

ANALYTICITY OF THE SRB MEASURE FOR HOLOMORPHIC FAMILIES OF QUADRATIC-LIKE COLLET-ECKMANN MAPS

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ABSTRACT. We show that if f_t is a holomorphic family of quadratic-like maps with all periodic orbits repelling so that for each real t the map f_t is a real Collet-Eckmann S -unimodal map then, writing μ_t for the unique absolutely continuous invariant probability measure of f_t , the map

$$t \mapsto \int \psi d\mu_t$$

is real analytic for any real analytic function ψ .

1. INTRODUCTION AND STATEMENT OF THE THEOREM

If $t \mapsto f_t$ is a smooth one-parameter family of dynamics f_t so that f_0 admits a unique SRB measure μ_0 , it is natural to ask whether the map $t \mapsto \mu_t$, where t ranges over a set Λ of parameters such that f_t has (at least) one SRB measure μ_t , is differentiable at 0. Differentiability should be understood in the sense of Whitney if Λ does not contain a neighbourhood of 0, as suggested by Ruelle [16]. Katok, Knieper, Pollicott, and Weiss [7] gave a positive answer to this differentiability question in the setting of C^3 families of transitive Anosov flows, showing that $t \mapsto \int \psi d\mu_t$ is differentiable, for all smooth ψ . If f_0 is a C^3 mixing Axiom A attractor and the family $t \mapsto f_t$ is C^3 , Ruelle [15] not only proved that $t \mapsto \int \psi d\mu_t$ is differentiable, but also gave an explicit formula, the *linear response formula*, for the derivative. Of course, in the Anosov and Axiom A cases, Λ is a neighbourhood of 0.

Ruelle [16] suggested that this linear response formula, appropriately interpreted, should hold in much greater generality. Indeed, Dolgopyat [6] obtained the linear response formula for a class of partially hyperbolic diffeomorphisms. In a previous work [3, 4], we found that in the setting of piecewise expanding unimodal interval maps, the SRB measure is differentiable if and only if the path f_t is tangent to the topological class of f_0 , that is, if and only if $\partial_t f_t|_{t=0}$ is horizontal. We emphasize that this setting is not structurally stable. When differentiability holds, Ruelle's candidate for the derivative, as interpreted in [2], gives the linear response formula. (We refer to [2, 3, 4], which also contain conjectures about smooth, not necessarily analytic, Collet–Eckmann maps, for more information and additional references.) Then, Ruelle [17] proved the linear response formula for a class of nonrecurrent

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analytic unimodal interval maps f_t , assuming that all f_t stay in the topological class of f_0 . (Recall that f_t is nonrecurrent if $\inf_k d(f_t^k(c), c) > 0$, where c denotes the critical point.)

In the present work, we consider holomorphic families f_t of quadratic-like holomorphic Collet–Eckmann maps. By holomorphic, we mean complex analytic. Our assumptions imply, using classical holomorphic motions, that all f_t lie in the same conjugacy class. Generalising one of the arguments in [7], we are able to show that $t \mapsto \int \psi d\mu_t$ is real analytic for any real analytic function ψ , our main result.

Let us now state our result more precisely. Let $I = [-1, 1]$. A C^3 map $f : I \rightarrow I$ is an *S-unimodal* map if it has $c = 0$ as unique critical point, and f has nonpositive Schwarzian derivative, that is $\frac{f'''}{f''} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \leq 0$ except at c . An *S-unimodal* map is called Collet-Eckmann if there exist $C > 0$ and $\lambda_c > 1$ so that $|(f^n)'(f(c))| \geq C\lambda_c^n$ for all $n \geq 1$. In this paper, we shall only consider *S-unimodal* maps with $f''(c) \neq 0$.

