathfrak{r}, v]$ the oriented path (resp. segment) from $\mathfrak{r}$ to v in T (resp. Γ).

Consider the boundary ∂T of T, which is by definition, the set of all infinite oriented paths in T with starting vertex at the root $\mathfrak{r}$, and define the *extended boundary* $\widetilde{\partial} T$ as the union of ∂T with the set of all oriented paths in T from the root $\mathfrak{r}$ to a leaf of T. For any pair (T, ℓ) with metric realization Γ, define $\widetilde{\partial}(T, \ell) = \widetilde{\partial}\Gamma = \widetilde{\partial} T$. For any point $\rho \in \widetilde{\partial} T$, we denote by $E(\rho)$ the set of all the edges of T which are in ρ.

Any two different elements of ∂T have a finite number of edges in common. So we can define a non-negative real-valued *intersection pairing* $\langle\, ,\rangle$ on $\widetilde{\partial}\Gamma$ as follows. For any two points $\rho, \tau \in \widetilde{\partial}\Gamma$, with $\rho \neq \tau$ if both ρ and τ both belong to the boundary of T, let

$$\langle \rho, \tau \rangle := \ell(\rho \cap \tau) = \sum_{e \in E(\rho) \cap E(\tau)} \ell(e).$$

Consider the following *greedy procedure* in choosing a sequence of elements $\rho_0, \rho_1, \dots$ in $\widetilde{\partial}\Gamma$. Let $\rho_0 \in \widetilde{\partial}\Gamma$ be any arbitrary element of the extended boundary. Proceeding inductively on $n \in \mathbb{N}$, assume that $\rho_0, \dots, \rho_{n-1} \in \widetilde{\partial}\Gamma$ have been chosen, and choose ρ_n, if possible, arbitrarily

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among the set of all elements $\rho \in \widetilde{\Gamma} \setminus \{\rho_0, \dots, \rho_{n-1}\}$ which minimizes the sum $\sum_{j=0}^{n-1} \langle \rho, \rho_j \rangle$. Define $a_n := \sum_{j=0}^{n-1} \langle \rho_n, \rho_j \rangle$. We have

Theorem 1.1. *For any pair (T, ℓ) consisting of a rooted tree T and a length function ℓ on T, the sequence $\{a_n\}$ constructed above only depends on the metric realization Γ of (T, ℓ).*

We call the number a_n both the (T, ℓ) and Γ-*factorial* of n, and denote it by $n!_{(T,\ell)}$ or $n!_\Gamma$. We call the sequence $\{\rho_n\}$ in the construction above a *factorial-defining sequence* for (T, ℓ) and Γ. When ℓ is the standard length function, we simple write $n!_T$ for the factorials of the pair (T, ℓ).

The above definition is an extension to arbitrary (metric) trees of the (logarithmic) factorial sequence associated by Bhargava to subsets of the ring of valuation of a local field, that we now recall [4, 5].

Let K be a local field with discrete valuation val, with ring of valuation R, with maximal ideal $\mathfrak{m}$, and with residue field $\kappa = R/\mathfrak{m}$, which is thus a finite field. Let S be a subset of R. The logarithmic factorial sequence associated to S is obtained as follows. Choose $s_0 \in S$ arbitrary. Proceeding inductively, and assuming $s_0, \dots, s_{n-1}$ are already chosen, choose s_n among all $s \in S$ which minimizes the quantity $\mathrm{val}\big(\prod_{j=0}^{n-1}(s - s_j)\big)$. Define

$$n!_S := \mathrm{val}\Big(\prod_{j=0}^{n}(s_n - s_0)\dots(s_n - s_{n-1})\Big).$$

To any subset $S \subset R$ of K as above, one can associate its *adelic tree* T_S, which is a rooted locally finite tree with vertices of valence bounded by $|\kappa| + 1$, as follows. For each integer $h \in \mathbb{N}_*$, consider the projection $\phi_h : R \to R/\mathfrak{m}^h$, and define $V_h = \phi_h(S)$. The rooted tree T_S has vertex set $\sqcup_{h=0}^{\infty} V_h$, and has as root the unique element of V_0. The edge set of T_S is defined as follows. For any h, there exists a map $\pi_h : R/\mathfrak{m}^{h+1} \to R/\mathfrak{m}^h$, and we have $\phi_h = \pi_h \circ \phi_{h+1}$. A vertex u in V_h is adjacent to a vertex $v \in V_{h+1}$ if and only if $\pi_h(v) = u$.

In the case $S = R$, the tree T_R is the $|\kappa|$-regular tree, and obviously, for any subset $S \subset R$, the tree T_S is a subtree of T_R. Consider the closure $\overline{S}$ of S in K. The factorials of the tree T_S, as defined above, coincide with the factorials of the subset $\overline{S} \subset R$. Since the factorials of S and $\overline{S}$ are all easily seen to be equal, we get the following proposition.

Proposition 1.2. *Let K be a local field with valuation ring R. Let S be a subset of R with adelic tree T_S. We have $n!_S = n!_{T_S}$, where $n!_S$ denotes the Bhargava's S-factorial of n.*

The proof given by Bhargava of the well-definedness of the factorial sequence $n!_S$ is indirect and goes through the ring of integer valued polynomials on S. In order to prove Theorem 1.1, we give an alternative local definition of a sequence associated to a pair (T, ℓ), show by induction that it is well-defined and only depends on the metric realization Γ, and then prove the equivalence of that definition with the definition given above. Thus our proof leads to an alternative combinatorial proof of the well-definedness of the factorial sequence associated to a subset of local fields.

Note that we have not made so far any finiteness assumption on the valence of vertices of T. In fact, as we will explain in a moment, we can always reduce to the case of locally finite trees with a *capacity function* on leaves, so we next define such objects.

1.1. **Locally finite trees with a capacity function on leaves.** Let T be a locally finite rooted tree and let ℓ be a length function on $E(T)$. Denote by $L(T)$ the set of all leaves of T. By a *capacity function* on T we mean a function $\chi : L(T) \to \mathbb{N} \cup \{\infty\}$. We modify the definition of the factorial sequence given in the previous section by taking into account the capacity of leaves of T as follows. Assuming for an integer $n \in \mathbb{N}$ that $\rho_0, \dots, \rho_{n-1}$ are chosen, we choose ρ_n, if possible, among those $\rho \in \widetilde{\partial}\Gamma$ which minimizes the sum $\sum_{j=0}^{n-1} \langle \rho, \rho_j \rangle$, and which verify the capacity condition that, when ρ is a leaf of T, the number of times ρ appears in the sequence $\rho_0, \dots, \rho_{n-1}$ is strictly less than the capacity of ρ. So in the sequence $\rho_0, \rho_1, \dots$ each leaf of T can appear at most as many times as its capacity. We define

$$a_n := \sum_{j=0}^{n-1} \langle \rho_n, \rho_j \rangle. \tag{1.1}$$

Then we have the following Theorem.

Theorem 1.3. *The sequence* $\{a_n\}$ *only depends on the pair* (Γ, χ)*, where* Γ *is the metric realization of the pair* (T, ℓ)*.*

We call a_n the (Γ, χ) or (T, ℓ, χ)-factorial of n, and denote it by $n!_{(T,\ell,\chi)} = n!_{(\Gamma,\chi)}$. When ℓ is the standard length function, we simply write $n!_{(T,\chi)}$.

Let S be a subset of the valuation ring R of a local field K. Let $h \in \mathbb{N}$. In the adelic tree T_S of S consider the subtree $T_{S,h}$ of all the vertices at distance at most h from the root $\mathfrak{r}$ of T_S. Define the capacity function χ_h on leaves of $T_{S,h}$ as follows. For any leaf v of $T_{S,h}$, consider the subtree $T_{S,v}$ of T which consists of v and all its descendants, and define $\chi_h(v)$ as the number of elements in the extended boundary of $T_{S,v}$. We have the following proposition.

Proposition 1.4. *Notations as above, we have* $n!_{(T,\chi_h)} = n!_{S,h}$

Thus, the factorials in the presence of a capacity function generalizes factorials of order h for subsets of local fields in the terminology of [5].

1.2. **Reduction to locally finite trees.** Let (T, ℓ) be a pair consisting of a tree T and a length function ℓ on T. We define the *locally finite component* T_0 of T as follows. Consider the set V_0 of all vertices v of T with the property that all the interior vertices of the oriented path $[\mathfrak{r}, v]$ have bounded valence in T. So, for example, if the root $\mathfrak{r}$ has infinite valence, then V_0 consists of a single vertex $\mathfrak{r}$. Define the subtree T_0 of T as the tree induced by T on V_0. For any leaf of T_0 which is a vertex of valence infinity in T, define the capacity $\chi_0(v)$ of v to be infinity. For other leaves of T_0, which are thus also leaves of T, define $\chi_0(v) = 1$. Let ℓ_0 be the restriction of ℓ to the edges of T_0. Then we have

Proposition 1.5. *Notation as above, we have for all* n*,* $n!_{(T,\ell)} = n!_{(T_0,\ell_0,\chi_0)}$.

Let now T be a locally finite tree, ℓ a length function on T, and χ a capacity function. Define the tree T_1 by adding $\chi(v)$ disjoint infinite paths to any leaf v of T, and extend ℓ to a length function ℓ_1 on T_1 by assigning arbitrary lengths to the new edges of T_1. It is easy to see that for any n, we have $n!_{(T,\ell,\chi)} = n!_{(T_1,\ell_1)}$.

Therefore, in what follows, there is no restriction in assuming the tree T is locally finite, and, if necessary, a capacity function χ is given.

1.3. **Growth of the factorial sequence and equidistribution.** Let T be a locally finite rooted tree and ℓ be a length function on T. Denote by Γ the metric realization of (T,ℓ).

We will prove that

$$\forall m,n \in \mathbb{N}, \qquad (m+n)!_\Gamma \geq m!_\Gamma + n!_\Gamma.$$

Combining this with Fekete's lemma, we get the convergence of the sequence

$$\frac{1}{n} n!_\Gamma \to H(\Gamma).$$

The quantity $H(\Gamma)$, that we deliberately denote by $H(T,\ell)$ as well, is an invariant of Γ and one of our objectives in this paper will be to characterize it.

We first describe a necessary and sufficient condition for the finiteness of $H(T,\ell)$.

Define the *conductance* $c: E(T) \to \mathbb{R}_+$ given by $\forall uv \in E(T)$, $c(u,v) = c(v,u) := \frac{1}{\ell(uv)}$.

Consider the random walk $RW(T,\ell)$ on T which starts at the root $\mathfrak{r}$, and which has probability of going from a vertex u of the tree to any of its neighbors v in the tree given by $p_{uv} := \frac{c(u,v)}{\sum_{w\sim u} c(u,w)}$.

The following theorem relates the finiteness of the limit of logarithmic factorials to the transience of the random walk on the tree, and leads to an alternative characterization of transient trees in terms of the sequence of factorial numbers associated to the tree.

Theorem 1.6. *Let T be an infinite locally finite rooted tree and ℓ a length function on T. Assume that the pair (T,ℓ) is weakly complete. The following two statements are equivalent.*

- *The random walk $RW(T,\ell)$ is transient.*
- *The limit $H(T,\ell)$ is finite.*

Equivalently, the random walk $RW(T,\ell)$ is recurrent if and only if $H(T,\ell) = \infty$.

The condition that (T,ℓ) is weakly complete means any infinite oriented path P in T which entirely consists of valence two vertices has to be of infinite length in the metric realization Γ of (T,ℓ). In particular, this is the case if the length function is ϵ-away from zero for some $\epsilon > 0$. We refer to Section 4 for more details.

In the presence of a capacity function χ on the leaves of T, the normalized factorials $\frac{1}{n} n!_{(\Gamma,\chi)}$ still converge to a parameter $H(\Gamma,\chi)$, and the theorem above still holds if the values of χ are all finite as can be easily observed by the transformation (T_1,ℓ_1) of (T,ℓ,χ) described in the previous section. (Indeed, in this case, we will always have $H(T,\ell) = H(T,\ell,\chi)$.) On the other hand, when χ takes value ∞ at some leaves of T, then the value of $H(T,\ell,\chi)$ is always finite.

We now turn to the question of determining the value of $H(T,\ell)$. By the previous theorem, we can assume that the random walk $RW(T,\ell)$ is transient. We have the following theorem which in particular leads to an explicit way of sampling the harmonic measure on a transient tree.

Theorem 1.7. *Let T be a locally finite tree and ℓ a length function on T so that the random walk $RW(T,\ell)$ on T is transient. Let η be a the unit current flow on T and μ_{har} the corresponding harmonic measure on ∂T. Assume that (T,ℓ) is weakly complete. Then,*

- *any factorial determining sequence $\rho_0, \rho_1, \ldots$ of (T,ℓ) is equidistributed in ∂T with respect to the harmonic measure μ_{har}.*
- *we have $H(T,\ell) = \|\eta\|^2$, where $\|\eta\|^2$ is the energy of the unit current flow η on T.*

As an immediate corollary, we get the following equidistribution theorem for factorial-determining sequences of subsets of local fields. Let us call a subset S of the valuation ring R of a local field K *transient* if the adelic tree T_S of S is transient. For a transient subset S of R, we denote by μ_{har} the corresponding harmonic measure of S which has support in the closure $\overline{S}$ of S in R. Note that a transient set S is a set with *finite logarithmic capacity* and the corresponding harmonic measure is also called the *equilibrium measure* in the literature, see e.g. [1]. We have

Theorem 1.8. *Let K be a local field with valuation ring R, and let S be an infinite subset of R. The following two conditions are equivalent.*

- *The subset S of R is transient.*
- *The sequence $\frac{1}{n}n!_S$ converges to a finite $H(S) \in (0, \infty)$.*

Moreover, for a transient subset S of R, any factorial determining sequence $s_0, s_1, s_2, \dots$ of S is equidistributed in $\overline{S}$ with respect to the equilibrium measure, and we have

$$\begin{aligned} H(S) &= \int_{\substack{(x,y)\in\overline{S}\times\overline{S} \\ x\neq y}} \operatorname{val}(x-y)d\mu_{\text{har}}(x)d\mu_{\text{har}}(y) \\ &= \int_{\overline{S}} \operatorname{val}(x_0-y)d\mu_{\text{har}}(y) \qquad \textit{a.s. for } x_0 \in \overline{S}. \end{aligned}$$

In other words, $\exp(H(S))$ *is the logarithmic capacity of the set S.*

We note that in the presence of a capacity function on the leaves of T which takes values infinity, the limit $H(T, \ell, \chi)$ has a similar expression. Indeed, it will be enough to consider the modified tree T_1 obtained by adding a countable number of paths to any leaf v of T with $\chi(v) = \infty$, and define the conductance of all these new edges to be equal to ∞. The random walk on T_1 with these conductances is equivalent to a random walk on T with absorption on the leaves of capacity infinity, and the limit $H(T, \ell, \chi)$ is the squared norm of the unit current flow on T_1. For the special case where T is a finite tree and χ is a function on the leaves of T which takes value infinity at some points of T, we have the following explicit way of calculating $H(T, \ell, \chi)$.

