

# NONCOMMUTATIVE BATALIN-VILKOVISKY GEOMETRY AND MATRIX INTEGRALS.

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ABSTRACT. I study the new type of *supersymmetric* matrix models associated with any solution to the quantum master equation of the noncommutative Batalin-Vilkovisky geometry. The asymptotic expansion of the matrix integrals gives homology classes in the Kontsevich compactification of the moduli spaces, which I associated with the solutions to the quantum master equation in my previous paper. I associate with the Bernstein-Leites matrix superalgebra equipped with an odd differentiation, whose square is *nonzero*, the family of cohomology classes of the compactification. This family is the generating function for the products of the tautological classes. The simplest example of my matrix integrals in the case of dimension zero is a supersymmetric extension of the Kontsevich model of 2-dimensional gravity.

*Notations.* I work in the tensor category of super vector spaces, over an algebraically closed field  $k$ ,  $\text{char}(k) = 0$ . Let  $V = V^{\text{even}} \oplus V^{\text{odd}}$  be a super vector space. I denote by  $\bar{\alpha}$  the parity of an element  $\alpha$  and by  $\Pi V$  the super vector space with inversed parity. For a finite group  $G$  acting on a vector space  $U$ , I denote via  $U^G$  the space of invariants with respect to the action of  $G$ . Element  $(a_1 \otimes a_2 \otimes \dots \otimes a_n)$  of  $A^{\otimes n}$  is denoted by  $(a_1, a_2, \dots, a_n)$ . Cyclic words, i.e. elements of the subspace  $(V^{\otimes n})^{\mathbb{Z}/n\mathbb{Z}}$  are denoted via  $(a_1 \dots a_n)^c$

## 1. NONCOMMUTATIVE BATALIN-VILKOVISKY GEOMETRY.

**1.1. Even inner products.** Let  $B : V^{\otimes 2} \rightarrow k$  be an even symmetric inner product on  $V$ :

$$B(x, y) = (-1)^{\bar{x}\bar{y}} B(y, x)$$

I introduced in [1] the space  $F = \bigoplus_{n=1}^{n=\infty} F_n$

$$(1.1) \quad F_n = ((\Pi V)^{\otimes n} \otimes k[\mathbb{S}_n]')^{\mathbb{S}_n}$$

where  $k[\mathbb{S}_n]'$  denotes the super  $k$ -vector space with the basis indexed by elements  $(\sigma, \rho_\sigma)$ , where  $\sigma \in \mathbb{S}_n$  is a permutation with  $i_\sigma$  cycles  $\sigma_\alpha$  and  $\rho_\sigma = \sigma_1 \wedge \dots \wedge \sigma_{i_\sigma}$ ,  $\rho_\sigma \in \text{Det}(\text{Cycle}(\sigma))$ ,  $\text{Det}(\text{Cycle}(\sigma)) = \text{Symm}^{i_\sigma}(k^{0|1})$ , is one of the generators of the one-dimensional determinant of the set of cycles of  $\sigma$ , i.e.  $\rho_\sigma$  is an order on the set of cycles defined up to even reordering, and  $(\sigma, -\rho_\sigma) = -(\sigma, \rho_\sigma)$ . The group  $\mathbb{S}_n$  acts on  $k[\mathbb{S}_n]'$  by conjugation. The space  $F$  is naturally isomorphic to:

$$F = \text{Symm}(\bigoplus_{j=1}^{\infty} \Pi(\Pi V^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}})$$

The space  $F$  carries the naturally defined Batalin-Vilkovisky differential  $\Delta$  (see loc.cit. and references therein). It is the operator of the second order with respect

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<sup>2</sup>"Comptes Rendus Mathematique", French Academy of Science, DOI 10.1016/j.crma.2010.02.002; presented for publication by Academy member M.Kontsevich on May,20,2009.

