EXPLOSION AND LINEAR TRANSIT TIMES IN INFINITE TREES

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Abstract. Let T be an infinite rooted tree with weights w_e assigned to its edges. Denote by $m_n(T)$ the minimum weight of a path from the root to a node of the nth generation. We consider the possible behaviour of $m_n(T)$ with focus on the two following cases: we say T is explosive if

$$\lim_{n\to\infty} m_n(T) < \infty\,,$$

and say that T exhibits linear growth if

$$\liminf_{n\to\infty} \frac{m_n(T)}{n} > 0\,.$$

We consider a class of infinite randomly weighted trees related to the Poisson-weighted infinite tree, and determine precisely which trees in this class have linear growth almost surely. We then apply this characterization to obtain new results concerning the event of explosion in infinite randomly weighted spherically-symmetric trees, answering a question of Pemantle and Peres [23]. As a further application, we consider the random real tree generated by attaching sticks of deterministic decreasing lengths, and determine for which sequences of lengths the tree has finite height almost surely.

1. Introduction

Let i.i.d. random weights w_e be assigned to the edges of an infinite rooted tree T, and let $m_n(T)$ denote the minimum weight of a path from the root to a node of the nth generation. In the context of first passage percolation, looking at the weight of an edge as the transition time between the two corresponding nodes, $m_n(T)$ is the first passage time to the nth generation. We consider the possible behaviour of $m_n(T)$ with particular focus on the following cases: we say T is *explosive* if

$$\lim_{n\to\infty} m_n(T) < \infty\,,$$

and say that T exhibits *linear growth* if

$$\liminf_{n\to\infty} \frac{m_n(T)}{n} > 0\,.$$

In the case where the tree T is itself a random Galton-Watson tree conditioned on the survival, the quantity m_n also occurs as the minimal nth generation position in a

branching random walk in $\mathbb{R}$. The linear growth property goes at least back to the work of Hammersley [19], Kingman [21], and Biggins [11]. Many other, including very recent, results [2, 3, 4, 10, 14, 17, 20, 22, 24] on the behavior of m_n in the context of branching random walks are known, we refer to [10] and the discussion in the introduction there for a short survey. The literature on explosion is partially surveyed by Vatutin and Zubkov [25].

We assume now that the tree T is deterministic. Pemantle and Peres introduced the concept of stochastic dominance between trees and proved in [23] that amongst trees with a given sequence of generation sizes, explosion is most likely in the case that the tree is spherically symmetric. Recall that a tree T is called spherically symmetric if all the vertices at generation n have the same number $f(n)$ of children, for some function $f : \mathbb{N} \cup \{0\} \to \mathbb{N}$. Pemantle and Peres proved that for a spherically symmetric tree T with a non-decreasing branching function f, and with weights w_e independent exponential random variables of mean one, the probability of the event of explosion is 0 or 1 according to whether the sum $\sum_{n=0}^{\infty} f(n)^{-1}$ is infinite or finite. They also showed that the same statement holds for weight random variables with distribution function G satisfying $\lim_{t\to 0} G(t)t^{-\alpha} = c > 0$ for some $\alpha > 0$. Furthermore, they asked if the same simple explosion criterion holds for general edge weight distributions, under reasonable assumptions.

One of our aims in this paper is to answer, essentially completely, this question of Pemantle and Peres. In order to do so, we consider a class of infinite weighted trees related to the Poisson-weighted infinite tree, the PWIT, introduced by Aldous [6, 9]. Since its introduction, the PWIT has been identified as the limit object of the solutions of various combinatorial optimization problems. The survey by Aldous and Steele [9] provides a general overview with several examples of applications; see also [1, 13] for some more recent applications.

As a cornerstone of all our results, we determine precisely which trees in this class of generalized PWITs have linear growth almost surely. We then present two applications of this result:

− First, we provide in Sections 3 and 4 our results concerning the event of explosion in spherically-symmetric trees, generalizing the results of [23], and

− Second, we consider general classes of random real trees constructed via a stick breaking process on $\mathbb{R}^+$, in a similar way that Aldous' CRT [5] is constructed, and give in Section 5 a criterion for these random real trees to have finite diameter almost surely.

In the remainder of this introduction, after summarizing our notation, we state our main results.

Basic definitions and notation. Given edge weights w_e for each $e \in E(T)$, we write $w(\gamma)$ for the sum of the weights in a path γ. We write T_n for the nodes of the nth generation (i.e., at distance n from the root), and $\Gamma_n(T)$ for the family of paths from the root to T_n. In this notation, $m_n(T) := \min_{\gamma \in \Gamma_n(T)} w(\gamma)$.

Recall that a spherically symmetric tree with *branching function* $f : \mathbb{N}_0 \to \mathbb{N}$ is the rooted tree T_f in which the root has $f(0)$ children and each node of $(T_f)_n$ has $f(n)$ children (here $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$). We write $F(n)$ for $\prod_{i=0}^{n-1} f(i)$, and note that $F(n) = |(T_f)_n|$. We shall tend to focus on the case that f is non-decreasing.

A Poisson point process of intensity $\lambda \in \mathbb{R}^+$ is a point process P on the positive real line such that for each pair of disjoint intervals $[a, b]$, $[c, d]$ we have

(i) $|P \cap [a, b]|$ is distributed as $\mathrm{Po}(\lambda(b - a))$, and,
(ii) $|P \cap [a, b]|$ and $|P \cap [c, d]|$ are independent.

More generally, given a (measurable) function $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$, which is locally integrable and satisfies $\int_0^\infty \lambda = \infty$, the inhomogeneous Poisson point process P^λ is a point-process on the positive real line such that for each pair of disjoint intervals $[a, b]$, $[c, d]$ we have

(i) $|P^\lambda \cap [a, b]|$ is distributed as $\mathrm{Po}(\int_a^b \lambda)$, and,
(ii) $|P^\lambda \cap [a, b]|$ and $|P^\lambda \cap [c, d]|$ are independent.

We denote by $P(j)$ the position of the jth smallest particle of the point process P.

We now define a class of infinite trees that generalize the Poisson-weighted infinite tree (which corresponds to the case λ is the constant function with value 1). We shall use $\mathbb{N}^{<\omega}$ to denote the set of finite (ordered) sequences of natural numbers. A typical element of $\mathbb{N}^{<\omega}$ is denoted by $\mathbf{i}$, and for an integer $j \in \mathbb{N}$, the sequence $\mathbf{i}j$ is obtained from $\mathbf{i}$ by inserting j to the very right end of the sequence.

Definition 1.1. The P$^\lambda$WIT, which we denote by T^λ, has vertices labelled by $\mathbb{N}^{<\omega}$, with $\emptyset$ labelling the root, and edge set

$$\{\ \{\mathbf{i}, \mathbf{i}j\}\ : \mathbf{i} \in \mathbb{N}^{<\omega}, j \geq 1\}.$$

Associate to each vertex $\mathbf{i}$ an independent point process $P^\lambda_{\mathbf{i}}$ (distributed as P^λ), and give edge $\{\mathbf{i}, \mathbf{i}j\}$ of T^λ the weight $P^\lambda_{\mathbf{i}}(j)$.

1.1. **Linear growth in generalizations of the PWIT.** The Poisson-weighted infinite tree exhibits linear growth almost surely. It is therefore natural to ask how general this property is in the generalizations of the PWIT defined above. We answer this question completely. Furthermore, we provide exponential probability bounds for the event that $m_n(T)$ grows more slowly.

Theorem 1.2. *Let $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ be any locally integrable function with $\int_0^\infty \lambda = \infty$. Then we have the following dichotomy.*

(i) If either there exists some $t > 0$ such that $\int_0^t \lambda = 0$, or there exists some $C > 0$ so that $\int_0^x \lambda \leq C^x$ for all $x \in \mathbb{R}^+$, then there exists $\alpha > 0$ such that

$$\lim_{n\to\infty} \frac{m_n(T^\lambda)}{n} = \alpha$$

almost surely. In particular, T^λ has linear growth, almost surely.