Let $L_0 \subset L(T)$ be the set of all leaves v with $\chi(v) = \infty$, and define the connected graph $G = (V, E)$ obtained by identifying all the vertices in L_0 to a single vertex $\mathfrak{s}$. Let $C^0(G, \mathbb{R})$ be the space of real valued functions on the vertices of G. The length function ℓ induces a length function on the edges of G, to which we can associate a Laplcian operator $\Delta : C^0(G, \mathbb{R}) \to C^0(G, \mathbb{R})$ as follows. For any function $f \in C^0(G, \mathbb{R})$, the value of $\Delta(f) \in C^0(G, \mathbb{R})$ at a vertex v of $V(G)$ is given by

$$\Delta(f)(v) = \sum_{\{u,v\}\in E(G)} \frac{1}{\ell_e}\Big(f(u) - f(v)\Big).$$

For a vertex v of G, denote by $\mathbf{1}_v$ the characteric function of v which takes value one at v, and value zero outside v. Let F be the real-valued function on V which solves the Laplace equation $\Delta(F) = \mathbf{1}_{\mathfrak{r}} - \mathbf{1}_{\mathfrak{s}}$. By connectivity of G, up to additioning a constant function, F is unique.

Theorem 1.9. *Notations as above, we have $H(T, \ell, \chi) = F(\mathfrak{s}) - F(\mathfrak{r})$.*

As an immediate corollary, for any $h \in \mathbb{N}$, we get a limit theorem for the factorials of order h associated to subsets of local fields.

1.4. **Organization of the paper.** The local weighting process and the proofs of Theorem 1.1, Theorem 1.3, and Propositios 1.5, as well as basic properties of the weighting process and the tree factorials, are presented in Section 2. In Section 3 we consider the important question of how much information the factorial sequence gives about the tree. We show the realizability of any sufficiently biased sequence of non-negative numbers as the factorials of a pair (T, ℓ), and deduce from that construction, the existence of different trees with the same factorial sequence.

The growth of the factorial sequence is studied in Section 4. The equivalence Theorem 1.6, as well as the limit and equidistribution theorem 1.7 are proved in that section. We omit the proof of Theorem 1.9, which can be obtained by the same arguments.

Some concluding remarks are given in Section 5.

2. Definition and basic properties

In this section, we give the definition of the factorial sequence in terms of a *local exploration process* in the tree. We then show later that this definition is equivalent to the definition given in the introduction.

Let T be a locally finite rooted tree. Denote by $\mathfrak{r}$ the root of T. We orient T away from the root. For any vertex v there is a unique oriented path from $\mathfrak{r}$ to v that we denote by $[\mathfrak{r}, v]$, and denote by $|v|$ the length of $[\mathfrak{r}, v]$ which we call the *generation* of v. We write $u \leq v$ if u lies in the oriented path from $\mathfrak{r}$ to v. The *parent* of a vertex $v \neq \mathfrak{r}$ is the unique vertex u with $u \leq v$, and $|u| = |v| - 1$; it is denoted by $\overleftarrow{v}$. For two vertices u, v, we write $u \sim v$ if u and v are adjacent in the tree. The valence of a vertex v is the number of vertices $u \sim v$ in the tree.

Edges in this paper mean oriented edges, the edge uv is thus oriented from u toward v, and we have $u = \overleftarrow{v}$. If a vertex v is a descendant of another vertex u, we denote by $[u, v]$ the unique path from u to v. A *pending* edge of a vertex u is an edge which joins u to one of its children. For any vertex v in T, we denote by $\mathrm{br}(v)$ the number of children of v. A vertex v in T with $\mathrm{br}(v) \geq 2$ is called *branching*. The set of all the branching vertices of T is denoted by $\mathscr{B}(T)$.

By a *leaf* of a rooted tree T we mean any vertex $v \neq \mathfrak{r}$ of T of valence one if T is not reduced to a single vertex $\mathfrak{r}$. Otherwise, if T has a unique vertex $\mathfrak{r}$, then $\mathfrak{r}$ is a leaf of T. We denote by $L(T)$ the set of all the leaves of T.

An *internal* vertex of T is any vertex different from the leaves of T.

By a *strict path* in T we mean any oriented path P which starts from the root $\mathfrak{r}$, does not contain any branching vertex in its interior, and which is maximal with respect to this property (for the inclusion of paths). It follows that a strict path P either connects $\mathfrak{r}$ to a branching vertex of T, or connects $\mathfrak{r}$ to a leaf of T, or is infinite and $T \setminus \Big(P \setminus \{\mathfrak{r}\}\Big)$ is connected. In addition, for any pending edge $\mathfrak{r}u$ at $\mathfrak{r}$, there exists a unique strict path which contains u, and these are all the strict paths of T.

For any vertex v in T, we denote by T_v the subtree of T consisting of v and all of its descendants, rooted at v.

Let $\ell : E(T) \to \mathbb{R}_+$ be a *length function* on the edges of T which assigns to each edge e of T its length $\ell_e = \ell(e)$. Denote by Γ the metric tree associated to the pair (T, ℓ). Recall that Γ is the disjoint union of the vertex set $V(T)$ and open intervals I_e of length ℓ_e, for $e \in E$, with the identification of the end-points of I_e with the corresponding vertices in $V(T)$. The pair (T, ℓ) is called a *model* of Γ.

A great source of examples for what follows are trees coming from an arithmetic situation, in which case the length function ℓ is the constant function 1. We call the constant length function 1 the *standard* length function.

A *capacity* function on T is any function $\chi : L(T) \to \mathbb{N} \cup \{\infty\}$ giving a capacity to any leaf of T. The standard capacity function is the constant function 1, and if there is no mention of the capacity in what follows, it means all the leaves have capacity one.

A *weighted tree* (T, ω) in this paper means a tree T with a weight function $\omega : E(T) \to \mathbb{N} \cup \{\emptyset\}$, such that the set of edges e with $\omega(e) \neq \emptyset$ forms a connected subgraph of T. An edge e of (T, ω) with $\omega(e) = \emptyset$ is called *unweighted*; all the other edges are called *weighted*.

For a weighted tree (T, ω), we denote by T_ω the subtree of T which contains the root and all the weighted edges $e \in E(T)$. A vertex u of T is called *clear* if all the pending edges of u are unweighted. The clear vertices are all the vertices of T which are either a leaf of T_ω or does not belong to T_ω.

A vertex v of T_ω is called *unsaturated* if either v is an internal vertex of T_ω and there is an edge in $E(T) \setminus E(T_w)$ incident to v, or, in the presence of a capacity function on T, v is a leaf of T and $\omega(\overleftarrow{v}v) < \chi(v)$.

For a tree T with a length function ℓ and weight function ω, the *weighted length* ℓ_ω of a path P in T_ω is defined as

$$\ell_\omega(P) := \sum_{e \in E(P)} \omega(e)\ell_e.$$

We now describe a *weighting process* which provides an alternative equivalent definition of the factorial sequence.

Let T be a locally finite tree, ℓ a length function on T, and χ a capacity function on $L(T)$. Consider the weight function ω_0 which assign $\emptyset$ to any edge. Let $T_0 := T_{\omega_0}$, and note that all the edges of T are unweighted and we have $T_0 = \{\mathfrak{r}\}$. We recursively construct a sequence of weighted trees (T, ω_n) and a sequence of non-negative real numbers a_n. The construction will be so that for all $n \geq 1$

$(*)$ *all the clear vertices of* T_{ω_n} *are either branching or a leaf in* T.

Define $a_0 := 0$. Let $\mathfrak{r}v$ be any edge of T incident to $\mathfrak{r}$. If such an edge does not exist, i.e., if T is reduced to a single vertex $\mathfrak{r}$, then we stop, and define $x_n = \mathfrak{r}$ and $a_n = 0$ for all $1 \leq n < \chi(\mathfrak{r})$. Otherwise, let P be a strict path in T containing both $\mathfrak{r}$ and v. Define

$$\omega_1(e) := \begin{cases} 1 & \text{for all edges } e \in P \\ \omega_0(e) = \emptyset & \text{otherwise.} \end{cases}$$

Note that $(*)$ is clearly verified for T_{w_1}.

Proceeding by induction, suppose now that $n \in \mathbb{N}$, we are at stage n and we have a weighted tree (T, ω_n), a sequence of vertices $x_1, \ldots, x_{n-1}$ of T, and a sequence of integers $a_0, \ldots, a_{n-1}$. Let $T_n := T_{\omega_n}$ and denote by U_n the set of all the unsaturated vertices v of T_n. Assume that T_n verifies $(*)$.

If $U_n = \emptyset$, then we stop. Otherwise, if U_n is non-empty, choose a vertex x_n in U_n with minimum weighted length to the root $\mathfrak{r}$, i.e., so that $\ell_{\omega_n}([\mathfrak{r}, x_n]) = \min_{v \in U_n} \ell_{\omega_n}(\mathfrak{r}, v)$. Set

$a_n := \ell_{\omega_n}(x_n)$, and define the weighted tree (T, ω_{n+1}) as follows, depending on whether x_n is clear or not.

(1) Either x_n is not clear. In this case, choose a pending edge $x_n y$ at x_n with $\omega_n(x_n y) = \emptyset$. Let P_y be the unique strict path in the subtree T_{x_n} which contains y. Define

$$\omega_{n+1}(e) := \begin{cases} 1 & \text{if } e \text{ belongs to } P_y, \\ \omega_n(e) + 1 & \text{if } e \text{ belongs to the path } [\mathfrak{r}, x_n] \text{ in } T, \\ \omega_n(e) & \text{otherwise.} \end{cases}$$

(2) Or x_n is a clear vertex of T_n. By Property $(*)$, x_n is either branching in T or it belongs to $L(T)$.

(2.1) If x_n is branching, then choose any two pending edges $e_1 = x_n z$ and $e_2 = x_n w$ at x_n, and consider the two (disjoint) strict paths P_z and P_w in T_{x_n} with $z \in P_z$ and $w \in P_w$. Define

$$\omega_{n+1}(e) := \begin{cases} 1 & \text{if } e \text{ belongs to the union } P_z \cup P_w, \\ \omega_n(e) + 1 & \text{if } e \text{ belongs to the path } [\mathfrak{r}, x_n], \\ \omega_n(e) & \text{otherwise.} \end{cases}$$

(2.2) If x_n is a leaf of T, then since x_n is unsaturated, we have $\omega_n(\bar{\bar{x}}_n x_n) < \chi(x_n)$. Define

$$\omega_{n+1}(e) := \begin{cases} \omega_n(e) + 1 & \text{if } e \text{ belongs to the path } [\mathfrak{r}, x_n], \\ \omega_n(e) & \text{otherwise.} \end{cases}$$

Let $T_{n+1} := T_{\omega_{n+1}}$. Any clear vertex v of T_{n+1} is either a clear vertex of T_n, or an end-point of a strict path in the subtree T_{x_n} (one among P_y, P_z, P_w). It follows that v is either branching or a leaf in T. Thus, T_{n+1} verifies Property $(*)$, and the above definition results in a sequence of weighted trees (T, ω_i), a sequence of vertices x_i, and specially, a sequence of reals a_i.

Note that in the case the length function takes integer values, all the numbers a_i are integers.

By definition, it is easy to see that in the case χ is the standard capacity function, the sequence is infinite if and only if the number of branching vertices of T is infinite. More generally, define $N_{T,\chi}$ by

$$N_{T,\chi} := 1 + \sum_{v \in \mathscr{B}} (\mathrm{br}(v) - 1) + \sum_{v \in L(T)} (\chi(v) - 1). \tag{2.1}$$

One can see directly from the definition that (T, ω_n) and a_n are defined provided that $0 \leq n < N_{T,\chi}$, as in this case U_n is always non-empty. (See also the proof of Theorem 2.1 below).

The sequence (T, ω_n) is obviously not unique in general as it involves making a choice of a vertex $x_n \in U_n$ and strict paths in some subtrees at each stage. However, the sequence $\{a_i\}_{0 \leq i < N_{T,\chi}}$ only depends on (T, ℓ) (actually, only on the rooted metric tree Γ associated to (T, ℓ)).

Theorem 2.1. *(i) The sequence $a_0, a_1, a_2, \dots$ only depends on (T, ℓ, χ).*
(ii) For two pairs (T_1, ℓ_1) and (T_2, ℓ_2) with the same metric realization, any capacity function χ on T_1 induces a capacity function on T_2, and the triples (T_1, ℓ_1, χ) and (T_2, ℓ_2, χ) have the same factorial sequence.

This leads to the following definition.

Definition 2.2 (Logarithmic tree factorials)**.** • Let T be a rooted locally finite tree. For each integer $0 \leq n < N_T$, the integer a_n associated to the tree T with standard length function $\ell \equiv 1$ is called the T-factorial of n and is denoted by $n!_T$.

