SPECTRA OF DIFFERENTIABLE HYPERBOLIC MAPS

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Abstract. This note is about the spectral properties of transfer operators associated to smooth hyperbolic dynamics. In the first two sections, we state our new results [5] relating such spectra with dynamical determinants, first announced at the conference "Traces in Geometry, Number Theory and Quantum Fields" at the Max Planck Institute, Bonn, October 2005. In the last two sections, we give a reader-friendly presentation of some key ideas in our work in the simplest possible settings, including a new proof of a result of Ruelle on expanding endomorphisms. (These last two sections are a revised version of the lecture notes given during the workshop "Resonances and Periodic Orbits: Spectrum and Zeta functions in Quantum and Classical Chaos" at Institut Henri Poincaré, Paris, July 2005.)

1. A brief introduction

For smooth hyperbolic dynamical systems and smooth weights (smooth means C^r for $r > 1$), we announce new results from [5] relating Ruelle transfer operators with dynamical Fredholm determinants and dynamical zeta functions: First we establish bounds for the essential spectral radii of the transfer operator on new spaces of anisotropic distributions (Theorem 2.1 and Lemma 2.4), improving previous results (Theorem 2.5 from [4]), and giving variational expressions for the bounds. Then (Theorem 2.6), we give a new proof of Kitaev's [13] lower bound for the radius of the disc in which the dynamical Fredholm determinant admits a holomorphic extension, and, in addition, we show that the zeroes of the determinant in the corresponding disc are in bijection with the eigenvalues of the transfer operator on our spaces. The proofs are based on elementary Paley-Littlewood analysis in Fourier space, using (and improving) a decomposition of the Fourier space into stable and unstable cones, inspired by [1] and introduced in [4]. To prove the results on the dynamical determinants we introduce in [5] methods based on approximation numbers [15].

In Section 2, we give precise definitions and statements of our new results (proofs will appear elsewhere [5]), recalling also some previous results from [4]. Sections 3 and 4 contain a — hopefully — pedagogical presentation of several key ideas and techniques in the proofs in two simple, but nontrivial, cases (a few steps of the argument are left as exercises for the reader):

In Section 3 we discuss, as a warm-up, transfer operators associated to smooth (C^r for $r > 1$) expanding endomorphisms on a manifold X. The case of expanding, noninvertible, maps is easier than the case of hyperbolic, invertible, maps, because composition by each local inverse branch improves regularity, and a relevant Banach

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space is $C^r(X)$. The bounds on the essential spectral radius together with the connection with the dynamical determinants are well-known (see [17], [18], [11]). We give a new proof of of the bounds of the essential spectral radius (this proof is the only original material in this text). This allows us to recall the basic Paley-Littlewood (or dyadic) decomposition tools that are instrumental in [4] and [5].

In Section 4, we consider the simplest hyperbolic diffeomorphisms, Anosov maps, giving the definition of a Banach space of distributions suitable for the hyperbolic case, and explaining the key steps in the proof of the bounds in [4] and [5] on the essential spectral radius of the transfer operators.

2. New results on transfer operators and dynamical determinants

Let X be a d-dimensional C^∞ Riemann manifold, and let $T : X \to X$ be a diffeomorphism which is of class C^r for some $r > 1$. (If r is not an integer, this means that the derivatives of T of order $[r]$ satisfy an $r - [r]$ Hölder condition.) Assume that there exists a hyperbolic basic set $\Lambda \subset X$ for T. This means that Λ is T-invariant, transitive and that there exist a compact neighborhood V of Λ such that $\Lambda = \cap_{m\in\mathbb{Z}} T^m(V)$ and an invariant decomposition $T_\Lambda X = E^u \oplus E^s$ (with $E^u \neq 0$ and $E^s \neq 0$) of the tangent bundle over Λ, such that for some constants $C > 0$ and $0 < \lambda_s < 1$, $\nu_u > 1$, we have for all $m \geq 0$ and $x \in \Lambda$

$$\|DT^m|_{E^s}\| \leq C\lambda_s^m \quad \text{and} \quad \|DT^{-m}|_{E^u}\| \leq C\nu_u^{-m} . \tag{1}$$

For $s \geq 0$, let $C^s(V)$ be the set of complex-valued C^s functions on X with support contained in the interior of V. The Ruelle transfer operator associated to the dynamics T and the weight $g \in C^{r-1}(V)$ is defined by

$$\mathcal{L} = \mathcal{L}_{T,g} : C^{r-1}(V) \to C^{r-1}(V), \quad \mathcal{L}\varphi(x) = g(x) \cdot \varphi \circ T(x) .$$

Since T is hyperbolic, $\mathcal{L}$ is not smoothness improving, so that it is in fact not very interesting to let $\mathcal{L}$ act on spaces of smooth functions. One of our goals is to find a space of distributions on V which is not too small (it should contain all C^{r-1} functions) and not too large ($\mathcal{L}$ should be bounded, with some control on the [1] essential spectral radius, guaranteeing in particular that $\mathcal{L}$ is quasicompact, and that $\mathcal{L}$ has a spectral gap when g is strictly positive on V). In other words, we are aiming at yet another avatar of the Ruelle-Perron-Frobenius theory in infinite dimension. (See e.g. [2] for more classical examples.)

Our latest result in this direction improves the bounds of [4] and [10] on the spectrum of $\mathcal{L}$ (We refer to the introduction of [4] for historical comments and references to the previous works, [6], [3], and in particular the important paper of Gouëzel and Liverani [10].) To state it, we need some notation (see [22] for background on ergodic theory). For a T-invariant Borel probability measure μ on Λ, we write h_μ for the metric entropy of (μ, T), and $\chi_\mu(A) \in \mathbb{R} \cup \{-\infty\}$ for the largest Lyapunov exponent of a linear cocycle A over $T|_\Lambda$, with $(\log \|A\|)^+ \in L^1(d\mu)$. Let $\mathcal{M}(\Lambda, T)$ denote the set of T-invariant ergodic Borel probability measures on Λ.

Theorem 2.1 (Bounds on the essential spectral radius [5])**.** *Let $r > 1$, T, and $\Lambda \subset V$ be as above. For any real numbers $q < 0 < p$ so that $p - q < r - 1$, there exists a Banach space $\mathcal{C}^{p,q}(T, V)$ of distributions on V, containing $C^s(V)$ for any $s > p$, and contained in the dual space of $C^s(V)$ for any $s > |q|$, with the following properties:*

[1] See § 3 for a definition of the essential spectral radius.

For any $g \in C^{r-1}(V)$, the Ruelle operator $\mathcal{L}_{T,g}$ extends to a bounded operator on $\mathcal{C}^{p,q}(T,V)$. Its essential spectral radius on this space is not larger than

$$Q^{p,q}(T,g) =$$
$$\exp \sup_{\mu \in \mathcal{M}(\Lambda,T)} \Big\{ h_\mu + \chi_\mu \left(\frac{g}{\det(DT|_{E^u})} \right) + \max\big\{ p\chi_\mu(DT|_{E^s}), |q|\chi_\mu(DT^{-1}|_{E^u}) \big\} \Big\} .$$

See [7, §8]–[11] for a variational expression analogous to $Q^{p,q}(T,g)$ in the setting of C^r expanding endomorphisms. Note that $\chi_\mu(g/\det(DT|_{E^u})) = \int \log|g|\, d\mu - \int \log|\det(DT|_{E^u})|\, d\mu$, but the expression as a Lyapunov exponent is useful when g is replaced by a bundle automorphism (see [5]).

Remark 2.2 (Decay of correlations). Assume for a moment that Λ is attracting for T, i.e., $T(V) \subset \text{interior}(V)$. Once we have the estimates in Theorem 2.1, it is not difficult to see that the spectral radius of the pull-back operator $T^*\varphi = \varphi \circ T$ on $\mathcal{C}^{p,q}_*(T,V)$ is equal to one. (The constant function is a fixed function.) If (T,Λ) is in addition topologically mixing, then 1 is the unique eigenvalue on the unit circle, it is a simple eigenvalue, and the fixed vector of the dual operator to T^* gives rise to the SRB measure μ: This corresponds to exponential decay of correlations for C^s observables and μ, for $s > p$. (See Blank–Keller–Liverani [6, §3.2] for example.)

Remark 2.3 (Spectral stability). It is not difficult to see that there is $\epsilon > 0$ so that if $\widetilde{T}$ and $\widetilde{g}$, respectively, are ϵ-close to T and g, respectively, in the C^r, resp. C^{r-1}, topology, then the associated operator $\mathcal{L}_{\widetilde{T},\widetilde{g}}$ has the same spectral properties than $\mathcal{L}_{T,g}$ on *the same Banach spaces.* Spectral stability can then be proved, as it has been done in [6] or [10] for the norms defined there.

We next give an alternative expression for $Q^{p,q}(T,g)$. If $g \in C^0(V)$, we write

$$g^{(m)}(x) = \prod_{k=0}^{m-1} g(T^k(x)) , \qquad \forall m \in \mathbb{Z}_+ .$$

Put $\lambda = \max\{\lambda_s, \nu_u^{-1}\}$. We define local hyperbolicity exponents for $x \in \Lambda$ and $m \in \mathbb{Z}_+$ by

$$\begin{aligned} \lambda_x(T^m) &= \sup_{v \in E^s(x)\setminus\{0\}} \frac{\|DT^m_x(v)\|}{\|v\|} \le C\lambda^m \quad \text{and} \\ \nu_x(T^m) &= \inf_{v \in E^u(x)\setminus\{0\}} \frac{\|DT^m_x(v)\|}{\|v\|} \ge C^{-1}\lambda^{-m}. \end{aligned} \tag{2}$$

For arbitrary real numbers q, p and integer $m \ge 1$, set for $x \in \Lambda$

$$\lambda^{(p,q,m)}(x) = \max\big\{ (\lambda_x(T^m))^p, (\nu_x(T^m))^q \big\} . \tag{3}$$

We may extend $E^s(x)$ and $E^u(x)$ to continuous bundles on V (which are not invariant in general), so that the inequalities (1) hold for $x \in \cap_{k=0}^{m-1} T^{-k}(V)$, and for all $m \ge 0$, with some constant C. We may thus extend the definition of $\lambda_x(T^m)$, $\nu_x(T^m)$ and $\lambda^{(p,q,m)}(x)$ to $\cap_{k=0}^{m-1} T^{-k}(V)$. Letting dx denote Lebesgue measure on X, define for integers $m \ge 1$, and p, $q \in \mathbb{R}$

$$\rho^{p,q}(T,g,m) = \int_X |g^{(m)}(x)| \lambda^{(p,q,m)}(x)\, dx . \tag{4}$$

In [5], we also show:

Lemma 2.4. *For $r > 1$, T, and V as above, $g \in C^\delta(V)$ for some $\delta > 0$, and real numbers $q \le 0 \le p$, the limit $\rho^{p,q}(T,g) = \lim_{m\to\infty}(\rho^{p,q}(T,g,m))^{1/m}$ exists, and we have $Q^{p,q}(T,g) = \rho^{p,q}(T,g)$.*

Kitaev [13] proved existence of the limit $\rho^{p,q}(T,g)$, and showed that it gave a lower bound for the domain of holomorphic extension of a dynamical determinant (see also our Theorem 2.6 below).