Furthermore, for each $K > 0$, there exists $\delta > 0$ such that

$$\mathbb{P}\left(m_n(T^\lambda) < \delta n\right) \leq e^{-Kn}.$$

(ii) Otherwise, T^λ does not have linear growth, almost surely.

Note that if $\int_0^t \lambda = 0$ for some $t > 0$, then obviously T^λ has linear growth. The main part of the theorem thus concerns the existence or the non-existence of a constant $C > 0$ so that $\int_0^x \lambda \leq C^x$ for all $x \in \mathbb{R}^+$, which provides a dichotomy for linear growth in the trees T^λ. The proof appears in Section 2.

1.2. **Explosion in infinite trees.** Recall that we call a rooted weighted tree *explosive* if

$$\lim_{n\to\infty} m_n(T) < \infty.$$

Pemantle and Peres proved that amongst trees with a given sequence of generation sizes, explosion is most likely in the case that the tree is spherically symmetric.

Let $f : \mathbb{N}_0 \to \mathbb{N}$, and let T_f denote the spherically-symmetric tree in which each node v of generation n has $f(n)$ children. Given a distribution function $G : [0, \infty) \to [0, 1]$, let T_f^G denote the randomly weighted tree obtained by giving each edge of T_f an i.i.d. weight distributed according to G.

Definition 1.3. Given the distribution G and non-decreasing function $f : \mathbb{N}_0 \to \mathbb{N}$, we say that f is *G-explosive* if T_f^G is explosive almost surely.

It is easy to see that a sufficient condition for f being G-explosive is (a slightly stronger version of) the "local min-summability" condition (compare to the "global min-summability" condition considered in [10]): apply a greedy algorithm to construct an infinite path in the tree T_f^G by starting from the root, and by choosing recursively for the end vertex v_n of the already constructed path up to level n, the minimum weight edge $v_n v_{n+1}$ among the $f(n)$ adjacent edges to level $n+1$, see Proposition 3.1. This motivates the following definition.

Definition 1.4. Given the distribution G and a non-decreasing function $f : \mathbb{N} \to \mathbb{N}$, we say that f is *G-small* if

$$\sum_{n \geq 0} G^{-1}(f(n)^{-1}) \, < \, \infty \, . \tag{1}$$

One may now interpret Pemantle and Peres [23, Page 193] as asking the following.

Question 1.5 (Pemantle-Peres [23])**.** For which G does the equivalence

$$f \text{ is } G\text{-small} \Leftrightarrow f \text{ is } G\text{-explosive}$$

hold in the class of non-decreasing functions $f : \mathbb{N}_0 \to \mathbb{N}$?

Pemantle and Peres [23] showed that if G has a limit law at 0 in the sense that $\lim_{x \to 0} G(x) x^{-\alpha}$ exists and is positive, then this equivalence holds. They speculated that perhaps the equivalence holds provided G is continuous and strictly increasing. The following definition encodes robust versions of the properties of being continuous and strictly increasing.

Definition 1.6. The distribution G is *controlled near* 0 if

$$1 \, < \, \liminf_{x \to 0} \frac{G(cx)}{G(x)} \, \leq \, \limsup_{x \to 0} \frac{G(cx)}{G(x)} \, < \, \infty \, ,$$

for some constant $c > 1$.

Note that this is much weaker than the requirement that G be regularly varying around 0, which corresponds to the condition that the limit exists for any $c > 1$ and lies in $(1, \infty)$, which is in turn weaker than the requirement that G has a limit law at 0. Thus, the following generalizes the result of Pemantle-Peres [23] and, in spirit, confirms the validity of their speculation.

Theorem 1.7. *If G is controlled near* 0*, then the equivalence*

$$f \textit{ is } G\textit{-small} \Leftrightarrow f \textit{ is } G\textit{-explosive}$$

holds in the class of non-decreasing functions $f : \mathbb{N} \to \mathbb{N}$.

On the other hand, we give examples that demonstrate that the "controlled near 0" condition cannot be significantly weakened. Firstly, we show that the equivalence may fail in general in both directions even if we assume G continuous and strictly increasing:

Proposition 1.8. *There exist a continuous, strictly increasing weight distribution* G*, and non-decreasing functions* $f_1, f_2 : \mathbb{N}_0 \to \mathbb{N}$ *with the following properties.*

(i) The function f_1 *is* G*-small but not* G*-explosive.*
(ii) The function f_2 *is* G*-explosive but not* G*-small.*

Secondly, a counterexample of the form (ii) holds in fact for a rather large class of weight distributions:

Theorem 1.9. *Let* G *be any weight distribution satisfying either*

$$\limsup_{i\to\infty} \frac{G(x_i)}{G(x_i/c)} \;<\; \limsup_{i\to\infty} \frac{G(cx_i)}{G(x_i)} \;=\; \infty\,, \tag{2}$$

or

$$1 \;=\; \liminf_{i\to\infty} \frac{G(x_i)}{G(x_i/c)} \;<\; \liminf_{i\to\infty} \frac{G(cx_i)}{G(x_i)}\,, \tag{3}$$

for some constant $c > 0$ *and decreasing sequence* $x_i : i \geq 1$ *with limit* 0*. Then there exists a function* $f : \mathbb{N}_0 \to \mathbb{N}$ *which is* G*-explosive but not* G*-small.*

See Section 4 for more details.

While Theorem 1.7 and Theorem 1.9 together cover most naturally defined distribution functions G, we would like to stress that it is still open to answer Question 1.5 completely.

1.3. **Finite height criterion for stick breaking random real trees.** Consider the following method for constructing a random real tree. Given a sequence $\ell(i) : i \in \mathbb{N}$, define the real tree A_ℓ recursively as follows. Let $A_\ell(1)$ consist of a closed segment of length $\ell(1)$ rooted at one end, and for each $i \geq 1$, define $A_\ell(i+1)$ by attaching one end of a closed segment of length $\ell(i+1)$ to a uniformly randomly chosen point of the tree $A_\ell(i)$. Let

$$A_\ell^o := \bigcup_{i\geq 1} A_\ell(i)$$

and define A_ℓ as the completion of A_ℓ^o. The random real tree A_ℓ is referred to as the random real tree given by the stick breaking process obtained by cutting the positive real line according to the segment lengths sequence ℓ.

Note that in the case where the sequence $\ell(i)$ is the length of the segment $[P^\lambda(i), P^\lambda(i+1)]$ given by an inhomogeneous Poisson point process P^λ with intensity $\lambda(t) = t$ on $\mathbb{R}^+$, the random real tree A_ℓ is precisely the continuum random tree constructed by Aldous [5]. The inhomogeneous, and more general, versions of this construction are treated in [7, 8, 18].

Curien and Haas [16] have recently studied the geometric properties of such trees (such as compactness and Hausdorff dimension) in the case of deterministic lengths $\ell(i)$ which decay roughly like a power $\ell(i) \approx i^{-\alpha}$ for $\alpha > 0$. It was a question of Curien [15] that led us to consider Problem 1.10 below.

For a real tree A, we denote by $d(A)$ the height of A, i.e., the supremum of distances from points of A to the root. We denote by $\mathrm{diam}(A)$ the diameter of A. Note that $d(A) \leq \mathrm{diam}(A) \leq 2d(A)$.

The following is then a very natural problem:

Problem 1.10. Classify all sequences $\ell(i), i \in \mathbb{N}$, for which we have $d(A_\ell) < \infty$, or equivalently, $\mathrm{diam}(A_\ell) < \infty$, almost surely.

Note that the property of having bounded diameter almost surely is equivalent to the almost sure compactness of the random real tree A_ℓ [16].