• Let Γ be a rooted metric tree with a model (T, ℓ), where T is a rooted locally finite tree and ℓ a length function on $E(T)$. Let $\chi : L(T) \to \mathbb{N} \cup \{\infty\}$ be a capacity function. For each integer $0 \leq n < N_{T,\chi}$, the real number a_n associated to the tree T with length function ℓ and with capacity χ is called the (T, ℓ, χ)-factorial or (Γ, χ)-factorial of n and is denoted by $n!_{(T,\ell,\chi)} = n!_{(\Gamma,\chi)}$. If χ is the standard capacity function, we drop χ and simply write $n!_{(T,\ell)}$ or $n!_\Gamma$.

• For any pair (T, ℓ) with metric realization Γ and a capacity function χ, the sequence x_n in the weighting process described above is a called a *factorial-determining* or *factorial-defining* sequence for (T, ℓ, χ) and (Γ, χ). The sequence of trees T_n in the weighting process is called a sequence of *factorial trees* for (T, ℓ).

Note that neither the factorial-defining sequence nor the factorial trees are unique in general.

Proof of Theorem 2.1(i). We proceed by induction. Consider the following property $\mathscr{P}_n$:

($\mathscr{P}_n$) For any locally finite tree T, any length function $\ell : E(T) \to \mathbb{R}_+$, any capacity function χ on T, and any $0 \leq i \leq \min\{n, N_{T,\chi} - 1\}$, the number a_i only depends on (T, ℓ, χ).

By our definition, $a_0 = 0$ for any tree T, so obviously $\mathscr{P}_0$ holds. We show that $\mathscr{P}_n$ implies $\mathscr{P}_{n+1}$, from which the theorem follows.

Assume $\mathscr{P}_n$ holds. Let T be any locally finite tree T, $\ell : E(T) \to \mathbb{R}_+$ a length function, and χ a capacity function on T. If T is reduced to a single vertex $\mathfrak{r}$, then we have $a_i = 0$ for all $0 \leq i < N_{T,\chi}$, by definition, and so the property $\mathscr{P}_{n+1}$ obviously holds for T. Otherwise, consider the following two cases depending on whether $\mathrm{br}(\mathfrak{r}) > 1$ or $\mathrm{br}(\mathfrak{r}) = 1$.

(I) *Suppose* $\mathrm{br}(\mathfrak{r}) > 1$. Let $d = \mathrm{br}(\mathfrak{r})$, and denote by $u_1, \dots, u_d$ the children of $\mathfrak{r}$ in T. First note that the first d terms in any sequence $a_0, a_1, \dots$ produced by the weighting process are equal to 0, since, by the positivity of the values of the length function, the weighting process has to give weight one to all the pending edges at $\mathfrak{r}$ before giving weight to any other unweighted edge of T.

Consider the subtrees $T_{u_1}, \dots, T_{u_d}$ of T rooted at $u_1, \dots, u_d$, respectively, with capacity function χ_j defined as the restriction of χ to T_{u_j}, and set $n_i := \min\{n, N_{T_{u_i},\chi_j} - 1\}$. Since $\mathscr{P}_n$ is verified for all trees, we get for any $i = 1, \dots, d$, a well-defined sequence $a_0^{u_i}, \dots, a_{n_i}^{u_i}$. For each i, define the set

$$A_i := \left\{ a_0^{u_i}, a_1^{u_i} + \ell_{\mathfrak{r}u_i}, \dots, a_{n_i}^{u_i} + n_i \ell_{\mathfrak{r}u_i} \right\},$$

and let A be the multiset union of the sets A_i. Let $m := n_1 + \cdots + n_d + d - 1$ and note that A has size $m + 1$. Order the elements of A in an increasing order $b_0 \leq b_1 \leq b_2 \leq \cdots \leq b_m$.

First note that

Claim 2.3. *We have* $m \geq \min\{n + 1, N_{T,\chi} - 1\}$.

Proof. If there is an $1 \leq i \leq d$, such that $n \leq N_{T_{u_i},\chi_i} - 1$, we get $m \geq n + d - 1 \geq n + 1$. Otherwise, we have $n_i = N_{T_{u_i},\chi_i} - 1$ for all $i \in \{1, \dots, d\}$, and so using that $N_{T,\chi} = \sum_{i=1}^d N_{T_{u_i},\chi_i}$, which comes from the definition, we get $m = (\sum_{i=1}^d N_{T_{u_i},\chi_i}) - 1 = N_{T,\chi} - 1$, and the claim follows. ☐

The following claim proves that property $\mathscr{P}_{n+1}$ holds for any rooted tree T with $\mathrm{br}(\mathfrak{r}) \geq 2$.

Claim 2.4. *The first $n+2$ terms of any sequence $a_0, a_1, \dots$ associated to (T, ℓ, χ) by the weighting process coincide with $b_0, \dots, b_{n+1}$.*

Proof. For the sake of a contradiction suppose this is not the case, and consider a weighting process resulting in a sequence $\{a_i\}_{0\leq i<N_{T,\chi}}$ such that the claim does not hold, and let $0 \leq t \leq n+1$ be the smallest integer with $a_t \neq b_t$. Since $a_0 = \dots = a_{d-1} = 0$, and $b_0 = \dots = b_{d-1} = 0$, we have $t \geq d$. In the ordering $b_0 \leq b_1 \leq \dots \leq b_t$, each b_i comes from one of the sets $A_1, \dots, A_d$. Let $0 \leq t_1 \leq n_1, \dots, 0 \leq t_d \leq n_d$ be integer numbers so that the union of the smallest $t_i + 1$ terms in each A_i when reordered in an increasing order gives the sequence of b_is for $0 \leq i \leq t$. We note that the sequence $t_1, \dots, t_d$ is not necessarily unique as it might be repetitions among the members of different sets A_i. We have

$$d + \sum_{i=1}^{d} t_i = t + 1.$$

Consider the weight function ω_{t+1}, and set $s_i := \omega_{t+1}(\mathfrak{r}u_i) - 1$. We have

$$d + \sum_{i=1}^{d} s_i = t + 1,$$

Define $B_i := \left\{ a_0^{u_i}, a_1^{u_i} + \ell_{\mathfrak{r}u_i}, \dots, a_{s_i}^{u_i} + s_i\ell_{\mathfrak{r}u_i} \right\}$. Consider the mutiset union B of the sets $B_1, \dots, B_d$.

Claim 2.5. *The sequence $a_0, \dots, a_t$ coincides with the increasing sequence formed out of the elements of B.*

Proof. Follows directly from the definition of the weighting process. Indeed, in any of the cases (1) or (2) in the definition of the weightings and the sequence $\{a_i\}$, if the vertex x_n is in the subtree T_{u_i}, for $1 \leq i \leq d$, then we have $a_n = a^{u_i}_{\omega_n(\mathfrak{r}u_i)} + \omega_n(\mathfrak{r}u_i)\ell_{\mathfrak{r}u_i}$. ☐

Applying the above claim, since we assumed $a_t \neq b_t$, we infer that the two sequences $(t_1, \dots, t_d)$ and $(s_1, \dots, s_d)$ are different. Let $1 \leq i, j \leq d$ be the indices which give

$$a_t = a_{s_i}^{u_i} + s_i\ell_{\mathfrak{r}u_i}, \quad \text{and} \quad b_t = a_{t_j}^{u_j} + t_j\ell_{\mathfrak{r}u_j}.$$

We divide the rest of the proof in two parts depending on whether $a_t > b_t$ or $b_t > a_t$.

Suppose first $a_t > b_t$. Given that $a_{t_i}^{u_i} + t_i\ell_{\mathfrak{r}u_i}$ appears among $b_0, \dots, b_t$, we get $a_{t_i}^{u_i} + t_i\ell_{\mathfrak{r}u_i} \leq b_t < a_t = a_{s_i}^{u_i} + s_i\ell_{\mathfrak{r}u_i}$. Therefore, the sequence $\{a_j^{u_i}\}$ being increasing, we must have $t_i < s_i$. Since $\sum_{j=1}^{d} t_j = \sum_{j=1}^{d} s_j$, there exists an index h so that $s_h < t_h$. It follows that

$$a_t > b_t \geq a_{t_h}^{u_h} + t_h\ell_{\mathfrak{r}u_h} \geq a_{s_h+1}^{u_h} + (s_h+1)\ell_{\mathfrak{r}u_h}.$$

This leads to a contradiction. Indeed, the description of the weighting process, the choice of a_t, and the fact that $\omega_{t+1}(\mathfrak{r}u_h) = s_h$ implies in particular that $a_{s_h+1}^{u_h} + (s_h+1)\ell_{\mathfrak{r}u_h} \geq a_t$.

Suppose now that $b_t > a_t$. Then we have

$$a_{t_j}^{u_j} + t_j\ell_{\mathfrak{r}u_j} > a_t \geq a_{s_j}^{u_j} + s_j\ell_{\mathfrak{r}u_j},$$

which implies $t_j > s_j$. Therefore, there exists an index $1 \leq h \leq d$ such that we have $s_h > t_h$. It follows that

$$a_t \geq a_{s_h} + s_h\ell_{\mathfrak{r}u_h} \geq a_{t_h+1} + (t_h+1)\ell_{\mathfrak{r}u_h} \geq b_t,$$

which again leads to a contradiction. □

(II) *Suppose* $\mathrm{br}(\mathfrak{r}) = 1$. Let P be the unique strict path in T. If P is infinite, then $N_T = 0$, and we are done. Otherwise, let v be the other end of P. If v is a leaf, then T is a rooted path, and in this case $N_{T,\chi} = \chi(v)$, and we have $a_i = i\ell(P)$ for all $0 \le i < \chi(v)$, and again we are done. So we can suppose that v is branching in T. By case (I), since the root v of T_v is branching, the property $\mathscr{P}_{n+1}$ is verified for T_v. Let $h = \min\{n+1, N_{T_v,\chi}\}$. In particular, the sequence $a_0^v, \dots, a_h^v$ associated to T_v is well-defined.

We now note that by Formula (2.1), we have $N_{T,\chi} = N_{T_v,\chi}$. The following claim, which directly follows from the definition of the weighting process, shows that property $\mathscr{P}_{n+1}$ holds also for any tree T with $\mathrm{br}(\mathfrak{r}) = 1$.

Claim 2.6. *For each $0 \le i \le h$, we have $a_i = a_i^v + i\ell(P)$.*

This finishes the proof of part (i) of Theorem 2.1. □

Proof of Theorem 2.1(ii). Denote by Γ the metric realization of both (T_1, ℓ_1) and (T_1, ℓ_2), and let χ be a capacity function on T_1. Note that we have $L(T_1) = L(T_2)$ and $\mathscr{B}(T_1) = \mathscr{B}(T_2)$. It follows that χ is also a capacity function for T_2, and for any vertex $v \in \mathscr{B}(T_1) = \mathscr{B}(T_2)$, there is a bijection between the strict paths P_1 of the subtree $T_{1,v}$ of T_1 rooted at v, and the strict paths P_2 of the subtree $T_{2,v}$ of T_2 rooted at v, and moreover, under this bijection, we have $\ell_1(P_1) = \ell_2(P_2)$.

The choices of vertices producing the factorial sequence in the description of the weighting process for a tree T only depends on the root, branching and leaf vertices, and the length of strict paths of subtrees T_v for branching vertices v. It follows that any factorial-determining sequence x_n in (T_1, ℓ_1, χ) is also factorial-determining sequence in (T_2, ℓ_2, χ), from which the theorem follows. □

We state the following useful recursive *min-max formula* for the factorials obtained in the above proof.

Theorem 2.7. *Let (T, ℓ, χ) be a triple of a locally finite tree T rooted at $\mathfrak{r}$, a length function ℓ and a capacity function χ on T, respectively. Let $d = \mathrm{br}(\mathfrak{r})$ and denote by $u_1, \dots, u_d$ all the children of $\mathfrak{r}$. For each $j = 1, \dots, d$, let $N_j = N_{T_{u_j},\chi_j}$, for the restriction χ_j χ to T_{u_j}, and denote by ℓ_j the restriction of ℓ to T_{u_j}. Then, for all integers $0 \le n < N_{T,\chi}$, we have*

$$n!_{(T,\ell,\chi)} = \min_{\substack{(n_1,\dots,n_d)\in\mathbb{N}^d \\ \text{for all } j,\ 0\le n_j<N_j \\ n_1+\dots+n_d=n+1}} \max \Big\{ (n_j-1)!_{(T_{u_j},\ell_j,\chi_j)} + (n_j-1)\ell_{\mathfrak{r}u_j} \Big\}_{j=1}^{d}.$$

2.1. **Proofs of Theorem 1.1 and Theorem 1.3.** We first prove the equivalence of the definition of the factorial sequence given in Section 1.1 with the one given in this section, thus proving Theorem 1.3.

Theorem 2.8. *Let T be a locally finite rooted tree, ℓ a length function and χ a capacity function on T. Let Γ be the metric tree with a model (T, ℓ). Let $\alpha_0, \alpha_1, \dots$ be the sequence of numbers associated to (Γ, χ) as in (1.1). We have for all $n \in \mathbb{N}$, $(n!)_{(T,\ell,\chi)} = \alpha_n$.*

Proof. Let $\rho_0, \rho_1, \dots$ be a sequence of elements of the extended boundary $\widetilde{\partial}\Gamma$ producing $\alpha_0, \alpha_1, \dots$, as in (1.1) in the introduction, i.e.,

$$\alpha_n = \sum_{i=0}^{n-1} \langle \rho_n, \rho_i \rangle.$$

Proceeding by induction on n, we show how the sequence $\{\rho_i\}$ define a weighting sequence $\{\omega_i\}$, tree factorials T_{ω_i}, and vertices $\{x_i\}$, so that we have $a_n = \alpha_n$. The weighting sequence $\{\omega_i\}$ is defined in such a way that for each n, the property $\mathscr{X}_n$ is verified:

($\mathscr{X}_n$) • for each edge uv in T_{ω_n}, $\omega_n(uv)$ is the number of elements ρ in the sequence $\rho_0, \dots, \rho_{n-1}$ which belong to the extended boundary $\widetilde{\partial}T_v$, and

• For each leaf v of T_{ω_n}, we have $\omega_n(\bar{v}v) = 1$.

