

Zariski theorems and diagrams for braid groups

David Bessis*

Department of Mathematics, Yale University, P.O. Box 208283, New Haven, CT 06520-8283, USA (e-mail: david.bessis@yale.edu)

Oblatum 7-XII-2000 & 22-III-2001
Published online: 20 July 2001 – © Springer-Verlag 2001

Abstract. Empirical properties of generating systems for complex reflection groups and their braid groups have been observed by Orlik-Solomon and Broué-Malle-Rouquier, using Shephard-Todd classification. We give a general existence result for presentations of braid groups, which partially explains and generalizes the known empirical properties. Our approach is invariant-theoretic and does not use the classification. The two ingredients are Springer theory of regular elements and a Zariski-like theorem.

Introduction

Complex reflection groups share many properties with real reflection groups. One of the main difficulties however is that no simple combinatorial description of complex reflection groups (generalizing Coxeter systems) is known. Elementary questions, such as knowing how many reflections are needed to generate the group, do not have satisfactory answers. In [OS], Orlik and Solomon mention the following result (where we have modified the notations to be consistent with the ones used here: $d_1 \leq \dots \leq d_r$ are the degrees, $d_1^* \geq \dots \geq d_r^*$ are the codegrees):

(5.5) Theorem. *Let W be a finite irreducible unitary reflection group. Then the following conditions are equivalent:*

- (i) $d_i + d_i^* = d_r$ for $i = 1, \dots, r$,
- (ii) $\sum_{i=1}^r (d_i + d_i^*) = rd_r$,
- (iii) $d_i^* < d_r$ for $i = 1, \dots, r$,
- (iv) W may be generated by r reflections,

* The author thanks Lê Dũng Tráng, Fabien Napolitano and Bernard Teissier for useful conversations, Cédric Bonnafé and Michel Broué for their comments on a first version, and the referees for their careful readings and valuable suggestions.

- (v) If $\zeta = \exp(2i\pi/d_r)$ then there exist generating reflections s_1, \dots, s_r for W such that the element $c = s_1 \dots s_r$ has eigenvalues $\zeta^{d_1-1}, \dots, \zeta^{d_r-1}$ and the element c^{-1} has eigenvalues $\zeta^{d_1^*+1}, \dots, \zeta^{d_r^*+1}$.

However, Orlik and Solomon describe these equivalences as “surprising facts for which we have no further explanation”, the proof relying entirely on case-by-case study, using Shephard-Todd classification. Our construction provides a partial “explanation” for these facts, and reduces the use of the classification in the proof of the above theorem.

In [BMR], diagrams are listed which, for *almost all* irreducible reflection groups, symbolize presentations by generators and relations of the associated braid group. These presentations satisfy remarkable properties: the generators correspond to generators-of-the-monodromy (these elements are natural substitutes for reflections in braid groups; see Sect. 2.1 below, and [BMR] 2.15), the relations are positive and homogeneous in the generators and, in many cases, a generator of the center of the braid group is given by a certain power of the product of all the generators, taken in a certain order.

The following theorem gives an *a priori* invariant-theoretic explanation for the existence of such presentations. For simplicity, we state it only as an existence result, but as it will appear later the proof of the existence of \mathbf{S} is constructive. The proof does not use the classification.

Theorem 0.1. *Let W be an irreducible complex reflection group of rank r , with associated braid group $\mathbf{B}(W)$. Let d be one of the degrees of W . Assume that d is a Springer regular number. Let N be the number of reflections in W , N^* the number of reflecting hyperplanes. Let $n := (N + N^*)/d$. Then n is an integer and there exists a subset $\mathbf{S} = \{s_1, \dots, s_n\}$ of $\mathbf{B}(W)$, such that:*

- (a) *The elements s_1, \dots, s_n are generators-of-the-monodromy, and therefore their images s_1, \dots, s_n in W are reflections.*
- (b) *The set \mathbf{S} generates $\mathbf{B}(W)$, and therefore $S := \{s_1, \dots, s_n\}$ generates W .*
- (c) *The product $(s_1 \dots s_n)^d$ is central in $\mathbf{B}(W)$ and belongs to the pure braid group $\mathbf{P}(W)$.*
- (d) *Let $\zeta := e^{2i\pi/d}$. The product $c = s_1 \dots s_n$ is a ζ -regular element in W (and therefore the eigenvalues of c are $\zeta^{d_1-1}, \dots, \zeta^{d_r-1}$, and the eigenvalues of c^{-1} are $\zeta^{d_1^*+1}, \dots, \zeta^{d_r^*+1}$).*
- (e) *There exists a finite set \mathcal{R} of relations of the form $w_1 = w_2$, where w_1 and w_2 are positive words of equal length in the elements of \mathbf{S} , such that $\langle \mathbf{S} | \mathcal{R} \rangle$ is a presentation for $\mathbf{B}(W)$.*
- (f) *For $s \in S$, denote by e_s the order of s . Take \mathcal{R} as in (e) (but view it as a set of relations in s_1, \dots, s_n). The group W is presented by $\langle S | \mathcal{R} ; \forall s \in S, s^{e_s} = 1 \rangle$.*

A consequence of this theorem is that *all* irreducible complex reflection groups admit “good diagrams”. This is due to the fact that any irreducible complex reflection group admits at least one regular degree (this can be checked on the classification: see [C3] for tables of regular degrees; when W has rank > 2 , it is even possible to find non-central regular elements; when

W is a real reflection group, the Coxeter number is always regular). Note that more than one degree may be regular. This may be an indication that, taking into account the various complex symmetries of the discriminant, one should perhaps consider more than one “diagram” for a single reflection group.

Sometimes the largest degree d_r is regular; this is the case under Orlik-Solomon assumptions (i) or (ii), and these assumptions imply that the corresponding n is r . Then (iv) and (v) are consequences of our theorem, which also implies the existence of “good” minimal presentations for the braid group, as in [BMR].

For the exceptional 4-dimensional group G_{31} , the largest degree 24 is still regular, but Orlik-Solomon assumptions (i) or (ii) are not satisfied. However, we have analogs to (iv) and (v) where r has to be replaced by the corresponding n (5 in this example). In addition, the theorem gives the existence (not previously known) of a Broué-Malle-Rouquier-like presentation for the braid group of G_{31} .

1. Invariants, discriminant, regular degrees

Let V be \mathbf{C} -vector space of finite dimension r . A (complex) reflection of V is a finite order element $s \in \text{GL}(V)$ such that $\ker(s - \text{Id})$ is a hyperplane. A reflection group of V is a finite subgroup $W \subset \text{GL}(V)$ which is generated by reflections.

1.1. Invariants

Let W be a finite subgroup of $\text{GL}(V)$. Let $\mathbf{C}[V]$ be the algebra of polynomial functions on V , i.e., the symmetric algebra $S(V^*)$, with the natural grading (i.e., non-trivial linear forms are of degree 1). We denote by $\mathbf{C}[V]^W$ the subalgebra of functions which are invariant for the dual action of W . A classical theorem by Shephard-Todd ([ST]) states that $\mathbf{C}[V]^W \simeq \mathbf{C}[V]$ if and only if W is a reflection group.

From now on, we assume that W is an irreducible reflection group. Since $\mathbf{C}[V]^W \simeq \mathbf{C}[V]$, we may find r algebraically independent polynomial functions f_1, \dots, f_r such that $\mathbf{C}[V]^W = \mathbf{C}[f_1, \dots, f_r]$. We may actually require f_1, \dots, f_r to be homogeneous, and put in such an order that $\deg f_1 \leq \dots \leq \deg f_r$. Such a sequence (f_1, \dots, f_r) is called a system of basic invariants. The sequence $(d_1, \dots, d_r) := (\deg f_1, \dots, \deg f_r)$ is independent of the choice of a specific system of basic invariants (see 2.3 in [S2]). The numbers d_1, \dots, d_r are the degrees of W . Note that they form a multi-set, rather than a set, since there may be repetitions.

We denote by $\mathbf{C}[X_1, \dots, X_r]$ the polynomial algebra in r indeterminates. A monomial $M = X_1^{\alpha_1} \dots X_r^{\alpha_r}$ has degree $\deg(M) = \sum_{i=1}^r \alpha_i$. We also define its weight by $\text{wt}(M) := \sum_{i=1}^r d_i \alpha_i$. This gives rise to two distinct gradings on $\mathbf{C}[X_1, \dots, X_r]$. The one associated with the degree is the linear grading, the other is the weighted grading.

