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VARIATIONS ON VAN KAMPEN'S METHOD

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Introduction

In the 1930's, Van Kampen described a general technique for computing representations of fundamental groups of complements of complex algebraic curves. Though Van Kampen's original approach was essentially valid, some technical details were not entirely clear and were later reformulated in more modern and rigorous terms (see, e.g., the account by Chéniot, [3]).

It is possible to transform Van Kampen's "method" into an entirely constructive algorithm. To my knowledge, two implementations have been realized, one by Jorge Carmona, the other by Jean Michel and myself (GAP package VKCURVE, [4]).

The goal of the present note is to clarify some aspects which are usually neglected but must be addressed to obtain an efficient implementation. Also, the "Van Kampen's method" explained here differs from the classical one, which assumes the choice of a "generic" projection: our variant method works with an arbitrary projection. The reason for what may appear to be a superfluous sophistication (since "generic" projections always exist and are easy to find) is that working with a nongeneric projection may be computationally more efficient. The variant method explained here is implemented in VKCURVE and has already been used to find previously unknown representations.

Let $P \in \mathbb{C}[X,Y]$. The equation $P(X,Y)=0$ defines an algebraic curve $\mathcal{C} \in \mathbb{C}^2$. Our goal is to find a representation for the fundamental group of $\mathbb{C}^2 - \mathcal{C}$ (the method can be adapted to work with projective curves, as is briefly mentioned at the end of Sec. 2). Without loss of generality, we may (and will) assume that P is quadrafrei. View P as a polynomial in X depending on the parameter Y:

$$P = \alpha_0(Y)X^d + \alpha_1(Y)X^{d-1} + \cdots + \alpha_d(Y),$$

with $\alpha_0(Y) \neq 0$. To study $\mathbb{C}^2 - \mathcal{C}$, we decompose it according to the fibers of the projection $\mathbb{C}^2 \to \mathbb{C}, (x,y) \mapsto y$. Up to changing the variables, one could assume that d equals the total degree of P (the projection is then said to be "generic"); however, for reasons detailed below, we do not make this assumption that we have a generic projection. For all but a finite number of exceptional values for y_0, the equation $P(X,y_0)=0$ has exactly d distinct solutions in X. The main idea in Van Kampen's method is that to compute a representation, it essentially suffices to be able to track these d solutions when the parameter Y varies along certain loops (around the exceptional values). These d solutions form certain braids with d strings called *monodromy braids*.

In complexity terms, the most expensive part of the algorithm is the computation of the monodromy braids. Since the computation time increases with the number of strings, it is tempting to keep it reduced by working with nongeneric projections with a smaller number of strings. However, these projections frequently involve vertical asymptotes (to us, vertical lines are lines with equations of the form $Y=y_0$; thus our X-axis is vertical, and our Y-axis is horizontal; vertical asymptotes can only appear when α_0 is not a scalar). The classical Van Kampen method is not adapted to deal with vertical asymptotes, but, as we explain in Secs. 3 and 4, some corrections can be introduced to make it work.

The structure of this note is as follows: after some preliminaries in Sec. 1, we describe the main steps of the algorithm in Sec. 2. Sections 3 and 4 are devoted to two technical points; they contain the only original material of this note.

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This note does not cover all aspects of the effective implementation of Van Kampen's method. The most serious gap is that we do not explain how to perform step a of Procedure 12; this, along with other features of VKCURVE, will be described in a forthcoming joint paper with Jean Michel.

Note. A modification of Van Kampen's method dealing with vertical asymptotes is also proposed in [1]. Our approach is probably more or less equivalent to theirs, but formulated in a way which is closer to a fully automated procedure. The author thanks Jorge Carmona for useful electronic discussions.

1. Topological Preliminaries

Though we will use them only in dimension 2, we formulate the results of this section in arbitrary dimension, since it does not cost more; for the same reason, we work in the projective space.

Let n be a positive integer and let $\mathcal{H}$ be an algebraic hypersurface in the complex projective space $\mathbb{P}^n$.

Let x_0 be a base point in $\mathbb{P}^n - \mathcal{H}$. We are interested in generating $\pi_1(\mathbb{P}^n - \mathcal{H}, x_0)$. There is a natural class of elements of this group, the *meridians* (also called *generators of the monodromy*) in which to pick the generators. To construct a meridian, one needs to choose:

(1) a smooth point $x \in \mathcal{H}$;
(2) a path from x_0 to x intersecting $\mathcal{H}$ only at the endpoint x.

To these choices, one associates a loop as follows: start from x_0, follow γ; just before reaching x, make a positive full turn around $\mathcal{H}$ (by the local inversion theorem, in the neighborhood of a smooth point, the complement of a hypersurface "looks like" the complement of a hyperplane, and the local fundamental group is isomorphic to $\mathbb{Z}$, the standard orientation of $\mathbb{C}$ telling which generator is the positive one); return to x_0 following γ backwards.