For $n = 0$, the element ρ_0 of the extended boundary starts with a strict P_0 of T, with one end point $\mathfrak{r}$. Let ω_1 be the weighting associated to the choice of P_0 in the weighting process, and note that ($\mathscr{X}_1$) obviously holds. Proceeding recursively, suppose that for $n \geq 1$, $\omega_0, \dots, \omega_n$ and $x_1, \dots, x_{n-1}$ are defined, so that ($\mathscr{X}_n$) holds. Consider $\rho_n \in \widetilde{\partial}\Gamma$, and define x_n as the last vertex of T_{ω_n} on the path ρ_n. By the definition of the sequence ρ_i, the vertex x_n is an unsaturated vertex of T_{ω_n} and so belongs to U_n. In addition, by Property ($\mathscr{X}_n$), we have

$$\ell_{\omega_n} = \sum_{e \in [\mathfrak{r}, x_n]} \omega_n(e)\ell_e = \sum_{j=0}^{n-1} \langle \rho_n, \rho_j \rangle,$$

which shows that x_n is a vertex in U_n which minimizes the ℓ_{ω_n}-distance to the root $\mathfrak{r}$.

Two cases can happen. If x_n is not clear, let $x_n y$ be the pending edge at x_n which belongs to ρ_n, and let P_y be the corresponding strict path in T_{x_n}. Define ω_{n+1} as Case (1) in the definition of the weighting process. One easily verifies that ($\mathscr{X}_{n+1}$) holds.

Otherwise, x_n is a clear vertex. If x_n is a leaf of T, then define ω_{n+1} as in Case (2.1) in the the definition of the weighting process. Otherwise, x_n is branching, and is a leaf of T_{ω_n}. By Property ($\mathscr{X}_n$), we have $\omega_n(\bar{x}_n x_n) = 1$ and so there exists a unique $0 \leq j \leq n-1$ so that ρ_j is in $\widetilde{\partial}T_{x_n}$. Let $x_n z$ and $x_n w$ be the two pending edges at x_n which belong to ρ_j and ρ_n, respectively, and define ω_{n+1} as in Case (2.2) in the definition of the weighting process. One easily verifies that in both cases, Property ($\mathscr{X}_{n+1}$) is verified. □

To prove Theorem 1.1, it will be now enough to prove Proposition 1.5.

Proof of Proposition 1.5. Let (T_0, ℓ_0, χ_0) be associated to (T, ℓ) as in Section 1.2. The proof is based on the observation that for any vertex v of valence infinity in T, and for any sequence $\rho_0, \rho_1, \dots$ defining the sequence $a_0, a_1, \dots$ for the pair (T, ℓ) as in the introduction, any two different elements ρ_i and ρ_j, for $i \neq j$ which belong to $\partial\widetilde{T}_v$, must contain two different pending edges vw and vz at v. The sequence $\rho_0, \rho_1, \dots$ gives a sequence $\tau_0, \tau_1, \dots$, where τ_i is defined by intersecting ρ_i with T_0. Using the observation, one verifies that for any n,

$$a_n = \sum_{j=0}^{n-1} \langle \rho_n, \rho_j \rangle = \sum_{j=0}^{n-1} \langle \tau_n, \tau_j \rangle.$$

The Proposition now follows from Theorem 1.3, since $\tau_0, \tau_1, \dots$ determines the factorials of (T_0, ℓ_0, χ_0). □

In the rest of this section, we prove some basic fundamental results which will be used in the upcoming sections.

2.2. **The exhaustiveness of the weighting process.** The following proposition shows the weighting process eventually gives weight to any edge of the tree.

Proposition 2.9. *Let (T,ℓ) be a pair consisting of a locally finite tree with a length function ℓ, and let χ be a capacity function on T. Any sequence of weighting $\{\omega_n\}$ producing the logarithmic factorials of (T,ℓ,χ) eventually gives a weight to any edge of the tree T. In other words, the union of the trees in any sequence of factorial trees is the whole tree T.*

Proof. Let $N = N_{T,\chi}$. Let $T_0 = \bigcup_{i=0}^{N_T} T_{\omega_i}$, and for the sake of a contradiction, suppose $E(T)\setminus E(T_0) \neq \emptyset$. Since T_0 is a tree, there exists a vertex v in $V(T)\setminus V(T_0)$ adjacent to a vertex $u \in T_0$, so we have $\overline{v} = u$.

As already observed before, the first $\mathrm{br}(\mathfrak{r})$ terms of the factorial sequence are all 0, and the weighting consists in giving weights to the edges of the strict paths of T, in particular to all the pending edges of T at $\mathfrak{r}$.

Note that since u is branching, it remains unsaturated in all the trees T_{ω_i} which contain u. This is impossible if the factorial sequence is finite, so we can suppose that $N=\infty$.

To simplify the presentation, we will use the usual O notation in what follows: for a sequence of non-negative number $\{f_n\}_{n\in\mathbb{N}}$, we write $f_n = O(1)$ if there exists a constant $C>0$ such that for all $n\in\mathbb{N}$, we have $f_n \le C$.

Denote by $P = \mathfrak{r}u_1u_2\dots u_k$ the oriented path $[\mathfrak{r},u]$ from $\mathfrak{r}$ to $u_k = u$ in T.

Claim 2.10. *For any $1\le j\le k$, and any pending edge u_jx at u_j in T_0, the sequence $\Big\{\omega_n(u_jx)\,\Big|\, n\in\mathbb{N} \text{ with } u_jx \in T_{\omega_n}\Big\}$ verifies $\omega_n(u_jx) = O(1)$. The same statement holds for the sequence $\Big\{\omega_n(\mathfrak{r}u_1)\,\Big|\, n\in\mathbb{N} \text{ with } \mathfrak{r}u_1 \in T_{\omega_n}\Big\}$.*

We prove the first part of the claim by a reverse induction on j.

Let $x_0, x_1, \dots$ be the factorial-determining sequence corresponding to the edge weighing sequence $\omega_0,\omega_1,\dots$. Consider first the vertex $u_k = u$, and let u_kx be a pending edge at u_k in T_0. We claim that $\omega_n(u_kx)\le 1$ for all n with $u_kx\in T_{\omega_n}$. Suppose this is not the case, and consider the integer m such that x_m is chosen in the subtree T_x, and $\omega_m(u_kx)=1$ and $\omega_{m+1}(u_kx)=2$. We have $\ell_{\omega_m}\big([\mathfrak{r},x_m]\big) > \ell_{\omega_m}\big([\mathfrak{r},u_k]\big)$, which shows that the choice of v instead of x_n gives a strictly smaller value for a_n (in the weighting process). This contradiction proves the claim for $j=k$.

Suppose now that the claim holds for all integers i with $1\le j<i\le k$. We prove that the claim holds for integer j. Let u_jx be a pending edge at u_j in T_0, and suppose that $\omega_n(u_jx)$ tends to infinity, as n tends to infinity. By the hypothesis of the induction, we have $\omega_n(u_iu_{i+1}) = O(1)$ for all $j+1\le i\le k$ and for all large enough integers n. In addition, since for any large n, we have $\omega_n(u_ju_{j+1}) = \sum_{u_{j+1}x\in T_0}\omega_n(u_{j+1}x)$, we infer that $\omega_n(u_ju_{j+1}) = O(1)$. This in particular implies that $\ell_{\omega_n}([u_j,u_k])\le C$, for some constant $C>0$ and all large enough integers n. Let m be an integer such that $\omega_m(u_jx) > C/\ell(u_jx)$ and $\omega_{m+1}(u_jx) = \omega_m(u_jx)+1$, which exists by the assumption that $\omega_n(u_jx)$ tends to infinity.

The vertex x_m lies in the subtree T_x. We have

$$\begin{aligned}\ell_{\omega_n}\big([\mathfrak{r},x_m]\big) \geq \ell_{\omega_m}\big([\mathfrak{r},x]\big) &= \ell_{\omega_m}\big([\mathfrak{r},u_j]\big)+\ell_{\omega_m}\big([u_j,x]\big) = \ell_{\omega_m}\big([\mathfrak{r},u_j]\big)+\omega_m(u_jx)\ell(u_jx)\\ &> \ell_{\omega_m}\big([\mathfrak{r},u_j]\big)+C \geq \ell_{\omega_m}\big([\mathfrak{r},u_j]\big)+\ell_{\omega_m}\big([u_j,u_k]\big)\\ &= \ell_{\omega_m}\big([\mathfrak{r},u_k]\big),\end{aligned}$$

which is a contradiction with the choice of x_m. This proves the claim for all the pending edges at u_j, $j=1,\dots,k$. The boundedness of the sequence $\omega_n(\mathfrak{r}u_1)$ follows now from the fact that $\omega_n(\mathfrak{r}u_1)=\sum_{u_1x\in E(T_0)}\omega_n(u_1x)$ for all large enough integers n.

To finish the proof of the proposition, note that since $N_{T,\chi}=\infty$, we have $\sum_{\mathfrak{r}x\in E(T_0)}\omega_n(\mathfrak{r}x)\to\infty$, which shows that $n!_{(T,\ell,\chi)}\to\infty$ as n tends to infinity. On the other hand, by the claim we just proved, we have $\ell_{\omega_n}\big([\mathfrak{r},u_k]\big)\leq C$ for some constant $C>0$ and all large enough integers $n\in\mathbb{N}$. But this is impossible since at some stage n, the weighting process will have the better choice of v instead of x_n. □

2.3. **Super-additivity of the factorial sequence.** We now prove the following useful proposition.

Proposition 2.11. *For any pair triple (T,ℓ,χ) consisting of a locally finite tree T, a length function ℓ and a capacity χ on T, and for all non-negative integers $n\geq m$, we have*

$$n!_{(T,\ell,\chi)}\geq (n-m)!_{(T,\ell,\chi)}+m!_{(T,\ell,\chi)}.$$

Applying Fekete's Lemma, the proposition implies that

Corollary 2.12. *For any triple (T,ℓ,χ) with $N_{T,\chi}=\infty$, the limit of the sequence $\frac{1}{n}(n!)_{(T,\ell,\chi)}$ exists and belongs to the interval $(0,+\infty]$.*

We will later describe the value of the limit.

Proof of Proposition 2.11. We prove by induction the following property $\mathscr{Q}_M$, for non-negative integers M.

($\mathscr{Q}_M$) *For all triples (T,ℓ,χ) consisting of a rooted tree T, length function ℓ and capacity function χ on T, and for all non-negative integers $n\leq M$ and $0\leq m\leq n$, we have*

$$n!_{(T,\ell,\chi)}\geq (n-m)!_{(T,\ell,\chi)}+m!_{(T,\ell,\chi)}.$$

The result obviously holds for $M=0$. So suppose $\mathscr{Q}_n$ holds for all $n<M$. We prove $\mathscr{Q}_M$.

Using Claim 2.6, we can reduce to the case where the root is branching, i.e., $d:=\mathrm{br}(\mathfrak{r})\geq 2$. Denote by $u_1,\dots,u_d$ the children of $\mathfrak{r}$, and denote by ℓ_j and χ_j the restriction of χ to T_{u_j}, respectively. Let $N_j:=N_{T_{u_j},\chi_j}$. For $j=1,\dots,d$, denote by $S_j:=\big\{a_i^j\big\}_{0\leq i<N_j}$ the set of factorials of (T_{u_j},ℓ_j,χ_j) with $a_i^j:=i!_{(T_{u_j},\ell_j,\chi_j)}$. Define $A_j:=\big\{a_i^j+i\ell_{\mathfrak{r}u_j}\big\}_{0\leq i<N_j}$, as in the proof of Theorem 2.1.

Let $0\leq m\leq n\leq M$ be two integers. We show the inequality $n!_{(T,\ell,\chi)}\geq(n-m)!_{(T,\ell,\chi)}+m!_{(T,\ell,\chi)}$. We can suppose that $n=M$, as otherwise, the inequality follows from the validity of $\mathscr{Q}_{M-1}$.

By Claim 2.5, $M!_{(T,\ell,\chi)}$ is the $(M+1)$-st term in the multiset union A of the sets $A_1,\dots,A_d$ when the terms are put in an increasing order. We label each element of the multiset union A with the index j of the set A_j where it comes from, and fix an increasing order on the elements of A. In this way we can define positive integer numbers M_j, for $j = 1, \dots, d$, as the number of terms labeled with j among the first $M+1$ terms. In other words, for each $1 \le j \le d$, $a^j_i + i\ell_{\mathfrak{r}u_j}$ for $i = 0, \dots, M_j - 1$ are among the $M+1$ first terms of A. In particular, we have $\sum_{j=1}^d M_j = M+1$, and

$$\text{for all } 1 \le j \le d,\ \ M!_{(T,\ell,\chi)} \ge a^j_{M_j-1} + (M_j-1)\ell_{\mathfrak{r}u_j},\ \text{with equality for at least one } j. \tag{2.2}$$

Similarly, the $(m+1)$-st term in A is equal to $m!_{(T,\ell,\chi)}$, and we define m_j, for each $j = 1, \dots, d$, as the number of terms of A_j which appear in the first $m+1$ terms of A. We have $\sum_{j=1}^d m_j = m+1$, and

$$\text{for all } 1 \le j \le d,\ \ m!_{(T,\ell,\chi)} \ge a^j_{m_j-1} + (m_j-1)\ell_{\mathfrak{r}u_j}\ \text{with equality for at least one } j.$$

In addition, since all the terms of the form $a^j_{m_j} + m_j\ell_{\mathfrak{r}u_j}$, for $j = 1, \dots, d$, appear after the $(m+1)$-st term of the sequence of A, it follows that

$$\forall\ 1 \le j \le d, \qquad a^j_{m_j} + m_j\ell_{\mathfrak{r}u_j} \ge m!_{(T,\ell,\chi)}. \tag{2.3}$$

Suppose without loss of generality that $m!_{(T,\ell,\chi)} = a^1_{m_1-1} + (m_1-1)\ell_{\mathfrak{r}u_1}$. Note that we have $M_j \ge m_j$ for all j.