Choosing a system of basic invariants $f = (f_1, \dots, f_r)$ is the same as choosing a graded algebra isomorphism $\Phi_f : \mathbf{C}[X_1, \dots, X_r] \xrightarrow{\sim} \mathbf{C}[V]^W$, $X_i \mapsto f_i$, where $\mathbf{C}[X_1, \dots, X_r]$ is endowed with the weighted grading.

Let $\mathbf{C}[V]_+^W$ be the ideal of $\mathbf{C}[V]^W$ consisting of elements vanishing at 0. A classical theorem of Chevalley ([C2]) states that, as a CW-module, $\mathbf{C}[V]/\mathbf{C}[V]_+^W\mathbf{C}[V]$ is isomorphic to the regular representation. It is a graded version of that representation, the grading being inherited from the one on $\mathbf{C}[V]$. For $j \in \mathbf{Z}_+$, denote by $(\mathbf{C}[V]/\mathbf{C}[V]_+^W\mathbf{C}[V])_j$ the homogeneous summand of degree j . For any irreducible CW-module M , the *fake degree* of M is the Poincaré polynomial of M in $\mathbf{C}[V]/\mathbf{C}[V]_+^W\mathbf{C}[V]$:

$$R_M(t) := \sum_{j=0}^{\infty} (M | (\mathbf{C}[V]/\mathbf{C}[V]_+^W\mathbf{C}[V])_j) t^j.$$

By [S2] 2.7, we have

$$R_V(t) = \sum_{i=1}^r t^{d_i-1}.$$

The numbers $d_i - 1$ are called the *exponents* of W . By analogy, Orlik and Solomon defined *coexponents*. We prefer here to work with codegrees: the multiset of codegrees is the multiset $\{d_1^*, \dots, d_r^*\}$ uniquely defined by

$$R_{V^*}(t) = \sum_{i=1}^r t^{d_i^*+1}$$

(the coexponents are the $d_i^* + 1$). We will assume that the codegrees are ordered in **decreasing** order:

$$d_1^* \geq \dots \geq d_r^*.$$

Let \mathcal{A} be the set of reflecting hyperplanes of W . For $H \in \mathcal{A}$, we denote by e_H the order of the pointwise stabilizer W_H of H in W (W_H is the cyclic group generated by the reflections with hyperplane H). Let N be the number of reflections in W , N^* the number of hyperplanes. Obviously

$$N = \sum_{H \in \mathcal{A}} (e_H - 1) \quad \text{and} \quad N^* = \sum_{H \in \mathcal{A}} 1.$$

The following classical result follows, for example, from (1.1), (1.2) and (1.4) in [OS].

Lemma 1.1. *We have*

$$N = \sum_{i=1}^r (d_i - 1) \quad \text{and} \quad N^* = \sum_{i=1}^r (d_i^* + 1).$$

1.2. Discriminant

The set of *regular vectors* in V is

$$V^{\text{reg}} := V - \cup_{H \in \mathcal{A}} H.$$

For each $H \in \mathcal{A}$, choose a linear form $l_H \in V^*$ with kernel H . An equation for V^{reg} in V is $\prod_{H \in \mathcal{A}} l_H \neq 0$. The function $\prod_{H \in \mathcal{A}} l_H^{e_H}$ is invariant, so $\prod_{H \in \mathcal{A}} l_H^{e_H} \neq 0$ is a W -invariant equation for V^{reg} .

Definition 1.2. *The discriminant variety of W is the closed subvariety of V/W given by the equation $\prod_{H \in \mathcal{A}} l_H^{e_H} = 0$.*

The discriminant polynomial of W with respect to a system of basic invariants f is the multivariable polynomial $\Delta_f := \Phi_f^{-1}(\prod_{H \in \mathcal{A}} l_H^{e_H})$.

It is easy to check that Δ_f is reduced (i.e., $\mathbf{C}[X_1, \dots, X_r]/(\Delta_f)$ has no nilpotent elements).

The polynomial Δ_f is weighted homogeneous of weight $N + N^*$, but is (in general) not linearly homogeneous. Denote by $\text{deg}(\Delta_f)$ the degree of a highest degree monomial among those involved in Δ_f , and by $\text{val}(\Delta_f)$ the degree of a lowest degree monomial (such a monomial is called a *valuation monomial*). The integer $\text{deg}(\Delta_f)$ depends on the choice of f , but $\text{val}(\Delta_f)$ is independent from that choice (the system-change morphisms $\Phi_{f'}^{-1}\Phi_f$ can only increase the valuation and are invertible, therefore they preserve the valuation). We have

$$\text{val}(\Delta_f) \geq \frac{N + N^*}{d_r},$$

since, for any monomial $M = X_1^{\alpha_1} \dots X_r^{\alpha_r}$, we have $\text{wt}(M) = \sum_{i=1}^r d_i \alpha_i \leq d_r \sum_{i=1}^r \alpha_i = d_r \text{deg}(M)$.

Choose a base point y_0 in V^{reg} , take its image x_0 as base point in V^{reg}/W ; the pure braid group $\mathbf{P}(W)$ and the braid group $\mathbf{B}(W)$ associated with W are, by definition, $\mathbf{P}(W) := \pi_1(V^{\text{reg}}, y_0)$ and $\mathbf{B}(W) := \pi_1(V^{\text{reg}}/W, x_0)$. A (once again classical) theorem by Steinberg ([S3]) implies that $V^{\text{reg}} \twoheadrightarrow V^{\text{reg}}/W$ is unramified, thus we have an exact sequence

$$1 \longrightarrow \mathbf{P}(W) \longrightarrow \mathbf{B}(W) \longrightarrow W \longrightarrow 1.$$

1.3. Regular degrees

The theory of regular elements has been initiated by Springer ([S2]).

Definition 1.3. *Let $\zeta \in \mathbf{C}$ a root of unity. An element $w \in W$ is ζ -regular if and only if $\ker(w - \zeta \text{Id}) \cap V^{\text{reg}} \neq \emptyset$. An integer d is a regular number for W if and only if it is the order of a regular element.*

Note that, by Steinberg theorem, if w is ζ -regular, then w and ζ have the same order.

Lemma 1.4. *Let f be a system of basic invariants. Let $d \in \mathbf{N}$. Let \mathcal{J} be the ideal of $\mathbf{C}[X_1, \dots, X_r]$ generated by $\{X_j \mid d \nmid d_j\}$. Then*

$$d \text{ is a regular number for } W \Leftrightarrow \Delta_f \notin \mathcal{J}.$$

Proof. Let ζ be a root of unity of order d . Let J be the subset of $\{1, \dots, r\}$ of those j such that $d \nmid d_j$. By [S2] 3.2 (i), the ideal in $\mathbf{C}[V]$ of functions vanishing on $\cup_{w \in W} \ker(w - \zeta \text{Id})$ is the radical of $\sum_{j \in J} \mathbf{C}[V]f_j$ (actually, the latter is radical if and only if d is regular, by [LS] 4.22).

Saying that d is regular is the same as saying that

$$\forall m > 0, \left(\prod_{H \in \mathcal{A}} l_H^{e_H} \right)^m \notin \sum_{j \in J} \mathbf{C}[V]f_j.$$

As $\mathbf{C}[V]^W \cap \sum_{j \in J} \mathbf{C}[V]f_j = \sum_{j \in J} \mathbf{C}[V]^W f_j$, this is equivalent to

$$\forall m > 0, \left(\prod_{H \in \mathcal{A}} l_H^{e_H} \right)^m \notin \sum_{j \in J} \mathbf{C}[V]^W f_j,$$

which, using the isomorphism Φ_f , is equivalent to

$$\forall m > 0, (\Delta_f)^m \notin \mathcal{J}.$$

The lemma then follows from the fact that \mathcal{J} is radical. □

Let $i \in \{1, \dots, r\}$. There is a canonical isomorphism $\mathbf{C}[X_1, \dots, X_r] \simeq \mathbf{C}[X_1, \dots, \hat{X}_i, \dots, X_r][X_i]$. For any $P \in \mathbf{C}[X_1, \dots, X_r]$, we denote by P_{X_i} the one-variable polynomial with coefficients in $\mathbf{C}[X_1, \dots, \hat{X}_i, \dots, X_r]$ corresponding to P .