to the multiplication and is completely determined by its action on the second power of  $\bigoplus_{j=1}^{\infty} \Pi(\Pi V^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}}$ . If one chooses a basis  $\{a_i\}$  in  $\Pi V$ , in which the antisymmetric even inner product defined by  $B$  on  $\Pi V$  has the form  $(-1)^{\bar{a}_i} B(\Pi a_i, \Pi a_j) = b_{ij}$ , then the operator  $\Delta$  sends a product of two cyclic words  $(a_{\rho_1} \dots a_{\rho_r})^c (a_{\tau_1} \dots a_{\tau_t})^c$ , to

$$(1.2) \quad \sum_{p,q} (-1)^{\varepsilon_1} b_{\rho_p \tau_q} (a_{\rho_1} \dots a_{\rho_{p-1}} a_{\tau_{q+1}} \dots a_{\tau_{q-1}} a_{\rho_{p+1}} \dots a_{\rho_r})^c + \\ + \sum_{p \pm 1 \neq q \bmod r} (-1)^{\varepsilon_2} b_{\rho_p \rho_q} (a_{\rho_1} \dots a_{\rho_{p-1}} a_{\rho_{q+1}} \dots a_{\rho_r})^c (a_{\rho_{p+1}} \dots a_{\rho_{q-1}})^c (a_{\tau_1} \dots a_{\tau_t})^c \\ + \sum_{p \pm 1 \neq q \bmod r} (-1)^{\varepsilon_3} b_{\tau_p \tau_q} (a_{\rho_1} \dots a_{\rho_r})^c (a_{\tau_1} \dots a_{\tau_{p-1}} a_{\tau_{q+1}} \dots a_{\tau_t})^c (a_{\tau_{p+1}} \dots a_{\tau_{q-1}})^c$$

where  $\varepsilon_i$  are the standard Koszul signs, which take into the account that the parity of any cycle is opposite to the sum of parities of  $a_i$ :  $\overline{(a_{\rho_1} \dots a_{\rho_r})^c} = 1 + \sum \bar{a}_{\rho_i}$ . It follows from the loc.cit., prop. 2, that  $\Delta$  defines the structure of Batalin-Vilkovisky algebra on  $F$ , in particular  $\Delta^2 = 0$ . The solutions of the *quantum master equation* in  $F$

$$(1.3) \quad \hbar \Delta S + \frac{1}{2} \{S, S\} = 0, \quad S = \sum_{g \geq 0} \hbar^{2g-1+i} S_{g,i,n}, \quad S_{g,i,n} \in \text{Symm}^i \cap F_n^{\text{even}},$$

with  $S_{0,1,1} = 0$ , are in one-to one correspondence, by the loc.cit., theorem 1, with the structure of  $\mathbb{Z}/2\mathbb{Z}$ -graded *quantum  $A_{\infty}$ -algebra* on  $V$ , i.e. the algebra over the  $\mathbb{Z}/2\mathbb{Z}$ -graded modular operad  $\mathcal{F}_{\mathcal{K}}\mathbb{S}$ , where  $\mathbb{S}$  is the  $\mathbb{Z}/2\mathbb{Z}$ -graded version of the twisted modular  $\mathcal{K}$ -operad  $\widehat{\mathfrak{S}}\mathbb{S}[t]$ , with components  $k[\mathbb{S}'_n][t]$ , described in loc.cit. The  $(g = 0, i = 1)$ -part is the cyclic  $\mathbb{Z}/2\mathbb{Z}$ -graded  $A_{\infty}$ -algebra with the *even* invariant inner product on  $\text{Hom}(V, k) \xrightarrow{B} V$ . Recall, see loc.cit., that for any solution to (1.3), with  $S_{0,1,1} = S_{0,1,2} = 0$ , the value of partition function  $c_S(G)$  on a stable ribbon graph  $G$ , with no legs, is defined by contracting the product of tensors  $\bigotimes_{v \in \text{Vert}(G)} S_{g(v), i(v), n(v)}$  with  $B^{\otimes \text{Edge}(G)}$  with appropriate signs.