Next, we compare Theorem 2.1 to our previous results, using the above lemma. (It is convenient to put $a^{1/\infty} = 1$ for $a \in \mathbb{R}^*_+$.) In [4], we proved:

Theorem 2.5. *Let $r > 1$, T, and $\Lambda \subset V$ be as above. For any real numbers $q < 0 < p$ so that $p - q < r - 1$, there exist a Banach space $\mathcal{C}^{p,q}_*(T,V)$ of distributions on V, and for each $1 < t < \infty$, a Banach space $W^{p,q,t}(T,V)$ of distributions on V, with the following properties:*

$\mathcal{C}^{p,q}_*(T,V)$ and $W^{p,q,t}(T,V)$ both contain $C^s(V)$ for any $s > p$, and they both are contained in the dual space of $C^s(V)$ for any $s > |q|$.

For any $g \in C^{r-1}(V)$, the operator $\mathcal{L}_{T,g}$ extends boundedly to $\mathcal{C}^{p,q}_*(T,V)$, with essential spectral radius not larger than $R^{p,q,\infty}(T,g)$, and $\mathcal{L}_{T,g}$ extends boundedly to $W^{p,q,t}(V)$ with essential spectral radius not larger than $R^{p,q,t}(T,g)$, with

$$R^{p,q,t}(T,g) = \lim_{m\to\infty}\left(\sup_\Lambda |\det DT^m|^{-1/t}|g^{(m)}(x)|\lambda^{(p,q,m)}(x)\right)^{1/m}.$$

Since $\rho^{p,q}(T,g) \le \inf_{1<t\le\infty} R^{p,q,t}(T,g)$, and the inequality can be strict, Theorem 2.1 can be viewed as an improvement of Theorem 2.5. Note however that (versions of) the anisotropic Sobolev spaces $W^{p,q,t}(T,V)$ have applications to situations with less hyperbolicity, such as time-one maps of expanding semi-flows [21].

We next turn to dynamical Fredholm determinants. The dynamical Fredholm determinant $d_{\mathcal{L}}(z)$ corresponding to the Ruelle transfer operator $\mathcal{L} = \mathcal{L}_{T,g}$ is

$$d_{\mathcal{L}}(z) = \exp\left(-\sum_{m=1}^\infty \frac{z^m}{m}\sum_{T^m(x)=x}\frac{g^{(m)}(x)}{|\det(1 - DT^m(x))|}\right). \tag{5}$$

The power series in z which is exponentiated converges only if $|z|$ is sufficiently small. The main new result in [5] is about the analytic continuation of $d_{\mathcal{L}}(z)$:

Theorem 2.6. *Let $r > 1$, T, V, and $g \in C^{r-1}(V)$ be as above.*
The function $d_{\mathcal{L}}(z)$ extends holomorphically to the disc of radius $(\rho_r(T,g))^{-1}$ with

$$\rho_r(T,g) = \inf_{q<0<p,\, p-q<r-1}\rho^{p,q}(T,g).$$

For any real numbers $q < 0 < p$ so that $p - q < r - 1$, and each z with $|z| < (\rho^{p,q}(T,g))^{-1}$, we have $d_{\mathcal{L}}(z) = 0$ if and only if $1/z$ is an eigenvalue of $\mathcal{L}$ on $\mathcal{C}^{p,q}(T,V)$, and the order of the zero coincides with the algebraic multiplicity of the eigenvalue.

Remark 2.7. The proof in [5] implies that for any real numbers $q < 0 < p$ so that $p - q < r - 1$, each $1 < t < \infty$, and each z with $|z| < (R^{p,q,t}(T,g))^{-1}$, respectively $|z| < (R^{p,q,\infty}(T,g))^{-1}$, we have $d_{\mathcal{L}}(z) = 0$ if and only if $1/z$ is an eigenvalue of $\mathcal{L}$ on $W^{p,q,t}(T,V)$, respectively $\mathcal{C}^{p,q}_*(T,V)$, and the order of the zero coincides with the algebraic multiplicity of the eigenvalue.

Note that for analytic hyperbolic diffeomorphisms and weights, it has been known for 30 years that $d_{\mathcal{L}}(z)$ is an entire function when the dynamical foliations are analytic [16]. More recently, Rugh and Fried [19, 9] studied $d_{\mathcal{L}}(z)$ in this analytic framework, without any assumption on the foliations, giving a spectral interpretation of its zeroes.

In the case of finite differentiability r, the connection between transfer operators and dynamical determinants has been well understood in the easier setting of expanding endomorphisms since 15 years ago (see [18]). The case of hyperbolic diffeomorphisms has only been studied recently. In an important and pioneering article [13], Kitaev obtained the first claim of our Theorem 2.6, without the spectral interpretation of the zeroes of $d_{\mathcal{L}}(z)$. Our proof is different and gives the spectral interpretation of the zeroes of $d_{\mathcal{L}}(z)$ contained in the second claim of Theorem 2.6. Note that a spectral interpretation of the zeroes (in a smaller disc, depending on the dimension d) has been obtained previously by Liverani [14], using Banach spaces of [10].

We refer to [5] for the proof of Theorem 2.6.

3. A toy model: expanding endomorphisms

In order to give in the next section the key ideas in the proof of Theorem 2.5 in [4] and Theorems 2.1 and 2.6 in [5], we revisit in this section the much easier (and well understood) situation of expanding endomorphisms. We first recall a definition and a few elementary facts, which will also be used in Section 4.

Definition (Essential spectral radius)**.** The essential spectral radius $r_{ess}(\mathcal{L}|_{\mathcal{B}})$ of a bounded operator $\mathcal{L}$ on a Banach space $\mathcal{B}$ is the infimum of the real numbers $\rho > 0$ so that, outside of the disc of radius ρ, the spectrum of $\mathcal{L}$ on $\mathcal{B}$ consists of isolated eigenvalues of finite multiplicity.

The following basic fact will be at the very center of our proof (it is behind most techniques to estimate the essential spectral radius: Lasota-Yorke or Doeblin-Fortet bounds, Hennion's theorem, the Nussbaum formula, see e.g. [2]):

Compact perturbation. If $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_0$ where $\mathcal{L}_1$ is compact on $\mathcal{B}$ and $\mathcal{L}_0$ is bounded on $\mathcal{B}$, then the essential spectral radius of $\mathcal{L}$ acting on $\mathcal{B}$ is not larger than the spectral radius of $\mathcal{L}_0$ acting on $\mathcal{B}$. (See e.g. [8].)

Not surprisingly, our main tool is integration by parts:

Integration by parts. By "integration by parts on w," we will mean application, for $f \in C^2(\mathbb{R}^d)$ with $\sum_{j=1}^d (\partial_j f(w))^2 \neq 0$ and a compactly supported $g \in C^1(\mathbb{R}^d)$, of the formula

$$
\begin{aligned}
(6) \qquad \int e^{if(w)} g(w) dw &= -\sum_{k=1}^{d} \int i(\partial_k f(w)) e^{if(w)} \cdot \frac{i(\partial_k f(w)) \cdot g(w)}{\sum_{j=1}^d (\partial_j f(w))^2} dw \\
&= i \cdot \int e^{if(w)} \cdot \sum_{k=1}^{d} \partial_k \left(\frac{\partial_k f(w) \cdot g(w)}{\sum_{j=1}^d (\partial_j f(w))^2} \right) dw \,,
\end{aligned}
$$

where $w = (w_k)_{k=1}^d \in \mathbb{R}^d$, and ∂_k denotes partial differentiation with respect to w_k. (Note that if f is C^r we can only integrate by parts $[r] - 1$ times in the above sense, even if g is C^r and compactly supported.)

Regularised integration by parts If $f \in C^{1+\delta}(\mathbb{R}^d)$ and $g \in C_0^\delta(\mathbb{R}^d)$, for $\delta \in (0,1)$, and $\sum_{j=1}^d (\partial_j f)^2 \neq 0$ on $\mathrm{supp}(g)$, we shall consider the following "regularised integration by parts:" Set, for $k = 1, \ldots, d$

$$h_k := \frac{i(\partial_k f(w)) \cdot g(w)}{\sum_{j=1}^d (\partial_j f(w))^2}.$$

Each h_k belongs to $C_0^\delta(\mathbb{R}^d)$. Let $h_{k,\epsilon}$, for small $\epsilon > 0$, be the convolution of h_k with $\epsilon^{-d} v(x/\epsilon)$, where the C^∞ function $v : \mathbb{R}^d \to \mathbb{R}_+$ is supported in the unit ball and satisfies $\int v(x)dx = 1$. There is C, independent of f and g, so that for each small $\epsilon > 0$ and all k,

$$\|\partial_k h_{k,\epsilon}\|_{L^\infty} \leq C\|h_k\|_{C^\delta}\epsilon^{\delta-1}, \quad \|h_k - h_{k,\epsilon}\|_{L^\infty} \leq C\|h_k\|_{C^\delta}\epsilon^\delta.$$

Finally, for every real number $\Lambda \geq 1$

$$\begin{aligned}(7) \qquad \int e^{i\Lambda f(w)} g(w)dw &= -\sum_{k=1}^d \int i\partial_k f(w) e^{i\Lambda f(w)} \cdot h_k(w)dw \\ &= \int \frac{e^{i\Lambda f(w)}}{\Lambda} \cdot \sum_{k=1}^d \partial_k h_{k,\epsilon}(w)dw \\ &\quad - \sum_{k=1}^d \int i\partial_k f(w) e^{i\Lambda f(w)} \cdot (h_k(w) - h_{k,\epsilon}(w))dw.\end{aligned}$$