Combining Inequalities (2.2) and (2.3), we get

$$\begin{aligned} M!_{(T,\ell,\chi)} - m!_{(T,\ell,\chi)} &\ge \Big(a^1_{M_1-1} + (M_1-1)\ell_{\mathfrak{r}u_1}\Big) - \Big(a^1_{m_1-1} + (m_1-1)\ell_{\mathfrak{r}u_1}\Big) \\ &= \Big(a^1_{M_1-1} - a^1_{m_1-1}\Big) + (M_1-m_1)\ell_{\mathfrak{r}u_1}, \end{aligned}$$

and for all values of $j \ge 2$ with $M_j > m_j$, we have

$$\begin{aligned} M!_{(T,\ell,\chi)} - m!_{(T,\ell,\chi)} &\ge \Big(a^j_{M_j-1} + (M_j-1)\ell_{\mathfrak{r}u_j}\Big) - \Big(a^j_{m_j} + m_j\ell_{\mathfrak{r}u_j}\Big) \\ &= \Big(a^j_{M_j-1} - a^j_{m_j}\Big) + (M_j-m_j-1)\ell_{\mathfrak{r}u_j}. \end{aligned}$$

Since $M_j \le M-1$ for all j, by property $\mathscr{Q}_{M-1}$ applied to the subtrees T_{u_j}, we get for all $j \ge 2$ with $M_j > m_j$,

$$\Big(a^j_{M_j-1} - a^j_{m_j}\Big) + (M_j-m_j-1)\ell_{\mathfrak{r}u_j} \ge a^j_{M_j-1-m_j} + (M_j-m_j-1)\ell_{\mathfrak{r}u_j}.$$

Moreover, for $j=1$, we have

$$\Big(a^1_{M_1-1} - a^1_{m_1-1}\Big) + (M_1-m_1)\ell_{\mathfrak{r}u_1} \ge a^1_{M_1-m_1} + (M_1-m_1)\ell_{\mathfrak{r}u_1}.$$

We infer that all the terms of the form $a^j_i + i\ell_{\mathfrak{r}u_j}$ for $j \ge 2$ and $0 \le i \le M_j - m_j - 1$, and all the terms $a^1_i + i\ell_{\mathfrak{r}u_1}$ for $0 \le i \le M_1 - m_1$ are bounded from above by $M!_{(T,\ell,\chi)} - m!_{(T,\ell,\chi)}$. This shows that in the multiset union $A = \bigcup_{j=1}^d A_j$, there are at least

$$M_1 - m_1 + 1 + \sum_{\substack{2\le j\le d\\ \text{such that } M_j>m_j}} (\,M_j - m_j\,) = 1 + \Big(\sum_{j=1}^d M_j - m_j\Big) = M - m + 1$$

terms bounded from above by $M!_{(T,\ell,\chi)} - m!_{(T,\ell,\chi)}$. Since the $(M-m+1)$-st term in the sequence of elements of A is $(M-m)!_{(T,\ell,\chi)}$, we finally get the required inequality

$$M!_{(T,\ell,\chi)} - m!_{(T,\ell,\chi)} \ge (M-m)!_{(T,\ell,\chi)}.$$

□

3. How much information factorial sequence gives about the tree?

It is natural to ask how much information about the tree T is captured by the factorial sequence and, in particular, whether the factorial sequence associated to T characterizes T uniquely? The question is intimately related to the question of characterizing the sequences of integers which can be realized as factorials associated to a tree. In this section we discuss these questions. Let us make the following definition.

Definition 3.1 (Factorial realizability)**.** Let $N \in \mathbb{N} \cup \{\infty\}$. A sequence $S = \{a_i\}_{0 \leq i < N}$ of increasing non-negative real numbers is *factorial realizable* if there exists a locally finite rooted tree T, a length function ℓ and a capacity function χ on T such that for each non-negative integer $0 \leq n < N$, we have $a_n = n!_{(T,\ell,\chi)}$.

3.1. **Realizability of sufficiently biased sequences.** Consider an infinite increasing sequence of positive numbers S. Let $d \in \mathbb{N}$ be a natural number. We can rewrite the elements of S in the form (in an increasing order)

$$a_{0,1}, \dots, a_{0,d}, a_{1,1}, \dots, a_{1,d}, a_{2,1}, \dots, a_{2,d^2}, \dots, a_{n,1}, \dots, a_{n,d^n}, \dots$$

consisting for each n of d^n reals $a_{n,1} \leq \cdots \leq a_{n,d^n}$.

Definition 3.2. An infinite increasing sequence S is called *d-sufficiently biased* if it satisfies:

$$\text{for each } n \geq 0, \qquad a_{n+1,1} > 2d^{n+1} \sum_{i=0}^{n} a_{i,d^i}.$$

Let $d \geq 2$ be an integer. Let $\mathscr{T}_d$ be the rooted d-regular tree, where every node has branching equal to d, and for each integer $n \geq 0$, choose an arbitrary total order $\leq_n$ on the vertices of the $\mathscr{T}_d$ at generation n. Let $\leq$ be the total order on the vertices of $\mathscr{T}_d$ induced by total orders $\leq_n$, and by declaring $u < v$ for two vertices u, v of $\mathscr{T}_d$ provided that the vertex u has generation strictly smaller than that of v.

Definition 3.3. Given a collection of total orders $\{\leq_n\}_{n=0}^{\infty}$ inducing a total order $\leq$ on the nodes of $\mathscr{T}_d$ as above, and given a length function $\ell : E(\mathscr{T}_d) \to \mathbb{N}$, we say that ℓ and $\leq$ are *coherent* if the following two properties hold in the construction of the factorials associated to (T, ℓ):

1. (1) for each non-negative integer n, all the vertices in generation $n+1$ are clear as far as there exists an unsaturated vertex in generation n;
2. (2) the order of weighting unweighted pending edges at vertices of generation $n-1$ of $\mathscr{T}_d$ in the weighting process coincides with the total order $\leq_n$ on generation n.

Note in particular that (1) implies that for any n, a vertex x_k of $\mathscr{T}_d$ which gives $k!_{(T,\ell)}$ in the construction of the factorial sequence lies in generation n provided that $d^n \leq k \leq d^{n+1}-1$.

We have the following theorem.

Theorem 3.4. *Let $d \geq 2$ be an integer. For any d-sufficiently biased sequence S as above with $a_{0,1} = a_{0,2} = \cdots = a_{0,d} = 0$, and any collection $\Big\{\leq_n\Big\}_{n\in\mathbb{N}}$ of total orders $\leq_n$ on the n-th generation of the d-regular tree $\mathscr{T}_d$, there is a length function $\ell : E(\mathscr{T}_d) \to \mathbb{R}_+$ such that*

- *ℓ and $\mathcal{O}$ are coherent, and*
- *the factorial sequence associated to the pair $(\mathscr{T}_d, \ell)$ coincides with S.*

Proof. We describe how to construct the length function by induction.

For each $n \in \mathbb{N}$, denote by T_n the subtree of $\mathscr{T}_d$ consisting of all the vertices at generation not exceeding n. Denote by $u_{n,1} <_n \cdots <_n u_{n,d^n}$ all the vertices of generation n in an increasing order with respect to the total order $\leq_n$. For each $1 \leq i \leq d^n$, denote by $e_{n,i}$ the unique edge of T joining a vertex of generation $n-1$ to $u_{n,i}$. So, for example, $u_{1,1} <_1 u_{1,2} < \cdots < u_{1,d}$ are the d vertices of $\mathscr{T}_d$ adjacent to the root $\mathfrak{r}$, and we have $e_{1,1} = \mathfrak{r}u_{1,1}, \dots, e_{1,d} = \mathfrak{r}u_{1,d}$.

First consider $n = 1$. Define for each $i = 1, \dots, d$, the length of $e_{1,i}$ by $\ell(e_{1,i}) := a_{1,i}$. Proceeding inductively, suppose that the length of all the edges in the tree T_n have been already assigned, and that the lengths of edges of T_n verify the following property $(\mathscr{L}_n)$

$$(\mathscr{L}_n) \qquad \forall\ 1 \leq j \leq n \text{ and } 1 \leq k \leq 2^j, \qquad \frac{a_{j,1}}{2} \leq \ell(e_{j,k}) \leq a_{j,k}.$$

We now explain how to define $\ell(e_{n+1,i})$ for all $1 \leq i \leq d^{n+1}$, ensuring the property $(\mathscr{L}_{n+1})$ as well.

Fix an $1 \leq i \leq d^{n+1}$, and let $P_i = v_0v_1v_2\dots v_nv_{n+1}$ be the unique path from the root $\mathfrak{r} = v_0$ to $v_{n+1} = u_{n+1,i}$. Note that $e_{n+1,i} = v_nv_{n+1}$. For each $1 \leq j \leq n$, let $f_{i,j}$ be the number of descendants of v_j among the vertices $u_{n+1,1}, \dots, u_{n+1,i-1}$. Obviously, we have $0 \leq f_{i,j} \leq d^{n+1-j}$ for any $1 \leq j \leq n$, and, we have $f_{1,1} = \dots, = f_{1,n} = 0$. Define $\ell(e_{n+1,i})$ by the recursive equation

$$\ell(e_{n+1,i}) + \sum_{j=1}^{n}\big(d^{n+1-j} + f_{i,j}\big)\ell(v_{j-1}v_j) = a_{n+1,i}.$$

By property $(\mathscr{L}_n)$, since S is d-sufficiently biased, we have

$$\sum_{j=1}^{n}\big(d^{n+1-j} + f_{i,j}\big)\ell(v_{j-1}v_j) \leq d^{n+1}\sum_{j=1}^{n} a_{j,d^j} < \frac{1}{2}a_{n+1,1} \leq a_{n+1,i},$$

which ensures that

$$\frac{a_{n+1,1}}{2} \leq \ell(e_{n+1,j}) \leq a_{n+1,i}.$$

To prove that ℓ and $\mathcal{O}$ are coherent, and that the factorial sequence associated to (T, ℓ) coincides with S one can proceed by induction. The details are straightforward and are left to the reader. □

3.2. **Module of definition of the length function.** Let S be an increasing sequence of positive numbers. Denote by $\mathbb{Z}\langle S\rangle$ the $\mathbb{Z}$-submodule of $\mathbb{R}$ generated by the elements of S. We have the following proposition.

Proposition 3.5. *Let S be an increasing sequence of positive numbers realizable by a pair (T, ℓ). For any edge $e \in E(T)$, we have $\ell(e) \in \mathbb{Z}\langle S\rangle$.*

Proof. Let $x_1, x_2, \dots$ be a factorial-determining sequence of vertices for (T, ℓ). By exhaustiveness of the weighting process, for each vertex v of the tree, there exists an integer n so that $x_n = v$, that we suppose in addition to be the smallest such n. Write the path $[\mathfrak{r}, v]$ in T as $v_0v_1\dots v_k$ with $v_0 = \mathfrak{r}$ and $v_k = v$. Since $\omega_n(v_{k-1}v_k) = 1$, we have $n!_{(T,\ell)} = \ell_{\omega_n}([\mathfrak{r}, v]) = \ell(v_{k-1}v_k) + \ell_{\omega_n}([\mathfrak{r}, v_{k-1}])$. Which gives

$$\ell(v_{k-1}v_k) = n!_{(T,\ell)} - \ell_{\omega_n}([\mathfrak{r}, v_{k-1}]).$$

Using this observation, a straightforward induction gives $\ell(e) \in \mathbb{Z}\langle S\rangle$ for any $e \in E(T)$. □

3.3. **Two non-isomorphic trees with the same factorial sequence.** The following direct corollary of Theorem 3.4 and Proposition 3.5 shows that the factorial sequence cannot determine the tree in general.

Proposition 3.6. *There are non-isomorphic trees T_1 and T_2 with the same factorial sequence.*

Proof. Consider a d-regular tree with $d \geq 2$. For any d-sufficiently biased sequence S of integers, and any collection of total orders on the n-th generation of the d-regular tree $\mathscr{T}_d$, for $n \in \mathbb{N}$, there is a length function ℓ such that factorial sequence of (T, ℓ) coincides with S. Changing the total orders $\leq_n$ associates another length function ℓ' with the same factorials. Note that by Proposition 3.5, the length function ℓ and ℓ' are integer valued.

However, one can easily construct a d-sufficiently biased sequence such that the two metric trees Γ and Γ' associated to (T, ℓ) and (T, ℓ'), respectively, become non-isomorphic. Each of Γ and Γ' has a model with a standard metric (i.e., with length function equal to one on edges). This results in two non-isomorphic trees T_1 and T_2 with the same factorials. $\square$

Remark 3.7. A variant of the construction of Theorem 3.4 leads to the following stronger statement. Let T_1, T_2 be any pair of infinite rooted locally finite trees, with roots $\mathfrak{r}_1$ and $\mathfrak{r}_2$, respectively, so that $\mathrm{br}(\mathfrak{r}_1) = \mathrm{br}(\mathfrak{r}_2)$. Suppose that all the vertices of T_1 and T_2 are branching. There exist length functions $\ell_1 : E(T_1) \to \mathbb{N}$ and $\ell_2 : E(T_2) \to \mathbb{N}$ so that the factorial sequences associated to (T_1, ℓ_1) and (T_2, ℓ_2) coincide. The proof goes as follows. One shows that for any infinite locally finite tree T in which every vertex is branching, and for any fixed total order on the vertices of generation n, and any appropriately biased sequence S of integers with respect to T (a modification of the definition for d-regular trees which takes into account the structure of T), there exists a length function $\ell : E(T) \to \mathbb{N}$ such that the factorial sequence associated to (T, ℓ) coincides with S. For any sequence which is biased for both T_1, T_2, this leads to the statement. Since we do not have any utility for this stronger version, we omit the detailed proof.

3.4. **The case of two trees one included in the other.** In this section, we prove that, not surprising, if one of two trees is included in the other one, and the two factorial sequences are the same, then the two trees are the same.