Definition 1.5. *A polynomial $P \in \mathbf{C}[X_1, \dots, X_r]$ is said to be monic in X_i if and only if the head coefficient of P_{X_i} is a scalar.*

This notion makes it easy to recognize, among the degrees of W , the ones which are regular:

Lemma 1.6. *Let $i_0 \in \{1, \dots, r\}$, let $d = d_{i_0}$.*

- (i) *The degree d is regular if and only there exists a system of basic invariants f such that Δ_f is monic in X_{i_0} . When this is the case, we have $\deg(\Delta_{f, X_{i_0}}) = (N + N^*)/d$.*
- (ii) *Assume that $\forall i \in \{1, \dots, r\}, d \mid d_i \Rightarrow i = i_0$. Then the following assertions are equivalent:*
 - *d is regular,*
 - *there exists a system of basic invariants f such that Δ_f is monic in X_{i_0} ,*
 - *for all system of basic invariants f , Δ_f is monic in X_{i_0} .*

The monicity had already been observed by Saito when W is a Coxeter group and d is the Coxeter number ([S1], Sect. 3).

Proof. (i): Let f be a system of basic invariants. Let, as in the proof of the previous lemma, $J \subset \{1, \dots, r\}$ be the subset of those j such that d does not divide d_j , and \mathcal{J} the ideal generated by $\{X_j \mid j \in J\}$. If Δ_f is monic in X_{i_0} , then $\Delta_f \notin \mathcal{J}$, so d is regular (by Lemma 1.4).

Now assume that d is regular, *i.e.*, that $\Delta_f \notin \mathcal{J}$. Let us construct from f a system of basic invariants f' such that $\Delta_{f'}$ is monic in X_{i_0} .

Denote by I the complement of J (thus $i_0 \in I$). Let $a = (a_i)_{i \in \{1, \dots, r\}}$ be a family of complex numbers such that $i \in J \cup \{i_0\} \Rightarrow a_i = 0$. Let $f'_a := (f_i - a_i f_{i_0}^{d_i/d})_{i \in \{1, \dots, r\}}$. The requirements on a ensure that f'_a is a system of basic invariants. By replacing X_i by $X_i + a_i X_{i_0}^{d_i/d}$ in Δ_f , one obtains the discriminant $\Delta_{f'_a}$.

Let $\overline{\Delta}_f$ be the image of Δ_f by the composition

$$\mathbf{C}[X_1, \dots, X_n] \twoheadrightarrow \mathbf{C}[X_1, \dots, X_n]/\mathcal{J} \xrightarrow[\sim]{X_i + \mathcal{J} \mapsto X_i} \mathbf{C}[X_i, i \in I].$$

The polynomial $\overline{\Delta}_f$ is non-zero (by Lemma 1.4) and weighted homogeneous of weight $N + N^*$. There are two possibilities:

- Either $\overline{\Delta}_f \in \mathbf{C}[X_{i_0}]$. By weighted homogeneity, $\overline{\Delta}_f = b X_{i_0}^{(N+N^*)/d}$, with $b \neq 0$. Once again by weighted homogeneity, $b X_{i_0}^{(N+N^*)/d}$ must be the only monomial of Δ_f of highest degree in X_{i_0} , and thus Δ_f is monic in X_{i_0} .
- Or $\overline{\Delta}_f \notin \mathbf{C}[X_{i_0}]$. A direct computation shows that the coefficient of $X_{i_0}^{(N+N^*)/d}$ in $\Delta_{f'_a}$ is the value of $\overline{\Delta}_f$ evaluated at $X_{i_0} = 1$ and $X_i = a_i$, $i \in I - \{i_0\}$. As $\overline{\Delta}_f \notin \mathbf{C}[X_{i_0}]$, this coefficient, seen as a polynomial function with variables a_i , $i \in I - \{i_0\}$, is not constant. Thus it is possible to choose the a_i , $i \in I - \{i_0\}$ such that this coefficient is non-zero. By weighted homogeneity, the corresponding $\Delta_{f'_a}$ will be monic in X_{i_0} .

(ii): The additional assumption, in the notations of the proof of (i), is that $I = \{i_0\}$. We prove that the first assertion implies the last one, which is enough. Assume that d is regular, let f be a system of basic invariants. The proof of (i) also proves that Δ_f is monic, once it has been noticed that, in the final discussion, $\overline{\Delta}_f \in \mathbf{C}[X_{i_0}]$, since $\mathbf{C}[X_{i_0}] = \mathbf{C}[X_i, i \in I]$. □

Corollary 1.7. *The largest degree d_r is regular if and only if the valuation of the discriminant is equal to $(N + N^*)/d_r$.*

Proof. If d_r is regular, then by Lemma 1.6 (i) we can find f such that the monomial $X_r^{(N+N^*)/d_r}$ appears in Δ_f ; this monomial must be a valuation monomial.

Now assume that the valuation of the discriminant is $(N + N^*)/d_r$. Choose a system of basic invariants f . By weighted homogeneity, a valuation monomial in Δ_f can only involve those X_i for which $d_i = d_r$. Using Lemma 1.4, this implies that d_r is regular. □

2. A theorem à la Zariski

Let V a complex affine space, and let Z be an algebraic hypersurface of V . Roughly speaking, Zariski theorems describe how the homotopy groups of $V - Z$ can be compared to those of $H \cap (V - Z)$, where H is a “generic” hyperplane.

The situation we have in mind is when Z is the discriminant hypersurface of a complex reflection group, where we are looking for generating systems for the fundamental group. A natural way to map a free group to the fundamental group of $V - Z$ is by considering an inclusion $L \cap (V - Z) \subset V - Z$, where L is an affine line. To obtain a surjective morphism, one could try to apply recursively a Zariski theorem, like the one in [HL], to a suitable generic affine flag.

In our situation, certain directions are natural to consider: the monic directions of the discriminant which, by Lemma 1.6, correspond to regular degrees. But there are no obvious ways to embed them in complete flags. We state in this section a Zariski-like theorem which allows us to work directly with an affine line, skipping the recursive process; the genericity condition is explicit and elementary. The proof follows Zariski’s original strategy (see [C1] for a clear and modernized account).

2.1. Generators-of-the-monodromy

First, we ought to justify our use of dashes. To us, generators-of-the-monodromy are just particular elements of fundamental groups; we do not want to discuss what is the monodromy. This subsection is included for the convenience of the reader; it is almost “copy-pasted” from the appendix of [BMR], where more details can be found.

Let U be a smooth connected complex algebraic variety. Let Z be an algebraic hypersurface of U , let A be the set of irreducible components of Z . Fix a basepoint $x_0 \in U - Z$. A “path from x_0 to Z in $U - Z$ ” is a path $\gamma : [0, 1] \rightarrow U$ such that

- $\gamma(0) = x_0$,
- $\gamma([0, 1]) \subset U - Z$,
- $\gamma(1)$ is a smooth point of Z .

The reader should note (without being disturbed) that a path from x_0 to Z in $U - Z$ is not really a path in $U - Z$. Let γ be a path from x_0 to Z in $U - Z$. A smooth point of Z is a point which belongs to exactly one $D \in A$, and is smooth in that D . Denote by D_γ the “target divisor”, *i.e.*, the only $D \in A$ containing $\gamma(1)$. We will say that “ γ is a path from x_0 to D_γ (in $U - Z$)” (note that the notion really depends on the remaining $D \in A$).

To γ , we associate the element $s_\gamma \in \pi_1(U - Z, x_0)$ which has the following informal description: starting at x_0 , follow γ ; just before arriving at $\gamma(1)$, make one direct full turn around D_γ ; return to x_0 following γ . By *direct* full turn, we mean that if P is an irreducible equation for D_γ , then the

path, composed with P , represents the generator of $\pi_1(\mathbf{C}^*, P(x_0))$ positive with respect to the usual trigonometric convention. The reader should check for himself that, as $\gamma(1)$ is a smooth point of Z , the “local fundamental group” of $U - Z$ at $\gamma(1)$ is \mathbf{Z} , and s_γ is well-defined. An element s_γ obtained this way is a “generator-of-the-monodromy around Z in $U - Z$ ”, or, more precisely, “around D_γ in $U - Z$ ”.

Denote by $\Omega(U)$ the space of paths in U . Another path γ' from x_0 to D_γ is D_γ -homotopic to γ (relatively to $U - Z$) if and only if there is a homotopy $\varphi : [0, 1] \rightarrow \Omega(U)$ such that $\varphi(0) = \gamma, \varphi(1) = \gamma'$ and for all $t \in [0, 1], \varphi(t)$ is a path from x_0 to D_γ . The element s_γ depends only on the D_γ -homotopy class of γ .

The following lemma is certainly well-known.