3.1. **The result for locally expanding maps.** Let $T : X \to X$ be C^r for $r > 1$, where X is a d-dimensional compact manifold. In this section, we assume that T is a locally expanding map, i.e., there are $C > 0$ and $\lambda_s < 1$ so that for each x, all $m \geq 1$ and all $v \in T_x X$, we have $\|D_x T^m v\| \geq C\lambda_s^{-m}\|v\|$. The function g is assumed to be C^r. We study the operator

$$\mathcal{L}_{T^{-1},g} u(x) := \sum_{y:T(y)=x} g(y)u(y)\,.$$

(This is the transfer operator associated to the branches of T^{-1}, which contract by at least λ_s.) Note that

$$R(T^{-1}, g) := \lim_{m\to\infty} \Big(\sup_x \sum_{y:T^m(y)=x} |g^{(m)}(x)|\Big)^{1/m}$$

is the spectral radius of $\mathcal{L}_{T^{-1},g}$ acting on continuous functions. [2]

For $p > 0$, recall that the C^p norm of $u \in C^\infty(\mathbb{R}^d)$ is

$$\|u\|_{C^p} = \max\left\{\max_{|\alpha|\leq[p]} \sup_{x\in\mathbb{R}^d} |\partial^\alpha u(x)|, \max_{|\alpha|=[p]} \sup_{x\in\mathbb{R}^d} \sup_{y\in\mathbb{R}^d/\{0\}} \frac{|\partial^\alpha u(x+y) - \partial^\alpha u(x)|}{\|y\|^{p-[p]}}\right\}$$

where $\partial^\alpha u$ for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_+$ denotes the partial derivative $\partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} u$, and $|\alpha| = \sum_{j=1}^d \alpha_j$. For $\varphi \in C^p(X)$, the above norm can be used in charts to define a norm $\|\varphi\|_{C^p(X)}$.

We shall prove the following result:

[2] If $g = |\det DT|^{-1}$ then it is well-known that $R(T^{-1}, g) = 1$.

Theorem 3.1 (Essential spectral radius for expanding maps)**.** *Let T be C^r and expanding, and let g be C^r for $r > 1$. For any noninteger $0 < p \le r$, let $C^p_*(X)$ be the closure of $C^\infty(X)$ for the C^p norm. Then the operator $\mathcal{L}_{T^{-1},g}$ is bounded on $C^p_*(X)$ and*

$$r_{ess}(\mathcal{L}_{T^{-1},g}|_{C^p_*(X)}) \le R(T^{-1}, g) \cdot \lambda_s^p \, .$$

The main interest of the proof given here is that it can be generalised to the hyperbolic case. Note also that if p is an integer, the proof below gives the same bounds for a Zygmund space $C^p_*(X)$.

Exercise 3.2. Prove that for any noninteger $p > 0$ we have $C^p_*(X) \subset C^p(X)$, and that the inclusion is strict.

Ruelle [17] proved the statement of Theorem 3.1 for $C^p(X)$ instead of $C^p_*(X)$. (It is in fact possible to modify the definitions in Subsection 3.2, to get a new proof of Ruelle's result. This modification is cumbersome when dealing with distributions in the later sections, and we do not present it here. Adapting the argument in [5] to the case of expanding endomorphisms in the spirit of this section, it is even possible to recover the optimal bounds in [11].)

3.2. **Local definition of Hölder norms in Fourier coordinates.** We present here the "dyadic decomposition" approach to compactly supported Hölder functions in $\mathbb{R}^d$ (for $d \ge 1$). Fix a C^∞ function $\chi : \mathbb{R}_+ \to [0,1]$ with

$$\chi(s) = 1, \quad \text{for } s \le 1, \qquad \chi(s) = 0, \quad \text{for } s \ge 2.$$

Define $\psi_n : \mathbb{R}^d \to [0,1]$ for $n \in \mathbb{Z}_+$, by $\psi_0(\xi) = \chi(\|\xi\|)$, and

$$\psi_n(\xi) = \chi(2^{-n}\|\xi\|) - \chi(2^{-n+1}\|\xi\|) \, , \quad n \ge 1 \, .$$

We have $1 = \sum_{n=0}^\infty \psi_n(\xi)$, and $\mathrm{supp}(\psi_n) \subset \{\xi \mid 2^{n-1} \le \|\xi\| \le 2^{n+1}\}$ for $n \ge 1$. Also $\psi_n(\xi) = \psi_1(2^{-n+1}\xi)$ for $n \ge 1$. Thus, for every multi-index α, there exists a constant C_α such that $\|\partial^\alpha \psi_n\|_{L^\infty} \le C_\alpha 2^{-n|\alpha|}$ for all $n \ge 0$, and the inverse Fourier transform of ψ_n,

$$\widehat{\psi}_n(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\xi} \psi_n(\xi) d\xi \, , \quad x \in \mathbb{R}^d \, ,$$

decays rapidly in the sense of Schwartz. Furthermore we have

$$\widehat{\psi}_n(x) = 2^{d(n-1)} \widehat{\psi}_1(2^{n-1}x)$$

for $n \ge 1$ and all x, and

$$\sup_n \int_{\mathbb{R}^d} |\widehat{\psi}_n(x)| dx < \infty \, . \tag{8}$$

Exercise 3.3. Prove the above claims on ψ_n and $\widehat{\psi}_n$.

Fix a compact subset $K \subset \mathbb{R}^d$ with non-empty interior and let $C^\infty(K)$ be the space of complex-valued C^∞ functions on $\mathbb{R}^d$ supported on K. Decompose each $u \in C^\infty(K)$ as $u = \sum_{n \ge 0} u_n$, by defining for $n \in \mathbb{Z}_+$ and $x \in \mathbb{R}^d$

$$u_n(x) = \psi_n(D)u(x) := (2\pi)^{-d} \int_K \int_{\mathbb{R}^d} e^{i(x-y)\xi} \psi_n(\xi) u(y) dy d\xi \, . \tag{9}$$

Note that u_n is not necessarily supported in K, although it satisfies good decay properties when $\|x\| \to \infty$: we say that the operator $\psi_n(D)$ is not a "local" operator, but it is "pseudo-local." (See [4]. The pseudo-local estimates there are useful e.g. to show the compactness results in Propositions 3.6 and 4.4.)

Remark 3.4. The notation $a(D)$ for the operator sending a compactly supported $u \in C^\infty(\mathbb{R}^d)$ to

$$a(D)u(x) := (2\pi)^{-d} \int_K \int_{\mathbb{R}^d} e^{i(x-y)\xi} a(\xi) u(y) dy d\xi = (\widehat{a} * u)(x)\,,$$

associated to $a \in C^\infty(\mathbb{R}^d)$ so that $\partial^\alpha a(\xi) \le C_\alpha(a) \|\xi\|^{-|\alpha|}$ for each multi-index α, stands for the "pseudo-differential operator associated to the symbol" a. We shall not need any knowledge about pseudodifferential operators, and shall not require symbols depending on both x and ξ.

Definition (Little hölder space $C^p_*(K)$)**.** For a real number $p > 0$, define on $C^\infty(K)$ the norm

$$\|u\|_{C^p_*} = \sup_{n \ge 0} 2^{pn} \|u_n\|_{L^\infty(\mathbb{R}^d)}\,.$$

The space $C^p_*(K)$ is the completion of $C^\infty(K)$ with respect to $\|\cdot\|_{C^p_*}$.

Remark 3.5. It is known that if p is not an integer then the norm $\|u\|_{C^p_*}$ is equivalent to the C^p norm. (See [20, Appendix A].)

We shall not give a proof of the following, very standard, result (the proof is based on the Ascoli-Arzelà lemma; see Proposition 4.4 for an anisotropic analogue):

Proposition 3.6 (Compact embeddings)**.** *If $0 < p' < p$ the inclusion $C^p_*(K) \subset C^{p'}_*(K)$ is compact.*

3.3. **Compact approximation for local maps.** Let $r > 1$. Let $K, K' \subset \mathbb{R}^d$ be compact subsets with non-empty interiors, and take a compact neighbourhood W of K. Let $\mathcal{T} : W \to K'$ be a C^r diffeomorphism onto its image (the reader should think of $\mathcal{T}$ as being a local inverse branch of an expanding map T, in charts). Let $\gamma : \mathbb{R}^d \to \mathbb{C}$ be a C^{r-1} function supported in the interior of K. In this section we study a local transfer operator:

$$L : C^{r-1}(K') \to C^{r-1}(K), \qquad Lu(x) = \gamma(x) \cdot u \circ \mathcal{T}(x)\,.$$

We define a "weakest contraction[3]" exponent

$$\|\mathcal{T}\|_+ = \sup_{x \in K} \sup_{\xi \ne 0} \frac{\|D\mathcal{T}^{tr}_x(\xi)\|}{\|\xi\|}\,.$$

The following result is the key to the proof of Theorem 3.1:

Theorem 3.7. *For any real number $p > 0$ such that $p \le r - 1$, and every compact K_0 contained in the interior of K, there is a constant C, so that for each C^r map $\mathcal{T}$ as above and every γ in $C^{r-1}(K_0)$ there is a compact operator $L_1 : C^p_*(K') \to C^p_*(K)$ such that for any $u \in C^p_*(K')$*

$$\|Lu - L_1 u\|_{C^p_*} \le C \|\gamma\|_{L^\infty} \cdot \|\mathcal{T}\|^p_+ \|u\|_{C^p_*}\,.$$

If $\gamma \in C^r(K_0)$ then the condition on p may be relaxed to $0 < p \le r$.

[3] This is because we will consider contracting maps $\mathcal{T}$ in the application to Theorem 3.1.

We sketch how to deduce Theorem 3.1 from Theorem 3.7: Take a system of local charts $\kappa_i : V_i \to K_i \subset \mathbb{R}^d$, $1 \le i \le k$, and a C^∞ partition of unity $\phi_i : X \to [0,1]$ subordinate to the covering by V_i, that is, the support of ϕ_i is contained in the interior of V_i. (Then, $C^p_*(X)$ is embedded in the direct sum of the local $C^p_*(K_i)$ spaces.) Consider an iterate $\mathcal{L}^m$ of $\mathcal{L}$ and define the operators $\mathcal{L}_{ij} : C^{r-1}_*(X) \to C^{r-1}_*(X)$ by $\mathcal{L}^m_{ij}\varphi(x) = \phi_j \cdot \mathcal{L}^m(\phi_i \varphi)$ so that $\mathcal{L}^m = \sum_{i,j} \mathcal{L}^m_{ij}$. Since the operator $\mathcal{L}^m_{ij}$ may be viewed as an operator in local charts, we may apply Theorem 3.7 to $\mathcal{L}^m_{ij}$, taking $\mathcal{T}$ to be a branch of the inverse of $\kappa_i \circ T^m \circ \kappa_j^{-1}$, and taking γ to be $(\phi_j \cdot \phi_i \circ T^m \cdot g^{(m)}) \circ \kappa_j^{-1} \circ \mathcal{T}$. Then we get that the essential spectral radius of the operator $\mathcal{L}^m = \sum_{i,j} \mathcal{L}^m_{ij}$ is bounded by

$$R_m := C \cdot \lambda_s^{mp} \cdot \left(\sup_{x \in X} \sum_{y : T^m(y) = x} |g^{(m)}(y)| \right),$$

for some constant C independent of m and g. (It is crucial that the constant C in Theorem 3.7 is independent of $\mathcal{T}$ and thus of the iterate m.) Thus the essential spectral radius of $\mathcal{L}$ is bounded by $(R_m)^{1/m}$. Considering large m, we obtain Theorem 3.1.