As an application of our result on linear growth of P^λWIT, we answer this question completely in Section 5 for those length sequences $\ell(i)$ which are deterministic and decreasing.

Theorem 1.11. *Let $\ell(i), i \in \mathbb{N}$, be a decreasing sequence. Then $d(A_\ell) < \infty$ almost surely if and only if $\sum_{n\geq 1} \frac{\ell(n)}{n} < \infty$, or equivalently, if and only if $\sum_{n\geq 1} \ell(2^n) < \infty$.*

We note that the requirement that $\ell(i)$ be decreasing may be relaxed: writing $\bar{\ell}(i)$ for the average value of $\ell(j) : j \leq i$, we only need to assume that the ratio $\ell(i)/\bar{\ell}(i)$ is bounded.

It might be possible that a similar criterion (applied to the decreasing rearrangement of ℓ) remains valid for a general sequence $\ell(i)$, a question we leave open.

Finally, we mention that Curien and Haas have independently and simultaneously proved in [16], by using different tools, that if $\ell(i) \leq i^{-\alpha+o(1)}$ for some $\alpha > 0$, then the random real tree A_ℓ has bounded height almost surely. They further prove (amongst other things) that under the additional assumption on the average $\bar{\ell}(i) = i^{-\alpha+0(1)}$ for $\alpha \in (0, 1]$, the Hausdorff dimension of A_ℓ is α^{-1}, while for $\alpha > 1$, the Hausdorff dimension is one, almost surely.

2. Linear growth in P^λWIT trees

In this section, we will prove Theorem 1.2.

Hammersley [19], Kingman [21], and Biggins [11] give general conditions under which linear growth occurs in a branching random walk, as well as associated limit theorems. The version most suitable to our situation is Kingman [21] (see also Biggins [12]). Kingman's theorem holds for general point processes on $\mathbb{R}^+$, but we state it here only for the case relevant to us, that of inhomogeneous Poisson point processes.

Theorem 2.1 (Kingman [21], specialized to the P^λWIT)**.** *Let T^λ be a P^λ WIT, for some appropriate $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$. Define $\mu : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{\infty\}$ by*

$$\mu(a) := \inf_{D\geq 0} \left\{ e^{Da}\mathbb{E}\left[\sum_{j\geq 1} e^{-DP^\lambda(j)}\right]\right\}. \tag{4}$$

Assume $\mu(a) < \infty$ for some $a > 0$. Then

$$\lim_{n\to\infty} m_n(T^\lambda)/n \to \alpha \ \text{ almost surely,}$$

where $\alpha = \inf\{a : \mu(a) > 1\}$. In particular, if $\alpha > 0$, then T^λ exhibits linear growth almost surely. Moreover,

$$\mathbb{P}\left(m_n(T^\lambda) \leq an\right) \leq \mu(a)^n.$$

We begin with part (i) of Theorem 1.2. The following argument allows to treat only the case where $\int_0^t \lambda > 0$ for any $t > 0$. Indeed, if $\int_0^c \lambda = 0$ for some $c > 0$, then $\mathbb{P}\left(m_n(T^\lambda) < cn\right) = 0$ and linear growth holds trivially. In addition, if c is chosen maximum with respect to $\int_0^c \lambda = 0$, then the intensity function $\tilde{\lambda}$ defined by $\tilde{\lambda}(t) := \lambda(t + c)$ for any $t \in \mathbb{R}^+$ verifies $\int_0^t \tilde{\lambda} > 0$ for all $t > 0$, so that the limit assertion for λ follows from that of $\tilde{\lambda}$, guaranteed either by part (i) or part (ii) of the theorem.

So suppose that $\int_0^t \lambda > 0$ for any $t > 0$. Let $C > 0$ so that $\int_0^x \lambda \leq C^x$ for all $x \geq 1$. The key lemma is the following, after which we will be able to directly apply Kingman's result.

Lemma 2.2. *For every $\eta > 0$, there exists $D \in \mathbb{R}$ such that*

$$\mathbb{E}\left[\sum_{j\geq 1} e^{-DP^\lambda(j)}\right] < \eta\,.$$

Proof. Note that, by a straightforward coupling, if λ_1 and λ_2 are such that $\int_0^x \lambda_1 \geq \int_0^x \lambda_2$ for all $x \geq 0$, then

$$\mathbb{E}\left[\sum_{j\geq 1} e^{-DP^{\lambda_1}(j)}\right] \leq \mathbb{E}\left[\sum_{j\geq 1} e^{-DP^{\lambda_2}(j)}\right].$$

This allows us to assume λ is decreasing on $[0,1]$, and that $\lambda(y) = (\log C)C^y$ for $y \geq 1$. In particular, if we prove the lemma for such functions λ then this proves the lemma in general. In addition, we may obviously assume that $\eta \leq 1$.

We shall deal separately with points in the interval $[0,1]$ and those in $(1,\infty)$. We give a choice of D such that

$$\mathbb{E}\left[\sum_{j\geq 1: P^\lambda(j)\leq 1} e^{-DP^\lambda(j)}\right] \leq \frac{\eta}{2} \qquad \text{and} \qquad \mathbb{E}\left[\sum_{j\geq 1: P^\lambda(j)> 1} e^{-DP^\lambda(j)}\right] \leq \frac{\eta}{2}.$$

Let D_1 be large enough that

$$\mathbb{E}\left[e^{-D_1 P^\lambda(1)}\right] \leq \frac{\eta}{4}.$$

For each $j \geq 2$, the fact that λ is decreasing on $[0,1]$ implies that conditioned on $P^\lambda(j) \leq 1$, the increment $P^\lambda(j) - P^\lambda(j-1)$ stochastically dominates $P^\lambda(1)$, and so

$$\mathbb{E}\left[e^{-D_1 P^\lambda(j)}\right] \leq \left(\mathbb{E}\left[e^{-D_1 P^\lambda(1)}\right]\right)^j \leq \left(\frac{\eta}{4}\right)^j.$$

Summing the above inequalities, we obtain that for any $D \geq D_1$,

$$\mathbb{E}\left[\sum_{j\geq 1: P^\lambda(j)\leq 1} e^{-DP^\lambda(j)}\right] \leq \frac{\eta}{2}.$$

Now, for the points in $(1,\infty)$, we have

$$\mathbb{E}\left[\sum_{j\geq 1: P^\lambda(j)> 1} e^{-DP^\lambda(j)}\right] \leq \int_1^\infty (\log C)C^y e^{-Dy}\, dy \leq \frac{C\log C}{D - \log C}\, e^{-D}.$$

Therefore, taking $D \geq D_1$ large enough so that $C\log C e^{-D}/(D - \log C) \leq \eta/2$, we have

$$\mathbb{E}\left[\sum_{j\geq 1: P^\lambda(j)> 1} e^{-DP^\lambda(j)}\right] \leq \eta/2,$$

which completes the proof of the lemma. □

Note that the lemma ensures that $\mu(a) < \infty$ for any $a \in \mathbb{R}^+$. Thus by Theorem 2.1, $m_n(T^\lambda)/n \to \alpha$ as $n \to \infty$, where $\alpha := \inf\{a : \mu(a) > 1\}$. The lemma also immediately implies that $\mu(a) \to 0$ as $a \to 0$. So $\alpha > 0$, and thus T^λ exhibits linear growth almost surely. Moreover, again by Theorem 2.1,

$$\mathbb{P}\left(m_n(T^\lambda) \leq \delta n\right) \leq \mu(\delta)^n \leq e^{-Kn}$$

for any $\delta > 0$ s.t. $\mu(\delta) \leq e^{-K}$. This completes the proof of part (i) of Theorem 1.2.