**Proposition 1.** ([1], s.10,11) *The graph complex  $\mathcal{F}_{\mathcal{K}}\mathbb{S}((0, \gamma, \nu))$  (part of  $\mathcal{F}_{\mathcal{K}}\mathbb{S}$  with no legs) is naturally identified with the cochain CW-complex  $C^*(\overline{\mathcal{M}}'_{\gamma, \nu}/\mathbb{S}_{\nu})$  of the Kontsevich compactification of the moduli spaces of Riemann surfaces from ([7]). For any solution to the quantum master equation (1.3) in  $F$ , with  $S_{0,1,1} = S_{0,1,2} = 0$ , the partition function on stable ribbon graphs  $c_S(G)$  defines the characteristic homology class in  $H_*(\overline{\mathcal{M}}'_{\gamma, \nu}/\mathbb{S}_{\nu})$ .*

**1.2. Odd inner products.** Let  $B$  be an odd symmetric inner product,  $B : V^{\otimes 2} \rightarrow \Pi k$ ,  $B(x, y) = (-1)^{\overline{xy}} B(y, x)$ . The analog of the space  $F$  in this situation has components

$$\widetilde{F}_n = (V^{\otimes n} \otimes k[\mathbb{S}_n])^{\mathbb{S}_n}$$

where  $k[\mathbb{S}_n]$  is the group algebra of  $\mathbb{S}_n$ , and  $\mathbb{S}_n$  acts on  $k[\mathbb{S}_n]$  by conjugation. The space  $\widetilde{F}$  is naturally isomorphic in this case to:

$$(1.4) \quad \widetilde{F} = \text{Symm}(\bigoplus_{j=1}^{\infty} (V^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}})$$

The space  $\widetilde{F}$  carries the naturally defined second order differential defined by the formula (1.2) with  $a_i \in V$ ,  $b_{ij} = B(a_i, a_j)$  and the Koszul signs  $\varepsilon_i$ , which now correspond to the standard parity of cycles:  $\overline{(a_{\rho_1} \dots a_{\rho_r})^c} = \sum \bar{a}_{\rho_i}$ . Again, it

follows from the loc.cit., prop. 2, that  $\Delta^2 = 0$ , and that  $\Delta$  defines the structure of Batalin-Vilkovisky algebra on  $\tilde{F}$ .

The solutions of quantum master equation (1.3) in  $\tilde{F}$ , with  $S_{0,1,1} = 0$ , are in one-to-one correspondence, by the loc.cit., with the structure of algebra over the twisted modular operad  $\mathcal{F}\tilde{\mathcal{S}}$  on the vector space  $V$ . Here  $\tilde{\mathcal{S}}$  is the *untwisted*  $\mathbb{Z}/2\mathbb{Z}$ -graded version of  $\tilde{\mathfrak{S}}\mathcal{S}[t]$ . The components  $\tilde{\mathcal{S}}((n))$  are the spaces  $k[\mathbb{S}_n][t]$ , with the composition maps defined as in loc.cit., sect. 9. The Feynman transform  $\mathcal{F}\tilde{\mathcal{S}}$  is a  $\mathcal{K}$ -twisted modular operad, whose  $(g = i = 0)$ -part corresponds to the cyclic  $A_\infty$ -algebra with the *odd* invariant inner product on  $Hom(\Pi V, k) \stackrel{B}{\simeq} V$ .

**Proposition 2.** ([1], s.10,11) *The graph complexes  $\mathcal{F}\tilde{\mathcal{S}}((0, \gamma, \nu))$  (part of  $\mathcal{F}\tilde{\mathcal{S}}$  with no legs) are naturally identified with the cochain CW-complexes  $C^*(\overline{\mathcal{M}}'_{\gamma, \nu}/\mathbb{S}_\nu, \mathcal{L})$  of the Kontsevich compactification of the moduli spaces of Riemann surfaces with coefficients in the local system  $\mathcal{L} = Det(P_\Sigma)$ , where  $P_\Sigma$  is the set of marked points. For any solution to the quantum master equation (1.3) in  $\tilde{F}$ , with  $S_{0,1,1} = S_{0,1,2} = 0$ , the partition function on stable ribbon graphs  $c_S(G)$  defines the characteristic homology classes in  $H_*(\overline{\mathcal{M}}'_{\gamma, \nu}/\mathbb{S}_\nu, \mathcal{L})$ .*