Proof of Theorem 3.7. We need a couple more notations. Recall the function χ from Section 3.2. Define $\tilde\psi_\ell : \mathbb{R}^d \to [0,1]$ by

$$\tilde\psi_\ell(\xi) = \begin{cases} \chi(2^{-\ell-1}\|\xi\|) - \chi(2^{-\ell+2}\|\xi\|), & \text{if } \ell \ge 1, \\ \chi(2^{-1}\|\xi\|), & \text{if } \ell = 0. \end{cases}$$

Note that $\tilde\psi_\ell(\xi) = 1$ if $\xi \in \mathrm{supp}(\psi_\ell)$.

We write[4]

- $\ell \hookrightarrow n$ if $2^n \le \|\mathcal{T}\|_+ 2^{\ell+4}$,
- $\ell \not\hookrightarrow n$ otherwise.

By the definition of $\not\hookrightarrow$ there exists an integer $N(\mathcal{T}) > 0$ such that

$$\inf_x d(\mathrm{supp}(\psi_n), D\mathcal{T}_x^{tr}(\mathrm{supp}(\tilde\psi_\ell))) \ge 2^{\max\{n,\ell\} - N(\mathcal{T})} \quad \text{if } \ell \not\hookrightarrow n. \tag{10}$$

Let $h : \mathbb{R}^d \to [0,1]$ be a C^∞ function supported in K and $\equiv 1$ on $\mathrm{supp}(\gamma)$. Noting that $L(f)$ is well-defined if $f \in C^\infty(\mathbb{R}^d)$ because γ is supported in K, we may define L_1 and L_0 by $L_j(f) = (M \circ L'_j)(f)$ with $Mf = h \cdot f$, and $L'_j u = \sum_n (L'_j u)_{(n)}$ with

$$(L'_0 u)_{(n)} = \sum_{\ell : \ell \hookrightarrow n} \psi_n(D)(L\, u_\ell),$$

and

$$(L'_1 u)_{(n)} = \sum_{\ell : \ell \not\hookrightarrow n} \psi_n(D)(L\, \tilde\psi_\ell(D) u_\ell).$$

Since $\tilde\psi_\ell(D)u_\ell = u_\ell$ and $h \equiv 1$ on $\mathrm{supp}(g)$, we have $L_0 + L_1 = L$. By Proposition 3.6, it is enough to show the following three bounds: First, there is $C(h)$, which only depends on $\max_{0 \le |\alpha| \le [r]+1} \sup |\partial^\alpha h|$, so that

$$\|Mu\|_{C^p_*} \le C(h) \|u\|_{C^p_*}, \tag{11}$$

[4] By definition, if $\ell \not\hookrightarrow n$ then $n > \ell - n(\mathcal{T})$ for some $n(\mathcal{T})$ depending only on $\mathcal{T}$. This feature will not be present in the hyperbolic case.

second, there is C, which does not depend on $\mathcal{T}$ and γ, so that for each $u \in C^p(K')$

$$\|L_0' u\|_{C_*^p} \le C \|\mathcal{T}\|_+^p \|\gamma\|_{L^\infty} \|u\|_{C_*^p} \,,$$

and finally, for each $0 < p' < p$ there is $C(\mathcal{T}, \gamma)$ so that for each $u \in C^p(K')$

$$\|L_1' u\|_{C_*^p} < C(\mathcal{T}, \gamma) \|u\|_{C_*^{p'}} \,. \tag{12}$$

(Note that if a Banach space $\mathcal{B}_1'$ is compactly included in a Banach space $\mathcal{B}_0'$, then any bounded linear operator from $\mathcal{B}_0'$ to $\mathcal{B}_1$ is compact when restricted to $\mathcal{B}_1'$, using that the composition of a compact operator followed by a bounded operator is compact.)

Notice that there is C (independent of $\mathcal{T}$ and γ) so that

$$\sum_{\ell : \ell \hookrightarrow n} 2^{pn - p\ell} \le 2^{4p} \|\mathcal{T}\|_+^p \sum_{j=0}^{\infty} 2^{-j} \le C \|\mathcal{T}\|_+^p, \ \forall n \,. \tag{13}$$

Also notice that

$$\psi_m(D) \circ \psi_n(D) = (\psi_m \cdot \psi_n)(D) = 0 \text{ when } |m - n| \ge 5 \,. \tag{14}$$

The bound for L_0' is then easy:

$$\begin{aligned}
\|L_0' u\|_{C_*^p} &= \sup_m 2^{pm} \|\psi_m(D) \big(\sum_n (L_0' u)_{(n)} \big) \|_{L^\infty(\mathbb{R}^d)} \\
&\le \sup_m 2^{pm} \sum_{|n-m|<5} \sum_{\ell : \ell \hookrightarrow n} \|\psi_n(D)(L u_\ell)\|_{L^\infty} \\
&\le C \|\gamma\|_{L^\infty} \sup_m 2^{pm} \sum_{|n-m|<5} \sum_{\ell : \ell \hookrightarrow n} \|u_\ell\|_{L^\infty} \\
&\le C \|\gamma\|_{L^\infty} \sup_m \sum_{|n-m|<5} \big(\sum_{\ell : \ell \hookrightarrow n} 2^{pn - p\ell} \big) \|u\|_{C_*^p} \\
&\le C \|\gamma\|_{L^\infty} \|\mathcal{T}\|_+^p \|u\|_{C_*^p} \,.
\end{aligned}$$

We used $\psi_n(D) f = \widehat{\psi}_n * f$, which implies $\|\psi_n(D) f\|_{L^\infty} \le C \|f\|_{L^\infty}$ for all n, by Young's inequality for L^∞.

We will have to work a little harder for L_1'. Assume first that $p \le r - 1$. Then it is enough to prove that for each $f \in C^\infty(\mathbb{R}^d)$ with rapid decay, and all n

$$\|\psi_n(D)(L(\tilde\psi_\ell(D) f))\|_{L^\infty} \le C(\mathcal{T}, \gamma) 2^{-(r-1)\max\{n,\ell\}} \|f\|_{L^\infty} \text{ if } \ell \not\hookrightarrow n. \tag{15}$$

Indeed, using (14) as in the estimate for L_0', the above bound implies that

$$\|L_1' u\|_{C_*^p} \le C(\mathcal{T}, \gamma) \cdot \sup_n \left(\sum_{\ell : \ell \not\hookrightarrow n} 2^{pn - p'\ell - (r-1)\max\{n,\ell\}} \right) \|u\|_{C_*^{p'}} \,, \tag{16}$$

and the conditions $p \le r - 1$ and $p' > 0$ ensure that the supremum over n of the sum over ℓ such that $\ell \not\hookrightarrow n$ above is finite (recall the footnote 4).

To show (15), we note that

$$(\psi_n(D) L \, \tilde\psi_\ell(D) f)(x) = (2\pi)^{-2d} \int_{\mathbb{R}^d} V_n^\ell(x, y) \cdot f \circ \mathcal{T}(y) |\det D\mathcal{T}(y)| dy \,,$$

where we have extended $\mathcal{T}$ to a bilipschitz C^r diffeomorphism of $\mathbb{R}^d$ and

$$V_n^\ell(x, y) = \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} e^{i(x-w)\xi + i(\mathcal{T}(w) - \mathcal{T}(y))\eta} \gamma(w) \psi_n(\xi) \tilde\psi_\ell(\eta) dw d\xi d\eta \,. \tag{17}$$

Since $\|f \circ \mathcal{T} \cdot |\det D\mathcal{T}|\|_{L^\infty} \le C(\mathcal{T})\|f\|_{L^\infty}$, the inequality (15) follows if we show that there exists $C(\mathcal{T}, \gamma)$ such that for all $\ell \not\hookrightarrow n$ the operator norm of the integral operator

$$H_n^\ell : f \mapsto \int_{\mathbb{R}^d} V_n^\ell(x, y) f(y) dy$$

acting on $L^\infty(\mathbb{R}^d)$ is bounded by $C(\mathcal{T}, \gamma) \cdot 2^{-(r-1)\max\{n,\ell\}}$.

Define the integrable function $b : \mathbb{R}^d \to \mathbb{R}_+$ by

$$b(x) = 1 \quad \text{if } \|x\| \le 1, \qquad b(x) = \|x\|^{-d-1} \quad \text{if } \|x\| > 1. \tag{18}$$

The required estimate on H_n^ℓ follows if we show

$$|V_n^\ell(x, y)| \le C(\mathcal{T}, \gamma) 2^{-(r-1)\max\{n,\ell\}} \cdot 2^{d\min\{n,\ell\}} b(2^{\min\{n,\ell\}}(x - y)), \tag{19}$$

for some $C(\mathcal{T}, \gamma) > 0$ and all $\ell \not\hookrightarrow n$. Indeed, as the right hand side of (19) is written as a function of $x - y$, say $B(x - y)$, we have, by Young's inequality in $L^\infty(\mathbb{R}^d)$,

$$\begin{aligned}\|H_n^\ell f\|_{L^\infty} &\le \|B * f\|_{L^\infty} \le \|B\|_{L^1}\|f\|_{L^\infty} \\ &\le C(\mathcal{T}, \gamma) 2^{-(r-1)\max\{n,\ell\}} \cdot \|b\|_{L^1} \cdot \|f\|_{L^\infty}.\end{aligned}$$

(Note that, by Young's inequality for $L^t(\mathbb{R}^d)$ with $1 < t < \infty$, the operator H_n^ℓ acting on each $L^t(\mathbb{R}^d)$ is also bounded by $C(\mathcal{T}, \gamma) \cdot 2^{-(r-1)\max\{n,\ell\}}$. This is useful to control the essential spectral radius on anisotropic Sobolev spaces, see [4].)