The reader must convince himself that this makes sense and that we have defined an element $s_\gamma \in \pi_1(\mathbb{P}^n - \mathcal{H}, x_0)$. Of course, different choices may yield the same element. However, although the element $x \in \mathcal{H}_{\text{smooth}}$ is not uniquely determined by s_γ, it should be noted that it belongs to exactly one of the irreducible components of $\mathcal{H}$ (since intersections of components belong to the singular locus) and that this component D is uniquely determined by s_γ (to see it, integrate over s_γ the inverses of defining polynomials). We will say that s_γ is a *meridian of* $\mathcal{H}$ *around* D. It is important to note that this notion depends not only on D but also on the remaining components of $\mathcal{H}$, since these have to be avoided when choosing the path from x_0 to a point of D.

We will get rid of all topological technicalities by admitting without proof the following folk-lemma.

Lemma 1. *Let D be an irreducible component of $\mathcal{H}$. We have $\mathcal{H} = \mathcal{H}' \cup D$ with $D \not\subseteq \mathcal{H}'$, where $\mathcal{H}'$ is the union of the remaining components.*

(i) *The meridians of $\mathcal{H}$ around D form a single conjugacy class.*
(ii) *Consider the embedding $\mathbb{P}^n - \mathcal{H} \hookrightarrow \mathbb{P}^n - \mathcal{H}'$ and the associated morphism ϕ between fundamental groups. Then ϕ is surjective, and its kernel is generated by the meridians of $\mathcal{H}$ around D.*

One may already observe that this lemma has a meaning in terms of generators and relations. First, an induction from (ii) proves that $\pi_1(\mathbb{P}^n - \mathcal{H})$ is generated by meridians. Second, assume that we already know a representation of $\pi_1(\mathbb{P}^n - \mathcal{H})$ with generators corresponding to meridians; then, by simply forgetting (as generators, and in the relations) those generators which are meridians around D, we obtain a representation of $\pi_1(\mathbb{P}^n - \mathcal{H}')$. We will use this later.

One may also note that the complement of m points in $\mathbb{C}$ is relevant to the above discussion (with $n = 1$). Since we will need it in the next sections, let us fix the following *ad hoc* terminology.

Definition 2. Let $x_1, \ldots, x_m$ be m distinct points in $\mathbb{C}$. Let $x_0 \in \mathbb{C} - \{x_1, \ldots, x_m\}$. We define a *planar tree connecting* x_0 *to* $\{x_1, \ldots, x_m\}$ to be a subset of $\mathbb{C}$ homeomorphic to a tree containing $\{x_0, \ldots, x_m\}$ and such that $x_1, \ldots, x_m$ are leaves.

Assume we have fixed such a planar tree T. For each $i \in \{1, \ldots, m\}$, there is (up to reparametrization) a unique path in T connecting x_0 to x_i; it avoids the other leaves $(x_j)_{j \neq i}$. Therefore, it defines a meridian $s_i \in \pi_1(\mathbb{C} - \{x_1, \ldots, x_m\}, x_0)$. The *generating set associated with* T is $\{s_1, \ldots, s_m\}$ by definition.

It is clear, in the above definition, that s_i realizes an explicit isomorphism between $\pi_1(\mathbb{C} - \{x_1, \ldots, x_m\}, x_0)$ and the free group of m generators.

2. The Main Idea: A Fibration Argument

After these preliminaries, we move to the central matter. Let

$$P = \alpha_0(Y)X^d + \alpha_1(Y)X^{d-1} + \cdots + \alpha_d(Y)$$

be as in the Introduction. For a "generic" choice of y_0, the equation $P(X, y_0) = 0$ has d solutions in X. Here "generic" means that y_0 should be not a zero of the discriminant $\Delta \in \mathbb{C}[Y]$ of P.

Let $y_1, \ldots, y_r$ be distinct roots of Δ. Let $B := \mathbb{C} - \{y_1, \ldots, y_r\}$. Let $E := \{(x, y) \in \mathbb{C} \times B | P(x, y) \neq 0\}$. Using the classical fact that the roots of a polynomial are continuous functions of its coefficients, we see that the mapping $p : E \to B, (x, y) \mapsto y$ is a locally trivial fibration with fibers homeomorphic to the complement of d points in $\mathbb{C}$.