## 2. SUPERSYMMETRIC MATRIX INTEGRALS.

**2.1. Odd inner product.** Let  $S(a_i)$  be a solution to the quantum master equation (1.3) from  $\tilde{F}$ , with  $S_{0,1,2} = S_{0,1,1} = 0$ . Consider the vector space

$$M = Hom(V, End(U))$$

where  $\dim U = (d|d)$ . The supertrace functional on  $End(U)$  gives a natural extension to  $M$  of the odd symmetric inner product on  $Hom(V, k)$  dual to  $B$ . Let us extend  $S$  to a function  $S_{gl}$  on  $M$  so that each cyclically symmetric tensor goes to the supertrace of the product of the corresponding matrices from  $End_k(U)$

$$(a_{i_1}, \dots, a_{i_k})^c \rightarrow tr(X_{i_1} \dots X_{i_k})$$

and the product of cyclic words goes to the product of traces. The commutator  $I_\Lambda = [\Lambda, \cdot]$ , for  $\Lambda \in End(U)^{odd}$ , is an odd differentiation of  $End(U)$ . Notice that  $I_\Lambda^2 \neq 0$  for generic  $\Lambda$ . For such  $\Lambda$  there always exists an operator  $I_\Lambda^{-1}$  of regularized inverse:  $[I_\Lambda, I_\Lambda^{-1}] = 1$ , preserving the supertrace functional. A choice of nilpotent  $I_\Lambda^{-1}$  is in one-to-one correspondence with  $I_\Lambda^2$ -invariant lagrangian subspace in  $\Pi End(U)$ , corresponding to  $L = \{x \in M | I_\Lambda^{-1}(x) = 0\}$ .

Let  $\Lambda = \begin{pmatrix} 0 & Id \\ \Lambda_{01} & 0 \end{pmatrix}$ ,  $\Lambda_{01} = diag(\lambda_1, \dots, \lambda_d)$ , I take  $I_\Lambda^{-1} \begin{pmatrix} X_{00} & X_{10} \\ X_{01} & X_{11} \end{pmatrix} = \begin{pmatrix} M_\lambda X_{01} & M_\lambda (X_{00} + X_{11}) \\ 0 & -M_\lambda X_{01} \end{pmatrix}$  where  $M_\lambda E_i^j = (\lambda_i + \lambda_j)^{-1} E_i^j$ .

**Theorem 1.** *By the standard Feynman rules, the asymptotic expansion, at  $\hbar^{-1} \rightarrow 0$ , is given by the following sum over oriented stable ribbon graphs:*

$$\log \frac{\int_L \exp \frac{1}{\hbar} \left( -\frac{1}{2} tr \circ B^{-1}([\Lambda, X], X) + S_{gl}(X) \right) dX}{\int_L \exp \frac{1}{\hbar} \left( -\frac{1}{2} tr \circ B^{-1}([\Lambda, X], X) \right) dX} = const \sum_G \hbar^{-\chi_G} c_S(G) c_\Lambda(G)$$

where  $\chi_G$  the euler characteristic of the corresponding surface,  $c_S(G)$  is my partition function associated with the solution  $S$ ,  $c_\Lambda(G)$  is the partition function associated with  $(End(U), tr, I_\Lambda^{-1})$  and constructed using the propagator  $tr(I_\Lambda^{-1}, \cdot)$ .  $c_\Lambda(G)$  defines the cohomology class in  $H^*(\overline{\mathcal{M}}'_{\gamma, \nu}/\mathbb{S}_\nu, \mathcal{L})$ .

This follows from the standard rules ([6]) of the Feynman diagrammatics (compare with the formula (0.1) from [5]). In particular the combinatorics of the terms in  $S_{gl}(X)$  matches the data associated with vertices in the complex  $\mathcal{F}\widetilde{\mathcal{S}}$ , i.e. the symmetric product of cyclic permutations and the integer number. The construction of  $c_\Lambda(G)$  is studied in more details in ([3]).