Lemma 2.1. *Let U be a smooth connected complex irreducible variety, let A and B be two families of irreducible codimension 1 closed subvarieties. Assume that $A \cap B = \emptyset$. Choose $x_0 \in U - \cup_{D \in A \cup B} D$. Consider the natural morphism $\psi : \pi_1(U - \cup_{D \in A \cup B} D, x_0) \rightarrow \pi_1(U - \cup_{D \in A} D, x_0)$. The morphism ψ is surjective, and:*

- (i) *Let $D_0 \in A$. For any generator-of-the-monodromy s around D_0 in $\pi_1(U - \cup_{D \in A} D, x_0)$, $\psi^{-1}(s)$ contains a generator-of-the-monodromy around D_0 in $\pi_1(U - \cup_{D \in A \cup B} D, x_0)$.*
- (ii) *The kernel of ψ is the subgroup generated by the generators-of-the-monodromy around B in $\pi_1(U - \cup_{D \in A \cup B} D, x_0)$.*

Proof. As $U - \cup_{D \in A \cup B} D$ is obtained from $U - \cup_{D \in A} D$ by removing complex codimension 1 subvarieties, ψ is surjective (see for example [G], X 2.3: use the second statement of the reference to remove the singular locus of $\cup_{D \in B} D$ without affecting the fundamental group, and then the first statement).

In terms of paths, (i) is the following: let γ be a path in $U - \cup_{D \in A} D$ from x_0 to a divisor $D_\gamma \in A$; then in the D_γ -homotopy class of γ relatively to $\cup_{D \in A} D$, there exists a path avoiding $\cup_{D \in B} D$. This follows from standard general position arguments, as $\cup_{D \in B} D \cap (U - \cup_{D \in A} D)$ has complex codimension 1 in $U - \cup_{D \in A} D$ (whereas γ has real dimension 1), and $\cup_{D \in B} D \cap D_\gamma$ has complex codimension 1 in D_γ (whereas the target point $\gamma(1)$ has dimension 0).

(ii) is nothing more than an induction from Proposition A.1 in [BMR]. □

Note that the assertion (ii) applied to $U = \mathbf{C}^r$ and $A = \emptyset$ implies that the fundamental group of the complement of a hypersurface in \mathbf{C}^r is generated by (all) generators-of-the-monodromy. In particular, braid groups are generated by generators-of-the-monodromy.

2.2. The main tool

Many Zariski-like theorems are either of projective ([C1]) or local ([HL]) nature. The result proven here is truly of affine nature; it is both natural and

elementary, and we have been surprised not to find it in the literature (it may simply be that we did not look at the right place).

Let \mathcal{H} be an algebraic hypersurface in \mathbf{C}^r , defined by a reduced polynomial $P \in \mathbf{C}[X_1, \dots, X_r]$. Choose $X = X_{i_0}$ one of the indeterminates. To simplify notations, we use Y to refer collectively to the variables $(X_i)_{i \neq i_0}$, e.g. we write $\mathbf{C}[X, Y]$ instead of $\mathbf{C}[X_1, \dots, X_r]$. A point in \mathbf{C}^r is described by its coordinates $(x, y) \in \mathbf{C} \times \mathbf{C}^{r-1}$. Denote by p the fibration $\mathbf{C}^r \rightarrow \mathbf{C}^{r-1}$, $(x, y) \mapsto y$. The fibers of p are the lines of direction X ; for $y \in \mathbf{C}^{r-1}$, we denote by L_y the line $p^{-1}(y)$.

Notations 2.2. We denote by $\text{Disc}(P_X)$ the resultant of P'_X and P_X . We denote by $\text{hd}(P_X)$ the head coefficient of P_X . We denote by $\alpha(P_X)$ the gcd of the coefficients of P_X .

Since P is reduced, so is P_X , and $\text{Disc}(P_X) \neq 0$. We also have

$$\alpha(P_X) = \text{gcd}(P_X, \text{Disc}(P_X)).$$

Definition 2.3. We say that a line L_y of direction X is generic with respect to \mathcal{H} if and only if $\text{Disc}(P_X)(y) \neq 0$. A line L_y of direction X is bad with respect to \mathcal{H} if and only if $\alpha(P_X)(y) = 0$.

There are some lines which are neither generic nor bad. The line L_y is generic if and only if the intersection $L_y \cap \mathcal{H}$ has exactly $\text{deg}(P_X)$ points. It is bad if and only if $L_y \subset \mathcal{H}$. The remaining lines are “better” than generic, in the sense that $L_y \cap \mathcal{H}$ is finite with cardinal strictly less than $\text{deg}(P_X)$.

Let \mathcal{K} be the hypersurface in \mathbf{C}^r defined by $\text{Disc}(P_X)$. Let $\overline{\mathcal{K}}$ be the hypersurface in \mathbf{C}^{r-1} defined by $\text{Disc}(P_X)$. Let

$$E := \mathbf{C}^r - (\mathcal{H} \cup \mathcal{K}), \quad \overline{E} := \mathbf{C}^{r-1} - \overline{\mathcal{K}}.$$

The restriction of p makes E into a fiber bundle over \overline{E} , with fibers being complex lines with $\text{deg}(P_X)$ points removed.

Choose a basepoint $(x_0, y_0) \in E$, and take y_0 for basepoint in \overline{E} .

Lemma 2.4. Let $s \in \pi_1(\overline{E}, y_0)$ a generator-of-the-monodromy around $\overline{\mathcal{K}}$. Then there exists $\tilde{s} \in \pi_1(E, (x_0, y_0))$ a generator-of-the-monodromy around \mathcal{K} such that $p_*(\tilde{s}) = s$.

Proof. The space $\mathbf{C}^r - \mathcal{K}$ is a trivial bundle of fiber \mathbf{C} over \overline{E} , so s can be lifted to a generator s' of the monodromy around \mathcal{K} in $\pi_1(\mathbf{C}^r - \mathcal{K}, (x_0, y_0))$ (any lifting of a defining path suits). Let A be the set of irreducible components of \mathcal{K} , let B the set of irreducible components of \mathcal{H} which are not in \mathcal{K} . The point (i) of Lemma 2.1 applied to these A and B asserts that s' is the image of a generator-of-the-monodromy $\tilde{s} \in \pi_1(E, (x_0, y_0))$ around \mathcal{K} by the embedding morphism $\pi_1(E, (x_0, y_0)) \rightarrow \pi_1(\mathbf{C}^r - \mathcal{K}, (x_0, y_0))$. \square

Let $S \subset \pi_1(\mathbf{C}^{r-1} - \overline{\mathcal{K}}, y_0)$ be a generating set of generators-of-the-monodromy around $\overline{\mathcal{K}}$ (as noted after Lemma 2.1, it is always possible to generate the fundamental group of the complement of an algebraic hypersurface by some generators-of-the-monodromy). Using Lemma 2.4, lift S to a set $\tilde{S} \subset \pi_1(E, (x_0, y_0))$. Denote by L the fiber L_{y_0} . Using the fibration exact sequence

$$\dots \rightarrow \pi_1(L - L \cap \mathcal{H}, x_0) \xrightarrow{\iota_*} \pi_1(E, (x_0, y_0)) \xrightarrow{p_*} \pi_1(\overline{E}, y_0) \rightarrow 1,$$

we see that $\iota_*(\pi_1(L - L \cap \mathcal{H}, x_0)) \cup \tilde{S}$ generates $\pi_1(E, (x_0, y_0))$.

Now make the additional assumption that $\alpha(P_X) = 1$. Let A be the set of irreducible components of \mathcal{H} , let B be the set of irreducible components of \mathcal{K} . The polynomials P_X and $\text{Disc}(P_X)$ are coprime, so $A \cap B = \emptyset$. By Lemma 2.1 (ii), \tilde{S} belongs to the kernel of the surjective morphism $\pi_1(E, (x_0, y_0)) \rightarrow \pi_1(\mathbf{C}^r - \mathcal{H}, (x_0, y_0))$. This completes the proof of:

Theorem 2.5. *Let \mathcal{H} be an algebraic hypersurface in \mathbf{C}^r , defined by a reduced non-constant polynomial $P \in \mathbf{C}[X_1, \dots, X_r]$. Let $X = X_{i_0}$ be one of the indeterminates. Assume that the coefficients of P_X are all together coprime. Let L be a line of direction X , generic with respect to \mathcal{H} . Then the inclusion $L - L \cap \mathcal{H} \hookrightarrow \mathbf{C}^r - \mathcal{H}$ is π_1 -surjective.*

The coprimality assumption is that $\alpha(P_X) = 1$; in particular, this requires $\frac{\partial P}{\partial X} \neq 0$. The coprimality assumption is satisfied, for example, when P_X is monic (this will be used in the proof of Theorem 0.1) or when P is irreducible and $\frac{\partial P}{\partial X} \neq 0$ (we will use this later, in Sect. 4.1).