In Section 2 we shall define precisely the notion of a holomorphic (complex analytic) family of *quadratic-like* maps in a neighbourhood of I and what *all periodic orbits repelling* means for such maps, and prove the main result of this work:

Theorem 1.1. *Let $t \mapsto f_t$ be a holomorphic family of quadratic-like maps in a neighbourhood of I , with all periodic orbits repelling. Assume in addition that for each small real t the map f_t restricted to I is a (real) Collet-Eckmann *S-unimodal* map. Then there exists $\epsilon > 0$ so that for each real analytic $\psi : I \rightarrow \mathbb{C}$, the map*

$$t \mapsto \int \psi \rho_t dx,$$

where ρ_t is the invariant density of f_t , is real analytic on $(-\epsilon, \epsilon)$.

The quadratic-like assumption implies that $f_t''(c) < 0$. The fact that periodic orbits are repelling implies that f_t is topologically conjugated with f_0 : See our use of Mañé-Sad-Sullivan [10] in the beginning of the proof of the theorem in Section 2. Besides Mañé-Sad-Sullivan [10] the other main ingredient of our proof are the results and constructions of Keller and Nowicki [8] which allow us to exploit dynamical zeta functions, following the argument in the work of Katok–Knieper–Pollicott–Weiss [7, First proof of Theorem 1].

The extension from quadratic-like to polynomial-like is straightforward, and we stick to the nondegenerate case $f''(c) \neq 0$ for the sake of simplicity of exposition. As the proof uses only real-analyticity of the holomorphic motions $t \mapsto h_t$, it is conceivable that the conclusion of the theorem holds if f_t is a real analytic family of quadratic-like maps, using ideas of [1], but this generalisation appears to be nontrivial.

Lyubich’s work [9] implies that there are *many* nontrivial families f_t satisfying the assumptions of our theorem. Constructing *examples* of such families f_t is in fact easier, and we sketch the procedure next: Start from two topologically conjugated Collet-Eckmann quadratic-like maps f and g which are not differentiably conjugated. By the result of Przytycki and Rohde [14], they are quasi-conformally conjugated. Hence, using a Beltrami path, one can construct a complex analytic family f_t of Collet-Eckmann maps containing both f and g , with $f_0 = f$, say. If f and g are real and conjugated in the real line, one can ensure that f_t is real for real parameters t . If f has negative Schwarzian derivative, then f_t has negative Schwarzian derivative for t close to 0. See also [5] for a specific example, and numerical experiments.

2. PROOF OF THE THEOREM

Before we prove the theorem, let us define precisely the objects we are studying:

Definition. We say that f_t is a *holomorphic family of quadratic-like maps in a neighbourhood of I* if there exists a complex neighbourhood U of I so that $t \mapsto f_t$ is a holomorphic map from a complex neighbourhood of zero to the Banach space $B(U)$ of holomorphic functions on U extending continuously to \overline{U} , with the supremum norm, such that:

- For real t , the map f_t is real on $U \cap \mathbb{R}$, with $f_t(I) \subset I$ and $f_t(-1) = f_t(1) = -1$.
- There exist simply connected complex domains W and V , whose boundaries are analytic Jordan curves, with $I \subset V$, $\overline{V} \subset U$, $\overline{V} \subset W$, and so that $f_0 : V \rightarrow W$ is a degree two ramified covering, with $c = 0$ as a unique critical point. That is, $f_0 : V \rightarrow W$ is a quadratic-like restriction of f_0 .

If f_t is a holomorphic family of quadratic-like maps in a neighbourhood of I then it is easy to see that for small complex t , denoting by V_t the connected component of $f_t^{-1}(W)$ containing 0, then $f_t : V_t \rightarrow W$ is a quadratic-like restriction of f_t : Indeed, ∂W is an analytic Jordan curve, and f_0 has no critical point on ∂V . If $f_t \in B(U)$ is close to f_0 , there is a simply connected domain V_t close to V such that $f_t(V_t) = W$, and the boundary of ∂V_t is a Jordan curve, by the implicit function theorem. Then $f_t : V_t \rightarrow W$ is a quadratic-like extension. We may then give another definition:

Definition. We say that f_t is a holomorphic family of quadratic-like maps in a neighbourhood of I *with all periodic orbits repelling*, if f_t is a holomorphic family of quadratic-like maps in a neighbourhood of I so that, for each small complex t , the map f_t only has repelling periodic orbits in V_t .