We now move to part (ii). So suppose that for any $t > 0$, we have $\int_0^t \lambda > 0$, and λ does not satisfy the conditions of part (i): in other words, for any $C > 0$, there exists an $x \geq 1$ so that $\int_0^x \lambda > C^x$. It is not possible to directly apply the result of Kingman, since it turns out that $\mu(a) = \infty$ for all a. A truncation argument may be used, but we prefer to give here a direct proof.

We will prove that T^λ a.s. does not have linear growth by finding a value n_δ for every $\delta > 0$, with $n_\delta \to \infty$ as $\delta \to 0$, such that

$$\mathbb{P}\left(m_{n_\delta}(T^\lambda) \geq \delta \cdot n_\delta\right) \to 0 \quad \text{as} \quad \delta \to 0.$$

So fix any $\delta > 0$. Let $p = p_\delta := \min\{\mathbb{P}\left(P^\lambda(1) < \delta/2\right), \delta/2\}$. Note that $p > 0$ by the assumption that $\int_0^t \lambda > 0$ for all $t > 0$. Also let $C = C_\delta := p^{-8p^{-1}}$, and choose $x = x_\delta \geq 1$ so that $\int_0^x \lambda > C^x$. Let $n = n_\delta := \lceil 2x/\delta \rceil$. Note that $n_\delta \to \infty$ as $\delta \to 0$.

Let $\mathfrak{T}$ denote the connected component containing the root of the subforest of T^λ obtained by keeping those edges adjacent to the root with weight less than x and all remaining edges having weight less than $\delta/2$. We thus have

$$\begin{aligned}\mathbb{P}\left(m_n(T^\lambda) \geq \delta n)\right) &\leq \mathbb{P}\left(\mathfrak{T}_n = \emptyset\right) \\ &\leq \mathbb{P}\left(\mathfrak{T}_n = \emptyset \mid |\mathfrak{T}_1| \geq p^{-2n}\right) + \mathbb{P}\left(|\mathfrak{T}_1| < p^{-2n}\right).\end{aligned}$$

Given that a particular child v of the root is in $\mathfrak{T}$, the probability that v has a descendant in $\mathfrak{T}_n$ (generation n) is certainly at least p^{n-1}. Therefore,

$$\mathbb{P}\left(\mathfrak{T}_n = \emptyset \mid |\mathfrak{T}_1| \geq p^{-2n}\right) \leq \mathbb{P}\left(\mathrm{Bin}(\lceil p^{-2n} \rceil, p^{n-1}) = 0\right).$$

Also $|\mathfrak{T}_1|$ is Poisson distributed, with mean

$$\int_0^x \lambda \geq C^x = p^{-8p^{-1}x} \geq p^{-2p^{-1}n\delta} \geq p^{-4n}.$$

So clearly the above bound on $\mathbb{P}\left(m_n(T^\lambda \geq \delta n)\right)$ goes to zero as $\delta \to 0$, and the result is proved.

3. Explosion in spherically symmetric trees

In this section we prove Theorem 1.7. That is, we show that if G is controlled near 0, and $f : \mathbb{N}_0 \to \mathbb{N}$ is non-decreasing, then

$$f \text{ is } G\text{-small} \Leftrightarrow f \text{ is } G\text{-explosive}\,.$$

One direction of the equivalence is straightforward. If G is controlled near 0 and f is G-small, then one easily deduces that

$$\sum_{n\geq 1} G^{-1}\left(\frac{1+\varepsilon}{f(n)}\right) < \infty\,,$$

for some $\varepsilon > 0$. The first direction of the equivalence then follows from the following proposition.

Proposition 3.1. *Let G be an arbitrary weight distribution function, and let $f : \mathbb{N}_0 \to \mathbb{N}$ be non-decreasing. If*

$$\sum_{n\geq 1} G^{-1}\left(\frac{1+\varepsilon}{f(n)}\right) < \infty\,,$$

then f is G-explosive.

Proof. Note that the summability condition certainly implies that $f(n) \to \infty$. Consider the forest F in which edges from generation n to generation $n+1$ are kept if their weight is at most $G^{-1}((1+\varepsilon)/f(n))$. Since any path in F has finite weight it suffices to show it contains an infinite path with positive probability. Each node of generation n has $\mathrm{Bin}(f(n), (1+\varepsilon)/f(n))$ children, which stochastically dominates the distribution $\max\{\mathrm{Po}(1+\varepsilon/2), \varepsilon^{-1}\}$ when n is large. Since the Galton-Watson branching process with offspring distribution $\max\{\mathrm{Po}(1+\varepsilon/2), \varepsilon^{-1}\}$ survives with positive probability, the same is true for F. □

We now turn to the remaining direction of the equivalence. We prove that if G is controlled near 0 and f is not G-small, i.e.,

$$\sum_{n\geq 1} G^{-1}(f(n)^{-1}) = \infty\,,$$

then f is not G-explosive.

The idea is to compare weights along paths in T_f^G with the terms of the sequence $a(n) := G^{-1}(f(n)^{-1})$. Indeed, the key intermediate result will be Proposition 3.3, which claims that this re-normalized weighted tree has linear growth (at least after the removal of some extra heavy edges).

Definition 3.2. Given the weighted infinite tree T_f^G, and a sequence $(a(n))$, define the renormalized weighted tree $\hat{T}_f^G$ to have the same underlying graph as T_f^G, but with weights

$$\hat{w}_e := \frac{w_e}{a(|e|)} \qquad e \in E(\hat{T}_f^G) = E(T_f^G)\,,$$

where w_e is the weight of e in T_f^G and $|e|$ is the generation of the parent in the edge e.

Say that $\hat{T}_f^G$ has been η-trimmed, if all edges e with

$$\hat{w}_e \geq \frac{\eta}{a(|e|)}$$

have been removed. Note, this is equivalent to removing all edges of weight at least η in T_f^G before renormalizing.

Proposition 3.3. *Let G be controlled near 0 and let f be a non-decreasing function with $f(n) \to \infty$ as $n \to \infty$. Then there exists $\eta > 0$ such that the tree $\hat{T}$ obtained when $\hat{T}_f^G$ is η-trimmed exhibits linear-growth almost surely.*

Let us show how to complete the proof of Theorem 1.7, by using Proposition 3.3. The following lemma is essentially what we need.

Lemma 3.4. *Let $\epsilon > 0$ and consider a sequence of non-negative real numbers $b(i) : i \in \mathbb{N}$ such that $\sum_{i=1}^n \frac{b(i)}{a(i)} \geq \epsilon n$ for all sufficiently large n. Then $\sum_{i=1}^\infty b(i) = \infty$*

Proof. Let n_0 be such that the claimed inequality holds for all $n \geq n_0$. We prove that for any n, $\sum_{i=1}^n b(i) \geq \epsilon \sum_{j=n_0}^n a(j)$, which proves the lemma, since the later sum is assumed to be divergent. Let $c(i) := \frac{b(i)}{a(i)}$ and $S(j) := \sum_{i=1}^j c(i)$. By assumption, for any $j \geq n_0$, we have $S(j) \geq \epsilon j$. We now have that

$$\begin{aligned}
\sum_{i=1}^n b(i) = \sum_{i=1}^n c(i)a(i) &= \sum_{i=1}^n c(i)\big[a(n) + \sum_{j=i}^{n-1}(a(j) - a(j+1))\big] \\
&= S(n)a(n) + \sum_{j=1}^{n-1} S(j)(a(j) - a(j+1)) \\
&\geq S(n)a(n) + \sum_{j=n_0}^{n} S(j)(a(j) - a(j+1)) \\
&\geq \epsilon n a(n) + \sum_{j=n_0}^{n-1} \epsilon j(a(j) - a(j+1)) \geq \epsilon \sum_{j=n_0}^{n} a(j).
\end{aligned}$$

The penultimate inequality uses that $a(n)$ is a decreasing sequence. Since $\sum_{j=n_0}^\infty a(j) = \infty$, the lemma follows. $\square$

Proof of Theorem 1.7. Let G be controlled near 0 and $f : \mathbb{N}_0 \to N$ a non-decreasing function. The proof that f being G-small implies f is G-explosive follows directly from Proposition 3.1, as explained above.