We now prove (19). If $r \ge 2$ (otherwise we do nothing at this stage), integrating (17) by parts $[r] - 1$ times on w (recall (6)), we obtain

$$V_n^\ell(x, y) = \int e^{i(x-w)\xi + i(\mathcal{T}(w) - \mathcal{T}(y))\eta} F(\xi, \eta, w) \psi_n(\xi) \tilde{\psi}_\ell(\eta) dw d\xi d\eta, \tag{20}$$

where $F(\xi, \eta, w)$ is a $C^{r-[r]}$ function in w which is C^∞ in the variables ξ and η. The following exercise is an important (but straightforward) step in the proof:

Exercise 3.8. Using (10), check that if $\psi_n(\xi) \cdot \tilde{\psi}_\ell(\eta) \ne 0$ then

$$\|F(\xi, \eta, \cdot)\|_{C^{r-[r]}} \le C(\mathcal{T}, \gamma) 2^{-([r]-1)\max\{n,\ell\}}. \tag{21}$$

The estimate (21) looks promising, but applying it naively is not enough: since we are integrating over ξ in the support of ψ_n and over η in the support of $\tilde{\psi}_\ell$, we would get an additional factor $2^{dn+d\ell}$. In order to get rid of this factor, we shall use another exercise:

Exercise 3.9. Using (10), show that if $\psi_n(\xi) \cdot \tilde{\psi}_\ell(\eta) \ne 0$, then for all multi-indices α and β

$$\|\partial_\xi^\alpha \partial_\eta^\beta F(\xi, \eta, \cdot)\|_{C^{r-[r]}} \le C_{\alpha,\beta}(\mathcal{T}, \gamma) 2^{-n|\alpha| - \ell|\beta| - ([r]-1)\max\{n,\ell\}}. \tag{22}$$

Assume first that r is an integer (then, $r = [r] \ge 2$). Put

$$G_{n,\ell}(\xi, \eta, w) = F(\xi, \eta, w) \psi_n(\xi) \tilde{\psi}_\ell(\eta)$$

and consider the scaling $\widetilde{G}_{n,\ell}(\xi, \eta, w) = G_{n,\ell}(2^n \xi, 2^\ell \eta, w)$.

The estimate (22) implies that for all α and β

$$\|\partial_\xi^\alpha \partial_\eta^\beta \widetilde{G}_{n,\ell}(\xi, \eta, \cdot)\|_{C^{r-[r]}} \le C_{\alpha,\beta}(\mathcal{T}, \gamma) 2^{-([r]-1)\max\{n,\ell\}}, \forall \xi, \eta, n, \ell. \tag{23}$$

Then, denoting by $\mathcal{F}$ the inverse Fourier transform with respect to the variable (ξ, η), and setting $W_n^\ell(u,v,w) :=$

$$(\mathcal{F}\widetilde{G}_{n,\ell})(u,v,w) = (2\pi)^{-2d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{iu\xi} e^{iv\eta} \widetilde{G}_{n,\ell}(\xi,\eta,w)\, d\xi d\eta\,, \tag{24}$$

the bounds (23) imply that for any nonnegative integers k and k'

$$\big\| \|u\|^k \|v\|^{k'} W_n^\ell(u,v,\cdot)\big\|_{L^\infty} \le \widetilde{C}_{k,k'}(\mathcal{T},\gamma) 2^{-([r]-1)\max\{n,\ell\}}\,, \forall u,v,n,\ell\,. \tag{25}$$

(Just note that the integrand in (24) is supported in $\max\{\|\xi\|, \|\eta\|\} \le 2$, and integrate by parts with respect to ξ and η as many times as desired.) Applying (25) to k, k' in $\{0, d+1\}$, we get $C(\mathcal{T},\gamma)$ so that for each $w \in K$, and all n, ℓ, u, v

$$|W_n^\ell(u,v,w)| \le C(\mathcal{T},\gamma) 2^{-([r]-1)\max\{n,\ell\}} b(u) b(v)\,. \tag{26}$$

(For $w \notin K$ we have $W_n^\ell(u,v,w) = 0$ for all u, v, n, ℓ.) Therefore, since

$$(\mathcal{F}G_{n,\ell})(u,v,w) = 2^{dn+d\ell} W_n^\ell(2^n u, 2^\ell v, w)\,,$$

we get by definition,

$$\begin{aligned}
|V_n^\ell(x,y)| &\le \int_K |(\mathcal{F}G_{n,\ell})(x-w, \mathcal{T}(w)-\mathcal{T}(y), w)|\, dw\\
&\le C \int_K 2^{dn+d\ell} |W_n^\ell(2^n(x-w), 2^\ell(\mathcal{T}(w)-\mathcal{T}(y)), w)|\, dw\\
&\le C(\mathcal{T},\gamma) 2^{-([r]-1)\max\{n,\ell\}+dn+d\ell} \int_K b(2^n(x-w)) b(2^\ell(\mathcal{T}(w)-\mathcal{T}(y)) dw\,.
\end{aligned}$$

Next, using $u = 2^n(x-w)$, note $w_u = x - 2^{-n}u$, and write

$$\begin{aligned}
&\int_K 2^{dn+d\ell} b(2^n(x-w)) b(2^\ell(\mathcal{T}(w)-\mathcal{T}(y)))\, dw\\
&\qquad = \int_{\mathbb{R}^d} 2^{d\ell} b(u) b(2^\ell(\mathcal{T}(w_u)-\mathcal{T}(y)))\, du\,.
\end{aligned} \tag{27}$$

Since $\ell \le n + N(\mathcal{T})$ (see footnote 4), we get by using

$$\int b(u) b(2^\ell(\mathcal{T}(w_u)-\mathcal{T}(y)))\, du \le \int b(u)\, du < \infty\,, \tag{28}$$

that $|V_n^\ell(x,y)| \le C(\mathcal{T},\gamma) 2^{d\min\{n,\ell\}-([r]-1)\max\{n,\ell\}}$.

If $\|x-y\| > 2^{-\min\{n,\ell\}}$, we can improve the estimate: let $q_0 \le \min\{\ell, n\}$ be the integer so that $\|x-y\| \in [2^{-q_0}, 2^{-q_0+1})$. Taking large constants $C(\mathcal{T})$, we may assume that for each $u \in \mathbb{R}^d$ either of the following conditions holds:

$$\begin{aligned}
&\|u\| = 2^n \|x - w_u\| \ge 2^{-C(\mathcal{T})+n-q_0} \ge 2^{-C(\mathcal{T})+\ell-q_0}\\
&2^\ell \|\mathcal{T}(w_u) - \mathcal{T}(y)\| > 2^{-C(\mathcal{T})+\ell} \|w_u - y\| > 2^{-C(\mathcal{T})+\ell-q_0}\,.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\int b(u) b(2^\ell(\mathcal{T}(w_u)-\mathcal{T}(y)))\, du\\
&\qquad \le 2^{C(\mathcal{T})-(n-q_0)(d+1)} \int b(2^\ell(\mathcal{T}(w_u)-\mathcal{T}(y)))\, du\\
&\qquad\qquad\qquad + 2^{C(\mathcal{T})-(\ell-q_0)(d+1)} \int b(u)\, du\\
&\qquad \le 2^{C(\mathcal{T})-(d+1)(\ell-q_0)}.
\end{aligned}$$

With this, we conclude

$$|V_n^\ell(x,y)| \le C(\mathcal{T},\gamma) 2^{d\min\{n,\ell\}-([r]-1)\max\{n,\ell\}} 2^{(d+1)(q_0-\min\{n,\ell\})},$$

proving (19) for integer r .

If $r>1$ is not an integer, we start from (20) and rewrite $V_n^\ell(x,y)$ as

$$\int e^{i\Lambda(x-w)(\xi/\Lambda)+i\Lambda(\mathcal{T}(w)-\mathcal{T}(y))(\eta/\Lambda)} F(\xi,\eta,w)\psi_n(\xi)\tilde\psi_\ell(\eta)\,dw d\xi d\eta, \tag{29}$$

for $\Lambda = 2^{\max\{\ell,n\}}$. Recalling (7), we apply to (29) one regularised integration by parts for $\delta = r-[r]$ (noting that $\mathcal{T}$ is $C^{1+\delta}$). We get two terms $F_{1,\epsilon}(\xi,\eta,w)$ and $F_{2,\epsilon}(\xi,\eta,w)$. Choosing $\epsilon = \Lambda^{-1}$, we may apply the above procedure to each of them. The proof when γ is C^r and $r-1<p\le r$ is done in Appendix A.

It only remains to check (11) for the multiplication operator $Mu = hu$. Since $\mathcal{T}$ is replaced by the identity map which satisfies $\|\mathrm{id}\|_+ \le 1$, and γ is replaced by h, this can be done by a simplification of the above arguments for L_0' and L_1' (we can take $p'=p$), decomposing $M = M_0 + M_1$ according to the relation $\hookrightarrow$ associated to id. We leave details as an exercise for the reader. □

4. Bounding the essential spectral radius in the Anosov case

We now move to hyperbolic situations. We take $r>1$ and X a compact Riemann manifold, and assume that $T: X\to X$ is a C^r Anosov diffeomorphism. Recall that this means that $\Lambda = X$ is a hyperbolic set for T in the sense of Section 2. For $g\in C^0(X)$, set

$$R(T,g) = \lim_{m\to\infty} (\sup |g^{(m)}(x)|)^{1/m}$$

(the limit is well-defined and equal to the infimum, by a standard subbadditivity argument). We shall give the key steps in the proof of the following result (which is weaker than Theorem 2.5):

Theorem 4.1 (Essential spectral radius)**.** *Let $T: X\to X$ be a C^r Anosov diffeomorphism, and let g be a C^{r-1} function, with $r>1$. For all real numbers $q<0<p$ with $p-q<r-1$, there is a Banach space $C_*^{p,q}(T)$ of distributions on X, containing all C^s functions with $s>p$, and contained in the dual of $C^s(X)$ for all $s<|q|$, on which $\mathcal{L}_{T,g}$ extends boundedly and so that*

$$r_{ess}(\mathcal{L}_{T,g}|_{C_*^{p,q}(T)}) \le R(T,g)\max\{\lambda_s^p,\nu_u^q\}\,.$$

In particular the pull-back operator $T^*\varphi = \varphi\circ T$ satisfies for all $q<0<p$ with $p-q<r-1$

$$r_{ess}(T^*|_{C_*^{p,q}(T)}) \le \max\{\lambda_s^p,\nu_u^q\} < 1\,.$$

In the rest of this paper, we explain how to adapt the tools in Section 3 to prove the above theorem.