Proposition 3.8. *Let (T, ℓ, χ) and (T', ℓ', χ') be two triples consisting of rooted locally finite trees with length and capacity functions, so that $T \subseteq T'$, and the restriction of ℓ' (resp. χ') to T coincides with ℓ (resp. χ), and so that (T, ℓ, χ) and (T', ℓ', χ') have the same factorial sequence. Then the inclusion induces an isomorphism $T = T'$ (and so $\ell = \ell'$ and $\chi = \chi'$).*

Proof. For the sake of a contradiction, assume $T \subsetneq T'$ and $n!_{(T,\ell,\chi)} = n!_{(T',\ell',\chi')}$ for all $n \in \mathbb{N}$. There exists a vertex v of $T' \setminus T$ such that $\bar{v}$ belongs to T. Consider a sequence of weighting ω_n for T which provides the factorial sequence for (T, ℓ), as in Section 2. By the equality of the factorial sequences of (T, ℓ, χ) and (T', ℓ', χ'), and since $T \subset T'$ and the length and capacity functions coincide on T', the same weighting sequence provides the factorial sequence in T'. This is however impossible since by Proposition 2.9 any weighting sequence eventually gives a weight to any edge of the tree T', while the edge $\bar{v}v$ in T' clearly remains weightless in the sequence ω_n. $\square$

Let $d \geq 2$ be an integer. Consider the family $\mathcal{T}_d$ of all locally finite trees T containing only branching vertices with $2 \leq \mathrm{br}(v) \leq d$ for any vertex v.

Question 3.9. Prove or disprove: *for any pair of trees $T_1, T_2 \in \mathcal{T}_d$ with the same factorial sequence, the two trees T_1 and T_2 are isomorphic.*

4. Growth of the factorial sequence: transience and equidistribution

The examples given in the previous section of trees with any sufficiently biased sequence of reals as the factorial sequence show that the factorials might have any atypical behavior. In this section, we prove Theorems 1.6 and 1.7, which show however some asymptotic regularity behavior when n tends to infinity.

We first recall some basic definitions and results on random walks and flows on locally finite infinite trees.

Let (T, ℓ) be a pair consisting of a locally finite rooted tree and a length function $\ell : E \to \mathbb{R}_+$. We assume as before that the edges of T are oriented away from the root; $E(T)$ denotes the oriented edges of T with this orientation. A *flow* σ on T consists of an application $\sigma : E(T) \to \mathbb{R}_{\geq 0}$ such that for any vertex $v \neq \mathfrak{r}$ of T, we have

$$\sigma(\bar{v}v) = \sum_{vu \in E(T)} \sigma(vu).$$

The *total amount* of a flow σ is by definition the quantity $\sum_{\mathfrak{r}v \in E(T)} \sigma(\mathfrak{r}v)$, and if this sum is equal to one, then σ is called a *unit flow*. We denote by $\mathscr{F}_u(T)$ the set of all unit flows on T. Note that $\mathscr{F}_u(T)$ is non-empty if and only if T is infinite.

Denote, as before, by ∂T the boundary of T, which consists of infinite (oriented) paths s which start from the root $\mathfrak{r}$. The boundary comes with a natural topology induced by the tree structure. Recall that a basis of non-empty open sets in this topology are the sets B_v in bijection with the vertices of v, where for a vertex $v \in V(T)$, the set B_v contains all the elements $s \in \partial T$ which contain v, i.e.,

$$B_v := \Big\{ s \in \partial T \mid v \in s \Big\}.$$

Any unit flow $\theta \in \mathscr{F}_u(T)$ defines naturally a measure of total mass one on ∂T by

$$\forall v \in T, \qquad \theta(B_v) := \theta(\bar{v}\, v).$$

This association of measures to unit follows induces a bijection from $\mathscr{F}_u(T)$ to the set of Borel measures of total mass one on ∂T.

Consider the L^2-space of real-valued functions on the edges of the tree

$$L^2_\ell(E) := \Big\{ \theta : E \to \mathbb{R} \mid \sum_{e \in E} \ell(e)\theta(e)^2 < \infty \Big\},$$

with the scalar product $\langle \cdot\,, \cdot \rangle$ given by

$$\forall\ \theta_1, \theta_2 \in L^2(E) \qquad \langle \theta_1, \theta_2 \rangle := \sum_e \ell(e)\theta_1(e)\theta_2(e).$$

We may refer to the norm squared of an element $\theta \in L^2_\ell(E)$, defined by $||\theta||^2 := \langle \theta, \theta \rangle$, as the *energy* of θ. The space of unit flows of bounded energy on T is defined by

$$\mathscr{F}^b_u(T, \ell) := \mathscr{F}_u(T) \cap L^2_\ell(E).$$

The *conductance* c_{uv} of an edge $uv \in E(T)$ is defined as the inverse of the length $\ell(uv)$, i.e., $c_{uv} := \frac{1}{\ell(uv)}$. For $uv \in E(T)$, we define the conductance of the edge with reverse orientation vu by symmetry, $c_{vu} = c_{uv}$.

Consider the random walk $RW_{(T,\ell)}$ on the tree T which starts from the root, and which has, for any vertex $v \in V(T)$, a probability of transition p_{vu} from v to any of its neighbors $u \sim v$ in the tree given by

$$p_{vu} = \frac{1}{c_{vu}} \sum_{\substack{w \in V(T)\\ w \sim v}} c_{vw}.$$

In particular, for the standard length function $\ell \equiv 1$, there is an equal chance of moving from a vertex v to any of its neighbors, and $RW_{(T,1)} = RW(T)$ is the simple random walk on T.

Recall that a random walk on a tree is called *transient* if, almost surely, the walk returns to the root only a finite number of times. Otherwise, it is called *recurrent*. We call a pair (T, ℓ) *transient* (resp. *recurrent*) if the random walk $RW_{(T,\ell)}$ is transient (resp. recurrent). The following theorem gives a necessary and sufficient condition for the transience of the random walk $RW_{(T,\ell)}$, see e.g. [6, 11, 13, 14].

Theorem 4.1. *The random walk $RW_{(T,\ell)}$ on T is transient if and only if $\mathscr{F}^b_u(T,\ell) \neq \emptyset$.*

Suppose from now on that (T, ℓ) is a transient pair, so that we have $\mathscr{F}^b_u(T,\ell) \neq \emptyset$. This implies the existence and uniqueness of a flow of minimum energy $\eta \in \mathscr{F}^b_u(T,\ell)$, c.f. [10, 6]. The flow η is called the *unit current flow* on T. Although not necessary for what follows, we recall the following probabilistic interpretation of η: for any edge $uv \in E(T)$, $\eta(uv)$ is the expected *net* number of crossing of the edge uv for the random walk $RW_{(T,\ell)}$, where net means that a crossing of an edge $uv \in E(T)$ is counted with positive sign while the walk crosses the edge from u to v, and with negative sign if the edge is crossed from v to u.

The Borel measure associated to the unit current flow η is called the *harmonic measure* on (T, ℓ) and is denoted by μ_{har}. We recall the following useful property for the harmonic measure whose proof can be found e.g. in [10, 12]

Proposition 4.2. *Let (T, ℓ) be a pair consisting of a locally finite tree T with a length function ℓ. Suppose that the random walk $RW_{(T,\ell)}$ is transient, and let η and μ_{har} be the corresponding unit current flow and harmonic measure, respectively. Let θ be any flow of bounded energy in $\mathscr{F}^b_u(T,\ell)$, and let μ_θ be the Borel measure associated to θ. Then we have*

$$\lim_{\substack{v \in s\\ v \to \infty}} \ell_\eta([\mathfrak{r}, v]) = ||\eta||^2 \qquad \mu_\theta\text{-a.s. on } \partial T,$$

where $\ell_\eta([\mathfrak{r}, v]) = \sum_{e \in [\mathfrak{r},v]} \eta(e)\ell(e)$.

In other words, μ_θ-almost surely, the infinite rays in T have the same ℓ_η-length, equal to the energy of the unit current flow η.

We are now ready to state the main theorem of this section on the growth of the factorial sequence associated to the transient pairs (T, ℓ). Let Γ be the metric tree associated to (T, ℓ). We call Γ, and also (T, ℓ), *weakly complete* if it verifies the property that for any vertex v of T, any infinite strict path P of T_v has length infinite ℓ-length, i.e., $\ell(P) = \infty$. Note that if the values of the length function are ϵ-away from zero for some $\epsilon > 0$, e.g., for integer valued length functions such as the standard length function $\ell \equiv 1$, the pair (T, ℓ) is automatically weakly complete.

Let (T, ℓ) be a transient pair. Consider a weighting sequence ω_n for the edges as in Section 2, and denote by $T_n = T_{\omega_n}$ the corresponding sequence of factorial trees. For each integer n, denote by $\widetilde{\omega}_n$ the *normalized weight function* on T_n defined by

$$\forall e \in T_n \qquad \widetilde{\omega}_n(e) := \frac{1}{n}\omega(e),$$

that we extend by zero to all the edges in $E(T) \setminus E(T_n)$. Note that for all internal vertices u of the tree T_n, we have the flow property at u for $\widetilde{\omega}$

$$\sum_{uv \in E(T_n)} \widetilde{\omega}_n(uv) = \widetilde{\omega}(\breve{u}u).$$

In addition, at root $\mathfrak{r}$ we have

$$\sum_{\mathfrak{r}v \in E(T_n)} \widetilde{\omega}_n(\mathfrak{r}v) = 1.$$

We can rephrase this by saying that $\widetilde{\omega}_n$ is a *partial* unit follow on T.

Definition 4.3. A measure μ_n on the extended boundary $\widetilde{\partial}T$ is called consistent with $\widetilde{\omega}_n$ if for all internal vertex v of T_n we have $\mu_n(B_v) = \widetilde{w}_n(\breve{v}v)$.

In particular, for a choice of elements $\rho_0, \rho_1, \dots$ in the extended boundary $\widetilde{\partial}T$ in the definition of the factorial sequence in Section 1.1, the discrete averaging measures $\mu_n = \frac{1}{n}\sum_{j=0}^{n-1}\delta_{\rho_j}$ are consistent with $\widetilde{\omega}_n$.

Obviously, $\widetilde{\omega}_n$ depends on the choices we made at each step in constructing the factorial sequence. However, the following theorem shows, when the pair (T, ℓ) is weakly complete and transient, asymptotically, the behavior of $\widetilde{\omega}_n$ is independent of the choices. More precisely,

Theorem 4.4. *Let (T, ℓ) be a transient pair, and let Γ be the corresponding metric tree. Denote by η and μ_{har} the unit current flow on T and the harmonic measure on ∂T, respectively. Assume that Γ is weakly complete. Then*

(1) *the sequence $\widetilde{\omega}_n$ converges point-wise to η, i.e., for any edge e, we have*

$$\lim_{n\to\infty} \widetilde{\omega}_n(e) = \eta(e).$$

(2) *for any sequence of measures μ_n on ∂T with μ_n consistent with $\widetilde{\omega}_n$, the sequence μ_n converges weakly to the harmonic measure μ_{har}.*

(3) *the (logarithmic) factorials of (T, ℓ) satisfy the following asymptotic*

$$H(\Gamma) = \lim_{n\to\infty} \frac{1}{n}\, n!_{\Gamma} = ||\eta||^2.$$

Remark 4.5. The condition of being weakly complete is necessary as the following example shows. Consider a pair (T, ℓ) with T a rooted tree with $\mathfrak{r}$. Assume $\mathfrak{r}$ has two children u, v, T_u is an infinite path of finite ℓ-length, and (T_u, ℓ) is a recurrent pair. Then the unit current flow on (T, ℓ) is the unit flow on the strict path P_u which contains u. However, the limit of $\widetilde{\omega}_n(\mathfrak{r}u)$ is obviously zero, as $\omega_n(\mathfrak{r}u) = 1$ for all large n. This is a typical situation where, in absence of the weakly completeness assumption, the arguments of the next section fail.

The rest of this section is devoted to the proof of Theorem 4.4.

4.1. **Upper bound on the growth of factorials.** In this section, we assume T is a locally finite tree and ℓ is a length function on T so that the pair (T,ℓ) is transient, so $\mathscr{F}^b_u(T) \neq \emptyset$, and (T,ℓ) is weakly complete. Both the condition are necessary for what follows.

Let ω_n be a sequence of weightings resulting in the construction of the factorial sequence of (T,ℓ). Let $T_n = T_{\omega_n}$, and denote by U_n the set of all the unsaturated vertices of T_n, as before. Let $\widetilde{\omega}_n = \frac{1}{n}\omega_n$, that we extend by zero to all the edges $E(T) \setminus E(T_n)$.

Let $\theta \in \mathscr{F}^b_u(T,\ell)$ be a unite flow of bounded energy on T. We have the following proposition.

Proposition 4.6. *For each non-negative integer n, there exists a vertex $v \in U_n$ such that for all the edges e on the oriented path $[\mathfrak{r}, v]$ from $\mathfrak{r}$ to v, we have*

$$(4.1) \qquad 0 < \widetilde{\omega}_n(e) \leq \theta(e).$$

In particular, the path P is part of an infinite ray of T, and we have

$$(4.2) \qquad H(T,\ell) \leq \sum_{e\in P} \theta(e)\ell(e).$$

As an application of this proposition, we get the following interesting corollary.

Corollary 4.7. *For any transient and weakly complete pair (T,ℓ), we have*

$$\lim_{n\to\infty} \frac{1}{n}(n!_{(T,\ell)}) \leq \|\eta\|^2 < \infty,$$

where $\eta \in \mathscr{F}^b_u(T,\ell)$ is the unite current flow.

Proof. Take $\theta = \eta$ in Proposition 4.6, for the unite current flow η on (T,ℓ). By Proposition 4.2, μ_{har}-almost surely, all the infinite rays of T have the same length, equal to $\|\eta\|^2$, with respect to ℓ_η. For the path P in the proposition, since θ is positive on any edge of $[\mathfrak{r}, v]$, we get $\sum_{e\in P} \eta(e)\ell(e) \leq \|\eta\|^2$, and the corollary follows. □

Proof of Proposition 4.6. We construct the path P proceeding by induction and using a greedy procedure. We actually prove both the statements in the proposition simultaneously. Note that for all edges e in $E(T_n)$, we automatically have $\widetilde{\omega}_n(e) > 0$, by the definition of T_n.