For the other direction, suppose that f is not G-small, i.e.,

$$\sum_{n\geq 1} G^{-1}(f(n)^{-1}) \,=\, \infty\,,$$

we shall prove that f is not G-explosive.

In the case that f is bounded the proof is easy. Let D be the maximum value of $f(n)$, and let $\delta_0 > 0$ be such that $G(\delta_0) < 1/2D$. All components of light edges (i.e., with weights at most δ_0) are finite almost surely (since they correspond to sub-critical branching processes). So every infinite path contains infinitely many edges of weight at least δ_0, which demonstrates that f is not G-explosive.

If $f(n)$ is unbounded, we shall use Proposition 3.3. First observe that we may remove all edges of weight above some $\eta > 0$. Indeed, let E be the event T_f^G contains an infinite path of finite weight and let $E(\eta)$ be the event T_f^G contains such a path with all edge weights at most η. It is elementary that the ratio of $\mathbb{P}(E)$ and $\mathbb{P}(E(\eta))$ is a constant. So to prove that $\mathbb{P}(E) = 0$, it suffices to prove that $\mathbb{P}(E(\eta)) = 0$.

The renormalized version of the remaining tree is precisely the tree $\hat{T}$ obtained after $\hat{T}_f^G$ is η-trimmed. By Proposition 3.3, there exists (almost surely), a constant $\delta > 0$ such that $m_n(\hat{T}) \geq \delta n$ for all sufficiently large n. Now, let γ be any infinite path descending from the root in the tree T_f^G and using only edges with weight at most η. Writing $b(n)$ for the weights along this path, $a(n)$ for $G^{-1}(f(n)^{-1})$ and $c(n)$ for the ratio $b(n)/a(n)$, we have that $c(n)$ are exactly the weights on the corresponding path in $\hat{T}$ and so satisfy $\sum_{i=1}^n c(i) \geq \delta n$ for all sufficiently large n. We are now in the setting of Lemma 3.4, and we deduce that $\sum_{n\geq 0} b(n)$ is divergent.

Since the choice of the path γ was arbitrary it follows that (almost surely) T_f^G does not contain an explosive path with all weights at most η, as required. □

In the rest of this section, we provide the proof of Proposition 3.3.

3.1. **Proof of Proposition 3.3.** It will be convenient to relabel the vertices of $\hat{T}_f^G$ (and so also $\hat{T}$) by sequences in $\mathbb{N}^{<\omega}$. We label the root by $\emptyset$, and for any vertex $\mathbf{i} \in \mathbb{N}^{<\omega}$ which labels a vertex in $\hat{T}_f^G$, we label with $\mathbf{i}j$ the child of $\mathbf{i}$ for which $\hat{w}_{\mathbf{i},\mathbf{i}j}$ has the j'th smallest value, amongst all weights of edges to children of $\mathbf{i}$.

With Theorem 1.2 in mind, it will suffice to couple the tree $\hat{T}$ (obtained after $\hat{T}_f^G$ is η-trimmed) with a P$^\lambda$WIT (with λ controlled by an exponential) in such a way that weights

in $\hat{T}$ are at least the equivalent weights in the P$^\lambda$WIT. Such a coupling is provided by the following lemma.

Lemma 3.5. *Let G be controlled near 0, and let $f : \mathbb{N}_0 \to \mathbb{N}$ be a non-decreasing function with $f(n) \to \infty$ as $n \to \infty$. Then there exist constants $n_0 \in \mathbb{N}$ and $\eta, C > 0$, a function $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ with $\int_0^x \lambda \le C^x$ for all $x \in \mathbb{R}^+$, and a coupling between $\hat{T}$ (obtained from η-trimming $\hat{T}_f^G$) and a $P^\lambda WIT$ T^λ in which*

$$\hat{w}_{\mathbf{i},\mathbf{i}j} \ge P^\lambda_{\mathbf{i}}(j)$$

for any edge $\{\mathbf{i}, \mathbf{i}j\}$ in $\hat{T}$ with $\mathbf{i}$ at level at least n_0.

Proof. We defer for now the explicit definitions of n_0, C, η and λ. It is sufficient to show that for any $n \ge n_0$, and any vertex $\mathbf{i} \in \hat{T}_n$, we can couple the weights of the downward edges from $\mathbf{i}$ with the inhomogeneous Poisson point process P^λ. The lemma then follows by applying the coupling at every vertex at distance n_0 or more from the root.

The weights $\hat{w}_{\mathbf{i},\mathbf{i}j} : j = 1, \dots, f(n)$ are obtained by taking i.i.d. samples from the distribution G, dividing by $a(n)$, and then arranging in an increasing order. Equivalently, they are generated as

$$\hat{w}_{\mathbf{i},\mathbf{i}j} = \frac{G^{-1}(U(j))}{a(n)}$$

where $U(1), \dots, U(f(n))$ are i.i.d. Unif$(0,1)$ random variables arranged in increasing order.

Let $\eta_0 = G^{-1}(1/2)$. We will later choose $\eta \le \eta_0$, thus ensuring η to satisfy $G(\eta) \le 1/2$.

We can ignore all edges $\{\mathbf{i}, \mathbf{i}j\}$ for which $U(j) \ge G(\eta)$, since these will disappear when we form $\hat{T}$ by η-trimming $\hat{T}_f^G$. Let J denote the largest index $j \le f(n)$ s.t. $U(j) < G(\eta)$.

Thus one may couple the uniforms $U(1), \dots, U(f(n))$ with a Poisson point process Q of intensity $2f(n)$ on the positive real line so that

$$U(j) \ \ge \ Q(j) \wedge G(\eta) \qquad \text{for all } j \ge 1\,.$$

It follows that there is a coupling in which

$$\hat{w}_{\mathbf{i},\mathbf{i}j} \ \ge \ \frac{G^{-1}(Q(j))}{a(n)} \qquad \text{for all } j = 1, \dots, J\,. \tag{5}$$

So to prove the lemma, it suffices to demonstrate a coupling between Q and P^λ (for our choice of λ, which will be given shortly) such that

$$\frac{G^{-1}(Q(j))}{a(n)} \ \ge \ P^\lambda(j) \qquad \text{for all } j = 1, \dots, J.$$

The remainder is concerned with using the properties of G to prove this statement.

Since G is controlled near 0, there exist constants $c > 1$ and $K \geq \max\{1/\eta_0, 2\}$ such that

$$1 + \frac{1}{K} < \frac{G(cx)}{G(x)} < K\,, \qquad \text{for all } x \leq 1/K.$$

Let $\eta = 1/K$ (note that then $\eta \leq G^{-1}(1/2)$, as assumed earlier). It follows easily from the above, combined with the monotonicity of G, that for any $x_0 \in (0, \eta]$,

$$G(x) \leq K \cdot (x/x_0)^{\log(1+1/K)/\log c} G(x_0) \quad \forall x \in (0, x_0],$$
$$G(x) \leq K \cdot (x/x_0)^{\log K/\log c} G(x_0) \quad \forall x \in [x_0, \eta].$$

More succinctly, we have that

$$G(x) \leq h(x/x_0)G(x_0) \qquad \text{for all } x, x_0 \in (0, \eta], \tag{6}$$

where

$$h(z) = \begin{cases} K z^{\log(1+1/K)/\log c} & z \leq 1 \\ K z^{\log K/\log c} & z > 1 \end{cases}. \tag{7}$$