4.1. **Local definition of the anisotropic norms.** In this subsection we define the anisotropic norms in a compact domain of $\mathbb{R}^d$. (The Banach space in Theorem 4.1 will be constructed by patching together such local spaces in coordinate charts.) Let $\mathbf{C}_+$ and $\mathbf{C}_-$ be closed cones in $\mathbb{R}^d$ with nonempty interiors, such that $\mathbf{C}_+\cap\mathbf{C}_- = \{0\}$. Let then $\varphi_+,\varphi_- : \mathbf{S}^{d-1}\to[0,1]$ be C^∞ functions on the unit sphere $\mathbf{S}^{d-1}$ in $\mathbb{R}^d$ satisfying

$$\varphi_+(\xi) = \begin{cases} 1, & \text{if } \xi\in\mathbf{S}^{d-1}\cap\mathbf{C}_+,\\ 0, & \text{if } \xi\in\mathbf{S}^{d-1}\cap\mathbf{C}_-,\end{cases} \qquad \varphi_-(\xi) = 1-\varphi_+(\xi)\,. \tag{30}$$

(What the reader can have in mind is that $\mathbf{C}_+$ is a cone containing a stable bundle and $\mathbf{C}_-$ a cone containing an unstable bundle.

Except in Exercise 4.7, Theorem 4.8, and Exercise 4.9 below, we shall work in this subsection with a fixed pair of cones $\mathbf{C}_\pm$ and fixed functions $\varphi_\pm$, they will not appear in the notation for the sake of simplicity. Recall ψ_n and χ from Subsection 3.2. For $n \in \mathbb{Z}_+$ and $\sigma \in \{+,-\}$, we define

$$\psi_{n,\sigma}(\xi) = \begin{cases} \psi_n(\xi)\varphi_\sigma(\xi/\|\xi\|), & \text{if } n \ge 1, \\ \chi(\|\xi\|)/2, & \text{if } n = 0. \end{cases}$$

Exercise 4.2. Prove that the $\psi_{n,\sigma}$ enjoy similar properties as those of the ψ_n, in particular the L^1-norm of the rapidly decaying function $\widehat{\psi}_{n,\sigma}$ is bounded uniformly in n.

Fix $K \subset \mathbb{R}^d$ compact and with nonempty interior. For $u \in C^\infty(K)$, define for each $n \in \mathbb{Z}_+$, $\sigma \in \{+,-\}$, and $x \in \mathbb{R}^d$:

$$u_{n,\sigma}(x) = (\psi_{n,\sigma}(D)u)(x) = (\widehat{\psi}_{n,\sigma} * u)(x)\,.$$

Since $1 = \sum_{n=0}^\infty \sum_{\sigma=\pm} \psi_{n,\sigma}(\xi)$, we have $u = \sum_{n\ge 0}\sum_{\sigma=\pm} u_{n,\sigma}$.

Definition (Anisotropic hölder spaces $C_*^{p,q}(K)$)**.** Let $\mathbf{C}_\pm$ and $\varphi_\pm$ be fixed, as above. Let p and q be arbitrary real numbers. Define the anisotropic hölder norm $\|u\|_{C_*^{p,q}}$ for $u \in C^\infty(K)$, by

$$\|u\|_{C_*^{p,q}} = \max\left\{ \sup_{n\ge 0} 2^{pn}\|u_{n,+}\|_{L^\infty},\ \sup_{n\ge 0} 2^{qn}\|u_{n,-}\|_{L^\infty} \right\}. \tag{31}$$

Let $C_*^{p,q}(K)$ be the completion of $C^\infty(K)$ for the norm $\|\cdot\|_{C_*^{p,q}}$.

Remark 4.3. In our application, $p > 0$, and $q < 0$. Recalling Section 3 for contracting branches and $p > 0$, it is then natural that $\mathbf{C}_+$ and φ_+ be associated to a contracting (i.e., stable) cone for the dynamics and $\mathbf{C}_-$ and φ_- be associated to an expanding (i.e., unstable) cone. Elements of $C_*^{p,q}(K)$ are distributions which are at least p-smooth in the directions in $\mathbf{C}_+$ and at most q-"rough" in the directions of $\mathbf{C}_-$.

We shall not give a proof of the following result, referring instead to [4]. (The proof is based on the Ascoli-Arzelà theorem.)

Proposition 4.4 (Compact embeddings)**.** *If $p' < p$ and $q' < q$, the inclusion $C_*^{p,q}(K) \subset C_*^{p',q'}(K)$ is compact.*

4.2. **Compact approximation for local hyperbolic maps.** Let $r > 1$. Let $K, K' \subset \mathbb{R}^d$ be compact subsets with non-empty interiors, and take a compact neighborhood W of K. Let $\mathcal{T} : W \to K'$ be a C^r diffeomorphism onto its image. Let $\gamma : \mathbb{R}^d \to \mathbb{C}$ be a C^{r-1} function such that $\mathrm{supp}(\gamma)$ is contained in the interior of K. In this section we study the transfer operator

$$L : C^{r-1}(K') \to C^{r-1}(K), \qquad Lu(x) = \gamma(x)\cdot u \circ \mathcal{T}(x)\,.$$

For a pair of cones $\mathbf{C}_\pm$ as in Subsection 4.1, we make the following *cone-hyperbolicity* assumption on $\mathcal{T}$:

$$D\mathcal{T}_x^{tr}(\mathbb{R}^d \setminus \mathrm{interior}\,(\mathbf{C}_+)) \subset \mathrm{interior}\,(\mathbf{C}_-) \cup \{0\} \quad \text{for all } x \in W, \tag{32}$$

where $D\mathcal{T}_x^{tr}$ denotes the transpose of the derivative of $\mathcal{T}$ at x. (The above condition is sufficient in the neighbourhood of a hyperbolic fixed point. More generally, it will be useful to allow more flexibility and to work with two pairs of cones. See Exercise 4.7 below.)

Put

$$\|\mathcal{T}\|_+ = \sup_x \sup_{0\neq D\mathcal{T}_x^{tr}(\xi)\notin \mathbf{C}_-} \frac{\|D\mathcal{T}_x^{tr}(\xi)\|}{\|\xi\|} \quad \text{(the "weakest contraction")},$$

$$\|\mathcal{T}\|_- = \inf_x \inf_{0\neq \xi\notin \mathbf{C}_+} \frac{\|D\mathcal{T}_x^{tr}(\xi)\|}{\|\xi\|} \quad \text{(the "weakest expansion ")}.$$

The following result is the key to the proof of Theorem 4.1 (see also Exercise 4.7 and Theorem 4.8 below):

Theorem 4.5 (Estimates for local cone-hyperbolic maps)**.** *For any $q' < q < 0 < p' < p$ such that $p - q' < r - 1$, there exists a constant C so that for each C^r diffeomorphism $\mathcal{T}$ and each C^{r-1} function γ as above (assuming in particular (32)), there is a linear operator L'_1 such that for any $u \in C_*^{p,q}(K')$*

$$\|Lu - L'_1u\|_{C_*^{p,q}} \le C\|\gamma\|_{L^\infty} \cdot \max\{\|\mathcal{T}\|_+^p, \|\mathcal{T}\|_-^q\}\|u\|_{C_*^{p,q}}\,, \tag{33}$$

and, in addition, there is $C(\mathcal{T},\gamma)$ such that for any $u \in C_*^{p',q'}(K')$

$$\|L'_1u\|_{C_*^{p,q}} \le C(\mathcal{T},\gamma)\|u\|_{C_*^{p',q'}}\,.$$

It is essential that the constant C in (33) does not depend on $\mathcal{T}$ and γ.

Proof of Theorem 4.5. We need more notation. By (32) there exist a closed cone $\widetilde{\mathbf{C}}_+$ contained in the interior of $\mathbf{C}_+$ such that for all $x \in W$

$$D\mathcal{T}_x^{tr}(\mathbb{R}^d \setminus \text{interior}(\widetilde{\mathbf{C}}_+)) \subset \text{interior}(\mathbf{C}_-) \cup \{0\}\,. \tag{34}$$

Fix also a closed cone $\widetilde{\mathbf{C}}_-$ contained in the interior of $\mathbf{C}_-$ and let $\tilde\varphi_\pm : \mathbf{S}^{d-1} \to [0,1]$ be C^∞ functions satisfying

$$\tilde\varphi_-(\xi) = \begin{cases} 0, & \text{if } \xi \in \mathbf{S}^{d-1}\cap\widetilde{\mathbf{C}}_+,\\ 1, & \text{if } \xi \notin \mathbf{S}^{d-1}\cap\mathbf{C}_+,\end{cases} \qquad \tilde\varphi_+(\xi) = \begin{cases} 1, & \text{if } \xi \notin \mathbf{S}^{d-1}\cap\mathbf{C}_-,\\ 0, & \text{if } \xi \in \mathbf{S}^{d-1}\cap\widetilde{\mathbf{C}}_-.\end{cases}$$

Recalling $\tilde\psi_\ell$ from the beginning of the proof of Theorem 3.7, define for $\sigma \in \{+,-\}$

$$\tilde\psi_{\ell,\sigma}(\xi) = \begin{cases} \tilde\psi_\ell(\xi)\tilde\varphi_\sigma(\xi/\|\xi\|), & \text{if } \ell \ge 1,\\ \chi(2^{-1}\|\xi\|), & \text{if } \ell = 0.\end{cases}$$

Note that $\tilde\psi_{\ell,\tau}(\xi) = 1$ if $\xi \in \text{supp}(\psi_{\ell,\tau})$.

Up to slightly changing the cones $\widetilde{\mathbf{C}}_\pm$, we can guarantee that

$$\inf_{x\in K} \inf_{0\neq\xi\notin\widetilde{\mathbf{C}}_+} \frac{\|D\mathcal{T}_x^{tr}(\xi)\|}{\|\xi\|} \ge \|\mathcal{T}\|_-/2\,, \tag{35}$$

$$\sup_{x\in K} \sup_{0\neq D\mathcal{T}_x^{tr}(\xi)\notin\widetilde{\mathbf{C}}_-} \frac{\|D\mathcal{T}_x^{tr}(\xi)\|}{\|\xi\|} \le 2\|\mathcal{T}\|_+\,. \tag{36}$$

We write $(\ell,\tau) \hookrightarrow (n,\sigma)$ if either

- $(\tau,\sigma) = (+,+)$ and $2^n \le 2^{\ell+5}\|\mathcal{T}\|_+$, or
- $(\tau,\sigma) = (-,-)$ and $2^{\ell-5}\|\mathcal{T}\|_- \le 2^n$, or
- $(\tau,\sigma) = (+,-)$ and ($2^n \ge 2^5\|\mathcal{T}\|_-$ or $2^\ell \ge 2^5\|\mathcal{T}\|_+$).