Let $n \in \mathbb{N}$. Consider first the equation

$$(*) \qquad \sum_{\mathfrak{r}u\in E(T_n)} \omega_n(\mathfrak{r}v) = n = n \sum_{\mathfrak{r}u\in E(T)} \theta(\mathfrak{r}u).$$

On of the two following cases, (0,1) or (0.2), can happen.

(0.1) Either, there exists an edge $\mathfrak{r}u \in E(T)$ which does not belong to T_n and which satisfies $\theta(\mathfrak{r}u) > 0$. In this case, we have $\widetilde{\omega}_n(\mathfrak{r}u) = 0$. Let $v = \mathfrak{r}$. Then the path $[r, v]$ is reduced to a single vertex $\mathfrak{r}$, and Inequality (4.1) trivially holds. In addition, since $\theta(\mathfrak{r}u) > 0$, v must be part of an infinite path in T, and we have $n!_{(T,\ell)} = 0 \leq \sum_{e\in P} \theta(e)\ell(e)$, so that Inequality (4.2) holds as well.

(0.2) Otherwise, we have $\theta(\mathfrak{r}v) = 0$ for all $\mathfrak{r}v \in E(T) \setminus E(T_n)$, and Equation (*) gives

$$\sum_{\mathfrak{r}u\in E(T_n)} \omega_n(\mathfrak{r}v) = n = n \sum_{\mathfrak{r}u\in E(T_n)} \theta(\mathfrak{r}u).$$

Therefor there exists an edge $\mathfrak{r}z_1 \in E(T_n)$ with $0 < \widetilde{\omega}_n(\mathfrak{r}z_1) \leq \theta(\mathfrak{r}z_1)$. Let P_1 be a strict path in T which contains $\mathfrak{r}z_1$. Since $\theta(\mathfrak{r}z_1) > 0$, and the norm of θ is finite, the strict path P_1 has to be of finite ℓ-length, i.e., $\ell(P_1) < \infty$. By the assumption that (T,ℓ) is weakly complete, the path P_1 has to be a finite path in T. Denote by u_1 the other end-vertex of P_1. Note that u_1 is a vertex of T_n which is not a leaf of T, since, otherwise, we should have $\theta(e) = 0$ for all the edges in P_1.

Proceeding inductively on $k \in \mathbb{N}$, assume that we have a sequence of vertices $u_0 = \mathfrak{r}, u_1, u_2, \dots, u_k$ and $z_1, z_2, \dots, z_k$ such that there is a strict oriented path P_i in $T_{u_{i-1}}$ from u_{i-1} to u_i which contains the edge $u_{i-1}z_i \in E(T_n)$, for all $1 \le i \le k$, the vertex u_k is not a leaf of T, and $\theta(e) \ge \widetilde{\omega}_n(e) > 0$ for all edges in any path among the P_is. One of two following cases can happen

$(k.1)$ either there exists an edge $u_k u \in E(T)$ which does not belong to T_n so that $\theta(u_k u) > 0$. In this case, we let $v = u_k$ and let $P = [\mathfrak{r}, u_k]$ (the union of all the paths $P_1, \dots, P_k$). Since $\theta(u_1 u) > 0$, the path P is part of an infinite path which contains the edge $u_1 u$, and we have

$$\frac{1}{n} n!_T \le \sum_{e \in P} \widetilde{\omega}_n(e) \le \sum_{e \in P} \theta(e),$$

which proves the result.

$(k.2)$ Otherwise, for all the edges $u_k u \in E(T) \setminus E(T_n)$, we must have $\theta(u_k u) = 0$. Since $\theta(e) > 0$ for any $e \in [\mathfrak{r}, u_k]$, and θ is a flow, this implies that u_k is not a leaf of T_n. Therefore, we must have

$$\sum_{u_k u \in E(T_n)} \widetilde{\omega}_n(u_k u) = \widetilde{\omega}_n(\bar{u}_k u_k) = \widetilde{\omega}_n(u_{k-1} z_{k-1}) \le \theta(u_{k-1} z_{k-1}) = \theta(\bar{u}_k u_k) = \sum_{u_k u \in E(T_n)} \theta(u_k u).$$

In particular, there exists $u_k z_{k+1} \in E(T_n)$ with

$$0 < \widetilde{\omega}_n(u_k z_{k+1}) \le \theta(u_k z_{k+1}).$$

Let P_{k+1} the strict path in T_n starting from u_k which contains the edge $u_k z_{k+1}$. Since θ has bounded norm, and the value of θ on all edges of P_{k+1} are equal to $\theta(u_k z_{k+1}) > 0$, from the assumption that (T, ℓ) is weakly complete, we infer that the path P_{k+1} is finite in T. Let u_{k+1} be the other end-point of P_{k+1}, and note that u_{k+1} is not a leaf in T.

Since the tree T_n is finite, this process eventually stops, i.e., there is an m such that the case $(m.1)$ happens, and the proposition follows. □

4.2. **Point-wise convergence of $\widetilde{\omega}_n$ in the case $H(T, \ell)$ is finite.** In this section, we assume that the pair (T, ℓ) is so that the sequence $\frac{1}{n} n!_{(T,\ell)}$ converges to a finite number $H = H(T, \ell) < \infty$. In particular, by Corollary 4.7, what follows applies to transient weakly complete pairs (T, ℓ). Our main result is Theorem 4.9 which shows that for any edge e, the sequence $\widetilde{\omega}_n(e)$ converges to a number $[0, 1]$.

Without loss of generality, using Claim 2.6, we can assume that the root has branching $\mathrm{br}(\mathfrak{r}) = d \ge 2$.

Denote by $u_1, \dots, u_d$ all the children of $\mathfrak{r}$, and set $N_j = N_{T_{u_j}}$. Let $S_j = \left\{a_i^j\right\}_{0 \le i < N_j}$ be the factorial sequence of the subtree T_{u_j} (which might be finite), and define the sets $A, \dots, A_d$ as in the previous section $A_j = \left\{a_i^j + i\ell_{\mathfrak{r}u_j}\right\}_{0 \le i < N_j}$.

Enumerate the terms in the multiset union A of the sets A_j in a fixed increasing order induced by the weighting sequence ω_n, depending on to which subtree T_{u_j} the (factorial-determining) vertex x_n in the definition of the factorials belongs. So each element of A is labeled with an index among $1, \dots, d$. For each j, denote by $k_j(n) + 1$ the number of elements among the first $n+1$ terms in A labeled by j, i.e., the number of indices $0 \leq i < N_j$ with $a^j_i + i\ell_{\mathfrak{r}u_j}$ among the first $n+1$ terms of the sequence A. We have the following straightforward, but useful, inequalities

$$0 \leq k_j(n) - k_j(n-1) \leq 1,$$

$$a^j_{k_j(n)} + k_j(n)\ell_{\mathfrak{r}u_j} \leq n!_{(T,\ell)} \leq a^j_{k_j(n)+1} + \big(k_j(n)+1\big)\ell_{\mathfrak{r}u_j}, \tag{4.3}$$

for all $j = 1, \dots, n$, and all $0 \leq n < N_j$. Note that in addition, by exhaustiveness of the weighting process proved in Proposition 2.9, we have

the tree T_{u_j} is infinite if and only if $k_j(n) \to \infty$.

For an integer $1 \leq j \leq d$ with $N_j = \infty$, denote by H_{u_j} the limit

$$H_{u_j} := \lim_{k\to\infty} \frac{1}{k} a^j_k = H(T_{u_j}, \ell_{|T_{u_j}}),$$

which exists by Corollary 2.12, and which belongs to the interval $(0, \infty]$.

For all $1 \leq j \leq d$, and all non-negative integers $n < N_j$, we get from Equation (4.3)

$$\frac{k_j(n)}{n}\Big(\frac{a^j_{k_j(n)}}{k_j(n)} + \ell_{\mathfrak{r}u_j}\Big) \;\leq\; \frac{1}{n} n!_{(T,\ell)} \leq \frac{k_j(n)+1}{n}\Big(\frac{a^j_{k_j(n)+1}}{k_j(n)+1} + \ell_{\mathfrak{r}u_j}\Big) \tag{4.4}$$

Note that we have $\frac{k_j(n)}{n} = \widetilde{\omega}_n(\mathfrak{r}u_j)$. We distinguish the following three different cases:

1. We have $N_j < \infty$. In this case, since $k_j(n) < N_j$, we get $\widetilde{\omega}_n(ru_j) \to 0$ as n tends to infinity.
2. We have $N_j = \infty$ and $H_{u_j} < \infty$. In this case, making n tend to infinity, we get from (4.4) that the limit of $\widetilde{\omega}_n(e_j)$ exists and is equal to
$$\lim_{n\to\infty} \widetilde{\omega}_n(e_j) = \frac{H}{H_{u_j} + \ell_{\mathfrak{r}u_j}} > 0.$$
3. We have $N_j = \infty$ and $H_{u_j} = \infty$. In this case, since $\frac{1}{n} n!_{(T,\ell)}$ converges to a finite H, and the term $a^j_{k_j(n)}/k_j(n)$ in (4.4) converges to infinity, we must have
$$\lim_{n\to\infty} \widetilde{\omega}_n(e_j) = \frac{k_j(n)}{n} = 0.$$

We have thus proved the point-wise convergence of $\widetilde{\omega}_n$ at all the pending edges at the root of T. Since $\widetilde{\omega}_n$ is a partial flow, and the weighting is exhaustive, it follows that in the first and third cases, we actually have $\lim_{n\to\infty} \widetilde{\omega}_n(e) = 0$ for all the edges e in the subtree T_{u_j}.

Note that, again by the exhaustiveness of the weighting, and since $\widetilde{\omega}_n$ is a partial flow, we get the equation

$$H \sum_{j=1}^{d} \frac{1}{H_{u_j} + \ell_{\mathfrak{r}u_j}} = 1. \tag{4.5}$$

Definition 4.8. For any vertex $v \in V(T)$, define $H_v \in [0, \infty]$ as follows:

$$H_v := \begin{cases} 0 & \text{if the tree } T_v \text{ is finite,} \\ H(T_v, \ell_{|T_v}) = \lim_{n\to\infty} \frac{1}{n}(n!)_{(T_v, \ell_{|T_v})} & \text{otherwise.} \end{cases}.$$

Proceeding now by induction on the generation $|v|$ of vertices $v \in T$, we prove the following theorem.

Theorem 4.9 (Point-wise convergence of $\widetilde{\omega}_n$)**.** *For any edge $uv \in E(T)$, the limit, when n tends to infinity, of $\widetilde{w}_n(uv)$ exists. It is non-zero precisely when H_v lies in the interval $(0,\infty)$, in which case, the limit is given by*

$$\lim_{n\to\infty} \widetilde{\omega}_n(uv) = \prod_{\substack{w\in[\mathfrak{r},v]\\ w\neq \mathfrak{r}}} \frac{H_{\overleftarrow{w}}}{H_w + \ell_{\overleftarrow{w}w}}.$$

The rest of this section is devoted to the proof of this theorem. We proceed by induction on the generation $|v|$ of v. By what proceeded the statement of the theorem, we already proved the theorem for the pending edges at $\mathfrak{r}$, i.e., in the case $u = \mathfrak{r}$ and $|v| = 1$.

Let $m \in \mathbb{N}$, and assume that the statement holds for all vertices v with $|v| = m$. We prove the theorem for all vertices of generation $m+1$. So let $uv \in E(T)$ with $|v| = m+1$. Since $|u| = m$, the statement already holds for the edge $\overleftarrow{u}u \in E(T)$. Two cases can happen:

- Either, $\lim_{n\to\infty} \widetilde{\omega}_n(\overleftarrow{u}u) = 0$. In this case, for all edges in the subtree $T_{\overleftarrow{u}}$ we have $\widetilde{\omega}_n(e) \leq \widetilde{\omega}_n(\overleftarrow{u}u)$, and so $\lim_{n\to\infty} \widetilde{\omega}_n(e) = 0$. In particular, the limit when n tends to infinity of $\widetilde{\omega}_n(uv)$ exists and is equal to zero.
- Or, $\lim_{n\to\infty} \widetilde{\omega}_n(\overleftarrow{u}u) > 0$.

In this case, we have $\lim_{n\to\infty} \omega_n(\overleftarrow{u}u) = \infty$, and by the hypothesis of the induction, we have $H_{\overleftarrow{u}} \in (0,\infty)$ and the following equation holds:

$$\lim_{n\to\infty} \widetilde{\omega}_n(\overleftarrow{u}u) = \prod_{\substack{w\in[\mathfrak{r},u]\\ w\neq \mathfrak{r}}} \frac{H_{\overleftarrow{w}}}{H_w + \ell_{\overleftarrow{w}w}}.$$

In particular, the tree $T_{\overleftarrow{u}}$ is infinite.