Choose a base point $y_0 \in B$. Let F be the fiber over y_0. In F, choose a base point x_0 (this choice is not innocent; we will return to this in the next section). Since F is connected and $\pi_2(B) = 1$, the fibration exact sequence basically amounts to

$$1 \longrightarrow \pi_1(F, x_0) \longrightarrow \pi_1(E, (x_0, y_0)) \longrightarrow \pi_1(B, y_0) \longrightarrow 1.$$

Both $\pi_1(F, x_0)$ and $\pi_1(B, y_0)$ are isomorphic to free groups, of ranks d and r, respectively. Any short exact sequence landing on a free group is split; therefore, $\pi_1(E, (x_0, y_0))$ must be a semidirect product $F_d \rtimes F_r$.

Let us be more specific. Choose a planar tree connecting x_0 to the roots of $P(X, y_0)$. This spider defines free generators $f_1, \ldots, f_d$ of $\pi_1(F, x_0)$ each of them being a meridian around one of the roots.

Similarly, choose a planar tree connecting y_0 with the roots of Δ. We obtain r meridians $g_1, \ldots, g_r$ freely generating $\pi_1(B, y_0)$.

Whichever way we choose to lift $g_1, \ldots, g_r$ to elements $\tilde{g}_1, \ldots, \tilde{g}_r \in \pi_1(E, (x_0, y_0))$, we have a semidirect product structure. The conjugacy action $f_i \mapsto \tilde{g}_j^{-1} f_i \tilde{g}_j$ defines a *monodromy automorphism* $\phi_j \in \mathrm{Aut}(F_d)$ (we will explain in Sec. 4 how to compute these automorphisms). We obviously have the representation

$$\pi_1(E, (x_0, y_0)) \simeq \left\langle f_1, \ldots, f_d, \tilde{g}_1, \ldots, \tilde{g}_r | \tilde{g}_j^{-1} f_i \tilde{g}_j = \phi_j(f_i) \right\rangle$$

(where by $\phi_j(f_i)$ we mean the corresponding word in the $f_1, \ldots, f_d$ and their inverses).

For each root y_j of Δ, define L_j to be the line in $\mathbb{C}^2$ of equation $Y = y_j$. Clearly,

$$E = \mathbb{C}^2 - \mathcal{C} \cup L_1 \cup \cdots \cup L_r.$$

Nothing prevents some of the L_j from being included in $\mathcal{C}$. The space we are interested in, $\mathbb{C}^2 - \mathcal{C}$, is obtained from E by adding (or, more exactly, by forgetting to remove) the L_j which are not included in $\mathcal{C}$. What we are tempted to do is to use Lemma 1 (ii) and to forget the corresponding $\tilde{g}_j$ in the above representation. If we had chosen $\tilde{g}_j$ to be meridians, we would obtain a representation for $\pi_1(\mathbb{C}^2 - \mathcal{C})$. The following lemma proves that this strategy works.

Lemma 3. *Let $g_j \in \pi_1(B, y_0)$ be a meridian around y_j. There exists a meridian $\tilde{g}_j$ around L_j in $\pi_1(E, (x_0, y_0))$ such that $p_*(\tilde{g}_j) = g_j$.*

Proof. This is a particular case of [2, Lemma 2.4] (whose proof relies on an easy general-position argument). For a more constructive argument, see the next section.

Van Kampen's method is the following procedure, which summarizes the above discussion:

Procedure 4 ("Van Kampen's method"). Start from a (quadrafrei) polynomial $P \in \mathbb{C}[X, Y]$.

(1) Compute $\Delta \in \mathbb{C}[Y]$ and its roots $y_1, \ldots, y_r$.

(2) Choose a base point $y_0 \in \mathbb{C} - \{y_1, \dots, y_r\}$. Construct generating meridians $g_1, \dots, g_r$ in $\pi_1(\mathbb{C} - \{y_1, \dots, y_r\}, y_0)$. Choose $x_0 \in \mathbb{C}$ such that $P(x_0, y_0) \neq 0$. Lift $g_1, \dots, g_r$ to elements $\tilde{g}_1, \dots, \tilde{g}_r \in \pi_1(E, (x_0, y_0))$ which are meridians around L_j.

(3) Choose generating meridians $f_1, \dots, f_d$ for the fundamental group of the fiber over y_0. Compute the monodromy automorphisms $\phi_1, \dots, \phi_r$.