2.3. A refinement

When $\alpha(P_X) = 1$, the previous theorem yields generating sets with $\deg(P_X)$ generators. Even when $\alpha(P_X) \neq 1$, “small” generating sets can be constructed using the same strategy, but the proof is slightly more technical. We begin by a definition:

Definition 2.6. *Let $\sigma \in \mathfrak{S}_r$. We define an ordering \leq_σ on the set of r -tuples of integers: $(a_1, \dots, a_r) \leq_\sigma (b_1, \dots, b_r)$ if and only if*

$$(a_{\sigma(1)}, \dots, a_{\sigma(r)}) \leq (b_{\sigma(1)}, \dots, b_{\sigma(r)})$$

for the lexicographical order.

Let $P \in \mathbf{C}[X_1, \dots, X_r]$. A monomial $M = X_1^{a_1} \dots X_r^{a_r}$ involved in P is σ -dominant in P if and only if it is such that (a_1, \dots, a_r) is maximal for \leq_σ . We say that M is dominant in P if and only if it is σ -dominant for some $\sigma \in \mathfrak{S}_r$.

Example. The dominant monomials have been underlined in the following polynomial: $\underline{X_1^5 X_2} + X_1^4 X_2^2 + \underline{X_1 X_2^3} + X_2^3$.

For the sake of simplicity, the next proposition is stated only as an existence result, though the proof is actually constructive.

Proposition 2.7. *Let \mathcal{H} be a hypersurface in \mathbf{C}^r defined by a reduced polynomial P . Let $M = X_1^{a_1} \dots X_r^{a_r}$ be a monomial involved in P . Assume that M is dominant. Then the fundamental group of $\mathbf{C}^r - \mathcal{H}$ can be generated with $\deg(M)$ generators-of-the-monodromy.*

Proof. We describe an inductive construction procedure for such generating sets. We keep the notations used in the previous subsection. Let $\sigma \in \mathfrak{S}_r$ be such that M is σ -dominant. By reordering the variables, we may assume without loss of generality that σ is the identity.

If $r = 1$, then M is the head monomial of $P = P_X$, $\alpha(P_X) = 1$ and Theorem 2.5 gives generating sets with $\deg(P_X) = \deg(M)$ generators-of-the-monodromy.

Now assume $r > 1$. Let $i_0 := 1$, let $X := X_{i_0} = X_1$, as in the previous subsection, and let $a := a_1$. We have $\deg(P_X) = a$. The monomial M/X^a is dominant in the head coefficient $\text{hd}(P_X)$ of P_X . Let \mathcal{L} be the hypersurface in \mathbf{C}^r defined by $\text{hd}(P_X)$, and let $\overline{\mathcal{L}} := p(\mathcal{L})$. By the induction hypothesis, it is possible to generate $\pi_1(\mathbf{C}^{r-1} - \overline{\mathcal{L}}, y_0)$ by a set T_0 consisting of $\deg(M/X^a) = \deg(M) - a$ generators-of-the-monodromy. As $\text{hd}(P_X) \mid \text{Disc}(P_X)$, we have an exact sequence

$$1 \longrightarrow \ker \psi \longrightarrow \pi_1(\mathbf{C}^{r-1} - \overline{\mathcal{K}}, y_0) \xrightarrow{\psi} \pi_1(\mathbf{C}^{r-1} - \overline{\mathcal{L}}, y_0) \longrightarrow 1 .$$

Using Lemma 2.1 (i), lift T_0 to a same cardinality set T of generators-of-the-monodromy around $\overline{\mathcal{L}}$ in $\pi_1(\mathbf{C}^{r-1} - \overline{\mathcal{K}}, (x_0, y_0))$. Let $U \subset \pi_1(\mathbf{C}^{r-1} - \overline{\mathcal{K}}, y_0)$ be the set of all generators-of-the-monodromy around the irreducible components of $\overline{\mathcal{K}}$ which are not in $\overline{\mathcal{L}}$. By Lemma 2.1 (ii), we have $\ker \psi = \langle U \rangle$. Thus $\pi_1(\mathbf{C}^{r-1} - \overline{\mathcal{K}}, y_0)$ is generated by $T \cup U$.

Using Lemma 2.4, lift T and U to same cardinality sets \tilde{T} and \tilde{U} of generators-of-the-monodromy around \mathcal{K} in $\pi_1(E, (x_0, y_0))$. Choose L generic of direction X . The fibration argument used before still proves that $\iota^*(\pi_1(L - L \cap \mathcal{H}, x_0)) \cup \tilde{T} \cup \tilde{U}$ generates $\pi_1(E, (x_0, y_0))$. As $\alpha(P_X) = \text{gcd}(P_X, \text{Disc}(P_X))$, the common irreducible components of $\mathcal{H} \cap \mathcal{K}$ are irreducible components of the hypersurface $(\alpha(P_X) = 0)$, thus, as $\alpha(P_X) \mid \text{hd}(P_X)$, they are also irreducible components of \mathcal{L} . Using Lemma 2.1 (ii), this implies that elements of \tilde{U} are mapped to 1 in $\pi_1(\mathbf{C}^r - \mathcal{H}, (x_0, y_0))$. To generate $\pi_1(\mathbf{C}^r - \mathcal{H}, (x_0, y_0))$, it is enough to take the images in $\pi_1(\mathbf{C}^r - \mathcal{H}, (x_0, y_0))$ of a generating set of $\pi_1(L - L \cap \mathcal{H}, x_0)$ (of cardinal $\deg(P_X) = a$) and of \tilde{T} (of cardinal $\deg(M) - a$). □

Remarks.

- The above refinement is still not optimal, since $\alpha(P_X)$ may be reducible and may also strictly divide $\text{hd}(P_X)$, thus some elements of \tilde{T} may become trivial in the fundamental group of $\mathbf{C}^r - \mathcal{H}$.

- In the next sections, we will apply Theorem 2.5 and its refinement to situations where P is weighted homogeneous. In that case, $\pi_1(\mathbf{C}^r - \mathcal{H})$ is isomorphic to the local fundamental group at 0. Local Zariski theorems ([HL]) applied to locally generic affine flags yield generating sets with $\text{val}(P)$ generators. Discriminants of reflection groups happen to always have valuation monomials which are dominant (this is not true of any weighted homogeneous polynomial), so the two methods give the same number of generators. Our tool also works with directions which are not locally generic, e.g., for $X^2 - Y^3$, we may use either X or Y .
- Let Y_1, \dots, Y_r be a system of (non-necessarily linear) algebra generators of $\mathbf{C}[X_1, \dots, X_r]$; the above constructions can of course be applied after having rewritten P in terms of the Y_i . What is the minimum degree of dominant monomials obtained after such substitutions?

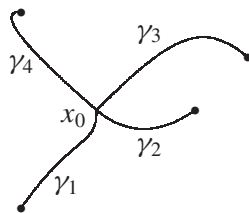
3. Presenting braid groups

We give in this section a constructive proof of Theorem 0.1. We return to the situation and notations of the first section, where W is a reflection group in $\text{GL}(V)$. Let $i \in \{1, \dots, r\}$. Let $d = d_i$, $X = X_i$. We make the following assumption:

The number d is assumed to be regular for W .

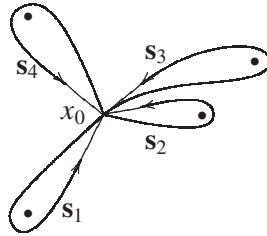
The construction of the generating set for $\mathbf{B}(W)$ depends on three successive choices:

- 1- Choose a system of basic invariants such that Δ_f is monic in X . This is possible thanks to Lemma 1.6 (i). The space V^{reg}/W is isomorphic to the complement of the hypersurface \mathcal{H} of \mathbf{C}^r defined by $\Delta_f = 0$.
- 2- Choose a generic line L of direction X . By Theorem 2.5, the embedding $L \cap (\mathbf{C}^r - \mathcal{H}) \hookrightarrow \mathbf{C}^r - \mathcal{H}$ is π_1 -surjective. The cardinality of the intersection of L with \mathcal{H} is equal to the degree of $\Delta_{f,X}$ which, according to Lemma 1.6 (i), is equal to $n := (N + N^*)/d$.
- 3- Choose a basepoint $x_0 \in L$ and a *planar spider* Γ from x_0 to $L \cap \mathcal{H}$. What we mean by a *planar spider* from x_0 to $L \cap \mathcal{H}$ is a collection $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ of non-intersecting (except of course at x_0) paths in L connecting x_0 to each of the points in $L \cap \mathcal{H}$.