**2.2. Even inner product.** Let now  $S$  denotes a solution to quantum master equation in the Batalin-Vilkovisky algebra (1.1), with  $S_{0,1,2} = S_{0,1,1} = 0$ . In this case my basic matrix algebra is the general *queer* superalgebra  $q(U)$  ([4]) with its *odd* trace. The associative superalgebra  $q(U)$  is the subalgebra of  $End(U \oplus \Pi U)$ ,

$$q(U) = \{X \in End(U \oplus \Pi U) | [X, \pi] = 0\}$$

where  $U$  is a purely even vector space and  $\pi : U \rightleftharpoons \Pi U$ ,  $\pi^2 = 1$  is the odd operator changing the parity. As a vector space  $q(U) = End(U) \oplus \Pi End(U)$ . The odd trace on  $q(U)$  is defined as  $otr(X) = \frac{1}{2}tr(\pi X)$ . Let us extend  $S$  to the function  $S_q$  on  $M = Hom(\Pi V, q(U))$ , so that each cyclically symmetric tensor in  $\Pi(\Pi V^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}}$  goes to the odd trace of the product of the corresponding elements from  $q(U)$

$$(a_{i_1}, \dots, a_{i_j})^c \rightarrow otr(X_{i_1} \dots X_{i_j})$$

and the product of cyclic words goes to the product of the odd traces. Let us denote by  $otr \circ B^{-1}$  the odd extension to  $M$ , which is defined using the *odd* pairing  $otr(XX')$  on  $q(U)$ , of the even symmetric inner product on  $Hom(V, k)$ . Let  $\Lambda = \Pi diag(\lambda_1, \dots, \lambda_N)$ ,  $\Lambda \in \Pi End(U)$  be the odd element from  $q(U)$ . The commutator  $I_\Lambda = [\Lambda, \cdot]$  is an odd differentiation of  $q(U)$ , and for generic  $\lambda_i$ ,  $I_\Lambda^2 \neq 0$ , and  $I_\Lambda$  is *invertible* outside of the even diagonal. Define the regularized inverse  $I_\Lambda^{-1}$ ,  $(I_\Lambda^{-1})E_i^j = (\lambda_i + \lambda_j)^{-1}\Pi E_i^j$ ,  $(I_\Lambda^{-1})\Pi E_i^j = 0$ ,  $[I_\Lambda, I_\Lambda^{-1}] = 1$ , and let  $L = \{x \in M | I_\Lambda^{-1}(x) = 0\}$ .

**Theorem 2.** *By the standard Feynman rules, the asymptotic expansion, at  $\Lambda^{-1} \rightarrow 0$ , is given by the following sum over oriented stable ribbon graphs:*

$$\log \frac{\int_L \exp \frac{1}{\hbar} \left( -\frac{1}{2}otr \circ B^{-1}([\Lambda, X], X) + S_q(X) \right) dX}{\int_L \exp \left( -\frac{1}{2\hbar}otr \circ B^{-1}([\Lambda, X], X) \right) dX} = const \sum_G \hbar^{-\chi_G} c_S(G) c_\Lambda(G),$$

where  $c_S(G)$  is the partition function from ([1]) and  $c_\Lambda(G)$  is the partition function, associated with  $(q(U), otr, I_\Lambda^{-1})$ , and constructed using the propagator  $otr^{dual}(I_\Lambda^{-1}, \cdot)$ .  $c_\Lambda(G)$  defines the cohomology class in  $H^*(\overline{\mathcal{M}}'_{\gamma, \nu} / \mathbb{S}_\nu)$ , it is the generating function for the products of tautological classes.

**Theorem 3.** *Let us consider the case of the one-dimensional vector space  $V$  with even symmetric inner product. The solutions in this case are arbitrary linear combination of cyclic words  $X^3, X^5, \dots, X^{2n+1}$ . My matrix integral in this case is a supersymmetric extension of the Kontsevich matrix integral from ([7]).*

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