One can think of this as providing some uniform control over the shape of G when "zooming in" around some point x_0. We now state our definition of n_0: it is chosen such that $f(n_0) \geq K$ and $a(n_0) \leq \eta$. Let us see what the above tells us about the behaviour of G about $a(n)$, for $n \geq n_0$. It yields

$$G(x) \leq h(x/a(n))G(a(n)) = \frac{h(x/a(n))}{f(n)} \qquad \text{for all } x \in (0, \eta]. \tag{8}$$

Making the substitution $y = G(x)$ and applying h^{-1}, we obtain

$$h^{-1}(yf(n)) \leq \frac{G^{-1}(y)}{a(n)} \qquad \text{for all } y \in (0, G(\eta)].$$

Since $Q(j) \leq G(\eta)$ for all $j \leq J$,

$$h^{-1}\big(Q(j)f(n)\big) \leq \frac{G^{-1}\big(Q(j)\big)}{a(n)} \qquad \text{for all } j = 1, \ldots, J\,. \tag{9}$$

We are now ready to give our choice of λ, and finish the proof of the lemma: simply define $\lambda := 2\frac{dh}{dx}$. Note that by the explicit form of h given in (7), it is clear that h grows polynomially and so there clearly exists a constant $C > 0$ so that $\int_0^x \lambda = 2h(x) \leq C^x$ for all $x > 0$. Moreover, the Poisson point process P^λ with intensity λ can be generated by $P^\lambda(j) = h^{-1}(Q(j)f(n))$, because $Q(j)f(n)$ is a Poisson point process of intensity 2; this combined with (5) and (9) provide the required coupling. □

Take n_0, η, C and λ as guaranteed by the above lemma. Then for any vertex in $\mathbf{i} \in \hat{T}_{n_0}$, the subtree rooted at $\mathbf{i}$ has linear growth almost surely, by the above coupling and Theorem 1.2. Since this occurs for every vertex in $\hat{T}_{n_0}$, $\hat{T}$ itself exhibits linear growth almost surely, which completes the proof of Proposition 3.3.

4. Sharpness of the equivalence Theorem 1.7

In this section we give examples of pairs (f, G) where the equivalence between f being G-small and f being G-explosive fails. We begin by giving simple examples for each direction of the equivalence.

Note that this does not quite prove Proposition 1.8, since the choice of G used in these two simple examples differ. However, we will then prove Theorem 1.9, and since the choice of G that we use in demonstrating Proposition 1.8 (i) also satisfies the conditions of Theorem 1.9, Proposition 1.8 follows.

Recall that we write $F(n)$ for $\prod_{i=0}^{n-1} f(i)$, and note that $F(n) = |(T_f)_n|$.

4.1. A pair (f, G) where f is G-small but not G-explosive.

We shall define a continuous strictly increasing distribution function G and a non-decreasing function $f : \mathbb{N}_0 \to \mathbb{N}$, such that f is G-small but not G-explosive.

In fact it is possible to define a general class of such examples. Let $B_1, B_2, \ldots$ be any strictly increasing sequence of natural numbers. We may define f such that the image of f is precisely $\{B_i : i \geq 1\}$.

We first define G_0, a distribution function that is neither continuous nor strictly increasing (in fact it is a non-decreasing step function), with the property that f is G_0-small but not G_0-explosive. We then define G as a tiny perturbation of G_0 in such a way that G becomes continuous and strictly increasing while maintaining the properties that f is G-small but not G-explosive.

Our construction will involve defining a decreasing sequence a_i of positive reals and a non-decreasing sequence n_i of natural numbers. We will give an inductive construction of these sequences.

Set $a_0 = 1$ and $n_0 = 0$. Given a_{k-1} and n_{k-1}, consider the B_k-regular infinite tree T_k. Consider critical bond percolation on T_k in which bonds are open with probability $1/B_k$. It is well known that all open clusters are finite almost surely. It follows that, for any constant $\varepsilon > 0$, there exists a constant $\xi(k, \varepsilon)$ such that with probability at least $1 - \varepsilon$, every path from the root of T_k to generation $\xi(k, \varepsilon)$ uses at least $1/a_{k-1}$ closed edges. Set

$n_k := n_{k-1} + \xi(k, \varepsilon_k)$, where $\varepsilon_k := \frac{1}{2F(n_{k-1})}$, and

$$f(n) := B_k \quad \text{for all} \quad n_{k-1} \leq n < n_k \,.$$

Finally set $a_k := \frac{1}{2^k n_k}$. This defines the sequences a_i, n_i, and the function $f : \mathbb{N}_0 \to \mathbb{N}$.

Define now G_0 by

$$G_0(x) := \frac{1}{B_k} \quad \text{for all} \quad a_k \leq x < a_{k-1}$$

It is easily checked that f is G_0-small.

We now show that for each k there is probability at least $1/2$ that every path from generation n_{k-1} to generation n_k has weight at least 1. This is straightforward since $\xi(k, \varepsilon)$ was precisely chosen so that with probability at least $1 - \varepsilon$ every path from a fixed vertex of generation n_{k-1} to generation n_k has at least $1/a_{k-1}$ edges of weight at least a_{k-1}, and therefore has total weight at least 1. The choice of ϵ_k is precisely made to guarantee that a union bound over vertices of generation n_{k-1} complete the proof.

This property easily implies that explosion is a probability zero event, and so f is not G_0-explosive.

The essential content of the proof is unaffected if the atom of probability mass of G_0 at a_i is spread equally over the interval $[a_i, 2a_i]$. This makes the distribution function continuous. To make G strictly increasing, add between $2a_i$ and a_{i-1} probability mass with such a small total value that it is very unlikely that any edge in the first n_i generations has a weight between $2a_i$ and a_{i-1}. In this way the proof is again unaffected.

This defines a continuous and strictly increasing G such that f is G-small but not G-explosive.

4.2. **A pair (f, G) where f is G-explosive but not G-small.** We now define a continuous strictly increasing distribution function G and a non-decreasing function $f : \mathbb{N}_0 \to \mathbb{N}$, such that f is G-explosive but not G-small.

Again we may define a whole class of such examples. Let $B_1, B_2, \ldots$ be a sequence of natural numbers satisfying $B_{i+1} \geq 2B_i$ for each $i \geq 1$. We may define f such that the image of f is precisely $\{B_i : i \geq 1\}$.

As above, we first define an example in which G_0 has atoms and then obtain G using the same kind of perturbation trick as in the above construction.

Our construction will involve defining a decreasing sequence a_i of positive reals and a non-decreasing sequence n_i of natural numbers.

Set $a_k := \frac{1}{k!}$ and $n_k := (k-2)!$ for $k \geq 2$, $n_0 = 0$ and $n_1 = 1$. Define G_0 by

$$G_0(x) := \frac{1 - 1/(k-1)!}{B_k} \quad \text{for all } a_k \leq x < a_{k-1},$$

and note that $G_0^{-1}(1/B_k) = a_{k-1}$. Define $f : \mathbb{N}_0 \to \mathbb{N}$ by

$$f(n) := B_k \quad \text{for all } n_{k-1} \leq n < n_k\,.$$

It is easily checked that f is not G_0-small.

To see that f is G_0 explosive with positive probability (and therefore with probability 1 by the 0-1 law) consider the following strategy for finding an infinite path of finite weight. Define a forest $\mathfrak{T} \subseteq T_f^{G_0}$ by keeping the following edges: between generations $n_k = (k-2)!$ and $2(k-2)!$ keep all edges of weight at most $a_k = 1/k!$, and between generations $2(k-2)!$ and $n_{k+1} = (k-1)!$ keep all edges of weight at most $a_{k+1} = 1/(k+1)!$. It is clear that any infinite path in $\mathfrak{T}$ necessarily has finite weight. Writing p_k for the probability that a node of generation n_k has no descendent in $(\mathfrak{T})_{n_{k+1}}$, it suffices to prove that $\sum_k p_k < \infty$.