(4) Let $J := \{j \in \{1, \dots, r\} | L_j \not\subseteq \mathcal{C}\}$. We obtain the following representation for $\pi_1(\mathbb{C}^2 - \mathcal{C}, (x_0, y_0))$:

$$\left\langle f_1, \dots, f_d, \tilde{g}_1, \dots, \tilde{g}_r \;\middle|\; \begin{array}{c} \forall i \in \{1, \dots, d\}, \forall j \in \{1, \dots, r\}, \tilde{g}_j^{-1} f_i \tilde{g}_j = \phi_j(f_i) \\ \forall j \in J, \tilde{g}_j = 1 \end{array} \right\rangle.$$

In implementations of step 1, one only computes approximations of $y_1, \dots, y_r$; it is not difficult to make sure that they are good enough for our purposes (which is to find loops circling the actual roots). In the next two sections, we will describe explicit constructive versions of steps 2 and 3. To be really useful, step 4 should be followed by a procedure “simplifying” the initial representation, which is usually highly redundant and unpleasant. This problem has no general solution, though some heuristics, implemented in VKCURVE, happen to be quite effective in many practical examples; these heuristics will not be described here.

Projective Van Kampen method The above procedure explains how to compute the fundamental group of the complement of an affine complex algebraic curve. To deal with a projective curve $\mathcal{C} \subset \mathbb{CP}^2$, we proceed as follows: decompose $\mathbb{CP}^2$ as $\mathbb{C}^2 \cup \mathbb{CP}^1$; let $\mathcal{C}' := \mathcal{C} \cap \mathbb{C}^2$ with the above method and compute a representation for $\mathbb{C}^2 - \mathcal{C}' = \mathbb{CP}^2 - \mathcal{C} \cup \mathbb{CP}^1$; let $s_\infty \in \pi_1(\mathbb{CP}^2 - \mathcal{C} \cup \mathbb{P}^1)$ be a meridian of $\mathcal{C} \cup \mathbb{CP}^1$ around $\mathbb{P}^1$ (in most situations, if $f_1, \dots, f_d$ are as in Sec. 4, $s_\infty := (f_d f_{d-1} \dots f_1)^{-1}$ is suitable; see [1, Proposition 2.10]); by Lemma 1, by adding the relation $s_\infty = 1$, one obtains a representation for $\pi_1(\mathbb{CP}^2 - \mathcal{C})$.

3. Lifting Meridians

In this section, we discuss the problem of lifting generating meridians $g_1, \dots, g_r$ of $\pi_1(B, y_0)$ to elements $\tilde{g}_1, \dots, \tilde{g}_r$ of $\pi_1(E, (x_0, y_0))$. As explained in the previous section, we would like these $\tilde{g}_j$ to be meridians around L_j. Owing to Lemma 3, we know that this is possible. However, we would like to do this in a constructive manner; something we could easily instruct a computer to do.

Practically, to encode g_j, we choose representatives $\gamma_1, \dots, \gamma_r$ in the loop space $\Omega(B, y_0)$ (in VKCURVE, we actually work with piecewise-linear loops with endpoints in $\mathbb{Q}[i]$).

Thus, the most natural way of lifting g_j is to take the elements represented by the loops $\tilde{\gamma}_1, \dots, \tilde{\gamma}_r$ defined as follows: for all $t \in [0, 1]$ and all j, we set $\tilde{\gamma}_j(t) := (x_0, \gamma_j(t))$. Two possible problems arise from this idea:

- the first problem is that $\tilde{\gamma}_j$ may not be a loop in E: nothing prevents it from intersecting $\mathcal{C}$ (although, by general-position arguments, this should only happen for a finite number of unlucky choices for x_0);
- the second problem is a more serious one: $\tilde{\gamma}_j$ may not represent meridians.

The classical Van Kampen method assumes that $\mathcal{C}$ does not have vertical asymptotes. With this assumption, a compactness argument can be used to eliminate both problems: the supports of γ_j can be assumed to be all included in a disk D, and, if we choose x_0 large enough (in module), we can guarantee that $\mathcal{C}$ does not intersect the lifted disk (x_0, D). With such a choice, it easily follows that the $\tilde{\gamma}_j$ indeed represent meridians around L_j.

However, as explained in the Introduction, we do not want to assume the absence of vertical asymptotes. To ensure that none of the above two problems occurs, we rely on the following criterion (as in Procedure 4, we set $J := \{j \in \{1, \dots, r\} | L_j \not\subseteq \mathcal{C}\}$).

Lemma 5 (explicit step 2)**.** *Let* $Q := P \prod_{j \in J}(Y - y_j)$. *Let* $\nabla \in \mathbb{C}[X]$ *be the discriminant of* Q *viewed as polynomial in* Y *with coefficients in* $\mathbb{C}[X]$. *Choose* $x_0 \in \mathbb{C}$ *which is not a root of* ∇. *Denote by* S *the set of solutions in* Y *of* $Q(x_0, Y) = 0$. *For all* $j \in \{1, \dots, r\}$, *one has* $y_j \in S$.