(The suspicious reader may prefer to assume that all the paths considered in this section are piecewise linear.) We assume that the legs $\gamma_1, \dots, \gamma_n$ are

indexed in counterclockwise cyclic order. To each leg γ_i , we associate the corresponding generator-of-the-monodromy $s_i \in \pi_1(\mathbf{C}^r - \mathcal{H}, x_0)$.



Since $\mathbf{S} := \{s_1, \dots, s_n\}$ generates $\pi_1(L - (L \cup \mathcal{H}), x_0)$, it generates $\pi_1(\mathbf{C}^r - \mathcal{H}, x_0)$ which we identify with $\mathbf{B}(W)$ through the choice of f . Let $S := \{s_1, \dots, s_n\}$ be the image of \mathbf{S} in W . By [BMR] 2.14, the image in W of a generator-of-the-monodromy is a reflection, thus the elements of S are reflections.

The properties (a) and (b) of 0.1 are satisfied. Let us now deal with (c) and (d) (note that the statement in (d) about the eigenvalues is a standard property of regular elements, [S2] 4.5).

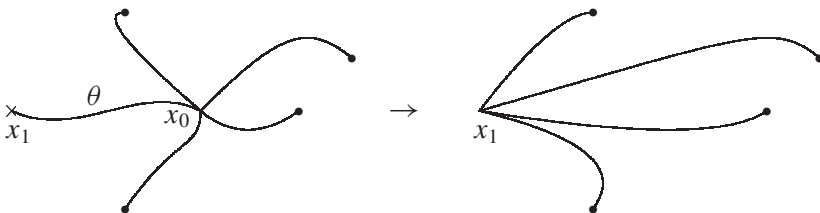
Lemma 3.1. *The product $(s_1 \dots s_n)^d$ is central in $\mathbf{B}(W)$, it belongs to the pure braid group $\mathbf{P}(W)$, and $c := s_1 \dots s_n$ is a $e^{2i\pi/d}$ -regular element of W .*

Proof. Note that $(s_1 \dots s_n)^d$ central implies

$$(s_1 \dots s_n)^d = (s_2 \dots s_n s_1)^d = \dots$$

and that both being pure and being regular are invariant by conjugation properties. Hence the statement is independent from the choice of the starting leg γ_1 (only the cyclic order matters). This will allow us, later in the proof, to choose γ_1 at our convenience.

Let us check that it is enough to prove the statement for only one basepoint chosen at our convenience. Let x_1 be the basepoint in $L - L \cap \mathcal{H}$ we prefer to the imposed x_0 . To compare the two situations, we can choose a path θ from x_0 to x_1 in $L - L \cap \mathcal{H}$. Along θ we have isomorphisms $\pi_1(L - L \cap \mathcal{H}, x_0) \simeq \pi_1(L - L \cap \mathcal{H}, x_1)$ and $\pi_1(\mathbf{C}^r - \mathcal{H}, x_0) \simeq \pi_1(\mathbf{C}^r - \mathcal{H}, x_1)$. We can drag our original spider along θ to get a spider with center x_1 :



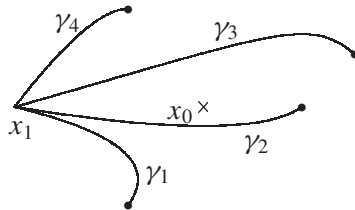
The isomorphism $\pi_1(\mathbf{C}^r - \mathcal{H}, x_0) \simeq \pi_1(\mathbf{C}^r - \mathcal{H}, x_1)$ maps the leg-generators of the original spider to the leg-generators of the new one. It maps central elements, pure elements, and elements whose image is regular to elements

with the same property. If the statement is proven for spiders with basepoint x_1 , then it follows for spiders with basepoint x_0 .

Consider the affine segment $[0, x_0]$ in \mathbf{C}^r . For each $x \in [0, x_0]$, we denote by L_x the affine line of direction X passing through x (we therefore have $L_{x_0} = L$). As Δ_f is monic in X , $L_0 \cap \mathcal{H} = \{0\}$, so L_0 is **not** generic, unless $r = 1$. The lemma is obvious when $r = 1$; we assume from now on that $r > 1$. This implies that $L \neq L_0$ and that the segment $[0, x_0]$ has non-zero length and is transverse to X . Let $E := \cup_{x \in [0, x_0]} L_x$. The space E is a fiber bundle over $[0, x_0]$, with fibers affine complex lines.

Choose trivializing affine coordinates $\phi : E \xrightarrow{\sim} [0, 1] \times \mathbf{C}$ and denote by p the projection $E \rightarrow \mathbf{C}$. Denote by ψ the multivalued map $[0, x_0] \rightarrow E$ defined by $x \mapsto L_x \cap \mathcal{H}$; the composition $p \circ \psi$ is continuous from $[0, x_0]$ to the set of subsets of \mathbf{C} of cardinal $\leq n$ (this set has a natural topology, with respect to which the set of roots of a polynomial of the form $X^n + \sum_{i=0}^{n-1} \alpha_i X^i$ depends continuously on the coefficients α_i ; the monicity of Δ_f ensures that $E \cap \mathcal{H}$ has an equation of this form, the coefficients depending continuously on $x \in [0, x_0]$). In particular, ψ is bounded, and it is possible to find $R > 0$ large enough such that $\forall x \in [0, x_0], L_x \cap \mathcal{H} \subset B(x, R)$ (where $B(x, R)$ is the open ball with center x and radius R).

Take $x_1 \in L$ at distance R from x_0 . We will prove the statement for spiders centered at x_1 .



To ease notations, let us assume that our affine coordinates

$$\phi : E \xrightarrow{\sim} [0, 1] \times \mathbf{C}$$

are such that $\phi(0) = (0, 0)$, $\phi(x_0) = (1, 0)$ and $\phi(x_1) = (1, R)$. If the starting leg is well-chosen, the product $s_1 \dots s_n \in \pi_1(\mathbf{C}^r - \mathcal{H}, x_1)$ is represented by the loop

$$\begin{aligned} [0, 1] &\longrightarrow \mathbf{C}^r - \mathcal{H} \\ t &\longmapsto \phi^{-1}((1, e^{2i\pi t} R)) \end{aligned}$$

Let $x_2 := \phi^{-1}((0, R))$. The affine segment $[x_1, x_2]$ yields an isomorphism $\pi_1(\mathbf{C}^r - \mathcal{H}, x_1) \simeq \pi_1(\mathbf{C}^r - \mathcal{H}, x_2)$. As R is large enough, the outer surface of the cylinder of radius R around $[x_2, x_1]$ does not intersect \mathcal{H} , and the isomorphism $\pi_1(\mathbf{C}^r - \mathcal{H}, x_1) \simeq \pi_1(\mathbf{C}^r - \mathcal{H}, x_2)$ maps the product $s_1 \dots s_n$ to the element \mathbf{b} represented by

$$\begin{aligned} [0, 1] &\longrightarrow \mathbf{C}^r - \mathcal{H} \\ t &\longmapsto e^{2i\pi t} x_2 \end{aligned}$$

(the \mathbf{C} -action used in the formula is the linear one, not the weighted action; in terms of weighted action, the formula would of course be $t \mapsto e^{2i\pi t/d} x_2$). From that description of \mathbf{b} , it is classical (see for example [BM], page 92) that \mathbf{b}^d is central in $\pi_1(\mathbf{C}^r - \mathcal{H}, x_2)$, that it is pure, and that the image b of \mathbf{b} in W is $e^{2i\pi/d}$ -regular. Thus $(s_1 \dots s_n)^d$ is central and pure in $\pi_1(\mathbf{C}^r - \mathcal{H}, x_1)$, and c , which is conjugate to the image b of \mathbf{b} in W , is a $e^{2i\pi/d}$ -regular element. \square

The remainder of Theorem 0.1 can be deduced from what we have obtained so far. Let $F_{\mathbf{S}}$ be the free group on \mathbf{S} . Let $\mathcal{R} \subset F_{\mathbf{S}} \times F_{\mathbf{S}}$ be a set of relations describing $\mathbf{B}(W)$, i.e., such that the canonical morphism $F_{\mathbf{S}} \rightarrow \mathbf{B}(W)$ is a solution of the presentation universal problem associated with \mathcal{R} .

We may assume that \mathcal{R} is finite. Indeed, the group $\mathbf{B}(W)$ is finitely presentable (the classical method of van Kampen, see [C1], can be used to construct finite presentations), and given any presentation $(\mathbf{S}, \mathcal{R})$ of a finitely presentable group, if \mathbf{S} is finite, we may extract a finite subset $\mathcal{R}' \subset \mathcal{R}$ such that $(\mathbf{S}, \mathcal{R}')$ is already a presentation (see, for example, [C4], p. 23, proposition 17). The author thanks the referee who pointed out this important precision.