We write $(\ell,\tau) \not\hookrightarrow (n,\sigma)$ otherwise.

Exercise 4.6. Let $\check{\mathbf{C}}_\pm$ be two closed cones with disjoint interiors, so that $\check{\mathbf{C}}_+ \cap \check{\mathbf{C}}_- = \{0\}$, and with

$$\text{closure}\,(\mathbb{R}^d \setminus \mathbf{C}_-) \subset \text{interior}\,(\check{\mathbf{C}}_+) \cup \{0\} \text{ and } \widetilde{\mathbf{C}}_- \subset \text{interior}\,(\check{\mathbf{C}}_-) \cup \{0\}\,.$$

Let $\check{\varphi}_\pm : \mathbf{S}^{d-1} \to [0,1]$, and $\check{\psi}_{\ell,\sigma} : \mathbb{R}^d \to [0,1]$, for $\sigma \in \{+,-\}$ and $\ell \in \mathbb{Z}_+$, be functions defined just like $\varphi_\pm$ and $\psi_{\ell,\sigma}$, but replacing the cones $\mathbf{C}_\pm$ by $\check{\mathbf{C}}_\pm$. Using (34) and (35), check that there exists an integer $N(\mathcal{T}) > 0$ such that for all $x \in \text{supp}(\gamma)$ and $\max\{n,\ell\} \ge N(\mathcal{T})$

$$d(\text{supp}(\check{\psi}_{n,\sigma}), D\mathcal{T}_x^{tr}(\text{supp}(\tilde{\psi}_{\ell,\tau}))) \ge 2^{\max\{n,\ell\}-N(\mathcal{T})} \quad \text{if } (\ell,\tau) \not\hookrightarrow (n,\sigma). \tag{37}$$

Hint: For $(\tau,\sigma) = (-,+)$, use (34). See [4] for further details.

Note that (37) is *exactly* the same lower bound as (10).

Define L'_1 and L'_0 by $L'_j u = \sum_{n,\sigma} (L_j u)_{(n,\sigma)}$ with

$$(L'_0 u)_{(n,\sigma)} = \sum_{(\ell,\tau):(\ell,\tau)\hookrightarrow(n,\sigma)} \check{\psi}_{n,\sigma}(D)(L\, u_{\ell,\tau})\,,$$

and

$$(L'_1 u)_{(n,\sigma)} = \sum_{(\ell,\tau):(\ell,\tau)\not\hookrightarrow(n,\sigma)} \check{\psi}_{n,\sigma}(D)(L\, \tilde{\psi}_{\ell,\tau}(D) u_{\ell,\tau})\,.$$

Since $\tilde{\psi}_{\ell,\tau}(D)u_{\ell,\tau} = u_{\ell,\tau}$, we have $L'_0 + L'_1 = L$. Note also that by definition of the cones $\check{\mathbf{C}}_\pm$, if $|n-m| > 5$ or $v = +$ and $\sigma = -$ then for $i=0$ and $i=1$:

$$\psi_{m,v}(D)(L_i u)_{(n,\sigma)} = 0\,. \tag{38}$$

The bound for L'_0 is easy, like in the proof of Theorem 3.7: Notice that there is C so that, setting $c(+) = p$, $c(-) = q$,

$$\sum_{(\ell,\tau):(\ell,\tau)\hookrightarrow(n,\sigma)} 2^{c(\sigma)n - c(\tau)\ell} \le C \max\{\|\mathcal{T}\|_+^p, \|\mathcal{T}\|_-^q\},\ \forall (n,\sigma)\,, \tag{39}$$

and recall that $\sup_{(n,\sigma)} \int |\widehat{\check{\psi}}_{n,\sigma}(x)|dx < \infty$.

Consider next L'_1. It is enough to prove that if $(\ell,\tau) \not\hookrightarrow (n,\sigma)$ then

$$\|\check{\psi}_{n,\sigma}(D)(L\,\tilde{\psi}_{\ell,\tau}(D)f)\|_{L^\infty} \le C(\mathcal{T},\gamma) 2^{-(r-1)\max\{n,\ell\}} \|f\|_{L^\infty}\,. \tag{40}$$

Indeed, setting $c'(+) = p'$, and $c'(-) = q'$, (40) and (38) imply that

$$\begin{aligned}
&\|L'_1 u\|_{C_*^{p,q}} \\
&\le \sup_{(m,v)} \sum_{(n,\sigma)} \sum_{(\ell,\tau):(\ell,\tau)\not\hookrightarrow(n,\sigma)} 2^{c(v)m} \|\psi_{m,v}(D)\check{\psi}_{n,\sigma}(D)(L\,\tilde{\psi}_{\ell,\tau}(D)u_{\ell,\tau})\|_{L^\infty} \\
&\le C(\mathcal{T},\gamma)\cdot \sup_{(n,\sigma)} \left(\sum_{(\ell,\tau):(\ell,\tau)\not\hookrightarrow(n,\sigma)} 2^{c(\sigma)n - c'(\tau)\ell - (r-1)\max\{n,\ell\}} \right) \|u\|_{C_*^{p',q'}}\,.
\end{aligned}$$

Then, since $p \le r-1$, $p - q' < r-1$, and thus $-q < r-1$, we see from the definition of $\not\hookrightarrow$ that

$$\sup_{(n,\sigma)} \left(\sum_{(\ell,\tau):(\ell,\tau)\not\hookrightarrow(n,\sigma)} 2^{c(\sigma)n - c'(\tau)\ell - (r-1)\max\{n,\ell\}} \right) < \infty\,. \tag{41}$$

(Note that $p - q \le r - 1$ is not enough to guarantee the above bound because of the case $(\tau, \sigma) = (-, +)$.)

To show (40), extend $\mathcal{T}$ to $\mathbb{R}^d$ as in the proof of Theorem 3.7, and rewrite

$$(\check\psi_{n,\sigma}(D)(L\,\tilde\psi_{\ell,\tau}(D)f)(x) = (2\pi)^{-2d} \int V^{\ell,\tau}_{n,\sigma}(x,y) \cdot f \circ \mathcal{T}(y) |\det D\mathcal{T}(y)| dy\,,$$

where

$$V^{\ell,\tau}_{n,\sigma}(x,y) = \int e^{i(x-w)\xi + i(\mathcal{T}(w)-\mathcal{T}(y))\eta} \gamma(w) \check\psi_{n,\sigma}(\xi) \tilde\psi_{\ell,\tau}(\eta) dw d\xi d\eta\,. \tag{42}$$

Recall b from (18). If we show

$$|V^{\ell,\tau}_{n,\sigma}(x,y)| \le C(\mathcal{T},\gamma) 2^{-(r-1)\max\{n,\ell\}} \cdot 2^{d \min\{n,\ell\}} b(2^{\min\{n,\ell\}}(x-y))\,, \tag{43}$$

for some $C(\mathcal{T},\gamma) > 0$ and all $(\ell,\tau) \not\hookrightarrow (n,\sigma)$ then (40) follows from Young's inequality, as in the expanding case from Section 3.3.

Finally, the proof of (43) is *exactly* the same as the proof of (19), up to using the change of variable $v = 2^\ell(\mathcal{T}(w) - \mathcal{T}(y))$ instead of $u = 2^n(x - w)$ in (27) if $\ell > n$. □

Exercise 4.7. Consider now two pairs of cones $\mathbf{C}_\pm$ and $\mathbf{C}'_\pm$, and construct, for each p and q, two norms $\|\cdot\|_{C^{p,q}_*}$ and $\|\cdot\|_{(C'_*)^{p,q}}$ (by choosing $\varphi_\pm$ and $\varphi'_\pm$ as above). Introduce a more general condition for $\mathcal{T}$:

$$D\mathcal{T}^{tr}_x(\mathbb{R}^d \setminus \operatorname{interior}(\mathbf{C}'_+)) \subset \operatorname{interior}(\mathbf{C}_-) \cup \{0\} \quad \text{for all } x \in W. \tag{44}$$

Put

$$\|\mathcal{T}\|_+ = \sup_{x\in K} \sup_{0\ne D\mathcal{T}^{tr}_x(\xi)\notin \mathbf{C}_-} \frac{\|D\mathcal{T}^{tr}_x(\xi)\|}{\|\xi\|} \quad (\text{the “weakest contraction”})\,,$$
$$\|\mathcal{T}\|_- = \inf_{x\in K} \inf_{0\ne \xi\notin \mathbf{C}'_+} \frac{\|D\mathcal{T}^{tr}_x(\xi)\|}{\|\xi\|} \quad (\text{the “weakest expansion”})\,.$$

Check that a small modification of the proof of Theorem 4.5 gives:

Theorem 4.8. *For any $q' < q < 0 < p' < p$ such that $p - q' < r - 1$ there exist a constant C so that for each C^r diffeomorphism $\mathcal{T}$ and C^{r-1} function γ, assuming (44), there exists a linear operator L'_1 such that for any $u \in (C'_*)^{p,q}(K')$*

$$\|Lu - L'_1 u\|_{C^{p,q}_*} \le C\|\gamma\|_{L^\infty} \cdot \max\{\|\mathcal{T}\|^p_+, \|\mathcal{T}\|^q_-\} \|u\|_{(C'_*)^{p,q}}\,,$$

and, in addition, there is $C(\mathcal{T},\gamma)$ so that for any $u \in (C'_*)^{p',q'}(K')$

$$\|L'_1 u\|_{C^{p,q}_*} \le C(\mathcal{T},\gamma) \|u\|_{(C'_*)^{p',q'}}\,.$$

(See [4].)

Remark 4.9. Theorem 4.8 may be applied to $\mathcal{T}$ the identity map, i.e., the operator $Mu = h \cdot u$ of multiplication by a smooth function h, up to taking suitable pairs of cones in order to guarantee cone-hyperbolicity.

4.3. **Transfer operators for Anosov diffeomorphisms.** We prove Theorem 4.1 by reducing to the model of Subsections 4.1 and 4.2.