Choose $y_0 \in \mathbb{C} - S$, and for all $j \in \{1, \ldots, r\}$, choose a loop $\gamma_j \in \Omega(\mathbb{C} - S, y_0)$ representing a meridian of S around y_j. Then, for all j, the path $\tilde{\gamma}_j := (x_0, \gamma_j)$ represents a meridian of $\mathcal{C} \cup L_1 \cup \cdots \cup L_r$ around L_j.

Proof. For all $j \in \{1, \ldots, r\}$, either $j \in J$ or $L_j \subseteq \mathcal{C}$; in both cases, it is clear that $y_j \in S$. The path $\tilde{\gamma}_j$ avoids $\mathcal{C} \cup L_1 \cup \cdots \cup L_r$, since the intersection of this curve with the line $X = x_0$ is precisely described by S. Proving that $\tilde{\gamma}_j$ represents a meridian of $\mathcal{C} \cup L_1 \cup \cdots \cup L_r$ around L_j essentially amounts to verification that (x_0, y_j) is a smooth point of $\mathcal{C} \cup L_1 \cup \cdots \cup L_r$ or, by an immediate reformulation, that L_j is the only component of $\mathcal{C} \cup L_1 \cup \cdots \cup L_r$ in which (x_0, y_j) lies. If this was not satisfied, the polynomial $Q(x_0, Y)$ would have multiple roots in Y, which contradicts the assumption on x_0.

4. Computing the Monodromy Automorphisms

At the end of the second step of Procedure 4, as detailed in Lemma 5, we are provided with:

- a base point $(x_0, y_0) \in E$;
- loops $\gamma_1, \ldots, \gamma_r \in \Omega(B, y_0)$ such that each horizontally lifted loop $\tilde{\gamma}_j = (x_0, \gamma_j)$ is in $\Omega(E, (x_0, y_0))$ and represents a meridian of $\mathcal{C} \cup L_1 \cup \cdots \cup L_r$ around L_j.

As was explained in the previous section, in the classical Van Kampen method, where one assumes the absence of vertical asymptotes, one can get rid of many problems by choosing x_0 far enough; then, to compute the *monodromy automorphism* ϕ_j corresponding to $\tilde{\gamma}_j$, it suffices to track the solutions in X of $P(X, Y)$ when Y moves along γ_j: this defines a *monodromy braid* b_j on d strings, from which the automorphism ϕ_j can be deduced using the standard Hurwitz action of the braid group on the free group (see Definition 9 and Lemma 10).

However, since we have decided to work in a situation allowing vertical asymptotes, we may no longer assume that x_0 is "far enough"; in particular, it may occur that the strings of the monodromy braid turn around x_0, in which case the monodromy automorphism cannot be computed by Hurwitz formulas (the example given at the end of this section should convince the reader that there is a serious obstruction; this is not just a matter of being smart when choosing x_0). However, by adding to the monodromy braid an additional string fixed at x_0, one obtains extra information which can be used to modify the Hurwitz formulas in a suitable way. This is what we detail in the present section.

To simplify the notation, we fix some $j \in \{1, \ldots, r\}$ and write γ and $\tilde{\gamma}$ instead of γ_j and $\tilde{\gamma}_j$; our goal is to compute the corresponding monodromy automorphism ϕ.

Let $\{x_1, \ldots, x_{d+1}\}$ be the set of solutions in X of $(X - x_0)P(X, y_0) = 0$. For the sake of simplicity, we assume that the x_i have distinct real parts (this is always true up to rescaling) and that $\Re(x_1) < \Re(x_2) < \cdots < \Re(x_{d+1})$ (this is always true up to reordering). Among the x_i, there is x_0, say $x_0 = x_{i_0}$.

Let X_{d+1} be the configuration space of $d+1$ points in the complex line. Tracking the solutions of $(X - x_0)P(X, \gamma(t)) = 0$ for $t \in [0, 1]$, we obtain an element of $\Omega(X_{d+1}, \{x_1, \ldots, x_{d+1}\})$ which represents an element b_γ of the braid group $\pi_1(X_{d+1}, \{x_1, \ldots, x_{d+1}\})$.

For all $i \in \{1, \ldots, d+1\}$, we denote by $x_i(t)$ the string of the monodromy braid starting at x_i, i.e., the unique continuous path $[0, 1] \to \mathbb{C}$ such that $\forall t \in [0, 1], P(x_i(t), \gamma(t)) = 0$ and $x_i(0) = x_i$.

Let $F := \mathbb{C} - \{x_1, \ldots, x_{i_0-1}, x_{i_0+1}, \ldots, x_{d+1}\}$. We view x_0 as a base point for F. We also introduce a secondary base point x_∞, which will be used in our argumentation but will not appear in the formulation of the final result. This secondary base point is assumed to have a "negative enough" imaginary part in the sense that it satisfies the following conditions:

$$\forall t \in [0, 1], \forall i \in \{1, \ldots, d+1\}, \Im(x_\infty) < \Im(x_i(t))$$

and

$$\forall i \in \{1, \ldots, d\}, \Re\left(\frac{x_{i+1} - x_\infty}{x_i - x_\infty}\right) > 0.$$

It is not difficult to figure out why such x_∞ exist; let us fix one.