We have to prove that \mathcal{R} can be modified such that the relations are in the free monoid on the alphabet \mathbf{S} and, in any relation, the two sides have equal length.

Up to adding them (since, by the previous lemma, they are true), we may assume that \mathcal{R} contains the relations

$$(s_1 \dots s_n)^d = (s_2 \dots s_n s_1)^d, \quad (s_2 \dots s_n s_1)^d = (s_3 \dots s_n s_1 s_2)^d, \quad \dots$$

There is a natural notion of length in braid groups (see 2.19 in [BMR]). Namely, the map $\mathbf{B}(W) \rightarrow \mathbf{Z}$ which maps each generator-of-the-monodromy to 1 extends to a unique morphism $l : \mathbf{B}(W) \rightarrow \mathbf{Z}$. As each element of \mathbf{S} has length 1, any relation $R \in \mathcal{R}$

$$(R) \quad s_{i_1}^{\varepsilon_1} s_{i_2}^{\varepsilon_2} \dots s_{i_k}^{\varepsilon_k} = s_{j_1}^{\varepsilon'_1} s_{j_2}^{\varepsilon'_2} \dots s_{j_l}^{\varepsilon'_l},$$

(where the exponents are taken in $\{\pm 1\}$) must be homogeneous:

$$\sum_{p=1}^k \varepsilon_p = \sum_{q=1}^l \varepsilon'_q.$$

If any of the ε_p or ε'_q is negative, then by multiplying both sides of R by $(s_1 \dots s_n)^d$, one gets an equivalent relation in which the number of negative exponents has been decreased (use the fact that $(s_1 \dots s_n)^d$ is central thus can be moved anywhere inside a word, and that it can be written starting with any $\mathbf{s} \in \mathbf{S}$). After a certain number of iterations, one gets a relation R' between positive words, which by homogeneity must have equal length. The set $\mathcal{R}' := (\mathcal{R} - \{R\}) \cup \{R'\}$ still describes $\mathbf{B}(W)$. By iteration, this proves property (e). Now (f) is a consequence of (e), using Proposition 2.18 from [BMR].

4. Complements and applications

4.1. When the largest degree is regular

The smallest generating sets obtained from Theorem 0.1 are obtained with d maximal among regular degrees. The ideal situation is when the maximal degree is regular. By compiling the information in [C3], we obtain:

Proposition 4.1. *Let W be an irreducible complex reflection group. Let d_r be the largest degree of W .*

- *If W is a Coxeter group, or $G(d, 1, r)$, or $G(2d, 2, 2)$, or $G(e, e, r)$, or an exceptional group other than G_{15} , then d_r is regular.*
- *If W is $G(de, e, r)$ with $e > 2$, $d, r \geq 2$, or $G(2d, 2, r)$ with $d \geq 2$, $r > 2$, or the exceptional group G_{15} , then d_r is not regular.*

The good news is that d_r is regular for almost all exceptional groups, including G_{24} , G_{27} , G_{29} , G_{31} , G_{33} and G_{34} . No presentations are known for the six corresponding braid groups. Theorem 0.1 proves the existence of nice presentations. Moreover, in each case, $n = (N + N^*)/d_r$ is the minimal number of reflections needed, which is either r (for G_{24} , G_{27} , G_{29} , G_{33} and G_{34}) or $r + 1$ (for G_{31}), so the presentations are “optimal”.

A criterion for deciding whether a number is regular or not has been discovered by Lehrer and Springer ([LS], 5.1): d is regular if and only if it divides as many degrees as codegrees. Unfortunately, the “if” implication is proven by case-by-case inspection (the “only if” is a consequence of elementary properties of fake degrees). Using Theorem 0.1 and the Lehrer-Springer criterion, one sees that, in the Orlik-Solomon theorem quoted in the introduction, assertion (i) implies all the others. Use of the classification is hidden in the Lehrer-Springer criterion. A direct proof of this criterion, which is purely invariant-theoretic, would be desirable, as well as a direct proof of the fact that any irreducible complex reflection group has at least one regular degree (and that, if the rank is > 2 , one can find a non-central regular element).

As an example of a partial result which can be obtained without using the classification (not even hidden in the Lehrer-Springer criterion), let us mention the following. Let W be an irreducible complex reflection group of rank r . Assume that the discriminant of W is irreducible (this is equivalent, by classical results, to the assumption that the W -action is transitive on the set of reflecting hyperplanes; in other words, this is the analog for complex reflection groups of the ADE case for Weyl groups), and assume that $\frac{\partial \Delta_f}{\partial X_r} \neq 0$. Then, in the Orlik-Solomon theorem quoted in the introduction, assertion (ii) implies all the others. Indeed, since the discriminant Δ_f is irreducible, we can apply Theorem 2.5 (even without *a priori* knowing that d_r is regular) with a generic line of direction X_r . This yields a generating set for the braid group of W consisting of $\deg(\Delta_{f, X_r})$ generators-of-the-monodromy. Thus W can be generated by $\deg(\Delta_{f, X_r})$ reflections.

Using the assumption (ii) and weighted homogeneity, it is readily seen that $\deg(\Delta_{f,X_r}) \leq r$, the equality being only possible when Δ_f is monic in X_r . Since an irreducible group of rank r cannot be generated by less than r reflections, we see that $\deg(\Delta_{f,X_r}) = r$, which implies assertion (iv), and also that Δ_f is monic in X_r . By Lemma 1.6, d_r is regular. We can now apply Theorem 0.1, which gives (v). The assertions (i) and (iii) are easily obtained by considering the eigenvalues of a d_r -regular element of W in the canonical representation and in the dual representation.

4.2. When the largest degree is not regular

As an example, we discuss the case of the exceptional group G_{15} . The degrees are 12 and 24, the codegrees 24 and 0. As $d_1 = 12$ is regular, we may choose a X_1 -monic discriminant polynomial:

$$X_1^5 + \alpha X_1^3 X_2 + \beta X_1 X_2^2.$$

Up to replacing X_2 by $X_2 + \lambda X_1^2$, we may assume $\beta \neq 0$. Applying Theorem 0.1 with $d = 12$ yields a presentation with 5 generators. Using the refined Zariski theorem mentioned in 2.7, one gets a generating system associated with the dominant monomial $\beta X_1 X_2^2$, thus with 3 generators, which is optimal.

The other cases can be handled the same way, as discriminants of complex reflection groups happen to always have a dominant valuation monomial. More precisely, one can easily check on the classification that if W is an irreducible complex reflection group of rank r with non-regular highest degree, then, for a suitable system of basic invariants f , the discriminant of W has a factorization $\Delta_f = X_i Q$, where $i \in \{1, \dots, r - 1\}$ and Q is monic in X_r .

4.3. The minimum number of generators

The situation can be summarized by the following proposition, which partially follows from Theorem 0.1, Corollary 1.7 and Proposition 2.7, and partially from case-by-case analysis.

Proposition 4.2. *Let W be an irreducible complex reflection group. The following integers are equal:*

- *The minimum number of reflections needed to generate W .*
- *The minimum number of generators-of-the-monodromy needed to generate $\mathbf{B}(W)$.*
- *The valuation of the discriminant.*
- $\lceil (N + N^*)/d_r \rceil$.

The degree d_r is regular if and only if $(N + N^)/d_r$ is an integer; when this is the case, minimal presentations are described by Theorem 0.1.*

4.4. Coxeter groups

Let W be an irreducible Coxeter group, seen as a complex reflection group by complexifying the natural real reflection representation. Examples of regular elements are w_0 (the longest element) and c (a Coxeter element). The corresponding regular numbers are the degrees 2 and d_r (d_r is the Coxeter number, also denoted by h). Generating sets corresponding to the standard Brieskorn presentation for $\mathbf{B}(W)$ satisfy the properties of Theorem 0.1 with $d = d_r$.

Applying Theorem 0.1 with $d = 2$ yields generating sets with N elements. An example of such a presentation is the Birman-Ko-Lee presentation for $\mathbf{B}(\mathfrak{S}_n)$ ([BKL]). Presentations of $\mathbf{B}(W)$ with N generators, with W any irreducible Coxeter group, are constructed in [B]. They share with Brieskorn and Birman-Ko-Lee presentations the following property: the monoid presented with the same relations imbeds in the braid group.

Even when W is not a Coxeter group, any presentation obtained by Theorem 0.1 also defines a monoid presentation, since the relations are positive. The above embedding property is in general **not** satisfied.

4.5. Quotients and extensions of reflection groups

A certain class of surjective morphisms between reflection groups is studied in [BBR]. Let us prove that, as announced in [BBR], the morphisms are induced from surjective morphisms between the corresponding braid groups.