Proof of Theorem 4.1. We first define the space $C_*^{p,q}(T)$ by using local charts to patch the anisotropic spaces from Subsection 4.1. Fix a finite system of C^∞ local charts $\{(V_j, \kappa_j)\}_{j=1}^J$ that cover X, and a finite system of pairs of closed cones [5] $\{(\mathbf{C}_{j,+}, \mathbf{C}_{j,-})\}_{j=1}^J$ in $\mathbb{R}^d$ with the properties that for all $1 \le j, k \le J$:

(a) The closure of $\kappa_j(V_j)$ is a compact subset K_j of $\mathbb{R}^d$.
(b) $\mathbf{C}_{j,+} \cap \mathbf{C}_{j,-} = \{0\}$.
(c) If $x \in V_j$, the cones $(D\kappa_j)^*(\mathbf{C}_{j,+})$ and $(D\kappa_j)^*(\mathbf{C}_{j,-})$ in the cotangent space contain the normal subspaces of $E^u(x)$ and $E^s(x)$, respectively.
(d) If $T^{-1}(V_k) \cap V_j \neq \emptyset$, setting $U_{jk} = \kappa_j(T^{-1}(V_k) \cap V_j)$, the map in charts $\mathcal{T}_{jk} := \kappa_k \circ T \circ \kappa_j^{-1} : U_{jk} \to \mathbb{R}^d$ enjoys the cone-hyperbolicity condition:

$$D\mathcal{T}^{tr}_{jk,x}(\mathbb{R}^d \setminus \text{interior } (\mathbf{C}_{k,+})) \subset \text{interior}(\mathbf{C}_{j,-}) \cup \{0\}, \quad \forall x \in U_{jk}. \tag{45}$$

The fact that such systems of cones exist is standard for Anosov maps, see e.g. [12].

Choose C^∞ functions $\varphi_j^+, \varphi_j^- : \mathbf{S}^{d-1} \to [0,1]$ for $1 \le j \le J$ which satisfy (30) with $\mathbf{C}_\pm = \mathbf{C}_{j,\pm}$, as in Section 4.1. This defines for each j a local space denoted $C_*^{p,q,j}$. Choose finally a C^∞ partition of the unity $\{\phi_j\}$ subordinate to the covering $\{V_j\}_{j=1}^J$, that is, the support of each $\phi_j : X \to [0,1]$ is contained in the interior of V_j, and we have $\sum_{j=1}^J \phi_j \equiv 1$ on X.

Definition. We define the Banach spaces $C_*^{p,q}(T)$ to be the completion of $C^\infty(X)$ for the norm

$$\|u\|_{C_*^{p,q}(T)} := \max_{1 \le j \le J} \|(\phi_j \cdot u) \circ \kappa_j^{-1}\|_{C_*^{p,q,j}} .$$

By definition, $C_*^{p,q}(T)$ contains $C^s(X)$ for $s > p$. If $0 \le p' < p$ and $q' < q$, Lemma 4.4 and a finite diagonal argument over $\{1, \ldots, J\}$, imply that the inclusion $C_*^{p,q}(T) \subset C_*^{p',q'}(T)$ is compact.

For $m \ge 1$ and j, k so that

$$V_{m,jk} := T^{-m}(V_k) \cap V_j \neq \emptyset ,$$

we may consider the map in charts

$$\mathcal{T}^m_{jk} = \kappa_k \circ T^m \circ \kappa_j^{-1} : \kappa_j(V_{m,jk}) \to \mathbb{R}^d .$$

Note that (45) implies that

$$(D\mathcal{T}^m_{jk,x})^{tr}(\mathbb{R}^d \setminus \text{interior } (\mathbf{C}_{k,+})) \subset \text{interior}(\mathbf{C}_{j,-}) \cup \{0\} , \ \forall x \in \kappa_j(V_{m,jk}) .$$

Set

$$R_m = \max_{j,k} \sup_{x \in \kappa_j(V_{m,jk})} |g^{(m)} \circ \kappa_j^{-1}(x)| \cdot \max\{\|\mathcal{T}^m_{jk}\|_+^p, \|\mathcal{T}^m_{jk}\|_-^q\} ,$$

where

$$\|\mathcal{T}^m_{jk}\|_+ = \sup_{x \in \kappa_j(V_{m,jk})} \sup \left\{ \frac{\|(D\mathcal{T}^m_{jk})^{tr}_x(\xi)\|}{\|\xi\|} \; ; \; 0 \neq (D\mathcal{T}^m_{jk})^{tr}_x(\xi) \notin \mathbf{C}_{j,-} \right\} ,$$

[5] We regard $\mathbf{C}_{j,\pm}$ as constant cone fields in the *cotangent* bundle $T^*\mathbb{R}^d$.

and

$$\|\mathcal{T}^m_{jk}\|_- = \inf_{x\in\kappa_j(V_{m,jk})} \inf \left\{ \frac{\|(D\mathcal{T}^m_{jk})^{tr}_x(\xi)\|}{\|\xi\|} \; ; \; 0 \ne \xi \notin \mathbf{C}_{k,+} \right\} .$$

A standard argument in uniformly hyperbolic dynamics gives

$$\lim_{m\to\infty} (\|\mathcal{T}^m_{jk}\|_+)^{1/m} \le \lambda_s \, ,$$

and

$$\lim_{m\to\infty} (\|\mathcal{T}^m_{jk}\|_-)^{1/m} \ge \nu_u \, .$$

Therefore

$$\limsup_{m\to\infty} (R_m)^{1/m} \le R(T,g) \max\{\lambda_s^p, \nu_u^q\} \, . \tag{46}$$

Since $p-q<r-1$, we can apply Theorem 4.8 to $\mathcal{T}^m_{jk}$ and $\gamma_j = (\phi_j g^{(m)}) \circ \kappa_j^{-1}$ to obtain C so that, setting $L^{(m)}_{jk} u = \gamma_j \cdot (u(\phi_k \circ \kappa_k^{-1})) \circ \mathcal{T}^m_{jk}$ for $u \in C^{p,q,k}_*(K_k)$,

$$\|L^{(m)}_{jk} u - (L^{(m)}_{jk})'_1 u\|_{C^{p,q,j}_*} \le C R_m \cdot \|u\|_{C^{p,q,k}_*} \, , \quad \forall m \, .$$

with $\|(L^{(m)}_{jk})'_1(u)\|_{C^{p,q,j}_*} \le C(\mathcal{T}^m_{jk}, \gamma_j)\|u\|_{C^{p',q',k}_*}$. Using Remark 4.9 and postcomposition by the multiplication operator $M_j u = h_j \cdot u$ where $h_j : \mathbb{R}^d \to \infty$ is C^∞, supported in K_j and $h_j \equiv 1$ on the support of $\phi_j \circ \kappa_j^{-1}$, similarly as in the last paragraph of the proof of Theorem 3.7 (details are left to the reader), this implies the claimed upper bound for the essential spectral radius of $\mathcal{L}_{T,g}$. □

Remark 4.10. Though it is not explicit in our notation, choosing a different system of local charts, a different partition of unity, or a different set of cones or functions $\varphi_\pm$, does not a priori give rise to equivalent norms. This is a little unpleasant, but does not cause problems.

Appendix A. Theorem 3.7 when both T and g are C^r

Proof. We only need to adapt the estimate (12) on L_1 to the case when γ is C^r and $r-1<p\le r$, for $r>1$, for some $0<p'<p$. Recall V^ℓ_n from (17) and b from (18). We shall show

$$|V^\ell_n(x,y)| \le C(\mathcal{T},\gamma) 2^{-r\max\{n,\ell\}} \cdot 2^{(d+1)\min\{n,\ell\}} b(2^{\min\{n,\ell\}}(x-y)), \tag{47}$$

for some $C(\mathcal{T},\gamma)>0$ and all $\ell \not\hookrightarrow n$.

Exercise A.1. Show that (47) combined with

$$\sup_n \left(\sum_{\ell : \ell \not\hookrightarrow n} 2^{pn - p'\ell + \min\{n,\ell\} - r\max\{n,\ell\}} \right) < \infty \, ,$$

gives the claim. (Recall footnote 4 and take $p'<p$ very close to p.)

Define for each y a C^r function:

$$A_y(w) = \mathcal{T}(w) - \mathcal{T}(y) - D\mathcal{T}(y)(w-y) \, .$$

We may rewrite (17) as

$$V^\ell_n(x,y) = \int e^{i(x-w)\xi + iD\mathcal{T}(y)(w-y)\eta} \big(e^{iA_y(w)\eta}\gamma(w)\big) \psi_n(\xi) \tilde{\psi}_\ell(\eta) \, dw d\xi d\eta \, .$$

Integrating (17) by parts once on w, we obtain

$$V_n^\ell(x,y) = \int e^{i(x-w)\xi + i(\mathcal{T}(w)-\mathcal{T}(y))\eta} \check{F}(\xi,\eta,w)\psi_n(\xi)\tilde{\psi}_\ell(\eta) dw d\xi d\eta \,, \tag{48}$$

where $\check{F}(\xi,\eta,w)$ is a C^{r-1} function in w which is C^∞ in the variables ξ and η. (We used properties of the derivative of an exponential to "reconstruct" $e^{i(\mathcal{T}(w)-\mathcal{T}(y))\eta}$.) Then, integrate (48) $[r]-1$ times by parts on w, giving

$$V_n^\ell(x,y) = \int e^{i(x-w)\xi + i(\mathcal{T}(w)-\mathcal{T}(y))\eta} \widetilde{F}(\xi,\eta,w)\psi_n(\xi)\tilde{\psi}_\ell(\eta) dw d\xi d\eta \,, \tag{49}$$

with $\widetilde{F}(\xi,\eta,w)$ a $C^{r-[r]}$ function in w which is C^∞ in the variables ξ and η. By (10), if $\psi_n(\xi) \cdot \tilde{\psi}_\ell(\eta) \neq 0$, then we have for all α and β

$$\|\partial_\xi^\alpha \partial_\eta^\beta \widetilde{F}\|_{C^{r-[r]}} \le C_{\alpha,\beta}(\mathcal{T},\gamma) 2^\ell 2^{-n|\alpha|-\ell|\beta|-[r]\max\{n,\ell\}} \,. \tag{50}$$

(The price we have to pay for the first integration by parts is the factor 2^ℓ. What we gained is $2^{-[r]\max\{n,\ell\}}$, with $[r]$ instead of $[r]-1$.) Then (50) implies (47), just like in Section 3.3 (recall that $\ell \le n$). □

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