We consider the planar tree $\bigcup_{i=1}^{d+1}[x_\infty, x_i]$ (where $[x_\infty, x_i]$ denotes the linear segment between x_∞ and x_i). We use this tree to describe generators for various fundamental groups (see Definition 2):

- being a tree connecting x_∞ to $\{x_1, \dots, x_{d+1}\}$, it defines generating meridians
$$e_1, \dots, e_{d+1}$$
for $\pi_1(F - \{x_0\}, x_\infty)$;
- being a tree connecting x_∞ with $\{x_1, \dots, x_{i_0-1}, x_{i_0+1}, \dots, x_{d+1}\}$, it defines generating meridians for $\pi_1(F, x_\infty)$. Conveniently abusing the notation, we still denote them by
$$e_1, \dots, e_{i_0-1}, e_{i_0+1}, \dots, e_{d+1};$$
- being a tree connecting $x_0 = x_{i_0}$ to $\{x_1, \dots, x_{i_0-1}, x_{i_0+1}, \dots, x_{d+1}\}$, it defines generating meridians
$$f_1, \dots, f_{i_0-1}, f_{i_0+1}, \dots, f_{d+1}$$
for $\pi_1(F, x_0)$.

Let E_γ be the pullback over $[0,1] \xrightarrow{\gamma} B$ of the fiber bundle $E \overset{p}{\twoheadrightarrow} B$; in other words, the fiber E_t of E_γ over t is the complement in $\mathbb{C}$ of $\{x_1(t), \dots, x_{i_0-1}(t), x_{i_0+1}(t), \dots, x_{d+1}(t)\}$. The space F defined above coincides with the fiber over 0 (or, equivalently, 1).

Lemma 6. *There exists a trivialization*
$$\Psi : E_\gamma \simeq F \times [0,1]$$
such that for all $t \in [0,1]$, $\Psi((x_0, \gamma(t))) = (x_0, t)$ *and* $\Psi((x_\infty, \gamma(t))) = (x_\infty, t)$. *In particular, the induced homeomorphism*
$$\psi : F = E_0 \xrightarrow{\sim} E_1 = F$$
satisfies $\psi(x_0) = x_0$ *and* $\psi(x_\infty) = \psi(x_\infty)$.

Proof. This is an elementary variation of the standard construction of the mapping from the braid group to the mapping class group of the punctured plane. Basically, one has to imagine that the plane is a piece of rubber, pinned to a desk at x_0 and x_∞, and that we force d other points to move according to b_γ. We leave the details to the reader.

We choose Ψ and ψ as in the lemma. Since ψ fixes both x_0 and x_∞, it induces automorphisms $\psi_{*0} \in \mathrm{Aut}(\pi_1(F, x_0))$ and $\psi_{*\infty} \in \mathrm{Aut}(\pi_1(F, x_\infty))$. We will need a third automorphism: since $\psi(x_0) = x_0$, ψ is restricted to a homeomorphism $\tilde{\psi}$ of $F - \{x_0\}$ and induces an element $\tilde{\psi}_{*\infty} \in \mathrm{Aut}(\pi_1(F - \{x_0\}, x_\infty))$.

Let $\tilde{\gamma}$ be the lifted path $(x_0, \gamma) \in \pi_1(E, (x_0, y_0))$. Let $\phi \in \mathrm{Aut}(\pi(F, x_0))$ be the associated monodromy automorphism (see Sec. 2).

Lemma 7. *We have* $\phi = \psi_{*0}$.

Proof. Consider the loop $\tilde{\gamma} = (x_0, \gamma) \in \Omega(E, (x_0, y_0))$. In the pull-back E_γ, it corresponds to the horizontal path $t \mapsto (x_0, t)$. For any loop $\omega \in \Omega(F, x_0)$, we may use the trivialization of Lemma 6 to construct a homotopy in $\Omega(E_\gamma, (x_0, 0))$ between ω (viewed as a loop in the fiber of E_γ over 0) and $\tilde{\gamma}\psi(\omega)\tilde{\gamma}^{-1}$ (where $\psi(\omega)$ is viewed as a loop in the fiber of E_γ over 1). Pushed back in E, this homotopy shows that conjugating by $\tilde{\gamma}$ is the same as applying ψ_{*0}.