Proof. Since we assumed that all periodic points of f_t are repelling, [10, Theorem B] implies that there exists a holomorphic motion of the Julia set $K(f_0)$ of f_0 , that is, a map $h : D \times K(f_0) \rightarrow C$ where $D = \{z \in \mathbb{C} \mid |z| < \epsilon_0\}$ for some $\epsilon_0 > 0$, such that for each $x \in K(f_0)$ the map $t \mapsto h_t(x)$ is holomorphic, and for every $t \in D$ the function $x \mapsto h_t(x)$ is continuous and injective on $K(f_0)$, with

$$h_t \circ f_0 = f_t \circ h_t.$$

In particular, h_t is a homeomorphism from $K(f_0)$ to $K(f_t)$. Note that [10, Theorem B] is quoted for polynomial maps, but the proof immediately extends to polynomial-like. Our assumptions imply that $[f_0^2(0), f_0(0)] = K(f_0) \cap \mathbb{R}$ and $h_t(K(f_0) \cap \mathbb{R}) = K(f_t) \cap \mathbb{R} = [f_t^2(0), f_t(0)]$. From now on, we only use real analyticity of $t \mapsto f_t(x)$ and $t \mapsto h_t(x)$ for $x \in [f^2(0), f(0)]$.

We next claim that our assumptions guarantee that each f_t satisfies the technical requirement needed by Keller and Nowicki [8, (1.2)]. Denoting by $\text{var}_J \phi$ the total variation of a function ϕ on an interval J , and writing $f = f_t$, we need to check that there is that a constant $M > 0$ such that:

- $M^{-1} < \sup_I \frac{|x-c|}{|f'(x)|} + \text{var}_I \frac{|x-c|}{|f'(x)|} < M$,
- $\text{var}_{J_u} \frac{|f(x)-f(u)|}{|x-u||f'(x)|} < M$ where $J_u = [-1, u]$ if $u < c$ and $= [u, 1]$ if $u > c$.

Let $\delta_1 > 0$ be so that $|f''(y)| > |f''(c)|/2$ if $|y - c| < \delta_1$. It suffices to prove (a.) and (b.) for $|x - c| < \delta_1$ and $|u - c| < \delta_1$, and we restrict to such points. Noting

that for every such $x \neq c$ there exist y_x, z_x , and \tilde{z}_x , between x and c , so that

$$\frac{|x-c|}{|f'(x)|} = -\frac{x-c}{f'(x)-f'(c)} = -\frac{1}{f''(y_x)},$$

and, using $f''(x) = f''(c) + f^{(3)}(z_x)(x-c)$ and $f'(x) = f'(c) + f^{(3)}(\tilde{z}_x)\frac{(x-c)^2}{2}$,

$$\partial_x \frac{|x-c|}{|f'(x)|} = \frac{-f'(x) + (x-c)f''(x)}{(f'(x))^2} = \frac{(x-c)^2}{(f'(x))^2} \left(f^{(3)}(z_x) - \frac{f^{(3)}(\tilde{z}_x)}{2} \right),$$

the first two conditions hold because f is C^3 . For the third condition, consider $x \geq u > c$ (the other case is symmetric). Since

$$\frac{f(x)-f(u)}{(x-u)f'(x)} = 1 + \frac{x-u}{f'(x)} \frac{f''(z_x)}{2} = 1 + \frac{x-u}{f'(x)} \frac{f''(z_x)}{2f''(y_x)},$$

and $0 < -\frac{x-u}{f'(x)} < -\frac{x-c}{f'(x)}$, we get that $|\frac{f(x)-f(u)}{(x-u)f'(x)}|$ is bounded on $[u, 1]$, uniformly in u . Finally, since

$$\partial_x \frac{x-u}{f'(x)} = \frac{f'(x) - (x-u)f''(x)}{(f'(x))^2},$$

analyticity of f implies that $\partial_x \frac{x-u}{f'(x)}$ changes signs finitely many times, uniformly in u , proving (b).