The situation is as follows. Let \tilde{W} be a reflection group acting on a complex vector space \tilde{V} . Let G be a normal subgroup of \tilde{W} . Let us assume that G contains no reflection. The action of \tilde{W} on \tilde{V} induces an action on the variety \tilde{V}/G . Choose an imbedding of \tilde{V}/G in its tangent space V at 0. Let us assume that G is “good” in \tilde{W} , as defined in [BBR], Sect. 3.1. This condition implies that the action on \tilde{V}/G is the restriction of an action of $W := \tilde{W}/G$ as a reflection group on V . Moreover, it is possible to choose a system of basic invariants $f = (f_1, \dots, f_r)$ for V (i.e., an identification $V/W \simeq \text{MaxSpec}(\mathbf{C}[X_1, \dots, X_r])$) and a subset $I \subset \{1, \dots, r\}$ such that $\tilde{f} := (\tilde{f}_i)_{i \in I}$ is a system of basic invariants for \tilde{W} , where \tilde{f}_i is the composition $\tilde{V} \rightarrow \tilde{V}/G \hookrightarrow V \xrightarrow{f_i} \mathbf{C}$.

Let $J := \{1, \dots, r\} - I$. The canonical embedding

$$\text{Spec}(\mathbf{C}[X_1, \dots, X_r]/(X_j)_{j \in J}) \hookrightarrow \text{Spec}(\mathbf{C}[X_1, \dots, X_r])$$

identifies \tilde{V}/\tilde{W} with the linear subspace of V/W defined by the equations $X_j = 0$ for $j \in J$. As observed in [BBR], paragraph 3.2.3, $\tilde{V}^{\text{reg}}/\tilde{W} = (V^{\text{reg}}/W) \cap \tilde{V}/\tilde{W}$.

Let us choose a basepoint $x_0 \in \tilde{V}^{\text{reg}}/\tilde{W}$. Define the braid groups $\mathbf{B}(\tilde{W})$ and $\mathbf{B}(W)$ with respect to x_0 .

Proposition 4.3. *The natural morphism $\mathbf{B}(\tilde{W}) \rightarrow \mathbf{B}(W)$ is surjective.*

Proof. It is enough to prove the proposition when \tilde{W} is irreducible: if \tilde{W} is reducible, it is a direct product of irreducible groups; except in a few degenerate and straightforward cases, G decomposes as a corresponding direct product; then W and $\mathbf{B}(W)$ also decompose, and the reduction to the irreducible case follows.

Choose $d = d_{i_0}$ a regular degree for \tilde{W} (one can check that any irreducible reflection group admits at least one regular degree).

We may assume that the discriminant $\tilde{\Delta}_{\tilde{f}}$ is monic in X_{i_0} . Indeed, by Lemma 1.6 (i), there exists a system of basic invariants \tilde{f}' of \tilde{W} such that the discriminant $\tilde{\Delta}_{\tilde{f}'}$ of \tilde{W} is monic in a variable X . The system \tilde{f}' is obtained from \tilde{f} by a certain sequence of algebraic substitutions. By performing the same substitutions among the corresponding elements of f (and leaving the remaining invariants unchanged), we get a new system f' which satisfies the monicity assumption (in addition to the defining properties of f).

The discriminant $\tilde{\Delta}_{\tilde{f}}$ of \tilde{W} is obtained from the discriminant Δ_f of W by “forgetting” all monomials involving one or more of the indeterminates $X_j, j \in J$ (this operation is the composition of $\mathbf{C}[X_1, \dots, X_r] \rightarrow \mathbf{C}[X_1, \dots, X_r]/(X_j)_{j \in J}$ with $\mathbf{C}[X_1, \dots, X_r]/(X_j)_{j \in J} \simeq \mathbf{C}[(X_i)_{i \in I}]$). As $\tilde{\Delta}_{\tilde{f}}$ is monic in X_{i_0} , and using weighted homogeneity, it is readily seen that Δ_f was already monic in X_{i_0} , and that $\tilde{\Delta}_{\tilde{f}}$ and Δ_f have the same degree in X_{i_0} .

Let $\tilde{\mathcal{H}}$ be the hypersurface in \tilde{V}/\tilde{W} defined by $\tilde{\Delta}_{\tilde{f}} = 0$. Let \mathcal{H} be the hypersurface in V/W defined by $\Delta_f = 0$. Let L be a $\tilde{\mathcal{H}}$ -generic line of direction X_{i_0} in \tilde{W}/\tilde{V} . The cardinal of $L \cap \tilde{\mathcal{H}} = L \cap \mathcal{H}$ is $\deg(\tilde{\Delta}_{\tilde{f}, X_{i_0}}) = \deg(\Delta_{f, X_{i_0}})$, thus L is generic relatively to \mathcal{H} in V/W . By Theorem 2.5, the inclusion $L - L \cap \mathcal{H} \hookrightarrow V/W - \mathcal{H}$ is π_1 -surjective. As it factors through $\tilde{V}/\tilde{W} - \tilde{\mathcal{H}} \hookrightarrow V/W - \mathcal{H}$, the latter map is also π_1 -surjective. \square

Let $\tilde{\mathcal{D}}$ be a diagram for $\mathbf{B}(\tilde{W})$, symbolizing a presentation by generators and relations as in Sect. 3, *i.e.*, corresponding to the choice of a generic line of regular direction and of a “planar spider”. The final argument from the above proof makes it clear that, by adding some relations to $\tilde{\mathcal{D}}$, one gets a diagram \mathcal{D} for $\mathbf{B}(W)$. The two diagrams are compatible, *i.e.*, the generators associated to $\tilde{\mathcal{D}}$ are sent to those of \mathcal{D} . This explains why the interpretation in [BBR] of $\tilde{W} \twoheadrightarrow W$ as a “morphism of diagrams” is actually valid at the level of braid groups.

References

[B] D. Bessis, The dual braid monoid, arXiv:math.GR/0101158
 [BBR] D. Bessis, C. Bonnafé, R. Rouquier, Quotients et extensions de groupes de réflexions, arXiv:math.GR/0102137
 [BKL] J. Birman, K.H. Ko, S.J. Lee, A new approach to the word and conjugacy problems in the braid groups, Adv. Math. **139** (1998), 322–353

- [BM] M. Broué, J. Michel, Sur certains éléments réguliers des groupes de Weyl et les variétés de Deligne-Lusztig associées, Proceedings de la Semaine de Luminy "Représentations des groupes réductifs finis", 73–139, Birkhäuser, 1996
- [BMR] M. Broué, G. Malle, R. Rouquier, Complex reflection groups, braid groups, Hecke algebras, *J. Reine Angew. Math.* **500** (1998), 127–190
- [C1] D. Chéniot, Une démonstration du théorème de Zariski sur les sections hyperplanes d'une hypersurface projective et du théorème de Van Kampen sur le groupe fondamental du complémentaire d'une courbe projective plane, *Compositio Math.* **27** (1973), 141–158
- [C2] C. Chevalley, Invariants of finite groups generated by reflections, *Amer. J. Math.* **77** (1955), 778–782
- [C3] A.M. Cohen, Finite complex reflection groups, *Ann. Sci. École Norm. Sup. (4)* **9** (1976), 379–436
- [C4] D.E. Cohen, Combinatorial group theory: a topological approach, London Math. Soc. Student Texts **14**, Cambridge University Press, 1989
- [G] C. Godbillon, Éléments de topologie algébrique, Hermann, Paris, 1971
- [HL] H.A. Hamm, Lê Dũng Tráng, Un théorème de Zariski du type de Lefschetz, *Ann. Sci. École Norm. Sup. (4)* **6** (1973), 317–355
- [LS] G.I. Lehrer, T.A. Springer, Reflection subquotients of unitary reflection groups, *Canad. J. Math.* **51** (1999), 1175–1193
- [OS] P. Orlik, L. Solomon, Unitary reflection groups and cohomology, *Invent. math.* **59** (1980), 77–94
- [S1] K. Saito, On a linear structure of the quotient variety by a finite reflection group, *Publ. Res. Inst. Math. Sci. Kyoto Univ.* **29** (1993), 535–579
- [S2] T.A. Springer, Regular elements of finite reflection groups, *Invent. math.* **25** (1974), 159–198
- [S3] R. Steinberg, Differential equations invariant under finite reflection groups, *Trans. Amer. Math. Soc.* **112** (1964), 392–400
- [ST] G.C. Shephard, J.A. Todd, Finite unitary reflection groups, *Canad. J. Math.* **6** (1954), 274–304