We may now explain our strategy for computing ϕ. First, we compute $\tilde{\psi}_{*\infty}$. As announced earlier, since x_∞ is "far enough," this can be done by using the Hurwitz formulas. Then we use this intermediate result in two ways: first we deduce $\psi_{*\infty}$; then we compute the discrepancy between $\psi_{*\infty}$ and ψ_{*0} coming from the change of base point.

Since this is a convenient setting for implementing the method, we will suppose that we are able to write b_γ as a word in the standard generators of the braid group.

Definition 8 (abstract braid group)**.** We denote by B_{d+1} the group given by the abstract representation

$$\left\langle \sigma_1, \ldots, \sigma_d \;\middle|\; \begin{array}{c} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for all } i \\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for all } i, j \text{ with } i - j > 1 \end{array} \right\rangle.$$

It is well known that B_{d+1} is isomorphic to the fundamental group of X_{d+1}. Let us be more specific. For each $i \in \{1, \ldots, d\}$, the positive twist of two consecutive strings along the segment $[x_i, x_{i+1}]$ defines an element $\sigma_i \in \pi_1(X_{d+1}, \{x_1, \ldots, x_{d+1}\})$; these elements satisfy the relations of the above definition and realize an explicit isomorphism (this is nothing but the usual way of considering braids via their real projection).

Definition 9 (Hurwitz action)**.** For all $i \in \{1, \ldots, d\}$ and $j \in \{1, \ldots, d+1\}$, set

$$H_i(e_j) := \begin{cases} e_{i+1} & \text{if } i = j, \\ e_{i+1}e_ie_{i+1}^{-1} & \text{if } i = j+1, \\ e_j & \text{otherwise.} \end{cases}$$

This defines an automorphism H_i of the free group $\langle e_1, \ldots, e_{d+1}\rangle$. The H_i satisfy the defining relations of B_{d+1} and induce a morphism $H : B_{d+1} \to \mathrm{Aut}(\langle e_1, \ldots, e_{d+1}\rangle)$.

Remark For compatibility with the usual conventions for multiplication in fundamental groups, we assume that groups of automorphisms act on the right in the above definition and throughout this section.

Lemma 10. *The automorphism* $\tilde{\psi}_{*\infty}$ *induced by* $\tilde{\psi}$ *on* $\langle e_1, \ldots, e_{d+1}\rangle = \pi_1(F - \{x_0\}, x_\infty)$ *is* $H(b_\gamma)$.

Proof. As in Lemma 6, for all braids $b \in B_{d+1}$, we can construct a homeomorphism of the pointed space $(F - \{x_0\}, x_\infty)$. This obviously induces a morphism $B_{d+1} \to \mathrm{Aut}(\langle e_1, \ldots, e_{d+1}\rangle)$. This morphism coincides with the Hurwitz action (it is enough to verify this for the standard generators of B_{d+1}, which is easy and classical). The result follows as a particular case.

Via the inclusion $F - \{x_0\} \hookrightarrow F$, we have

$$\pi_1(F, x_\infty) = \langle e_1, \ldots, e_{d+1}\rangle / e_{i_0}.$$

As mentioned above, we will still denote by e_i the image of e_i in the quotient. The braid b_γ is x_0-pure (since the x_0-strand is constant); therefore, the automorphism $H(b_\gamma)$ sends e_{i_0} to a conjugate of e_{i_0}. In particular, $H(b_\gamma)$ induces an endomorphism of $\langle e_1, \ldots, e_{d+1}\rangle / e_{i_0}$, which is nothing but the automorphism of $\pi_1(F, x_\infty)$ induced by ψ.

An automorphism of a topological space yields a natural automorphism of the functor from the fundamental groupoid into the category of groups, which associates to each point the fundamental group at this point. We have the following commutative diagram of isomorphisms, where the vertical arrows are isomorphisms associated to paths connecting the two base points:

$$\begin{tikzcd}
\pi_1(F, x_0) \arrow[r, "\psi_{*0}=\phi"] \arrow[d, "h_{[x_0,x_\infty]}"'] & \pi_1(F, x_0) \arrow[d, "h_{\psi([x_0,x_\infty])}"] \\
\pi_1(F, x_\infty) \arrow[r, "\psi_{*\infty}"] & \pi_1(F, x_\infty)
\end{tikzcd}$$

Our goal is to compute the monodromy automorphism

$$\phi = h^{-1}_{\psi([x_0,x_\infty])}\psi_{*\infty}h_{[x_0,x_\infty]} = (h^{-1}_{\psi([x_0,x_\infty])}h_{[x_0,x_\infty]})(h^{-1}_{[x_0,x_\infty]}\psi_{*\infty}h_{[x_0,x_\infty]}).$$

Since $h_{[x_0,x_\infty]}$ is the isomorphism sending f_i to e_i, the automorphism $h^{-1}_{[x_0,x_\infty]}\psi_{*\infty}h_{[x_0,x_\infty]}$ is given by the Hurwitz action (Lemma 10; of course, after replacing e_i with f_i).