Also, the results of Nowicki–Sands [13] and Nowicki–Przytycki [12] ensure (see Appendix A) that there exist $\lambda_c > 1$, $\lambda_{per} > 1$, $\lambda_\eta > 1$, and $\epsilon_1 > 0$ so that, for each $|t| < \epsilon_1$, there is $C_t > 0$ with

$$(1) \quad |(f_t^n)'(f_t(0))| \geq C_t \lambda_c^n, \forall n \geq 1,$$

and so that for each $x \in I$ so that $f_t^p(x) = x$ for some $p \geq 1$, we have

$$(2) \quad |(f_t^p)'(x)| \geq C_t \lambda_{per}^p,$$

and, finally, setting

$$\lambda_\eta(t) := \liminf_{n \rightarrow \infty} \{ |\eta|^{-1/n} \mid \eta \subset I \text{ is the biggest monotonicity interval of } f_t^n \},$$

$$(3) \quad \inf_{|t| < \epsilon_1} \lambda_\eta(t) > \lambda_\eta.$$

In other words, the hyperbolicity constants are uniform in t , guaranteeing uniformity when applying the results of Keller and Nowicki [8]. (We choose $\epsilon_1 < \epsilon_0$.)

We now adapt the strategy used in the first proof of [7, Theorem 1]. Fix ψ and, for $x \in I$ so that $f_0^p(x) = x$ for $p \geq 1$, and for small real s and t , consider

$$(4) \quad g_{s,t}(x) = \frac{e^{s\psi(h_t(x))}}{|f_t'(h_t(x))|}.$$

Since ψ is real analytic, the analyticity of $t \mapsto h_t$ and of $t \mapsto f_t$ together with (2) imply that there is $\epsilon_2 > 0$ so that, for every periodic point $x \in I$ of period $p \geq 1$ for f , the function

$$(t, s) \mapsto g_{s,t}^{(p)}(x) := \frac{e^{s \sum_{k=0}^{p-1} \psi(h_t(f^k(x)))}}{|(f_t^p)'(h_t(x))|}$$

is real analytic in $|s| < \epsilon_2$ and $|t| < \epsilon_2$, uniformly in x . We take $\epsilon_2 < \epsilon_1$.

Therefore, the dynamical zeta function defined by

$$(5) \quad \zeta(s, t, z) := \exp \sum_{p=1}^{\infty} \frac{z^p}{p} \sum_{x \in I: f_0^p(x)=x} g_{s,t}^{(p)}(x)$$

has the following property: There exists $\delta_2 > 0$ so that for each $|z| < \delta_2$ the function $\zeta(s, t, z)$ is real analytic in $|t| < \epsilon_2$, $|s| < \epsilon_2$, and so that for each (s, t) with $|t| < \epsilon_2$, $|s| < \epsilon_2$ the map $\zeta(s, t, z)$ is holomorphic and nonvanishing in $|z| < \delta_2$.

Now, $h_t \circ f_0 = f_t \circ h_t$ immediately implies

$$(6) \quad \zeta(s, t, z) = \exp \sum_{p=1}^{\infty} \frac{z^p}{p} \sum_{y \in I: f_t^p(y)=y} \frac{e^{s \sum_{k=0}^{p-1} \psi(f_t^k(y))}}{|(f_t^p)'(y)|}.$$

Before we proceed, we warn the reader that our parameter s is called t in [8], the parameter β in [8] is $\beta = 1$, and our parameter t corresponds to changing the dynamics.