Denote by $I_u = \Big\{p_0, p_1, p_2, \dots\Big\} \subset \mathbb{N}$ the set of all the non-negative integers n where an unweighted edge incident to a vertex of the subtree T_u is weighted in the description of the weighting process, enumerated in an increasing order, so $p_0 < p_1 < \dots$. The weighting sequence ω_{p_j} restricted to the subtree T_u produces a weighting sequence ω^u_j for $(T_u, \ell_{|T_u})$. Since $0 < H_u < \infty$, by what proceeded before the statement of the theorem applied to T_u, we get that

- for all edges $uv \in E(T_u)$, the limit when j tends to infinity of $\widetilde{\omega}^u_j = \omega^u_j/j$ exists; and
- this limit is non-zero precisely when $H_v \in (0,\infty)$, in which case, the limit is given by

$$\lim_{j\to\infty} \widetilde{\omega}^u_j(uv) = \frac{H_u}{H_v + \ell_{uv}}.$$

We now observe that for all edges $uv \in E(T)$,

$$\begin{aligned}\lim_{n\to\infty} \widetilde{\omega}_n(uv) &= \lim_{n\to\infty} \frac{1}{n}\omega_n(uv) = \lim_{n\to\infty} \Big(\frac{\omega_n(\overleftarrow{u}u)}{n} \cdot \frac{\omega_n(uv)}{\omega_n(\overleftarrow{u}u)}\Big)\\ &= \lim_{n\to\infty} \widetilde{\omega}_n(\overleftarrow{u}u)\,.\lim_{n\to\infty} \widetilde{\omega}^u_j(uv).\end{aligned}$$

Combing all these together, we finally get that for $uv \in E(T)$,

- the limit when n tends to infinity of $\widetilde{\omega}_n(uv)$ exists; and

- it is non-zero precisely when $\widetilde{\omega}_j^u(uv)$ has a non-zero limit when n tends to infinity, i.e., when $H_v \in (0,\infty)$, in which case we have

$$\lim_{n\to\infty} \widetilde{\omega}_n(uv) = \Big(\prod_{\substack{w\in[\mathfrak{r},u]\\ w\neq \mathfrak{r}}} \frac{H_{\bar{w}}}{H_w+\ell_{\bar{w}w}}\Big)\cdot \frac{H_u}{H_v+\ell_{uv}} = \prod_{\substack{w\in[\mathfrak{r},v]\\ w\neq \mathfrak{r}}} \frac{H_{\bar{w}}}{H_w+\ell_{\bar{w}w}}\ .$$

This finishes the proof of our theorem. □

4.3. **Equivalence of the transience of (T,ℓ) with finiteness of $H(T,\ell)$.** In this section, we prove Theorem 1.6. So let (T,ℓ) be a pair consisting of an infinite locally finite tree and a length function ℓ on T. Assume that (T,ℓ) is weakly complete.

Theorem 4.10. *The following two statements are equivalent.*

(*i*) *The pair (T,ℓ) is transient.*
(*ii*) *We have $H(T,\ell)<\infty$.*

The implication $(i)\Rightarrow(ii)$ is already proved in Corollary 4.7. We prove (ii) implies (i).

Let ω_n be a sequence of weighting for the pair (T,ℓ). Assume that $H(T,\ell)<\infty$. For each vertex $v\in T$, define H_v by Definition 4.8. By the results of the previous section, we have the point-wise convergence of the sequence $\widetilde{\omega}_n$ to some $\phi: E(T)\to\mathbb{R}_{\geq 0}$. Obviously, we have $\phi\in\mathscr{F}_u(T)$, and by what we proved in the previous section, the non-zero values of ϕ on edges are given by

$$\forall\ uv\in E(T) \text{ with } \phi(uv)\neq 0, \qquad \phi(uv) = \prod_{\substack{w\in[\mathfrak{r},v]\\ w\neq \mathfrak{r}}} \frac{H_{\bar{w}}}{H_w+\ell_{\bar{w}w}}.$$

The following claim finishes the proof of our theorem.

Claim 4.11. *The unit flow ϕ has bounded energy, and thus belongs to $\mathscr{F}_u^b(T,\ell)$.*

In order to prove this statement, we need some preliminaries. Define the function $F: V(T)\to\mathbb{R}$ as follows. Let $F(0)=0$, and for all vertices $v\in V(T)\setminus\{\mathfrak{r}\}$, define

$$F(v) := \sum_{\substack{w\in[\mathfrak{r},v]\\ w\neq r}} \phi(\bar{w}w)\ell_{\bar{w}w}.$$

Claim 4.12. *We have for all $v\in V(T)$, $F(v)\leq H(T,\ell)$.*

Proof. It will be enough to prove the result for any vertex $v\in V(T)$ with $0<H_v<\infty$. Let v be such a vertex and denote by $v_0=\mathfrak{r},v_1,\dots,v_k=v$ all the vertices on the path $[\mathfrak{r},v]$ from

$\mathfrak{r}$ to v, with $e_i := v_{i-1}v_i \in E(T)$ for $i = 1, \dots, k$. We have

$$\begin{aligned}
F(v) &= \sum_{\substack{u\in[\mathfrak{r},v]\\ u\neq\mathfrak{r}}} \phi(\overleftarrow{u}u)\ell(\overleftarrow{u}u) = \sum_{j=1}^{k} \ell(e_j) \prod_{i=1}^{j} \frac{H_{v_{i-1}}}{H_{v_i}+\ell(e_i)} \\
&= \sum_{j=1}^{k} \Big(H_{v_j} + \ell(e_j) - H_{v_j} \Big) \prod_{i=1}^{j} \frac{H_{v_{i-1}}}{H_{v_i}+\ell(e_i)} \\
&= \sum_{j=1}^{k} \Big(\big(H_{v_j} + \ell(e_j)\big) \prod_{i=1}^{j} \frac{H_{v_{i-1}}}{H_{v_i}+\ell(e_i)} - H_{v_j} \prod_{i=1}^{j} \frac{H_{v_{i-1}}}{H_{v_i}+\ell(e_i)} \Big) \\
&= H_{\mathfrak{r}} - H_{\mathfrak{r}} \cdot \frac{\prod_{j=1}^{k} H_{v_j}}{\prod_{j=1}^{k} (H_{v_j}+\ell(e_j))} \le H_{\mathfrak{r}} = H(T,\ell).
\end{aligned}$$

□

By the previous claim we can extend F to a function on the boundary ∂T. We have

Claim 4.13. *Denote by μ_ϕ the measure of mass one on ∂T associated to ϕ. We have $\|\phi\|^2 = \int_{\partial T} F\, d\mu_\phi$.*

Proof. The idea is that one can write

$$\sum_{uv\in E(T)} \phi(uv)^2 \ell(uv) = \sum_{uv\in E(T)} \phi(uv)(F(v) - F(u)).$$

Consider a finite cut set C of T with vertex set U and with complementary vertex set $W = V(T) \setminus U$. Using that $\phi \in \mathscr{F}_u(T)$, we have for the partial sum

$$\sum_{\substack{u\in U\\ uv\in E(T)}} \phi(uv)(F(v) - F(u)) = F(\mathfrak{r}) + \sum_{v\in\partial W} F(v)\phi(\overleftarrow{v}v) = \sum_{v\in\partial W} F(v)\mu_\phi(B_v),$$

where ∂W is the set of all vertices v with $\overleftarrow{v} \in U$, and B_v is the open subset of ∂T defined previously. The result now follows by tending the cut set C to infinity. □

Combining the two previous claims gives

$$\|\phi\|^2 = \int_{\partial T} F d\mu_\phi \le H(T,\ell) < \infty, \tag{4.6}$$

and finishes the proof of Claim 4.11. The proof of Theorem 4.10 is now complete.

4.4. **Proof of Theorem 4.4.** With what we proved in the previous sections, we can now complete the proof of Theorem 4.4.

Let (T,ℓ) be a transient pair, and denote by $\eta \in \mathscr{F}_u^b(T,\ell)$ the corresponding unit current flow on T. Denote by ϕ the point-wise limit of $\widetilde{\omega}_n$ for a weighting sequence ω_n.

By Corollary 4.7, we have $H(T,\ell) \le \|\eta\|^2$. On the other hand, by Inequality 4.6, we have $\|\phi\|^2 \le H(T,\ell)$. Since $\eta \in \mathscr{F}_u^b(T,\ell)$ is the flow of minimum energy, it follows that $\phi = \eta$, which is part (1) of Theorem 4.4. Part (2) is a direct consequence of part (1). Part (3) follows from the equality $\phi = \eta$ combined with the inequalities of Corollary 4.7 and Equation 4.6.

5. Concluding remarks

We include here a brief discussion of some results and questions complementary to what we presented in the previous sections.

5.1. **Removed version.** Let $t \in \mathbb{N}$. Let T be a locally finite tree, ℓ and χ a length and capacity function on T, respecitvly. One can define a t-removed version of the factorials associated to (T, ℓ, χ). Choose $\rho_0, \dots, \rho_{t-1} \in \widetilde{T}$ arbitrarily, in such a way that the capacity condition is verified. Assuming that $\rho_0, \dots, \rho_{n-1}$ are chosen, one chose $\rho \in \widetilde{\partial}T$ among those unsaturated elements $\rho \in \widetilde{\partial}T$ which minimizes the quantity

$$a_n(\rho) = \min_{\substack{A \subset \{0,\dots,n-1\} \\ |A|=n-t}} \sum_{j \in A} \langle \rho, \rho_j \rangle,$$

and define $a_n = a_n(\rho_n)$.

Theorem 5.1. *The sequence $\{a_n\}$ only depends on (Γ, χ) and t.*

The proof is similar to the proof of Theorem 2.1, and leads to a combinatorial proof of a generalization of [5]. Define $n!^{\{t\}}_{(\Gamma,\chi)} := a_n$. We have the following theorems.

Theorem 5.2. *Let (T, ℓ) be a pair of a locally finite tree T rooted at r, and a length function ℓ on T. Let $d = \mathrm{br}(r)$ and denote by $u_1, \dots, u_d$ all the children of r. Let Γ_j be the metric tree associated to the pair $(T_{u_j}, \ell_{|T_{u_j}})$, and let χ_j be the restriction of χ to T_{u_j}. We have for all $0 \le n < N_{T,\chi}$,*

$$n!^{\{t\}}_{(\Gamma,\chi)} = \min_{\substack{(t_1,\dots,t_d) \in \mathbb{N}^d_* \\ t_1+\dots+t_d=t}} \min_{\substack{(n_1,\dots,n_d) \in \mathbb{N}^d_* \\ n_1+\dots+n_d=n+1}} \max \Big\{ (n_j - 1)!^{\{t_j\}}_{(\Gamma_j,\chi_j)} + (n_j - 1)\ell(\mathfrak{r}u_j) \Big\}_{j=1}^{d}.$$

Theorem 5.3. *For any pair (T, ℓ, χ) with $N_{T,\chi} = \infty$, and any $t \in \mathbb{N}$, we have*

$$\lim_{n\to\infty} \frac{1}{n} n!^{\{t\}}_{(T,\ell,\chi)} = \lim_{n\to\infty} \frac{1}{n} n!_{(T,\ell,\chi)}.$$

5.2. **Factorial-based characterization of the branching number.** Let T be a countable infinite locally finite tree. A cut-set in T is a subset C of vertices such that any infinite (oriented) part from root meet a vertex of C. Recall that the *branching number* of T is defined as

$$\mathrm{br}(T) := \sup\Big\{\lambda : \inf_{C \text{ cut-set}} \sum_{v \in C} \lambda^{-|v|} > 0 \Big\},$$

where for a vertex v, the distance of v to $\mathfrak{r}$ in the tree T is denoted by $|v|$.

For any $\lambda > 0$, denote by ℓ_λ the length function on T which associates the length $\lambda^{|u|}$ to any oriented edge $uv \in E(T)$. Combining a theorem of Lyons [10] with our Theorem 1.6, we get the following characterization of the branching number in terms of tree factorials.

Theorem 5.4. *The branching number of T is the supremum of λ so that the normalized factorials of the pair (T, ℓ_λ) have a finite limit.*

5.3. **Subsets of $\mathbb{Z}$ versus p-trees.** Let $\mathcal{P}$ be the set of prime numbers in $\mathbb{Z}$. For any subset $X \subset \mathbb{Z}$ of integers, and any prime p, denote by $T_{X,p}$ the tree associated to $X \subset \mathbb{Z} \subset \mathbb{Z}_p$. By the discussion in the introduction, we have

$$n!_X = \prod p^{n!_{T_{X,p}}},$$

where $n!_X$ denotes the Bhargava's factorial of n for the set X. Note in particular that for two subsets $X, Y \subset \mathbb{Z}$, we have $n!_X = n!_Y$ if and only if $n!_{T_{X,p}} = n!_{T_{Y,p}}$ for all $p \in \mathcal{P}$, and any $n \in \mathbb{N}$. In particular, two subsets of integers with the same p-trees, have the same factorials.

In this regard, it seems natural to wonder (1) how can two subsets of the integers have the same collection of p-trees ? (2) what can be said about the p-trees of two subsets X and Y if they have same factorials? and (3) which collections of p-trees, one for each $p \in \mathcal{P}$, come from a subset X of $\mathbb{Z}$? The following proposition shows that in general two sets X and Y with the same factorial sequence can be very different.

Proposition 5.5. *Let $\epsilon \in (0,1)$ be a fixed positive real number. Let X be a random subset of $\mathbb{Z}$ obtained by choosing any integer $k \in \mathbb{Z}$ with probability ϵ independently at random. For all $n \in \mathbb{N}$, we have $n!_X = n!$.*

Proof. For any prime p, the tree $T_{X,p}$ is the regular p-tree $\mathscr{T}_p$. It follows that $n!_X = n!$. □

In particular, it seems very unlikely to have an answer to (1) without any further assumption on X and Y.

Regarding (3), by applying Chinese reminder lemma, we have the following proposition.

Proposition 5.6. *Let $\{T_p\}$ be a sequence of trees one for each prime p, with T_p a subtree of the p-regular tree $\mathscr{T}_p$ without any leaf. Assume there exists an integer n such that for all primes $p > n$, the tree T_p is the complete p tree. There exists a subset X of $\mathbb{Z}$ whose associated p-tree is equal to T_p for all prime p.*

Question (3) for more general collection of trees seems to be quite interesting on its own.

Regarding question (2) above, applying the above proposition, we infer from the existence of non-isomorphic trees with the same factorial sequence for a prime p, the existence of two subsets X and Y with the same factorial sequence and without necessary the same adelic trees. So again the answer to (2) seems to be rather delicate.

5.4. **Factorials of definable sets.** A natural classes of trees arising from an arithmetic situation are trees associated to definable sets over p-adic numbers. A structure theorem for such trees is proved by Halupczok in [8, 9]. It seems natural to expect that factorials of such trees should reflect their arithmetical properties, and must be thus related to other types of arithmetic functions associated to those sets. It thus appears to be an interesting problem to study the factorials of trees of definable sets.

5.5. **Relation to other adelic equidistribution theorems.** Is there any connection between the equidistribution theorem proved in this paper and other known adelic equidistribution theorems, e.g. for small points [3, 2, 1, 7] ?

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