The automorphism $h^{-1}_{\psi([x_0,x_\infty])}h_{[x_0,x_\infty]}$ is an inner automorphism. This is where we use the extra information provided by the x_0-string of b_γ.

Lemma 11. *The element $H(b_\gamma)(e_{i_0})$ is conjugate to e_{i_0} in $\langle e_1, \dots, e_{d+1}\rangle$. Let a be such that $H(b_\gamma)(e_{i_0}) = ae_{i_0}a^{-1}$. Let $\overline{a}$ be the image of a in $\langle f_1, \dots, f_{i_0-1}, f_{i_0+1}, \dots, f_{d+1}\rangle$ by the morphism sending $e_i, i \neq i_0$, to f_i and e_{i_0} to 1. Then $h^{-1}_{\psi([x_0,x_\infty])}h_{[x_0,x_\infty]}$ is the morphism $f \mapsto \overline{a}^{-1}f\overline{a}$.*

Proof. Left to the reader (hint: if true, this lemma provides a formula for ϕ; first, verify that this formula indeed defines a morphism $b_\gamma \mapsto \phi$; then verify it on generators of the group of x_0-pure braids on $d+1$ strings).

The following procedure is a summary of the results of this section. As promised, the exact choice of x_∞ does not matter; nor does it matter to have distinct notations for e_i and f_i.

Procedure 12 (explicit step 3)**.** Suppose γ is one of the loops γ_j constructed in Lemma 5. To compute the associated monodromy automorphism ϕ, one may proceed as follows.

(a) Compute (as a word in the standard generators, using real projection) the monodromy braid b_γ with $d+1$ strings.
(b) Compute the Hurwitz action $H(b_\gamma)$ on the free group $\langle f_1, \dots, f_{d+1}\rangle$.
(c) Identify the index i_0 of the x_0-string.
(d) Find a such that $H(b_\gamma)(f_{i_0}) = af_{i_0}a^{-1}$ (this is trivial to do: take a to be the first half of a reduced word for $H(b_\gamma)(f_{i_0})$).
(e) The composition of $H(b_\gamma)$ with the automorphism $f \mapsto a^{-1}fa$ is an automorphism of $\langle f_1, \dots, f_{d+1}\rangle$ fixing f_{i_0}. It induces an automorphism of the free group $\langle f_1, \dots, f_{d+1}\rangle / f_{i_0}$ of rank d: this is the monodromy automorphism ϕ.

Steps (b), (c), (d), and (e) are straightforward to implement, as soon as one works with software where braid groups, free groups, and group automorphisms are available (this is the case with GAP). Finding an efficient implementation of step (a) is the main issue.

In the classical method, one may assume that x_0 has a "large enough" real part; this implies that $i_0 = d+1$, that $H(b_\gamma)$ has no factor $\sigma_d^{\pm 1}$, and that $H(b_\gamma)(f_{d+1}) = f_{d+1}$: the Hurwitz action does not need a corrective term.

Complexity of the modified method As mentioned in the Introduction, we are interested in non-generic projections, because they reduce the number of strings ($d < \deg P$). To be able to work in this context, it is necessary to introduce an additional string. The complexity cost of this additional string is usually much smaller than the gain ($d+1 \leq \deg P$; also, the additional string, which is constant at x_0, is handled very efficiently). The cost of steps (c), (d), and (e) (which are not present in the classical Van Kampen method) is negligible. Actually, the only serious side effect of the variant method is not in Procedure 12 itself, but hidden in Lemma 5: one has to construct more complicated loops (for which step 3a will be more costly). However, the overall balance is positive (this is a purely empirical statement, we did not try to assess the theoretical complexity of our implementation).

To conclude, we illustrate by an example (inspired by [1, Example 3.1]) how the monodromy automorphism is affected by the choice of the base point. It is a good exercise to apply Procedure 12 to verify the claims.

Example Consider the example of the monodromy braid with two strings whose positions are $e^{-i\pi t}$ and $-e^{-i\pi t}$ for $t \in [0,1]$. This braid occurs in studying the monodromy of $(XY-1)(XY+1) = 0$ around the singular fiber $Y = 0$. The induced monodromy automorphism depends on the choice of the base point in $\mathbb{C} - \{\pm 1\}$:

(i) choosing $x_\infty := -2i$ as the base point for the fiber, the monodromy automorphism has infinite order;
(ii) choosing $x_0 := 0$ as the base point for the fiber, the monodromy automorphism has order 2. In particular, it cannot be described in terms of the usual Hurwitz action.

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