Recall (1, 2, 3) and take $\Theta \in (0, 1)$ with

$$\Theta^{-1} < \min\{\lambda_\eta, \sqrt{\min(\lambda_c, \lambda_{per})}\}.$$

Keller and Nowicki [8, Theorem 2.1] prove that, if $\epsilon_3 \in (0, \epsilon_2)$ is small enough, then for $|s| < \epsilon_3$ and $|t| < \epsilon_3$ the transfer operator

$$\mathcal{L}_{s,t}\varphi(x) = \sum_{\hat{f}_t(y)=x} \frac{\omega_t(y) \exp(s\psi(y))}{\omega_t(x) |f_t'(y)|} \varphi(y),$$

acting on functions of bounded variation on a suitable Hofbauer tower extension $\hat{f}_t : \hat{I} \rightarrow \hat{I}$ of f_t [8, Section 3], endowed with an appropriate [8, §6.2] cocycle ω_t , is a bounded operator. Note that the cocycle embodies the singularities along the postcritical orbit of f_t .

If $s = 0$ then the spectral radius $\lambda_{0,t}$ of $\mathcal{L}_{s,t}$ is equal to 1, it is a simple eigenvalue, whose eigenvector gives the invariant density ρ_t of f_t , and the rest of the spectrum is contained in a disc of strictly smaller radius. In addition, the essential spectral radius $\theta_{s,t}$ of $\mathcal{L}_{s,t}$ satisfies $\sup_{|t| < \epsilon_3, |s| < \epsilon_3} \theta_{s,t} < \Theta$, and for each $|t| < \epsilon_3$ the spectral radius $\lambda_{s,t} > \Theta$ of $\mathcal{L}_{s,t}$ is an analytic function [8, Prop. 4.2] of s .

Note that $\lambda_{s,t}$ is the exponential of the topological pressure of $s\psi - \log |f_t'|$ for f_t , and that $\rho_t dx$ is the equilibrium state for f_t and $-\log |f_t'|$. Now, perturbation theory gives (see [8, (5.2)])

$$(7) \quad \partial_s \log \lambda_{s,t}|_{s=0} = \int \psi \rho_t dx.$$

Keller and Nowicki also show [8, Theorem 2.2] that for $|t| < \epsilon_3$ and $|s| < \epsilon_3$ the power series $\zeta(s, t, z)$ defined by (6) extends meromorphically to the disc of radius Θ^{-1} , and its poles z_k in this disc are in bijection with the eigenvalues λ_k of $\mathcal{L}_{s,t}$, via $\lambda_k = z_k^{-1}$. In addition, the order of the pole coincides with the algebraic multiplicity of the eigenvalue. By [8, Prop. 4.3 and Lemma 4.5] $\zeta(s, t, z)$ does not vanish in the disc of radius Θ^{-1} . It follows that $z \mapsto \zeta(s, t, z)^{-1}$ is holomorphic in the disc of radius Θ^{-1} . This disc contains $\lambda_{s,t}^{-1}$, which is a simple zero.

To end the proof, recalling (7), it suffices to see that $(s, t) \mapsto \lambda_{s,t}$ is real analytic, but this easily follows from Shiffman's [18] real analytic Hartogs' theorem (see Appendix B or [7, Thm p. 589]) applied to $d(s, t, z) = \zeta(s, t, z)^{-1}$, which implies

that for each $(s, t) \in (-\epsilon_3, \epsilon_3) \times (-\epsilon_3, \epsilon_3)$ the map $z \mapsto d(s, t, z)$ is holomorphic in $|z| < \Theta^{-1}$. Indeed, by the implicit function theorem, the simple zeroes of $d(s, t, \cdot)$ depend real analytically on s and t .

We used the same ϵ_i discs for the s and t variables, but a more careful analysis shows that ϵ in the statement of the theorem may be selected independently of ψ . \square

APPENDIX A. UNIFORMITY OF THE HYPERBOLICITY CONSTANTS

Duncan Sands' explanations were instrumental towards writing this appendix, and we thank him for that.