Let us bound p_k from above. Let v be a node of $\mathfrak{T}_{n_k}$ and consider the tree of descendants of v in $\mathfrak{T}$. In the first $(k-2)!$ generations following n_k, each node has $\mathrm{Bin}(B_{k+1}, \frac{1-1/(k-1)!}{B_k}))$ children. Denoting by $Z(.)$ the branching process with offspring distribution $\mathrm{Bin}(B_{k+1}, \frac{1-1/(k-1)!}{B_k}))$, which has mean at least $3/2$, it is straightforward to verify that the probabilities

$$q_k \;:=\; \mathbb{P}\left(Z\big((k-2)!\big) < \left(\frac{4}{3}\right)^{(k-2)!} \right)$$

are summable. In addition, let us note that each node $u \in \mathfrak{T}_{2(k-2)!}$ has probability at least $(1 - 1/k!)^{(k-1)!} \geq e^{-1}$ to have a descendent in generation $n_{k+1} = (k-1)!$. The probability r_k that none of $(4/3)^{(k-2)!}$ nodes of generation $2(k-2)!$ has a descendent in generation n_{k+1} is at most $e^{-(4/3)^{(k-2)!}}$, which is clearly summable.

The proof is now complete since $p_k \leq q_k + r_k$.

4.3. **Proof of Theorem 1.9.** Let G be a distribution function satisfying (2), i.e., such that

$$\limsup_{i\to\infty} \frac{G(x_i)}{G(x_i/c)} \;<\; \limsup_{i\to\infty} \frac{G(cx_i)}{G(x_i)} \;=\; \infty\,,$$

for some constant $c > 0$ and a decreasing sequence $x_i : i \geq 1$ with limit 0. Restricting to a subsequence if necessary, we may assume that

$$\frac{G(x_i)}{G(x_i/c)} \;\leq\; K \;\leq\; K4^{4^i} \;\leq\; \frac{G(cx_i)}{G(x_i)}\,,$$

where K is a constant, and $x_{i-1} \geq 4^i x_i$ for each $i \geq 1$. We may also assume that $G(x_i) \leq 1/4$.

One may now define the sequence n_i by $n_0 := 0$ and $n_i := \lfloor 1/x_i \rfloor$ for $i \geq 1$, and the function $f : \mathbb{N}_0 \to \mathbb{N}$ by

$$f(n) := \left\lfloor \frac{1}{2G(x_i)} \right\rfloor \qquad \text{for } n_{i-1} < n \leq n_i \,.$$

This choice of $f(n)$ is designed to be around $1/G(x_i)$, note that it satisfies

$$\frac{1}{4G(x_i)} < f(n) < \frac{1}{G(x_i)} \,.$$

Using the second inequality above and the choice of the sequence n_i it is easily observed that f is not G-small.

Consider the forest $\mathfrak{T} \subseteq T_f^G$ obtained by keeping edges between generation n_{i-1} and $n_{i-1} + n_i/2^i$ with weight at most cx_i, and edges between generations $n_{i-1} + n_i/2^i$ and n_i with weight at most $x_i/2^i$. Clearly any infinite path in $\mathfrak{T}$ has weight at most $\sum_{i\geq 1} 2^{-i} + \sum_{i\geq 1} 2^{-i} < \infty$. Thus, to prove that f is G-explosive it suffices to prove that $\mathfrak{T}$ contains an infinite path with positive probability. The idea of the proof is that with very high probability a node of generation n_{i-1} will have a very large number of descendants in generation $n_{i-1} + n_i/2^i$, and each such node has a not so small probability of having a descendant in generation n_i. While we do not go through every detail of the proof, it suffices to note that between generations n_{i-1} and $n_{i-1} + n_i/2^i$ the number of children of each node stochastically dominates

$$\mathrm{Bin}\left(f(n), K4^{4^i} G(x_i)\right) \geq \mathrm{Bin}\left(f(n), \frac{K4^{4^i-1}}{f(n)}\right) ,$$

which has mean $K4^{4^i-1}$. It is therefore extremely likely that a node v of generation n_{i-1} has at least $e^{2^i n_i}$ descendants in generation $n_{i-1} + n_i/2^i$, denote this event E_v.

Between generations $n_{i-1}+n_i/2^i$ and n_i each node has at least one child with probability at least $K^{-2\lceil \log_c(2^i)\rceil} \geq e^{-K'i}$ for some constant K'. Therefore a node of generation $n_{i-1} + n_i/2^i$ has a descendant in generation n_i with probability at least $e^{-K'in_i}$. Thus, the number of descendants in generation n_i of a node v of generation n_{i-1} stochastically dominates

$$1_{E_v} \cdot \mathrm{Bin}((e^{2^i n_i}, e^{-K'in_i})$$

which is zero with very small probability.

Up to routine details this completes the proof that $\mathfrak{T}$ contains an infinite path with positive probability.

The proof in the case that G satisfies (3) is similar to the proof given in Section 4.2 – define a forest $\mathfrak{T}$ which grows exponentially for a period after generation n_{i-1}, then only slightly sub-critical from there until generation n_i. We omit the details.

5. Finite height criterion for stick breaking problem

Recall the definition of the random real tree A_ℓ constructed by a stick breaking process on $\mathbb{R}^+$ given by a sequence $\ell(i)$, for $i \in \mathbb{N}$: Given such a sequence, the stick breaking process defines a random real tree A_ℓ as follows. Let $A_\ell(1)$ consist of a closed segment of length $\ell(1)$, seen as a (rooted) real tree with one edge, rooted at one end, and for each $i \geq 1$ let $A_\ell(i+1)$ be obtained by attaching one end of a closed segment of length $\ell(i+1)$ to a uniformly random position on the real tree $A_\ell(i)$. Define A_ℓ as the completion of $A_\ell^o = \bigcup_{i\geq 1} A_\ell(i)$.

For any real tree A, denote by $d(A)$ the height of A. Our aim in this section is to prove Theorem 1.11, namely, to show that if ℓ is a decreasing sequence, then

$$d(A_\ell) < \infty \quad \text{almost surely if and only if} \quad \sum_{n\geq 1} \ell(2^n) \, < \, \infty \, .$$

The intuition behind the summability condition is roughly speaking as follows: traversing a path through A_ℓ the index of the segment which contains the nth edge used on the path should grow exponentially in n, so that the sum of the lengths of the segments containing the edges of the path should behave like

$$\sum_{n\geq 1} \ell(2^n) \, .$$

The following observation indeed allows us to focus on the sum of the lengths of the segments (i.e., $\ell(i)$s) rather than distances in the real tree. Given a path ξ in A_ℓ, let $td(\xi)$ denote the sum of the lengths of the segments which contain the edges of ξ. Let $td(A_\ell)$ be the maximum of $td(\xi)$ over all the paths in the rooted tree A_ℓ.

Observation 5.1. The inequality $d(A_\ell) \leq td(A_\ell)$ holds deterministically. On the other hand, if $td(A_\ell)$ is infinite then $d(A_\ell)$ is infinite almost surely.

The observation is indeed straightforward as the randomness used to position the endpoint of a segment e_j on a current segment e_i may be sampled independently. In the limit at least half of the edges connect at least half way along.

The following two lemmas will be useful in our proof of the theorem. The first is a straightforward monotonicity statement. The second will be used to show that when

$\sum_{n\geq 1}\ell(2^n)$ is divergent, it cannot be that $\ell(i)$ is always much smaller than the average of $\ell(j) : j \leq i$.