We start with a preliminary observation: Let g be an S -unimodal Collet–Eckman map (with $g''(0) < 0$, say). Denote by $\lambda_c(g)$, $\lambda_{per}(g)$, and $\lambda_\eta(g)$ the constants defined by (1, 2, 3) (replacing f_t by g). Nowicki and Sands [13] proved that if g is an S -unimodal map and $\lambda_{per}(g) > 1$ then $\lambda_c(g) > 1$. A careful study of their proof shows that $\lambda_c(g) > \lambda_{per}(g)^\alpha$, where the exponent $\alpha > 0$ only depends on the maximum length $N(g)$ of “almost-parabolic funnels” of g (see [13, Lemma 6.6] for a definition of $N(g)$, which can be bounded by a function of $1/\log(\lambda_{per}(g))$ and $\sup|g'|$). Since $N(g)$ is in fact invariant under topological conjugacy and f_t is topologically conjugated to f_0 , we conclude that $\lambda_c(f_t) > \lambda_{per}(f_t)^\alpha$, with $\alpha > 0$ uniform in small t .

Next, recall that Nowicki and Przytycki [12] proved that if g and \tilde{g} are S -unimodal maps, with $g''(c) \neq 0$ and $\tilde{g}''(c) \neq 0$, say, conjugated by a homeomorphism of the interval and g is Collet–Eckmann, then \tilde{g} is Collet–Eckmann. Take $g = f_0$ and $\tilde{g} = f_t$. In particular, f_t is C^2 close to f_0 and $t \mapsto h_t$ is smooth. Then it is not very difficult to see that the constants $M = M(f_t) > 0$, $P_4 = P_4(f_t) > 0$, and $\delta_4 = \delta_4(f_t) > 0$ from the topological characterisation (“finite criticality”) of Collet–Eckmann in [12, (4) p. 35]) are uniform in small t .

Recall that our assumptions imply $f_t''(c) \neq 0$ for all small t , so that the constant denoted l_c in [12] is $l_c = 2$. Section 2 of [12], and in particular the use of the Koebe principle there, implies that there exists a (universal) function $q : \mathbb{R}_*^+ \times (0, 1) \rightarrow (0, 1)$ with $q(M, 1/4) < 1/2$ for any M (see [12, Lemma 2.2]), and so that $\lambda_{per}(f_t) > (1 - 2q(M(f_t), 1/4))^{-1}$. Therefore, $\lambda_{per}(f_t) > 1$ is uniformly bounded away from 1 for small t . The preliminary observation then implies that $\lambda_c(f_t)$ is also uniformly bounded in t . By [11, Proposition 3.2] (see also [12, p. 35]), this implies a uniform lower bound for $\lambda_\eta(f_t)$. Indeed, in the notations of [11, §3], we have $\lambda_\eta = \lambda_5 = \lambda_4 \geq \lambda_3 = \lambda_1 \geq \sqrt{\lambda_c}$.

APPENDIX B. SHIFFMAN'S REAL ANALYTIC HARTOGS' EXTENSION THEOREM

Theorem B.1. [18] *Let $\delta > 0$ and $0 < r < R$. Assume that*

$$d : (-\delta, \delta)^2 \times \{z \in \mathbb{C} \mid |z| < R\} \rightarrow \mathbb{C}$$

satisfies the following conditions:

- *For each $(s, t) \in (-\delta, \delta)^2$ the map $z \mapsto d(s, t, z)$ is holomorphic in $|z| < R$.*
- *For each $|z| < r$ the map $(s, t) \mapsto d(s, t, z)$ is real analytic in $(-\delta, \delta)^2$.*

Then $d(s, t, z)$ is real analytic on $(-\delta, \delta)^2 \times \{|z| < R\}$.

Note that the above theorem fails if real analyticity is replaced by C^k for $k \leq \infty$.

The theorem holds because $|z| < r$ is not pluripolar in $|z| < R$. Shiffman's result is based on deep work of Siciak [19]

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