Lemma 5.2. *Suppose R, S are two subsets of $\mathbb{N}$ and for all $j \in \mathbb{N}$,*

$$|R \cap \{1, \ldots, j\}| \leq |S \cap \{1, \ldots, j\}|.$$

Then

$$\sum_{i\in R}\ell(i) \leq \sum_{i\in S}\ell(i)\,.$$

Proof. The sums are limits of the partial sums up to j. For any fixed j, we can find an injection $\mathfrak{i} : R \cap \{1, \ldots, j\} \hookrightarrow S \cap \{1, \ldots, j\}$ such that $\mathfrak{i}(i) \leq i$ for any $i \in R \cap \{1, \ldots, j\}$. Since ℓ is decreasing, we get

$$\sum_{i\in R\cap\{1,\ldots,j\}}\ell(i) \leq \sum_{i\in R\cap\{1,\ldots,j\}}\ell(\mathfrak{i}(i)) \leq \sum_{i\in S\cap\{1,\ldots,j\}}\ell(i)$$

for partial sums, from which the lemma follows. □

Lemma 5.3. *Let $\ell(i)$ be an integer sequence and let*

$$\mathfrak{D} := \left\{n \in \mathbb{N} : \ \ell(2^n) > \frac{\ell(2^m)}{2^{(n-m)/2}} \quad \textit{for all } m < n\right\}.$$

Then

$$\sum_{n\geq 1}\ell(2^n) \ \textit{ is divergent if and only if } \sum_{n\in\mathfrak{D}}\ell(2^n) \ \textit{ is divergent.}$$

Proof. For each $n \in \mathbb{N}$ define $\pi(n)$ to be the least natural number m such that

$$\ell(2^n) \leq \frac{\ell(2^m)}{2^{(n-m)/2}}\,.$$

Note that $n \in \mathfrak{D}$ if and only if $\pi(n) = n$, and $\pi(\pi(n)) = \pi(n)$ for any $n \in \mathbb{N}$, in other words, π is a projection from $\mathbb{N}$ to $\mathfrak{D}$. For each $m \in \mathfrak{D}$, let

$$H_m := \{n : \pi(n) = m\}\,.$$

The lemma follows immediately from the observations that

$$\mathbb{N} = \bigcup_{m\in\mathfrak{D}} H_m$$

and that for any $m \in \mathfrak{D}$,

$$\sum_{n\in H_m}\ell(2^n) \leq \sum_{n=m}^{\infty}\frac{\ell(2^m)}{2^{(n-m)/2}} \leq 4\ell(2^m)\,.$$

□

We are now ready to prove Theorem 1.11.

Proof. Neither direction is trivial. We begin by showing that if

$$\sum_{n\geq 1} \ell(2^n) \,<\, \infty\,,$$

then $d(A_\ell)$ is finite almost surely. Let us identify a path ξ in A_ℓ with the set of segments it uses, and further, identify ξ with the subset of $\mathbb{N}$ of indices of these segments. The length of the path ξ is obviously bounded by $\sum_{i\in\xi} \ell(i)$. By Lemma 5.2 it suffices to show that every path ξ in A_ℓ verifies

$$|\xi \cap \{1,\ldots,j\}| \,\leq\, |R \cap \{1,\ldots,j\}| \qquad \text{for all } j \in \mathbb{N}\,.$$

for some set R of the form $R = \{\lfloor e^{\delta n}\rfloor : n \geq 1\}$.

We prove this using our result, Theorem 1.2, on linear growth in generalizations of the PWIT. We can construct an infinite weighted (random) tree T associated with A_ℓ. As a (combinatorial) tree, T is the genealogy tree of A_ℓ, formally defined as follows: To each segment e_i (of length $\ell(i)$) used in A_ℓ we associate a vertex v_i of T. The vertex v_1 will be the root of T, and the vertex v_i is a child of another vertex v_j if $j < i$ and in the construction of A_ℓ, the segment e_i is attached to a point of e_j. The weight of the edge v_jv_i is defined to be $\log j - \log i$. For technical reasons we shall ignore the root and the edges to the children of the root. Consider the subtree beneath some vertex v_i that is a child of the root. The probability that v_j will be a child of v_i for $j > i$ is at most $1/i$, therefore the distribution of the number of children of v_i of weight at most $w > 0$ is stochastically dominated by the binomial distribution

$$\mathrm{Bin}(\lfloor e^w i - i\rfloor, 1/i) \,=\, \mathrm{Bin}(\lfloor i(e^w-1)\rfloor, 1/i) \,\leq\, \mathrm{Po}\left(\int_0^w C^t\,dt\right)$$

for an appropriately chosen constant $C > 1$ (in fact one may take $C = e^2$). Theorem 1.2 now tells us that all subtrees of the children of the root exhibit linear growth, and furthermore the probability of a path to generation n of weight less that δn (for some $\delta > 0$) is at most e^{-2n}. The same is true deterministically for paths beginning with an edge of weight at least n. Since at most e^n children of the root have weight less than n, a union bound yields that

$$m_n(T) \,\geq\, \delta n$$

with probability at least $1 - e^{-n}$. And so, almost surely, $m_n(T) \geq \delta n$ for all sufficiently large n

This translates into the fact that every path ξ in A_ℓ, when viewed as a subset of $\mathbb{N}$ has nth element at least $e^{\delta n}$ for all sufficiently large n. This completes the required comparison with a set R, and so completes this half of the proof.

For the other direction, let us assume that the sum

$$\sum_{n\geq 1} \ell(2^n)$$

is divergent. Applying Lemma 5.3, we have that

$$\sum_{n\in\mathfrak{D}} \ell(2^n)$$

is divergent, where

$$\mathfrak{D} := \left\{ n \in \mathbb{N} : \ell(2^n) \geq \frac{\ell(2^m)}{2^{(n-m)/2}} \text{ for all } m < n \right\} .$$

Let $\overline{\mathfrak{D}}$ denote the set $\bigcup_{n\in\mathfrak{D}}\{2^{n-1}+1,\ldots,2^n\}$, and consider the path ξ in A_ℓ generated as follows: the first segment of ξ is e_1, thereafter ξ chooses to connect to the segment e_i with minimal index $i \in \overline{\mathfrak{D}}$ that is a descendant of its latest segment. By Observation 5.1, we complete a proof of the theorem by proving that $td(\xi)$ is infinite almost surely.

Note that for a sequence of non-negative reals $\big(a(i)\big)$, if the sum $\sum_{i\in\mathbb{N}} a(i)$ diverges, then a sum of the form $\sum_{i\in S} a(i)$ for a random subset $S \subset \mathbb{N}$ which contains each index $n \in N$ independently with probability $p > 0$, will diverge almost surely. Similarly, if there is not necessarily independence between the inclusion of indices in S but S is constructed recursively so that the inclusion of n in S (given all the previous information) has probability at least $p > 0$, then again the sum $\sum_{i\in S} a(i)$ will diverge almost surely. For this reason, it suffices to show that with probability at least some constant $p > 0$ the path ξ will contain a segment e_i with $2^{n-1}+1 \leq i \leq 2^n$.

Given the path ξ up to inclusion of some edge $e_j : j \in \overline{\mathfrak{D}}$, let $m \in \mathfrak{D}$ be the integer such that $2^{m-1} < j \leq 2^m$, and let $n \in \mathfrak{D}$ be minimum integer in $\mathfrak{D}$ such that $2^{n-1} \geq j$. We complete the proof by bounding below the probability that ξ contains a segment e_i with $2^{n-1}+1 \leq i \leq 2^n$. Until 2^n segments have been attached in the construction of A_ℓ, the total length may be bounded above by

$$\sum_{k=1}^{n} 2^{k-1}\ell(2^{k-1}) \leq \sum_{k=1}^{n} 2^{k-1}2^{(n-k)/2}\ell(2^n) \leq 2^{n+2}\ell(2^n) \leq 2^{n+2}\ell(j) .$$

And so, each segment added has probability at least 2^{-n-2} of being attached to e_j. The probability that no segment e_i with $2^{n-1} < i \leq 2^n$ joins to e_j is therefore at most

$$\left(1 - \frac{1}{2^{n+2}}\right)^{2^{n-1}} \leq e^{-1/8}.$$

This completes the proof. □

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